International Journal of Analysis and Applications
ISSN 2291-8639
Volume 3, Number 2 (2013), 112-118
http://www.etamaths.com

COMMON FIXED POINT THEOREM IN CONE METRIC SPACE FOR RATIONAL CONTRACTIONS

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Abstract. In this paper we prove the common fixed point theorem in cone metric space for rational expression in normal cone setting. Our results generalize the main result of Jaggi [10] and Dass, Gupta [11].

1. Introduction

The Banach contraction principle with rational expressions have been expanded and some fixed and common fixed point theorems have been obtained in [1], [2]. Huang and Zhang [3] initiated cone metric spaces, which is a generalization of metric spaces, by substituting the real numbers with ordered Banach spaces. They have considered convergence in cone metric spaces, introduced completeness of cone metric spaces, and proved a Banach contraction mapping theorem, and some other fixed point theorems involving contractive type mappings in cone metric spaces using the normality condition. Later, various authors have proved some common fixed point theorems with normal and non-normal cones in these spaces [4], [5], [6], [7], [8]. Quite recently Muhammad arshad et al.[9] have introduced almost Jaggi and Gupta contraction in Partially ordered metric space to prove the fixed point theorem. In this paper we prove the common fixed point theorem in cone metric space for rational expression in normal cone setting. Our results generalize the main result of Jaggi [10] and Dass, Gupta [11].

1.1. **Basic facts and definitions.** Let E be a real Banach space and P a subset of E. P is called a cone if and only if
(i) P is closed, nonempty, and $P \neq \{0\}$;
(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$
(iii) $P \cap (-P) = \{0\}$
Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, int$P$ denotes the interior of P.
The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq M \|y\|$.
The least positive number satisfying above is called the normal constant of P.
In the following we always suppose E is a Banach space, P is a cone in E with int$P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

2010 *Mathematics Subject Classification.* 54E35, 54E50, 37C25, 54H25.
Key words and phrases. Cone metric space; Rational expression; Fixed point.

Definition 1.1. [3] *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*
(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 1.1. [3] *Let $E = R^2$, $P = \{(x, y) \in E | x, y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.*

Definition 1.2. [3] *Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then*
(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$.
(ii) $\{x_n\}$ is a cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

Definition 1.3. [3] *Let (X, d) is said to be a complete cone metric space, if every cauchy sequence is convergent in X.*

2. Main Results

Definition 2.1. [9] *Let (X, d) be a cone metric space. A self mapping T on X is called an almost Jaggi contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y) + L min\{d(x, Ty), d(y, Tx)\} \tag{1}$$

for all $x, y \in X$, where $L \geq 0$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Theorem 2.1. *Let (X, d) be a complete cone metric space and P a normal cone with normal constant M. Let $T : X \to X$ be an almost Jaggi contraction, for all $x, y \in X$ where $L \geq 0$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point in X.*

Proof:
Choose $x_0 \in X$. Set $x_1 = Tx_0$, $x_n = Tx_{n-1}$

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \left[\frac{\alpha d(x_{n-1}, Tx_{n-1})\, d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n)\right. \\
&\quad \left. + L min\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}\right] \\
&\leq \left[\frac{\alpha d(x_{n-1}, x_n)\, d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n)\right. \\
&\quad \left. + L min\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}\right] \\
&\leq d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) \\
(1 - \alpha)\, d(x_n, x_{n+1}) &\leq \beta d(x_{n-1}, x_n) \\
d(x_n, x_{n+1}) &\leq \frac{\beta}{1 - \alpha} d(x_{n-1}, x_n)
\end{aligned}$$

$k = \frac{\beta}{1-\alpha}, \ \ \alpha + \beta < 1 \ \ 0 < k < 1$
and by induction,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq kd(x_{n-1}, x_n) \\ &. \\ &. \\ &\leq k^n d(x_0, x_1) \end{aligned}$$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+m-1}, x_m) \\ &\leq \left(k^n + k^{n+1} + ... + k^{n+m-1}\right) d(x_0, x_1) \\ &\leq \frac{k^n}{1-k} d(x_0, x_1) \end{aligned}$$

We get $\|d(x_n, x_m)\| \leq M\frac{k^n}{1-k} \|d(x_0, x_1)\|$ which implies that $d(x_n, x_m) \to 0$ as $n \to \infty$. Hence x_n is a cauchy sequence, so by completeness of X this sequence must be convergent in X.

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + d(x_{n+1}, Tu) \\ &\leq d(u, x_{n+1}) + d(Tx_n, Tu) \\ &\leq d(u, x_{n+1}) + \frac{\alpha d(x_n, Tx_n) d(u, Tu)}{d(x_n, u)} + \beta d(x_n, u) \\ &\quad +Lmin\{d(x_n, Tu), d(u, Tx_n)\} \\ &\leq d(u, x_{n+1}) + \frac{\alpha d(x_n, x_{n+1}) d(u, u)}{d(x_n, u)} + \beta d(x_n, u) \\ &\quad +Lmin\{d(x_n, u), d(u, x_{n+1})\} \\ &\leq d(u, x_{n+1}) + \beta d(x_n, u) + Lmin\{d(x_n, u), d(u, x_{n+1})\} \end{aligned}$$

So using the condition of normality of cone

$$\|d(u, T(u))\| \leq M(\|d(u, x_{n+1})\| + \beta \|d(x_n, u)\| + Lmin \|d(x_n, u), d(u, x_{n+1})\|)$$

As $n \to 0$ we have $\|d(u, T(u))\| \leq 0$. Hence we get $u = Tu$, u is a fixed point of T.

Definition 2.2. [10] *Let (X, d) be a cone metric space. A self mapping T on X is called Jaggi contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d(x, y)} + \beta d(x, y) \tag{2}$$

for all $x, y \in X$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Corollary 2.1. *Let (X, d) be a complete cone metric space and P a normal cone with normal constant M. Let $T : X \to X$ be a Jaggi contraction*

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d(x, y)} + \beta d(x, y) \tag{3}$$

for all $x, y \in X$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point in X.
Proof: *Set $L = 0$ in theorem 2.1.*

Theorem 2.2. *Let (X, d) be a complete cone metric space and P a normal cone with normal constant M. Suppose the mappings S, T is called an almost Jaggi contraction if it satisfies the following condition:*

$$d(Sx, Ty) \leq \frac{\alpha d(x, Sx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y) + Lmin\{d(x, Ty), d(y, Sx)\} \tag{4}$$

for all $x, y \in X$ where $L \geq 0$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then each of S, T has a unique fixed point and these two fixed points coincide.

Proof:
Let $x_1 \in S(x_o)$ and $x_2 = T(x_1)$ such that $x_{2n+1} = S(x_{2n})$, $x_{2n+2} = T(x_{2n+1})$

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \left[\frac{\alpha d(x_{2n}, Sx_{2n})\, d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1})\right. \\
&\quad \left. + Lmin\{d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n})\}\right] \\
&\leq \left[\frac{\alpha d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1})\right. \\
&\quad \left. + Lmin\{d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\}\right] \\
d(x_{2n+1}, x_{2n+2}) &\leq \alpha d(x_{2n+1}, x_{2n+2}) + \beta d(x_{2n}, x_{2n+1}) \\
(1-\alpha)\, d(x_{2n+1}, x_{2n+2}) &\leq \beta d(x_{2n}, x_{2n+1}) \\
d(x_{2n+1}, x_{2n+2}) &\leq \frac{\beta}{1-\alpha} d(x_{2n}, x_{2n+1})
\end{aligned}$$

$$d(x_{2n+1}, x_{2n+2}) \leq k d(x_{2n}, x_{2n+1}) \tag{5}$$

where $k = \frac{\beta}{1-\alpha}$, $\alpha + \beta < 1$

$$\begin{aligned}
d(x_{2n+3}, x_{2n+2}) &= d(S(x_{2n+2}), T(x_{2n+1})) \\
(6) \qquad &\leq \left[\frac{\alpha d(x_{2n+2}, Sx_{2n+2})\, d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n+2}, x_{2n+1})} + \beta d(x_{2n+2}, x_{2n+1})\right. \\
&\quad \left. + Lmin\{d(x_{2n+2}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n+2})\}\right] \\
&\leq \left[\frac{\alpha d(x_{2n+2}, x_{2n+3})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n+2}, x_{2n+1})} + \beta d(x_{2n+2}, x_{2n+1})\right. \\
&\quad \left. + Lmin\{d(x_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\}\right] \\
d(x_{2n+3}, x_{2n+2}) &\leq \alpha d(x_{2n+2}, x_{2n+3}) + \beta d(x_{2n+2}, x_{2n+1}) \\
(1-\alpha)\, d(x_{2n+3}, x_{2n+2}) &\leq \beta d(x_{2n+2}, x_{2n+1}) \\
d(x_{2n+3}, x_{2n+2}) &\leq \frac{\beta}{1-\alpha} d(x_{2n+2}, x_{2n+1}) \\
d(x_{2n+3}, x_{2n+2}) &\leq k d(x_{2n+2}, x_{2n+1})
\end{aligned}$$

$k = \frac{\beta}{1-\alpha}$, $\alpha + \beta < 1$
Add equation (6) and (7) we get

$$\begin{aligned}
\sum_{n=1}^{\infty} d(x_n, x_{n+1}) &\leq \sum_{n=1}^{\infty} k^n d(x_0, x_1) \\
&= \frac{k}{1-k} d(x_0, x_1)
\end{aligned} \tag{7}$$

We get $\|d(x_n, x_{n+1})\| \leq M\frac{k}{1-k}\|d(x_0, x_1)\|$ which implies that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is a cauchy sequence, so by completeness of X this sequence must be convergent in X. We shall prove that u is a common fixed point of S and T.

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{2n+1}) + d(x_{2n+1}, Tu) \\ &\leq d(u, x_{2n+1}) + d(Sx_{2n}, Tu) \\ &\leq d(u, x_{2n+1}) + \left[\frac{\alpha d(x_{2n}, Sx_{2n}) d(u, Tu)}{d(x_{2n}, u)} + \beta d(x_{2n}, u)\right. \\ &\quad \left. + Lmin\{d(x_{2n}, Tu), d(u, Sx_{2n})\}\right] \\ &\leq d(u, x_{2n+1}) + \left[\frac{\alpha d(x_{2n}, x_{2n+1}) d(u, u)}{d(x_{2n}, u)} + \beta d(x_{2n}, u)\right. \\ &\quad \left. + Lmin\{d(x_{2n}, u), d(u, x_{2n+1})\}\right] \\ &\leq d(u, x_{2n+1}) + \beta d(x_{2n}, u) + Lmin\{d(x_{2n}, u), d(u, x_{2n+1})\} \end{aligned}$$

So using the condition of normality of cone

$$\|d(u, T(u))\| \leq M(\|d(u, x_{2n+1})\| + \beta\|d(x_{2n}, u)\| + Lmin\|d(x_{2n}, u), d(u, x_{2n+1})\|)$$

As $n \to 0$ we have $\|d(u, T(u))\| \leq 0$. Hence we get $u = Tu$, u is a fixed point of T. Similarly

$$\begin{aligned} d(u, S(u)) &\leq d(u, x_{2n+2}) + d(x_{2n+2}, Su) \\ &\leq d(u, x_{2n+2}) + d(Su, Tx_{2n+1}) \\ &\leq d(u, x_{2n+2}) + \left[\frac{\alpha d(u, Su) d(x_{2n+1}, Tx_{2n+1})}{d(u, x_{2n+1})} + \beta d(u, x_{2n+1})\right. \\ &\quad \left. + Lmin\{d(u, Tx_{2n+1}), d(x_{2n+1}, Su)\}\right] \\ &\leq d(u, x_{2n+2}) + \left[\frac{\alpha d(u, u) d(x_{2n+1}, x_{2n+2})}{d(u, x_{2n+1})} + \beta d(u, x_{2n+1})\right. \\ &\quad \left. + Lmin\{d(u, x_{2n+2}), d(x_{2n+1}, u)\}\right] \\ &\leq d(u, x_{2n+2}) + \beta d(u, x_{2n+1}) + Lmin\{d(u, x_{2n+2}), d(x_{2n+1}, u)\} \end{aligned}$$

So using the condition normality of cone

$$\|d(u, S(u))\| \leq M(\|d(u, x_{2n+1})\| + \beta\|d(u, x_{2n+1})\| + Lmin\|d(u, x_{2n+2}), d(x_{2n+1}, u)\|)$$

As $n \to 0$ we have $\|d(u, S(u))\| \leq 0$. Hence we get $u = Su$, u is a fixed point of S.

Definition 2.3. [9] *Let (X, d) be a cone metric space. A self mapping T on X is called Dass and Gupta contraction if it satisfies the following condition:*
(8)

$$d(Tx, Ty) \leq \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + Lmin\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, where $L \geq 0$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Theorem 2.3. *Let (X, d) be a complete cone metric space and P a normal cone with normal constant M. Let $T : X \to X$ be a Dass and Gupta contraction, for all $x, y \in X$ where $L \geq 0$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point in X.*

Proof:
Choose $x_0 \in X$. Set $x_1 = Tx_0$, $x_n = Tx_{n-1}$

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \left[\frac{\alpha d(x_n, Tx_n)\left[1 + d(x_{n-1}, Tx_{n-1})\right]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n)\right. \\
&\quad \left. + Lmin\left\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\right\}\right] \\
&\leq \left[\frac{\alpha d(x_n, x_{n+1})\left[1 + d(x_{n-1}, x_n)\right]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n)\right. \\
&\quad \left. + Lmin\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\right\}\right] \\
(1-\alpha) d(x_n, x_{n+1}) &\leq \beta d(x_{n-1}, x_n) \\
d(x_n, x_{n+1}) &\leq \frac{\beta}{1-\alpha} d(x_{n-1}, x_n)
\end{aligned}$$

$k = \frac{\beta}{1-\alpha}$, $\alpha + \beta < 1$ $0 < k < 1$
and by induction,

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq kd(x_{n-1}, x_n) \\
&. \\
&. \\
&\leq k^n d(x_0, x_1) \\
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots + d(x_{n+m-1}, x_m) \\
&\leq \left(k^n + k^{n+1} + \ldots.. + k^{n+m-1}\right) d(x_0, x_1) \\
&\leq \frac{k^n}{1-k} d(x_0, x_1)
\end{aligned}$$

We get $\|d(x_n, x_m)\| \leq M\frac{k^n}{1-k}\|d(x_0, x_1)\|$ which implies that $d(x_n, x_m) \to 0$ as $n \to \infty$. Hence x_n is a cauchy sequence, so by completeness of X this sequence must be convergent in X.

$$\begin{aligned}
d(u, T(u)) &\leq d(u, x_{n+1}) + d(x_{n+1}, Tu) \\
&\leq d(u, x_{n+1}) + d(Tx_n, Tu) \\
&\leq d(u, x_{n+1}) + \frac{\alpha d(u, Tu)\left[1 + d(x_n, Tx_n)\right]}{1 + d(x_n, u)} + \beta d(x_n, u) \\
&\quad + Lmin\left\{d(x_n, Tx_n), d(x_n, Tu), d(u, Tx_n)\right\} \\
&\leq d(u, x_{n+1}) + \frac{\alpha d(u, u)\left[1 + d(x_n, x_{n+1})\right]}{1 + d(x_n, u)} + \beta d(x_n, u) \\
&\quad + Lmin\left\{d(x_n, x_{n+1}) d(x_n, u), d(u, x_{n+1})\right\} \\
&\leq d(u, x_{n+1}) + \beta d(x_n, u) + Lmin\left\{d(x_n, x_{n+1}), d(x_n, u), d(u, x_{n+1})\right\}
\end{aligned}$$

So using the condition normality of cone

$\|d(u, T(u))\| \leq M\left(\|d(u, x_{n+1})\| + \beta\|d(x_n, u)\| + Lmin\|d(x_n, x_{n+1}), d(x_n, u), d(u, x_{n+1})\|\right)$

As $n \to 0$ we have $\|d(u, T(u))\| \leq 0$. Hence we get $u = Tu$, u is a fixed point of T.

Corollary 2.2. *Let (X, d) be a complete cone metric space and P a normal cone with normal constant M. Let $T : X \to X$ a Dass, Gupta rationl contraction*

$$d(Tx, Ty) \leq \frac{\alpha d(y, Ty)\left[1 + d(x, Tx)\right]}{1 + d(x, y)} + \beta d(x, y) \tag{9}$$

for all $x, y \in X$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point in X.

Proof: *Set $L = 0$ in theorem 2.4.*

Acknowledegement

This research work has been supported by University Grants Commission (UGC - SAP II) New Delhi, India.

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