International Journal of Analysis and Applications
ISSN 2291-8639
Volume 12, Number 2 (2016), 157-162
http://www.etamaths.com

APPLICATION OF HYPERGEOMETRIC DISTRIBUTION SERIES ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

TRAILOKYA PANIGRAHI*

Abstract. The object of the present paper is to give some characterizations for hypergeometric distribution series to be in various subclasses of analytic functions.

1. Introduction

Let $\mathcal{A}$ denote the family of all functions f analytic in $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ with the usual normalization condition $f(0) = f'(0) - 1 = 0$. Thus f has the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{l=2}^{\infty} a_l z^l. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions f of the form (1) which are univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in $k-\mathcal{UCV}$, the class of k-uniformly convex function $(0 \leq k < \infty)$ if $f \in \mathcal{S}$ along with the property that for every circular arc γ contained in $\mathbb{U}$ with center ξ where $|\xi| < k$, the image curve $f(\gamma)$ is a convex arc. It is well-known that [5] $f \in k-\mathcal{UCV}$ if and only if the image of the function p, where $p(z) = 1 + \frac{zf''(z)}{f'(z)}$ $(z \in \mathbb{U})$ is a subset of the conic region

$$\Omega_k = \{w = u + iv : u^2 > k^2(u-1)^2 + k^2v^2, 0 \leq k < \infty\}. \tag{2}$$

The class $k-\mathcal{ST}$ consisting of k-uniformly starlike functions is defined via $k-\mathcal{UCV}$ by the Alexander transform i.e.

$$f \in k-\mathcal{ST} \Longleftrightarrow g \in k-\mathcal{UCV} \text{ where } g(z) = \int_0^z \frac{f(t)}{t}dt.$$

The class $k-\mathcal{ST}$ and its properties were investigated in [6]. The analytic characterization of $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ are given as below:

$$k-\mathcal{UCV} = \{f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| (z \in \mathbb{U})\} \tag{3}$$

and

$$k-\mathcal{ST} = \{f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| (z \in \mathbb{U})\} \tag{4}$$

Note that for $k = 0$ and $k = 1$, we get $0-\mathcal{UCV} = \mathcal{K}$, $0-\mathcal{ST} = \mathcal{S}^*$, $1-\mathcal{UCV} = \mathcal{UCV}$ and $1-\mathcal{ST} = \mathcal{SP}$, where $\mathcal{K}$, $\mathcal{S}^*$, $\mathcal{UCV}$, $\mathcal{SP}$ are respectively the familiar classes of univalent convex functions, univalent starlike functions [3], uniformly convex functions [4] (also, see [7, 12]) and parabolic starlike functions [12].

For two analytic functions f and g in $\mathbb{U}$, the function f is said to be subordinate to g or g is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$ with $|w| \leq |z|$ such that $f(z) = g(w(z))$. In such case, we write $f \prec g$ or $f(z) \prec g(z)$. If the function g is univalent in $\mathbb{U}$, then $f \prec g \Longleftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (see, for detail [8]).

2010 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. analytic functions; k-uniformly convex functions; k-uniformly starlike functions; hypergeometric distribution series.

Making use of subordination between analytic functions, Aouf [1] introduced and studied the class $\mathcal{R}^{\lambda}(A,B,\alpha)$ as follows:

Definition 1.(see[1, with p=1]) For $-1 \le A < B \le 1,\ |\lambda| < \frac{\pi}{2}$ and $0 \le \alpha < 1$, we say that a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{R}^{\lambda}(A,B,\alpha)$ if it satisfies the following subordination condition:

$$e^{i\lambda} f'(z) \prec cos\lambda\big[(1-\alpha)\frac{1+Az}{1+Bz}+\alpha\big] + isin\lambda. \tag{5}$$

The subordination (5) is equivalent to the inequality (6) given below:

$$\left|\frac{e^{i\lambda}(f'(z)-1)}{Be^{i\lambda}f'(z)-[Be^{i\lambda}+(A-B)(1-\alpha)cos\lambda]}\right| < 1 \quad (z \in \mathbb{U}). \tag{6}$$

For particular values of parameters A, B, α and λ, we obtain various subclasses of analytic functions studied by different researchers (for details, see [2]).

In 1998, Ponnusamy and Ronning [10] introduced and studied the classes $\mathcal{S}^*_{\beta}$ and $\mathcal{C}_{\beta}$ consisting of functions of the form (1) satisfying the following conditions:

$$\mathcal{S}^*_{\beta} = \big\{f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| < \beta \ (z \in \mathbb{U},\ \beta > 0)\big\}, \tag{7}$$

and

$$\mathcal{C}_{\beta} = \big\{f \in \mathcal{A} : \left|\frac{zf''(z)}{f'(z)}\right| < \beta \ (z \in \mathbb{U},\ \beta > 0)\big\}. \tag{8}$$

It is worthy to mention here that

$$f \in \mathcal{C}_{\beta} \Longleftrightarrow zf' \in \mathcal{S}^*_{\beta} \ (\beta > 0).$$

Recently, we introduced a new series $H(M,N,n;z)$ whose coefficient are probabilities of hypergeometric distribution as follows:

$$H(M,N,n;z) = z + \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} \binom{M}{l-1}\binom{N-M}{n-l+1} z^l. \tag{9}$$

Let us define the linear operator $\mathcal{J}(M,N,n) : \mathcal{A} \longrightarrow \mathcal{A}$ given by

$$\mathcal{J}(M,N,n)f(z) = H(M,N,n;z) \star f(z) = z + \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} \binom{M}{l-1}\binom{N-M}{n-l+1} a_l z^l \quad (z \in \mathbb{U}), \tag{10}$$

where $\star$ denote the convolution or Hadamard product between two analytic functions.

Motivated by the works of [9, 10, 13], in this paper we investigate some characterization for hypergeometric distribution series to be in the subclasses $\mathcal{S}^*_{\beta}$ and $\mathcal{C}_{\beta}$ of analytic functions.

2. Preliminaries lemmas

To prove our main results, we need the following lemmas.

Lemma 1. (see [1], Theorem 4 with p=1) A sufficient condition for $f(z)$ defined by (1) to be in the class $R^{\lambda}(A,B,\alpha)$ is

$$\sum_{l=2}^{\infty} l(1+|B|)|a_l| \le (B-A)(1-\alpha)cos\lambda. \tag{11}$$

Lemma 2. (see [6]) Let $f(z) \in \mathcal{A}$. If for some k, the following inequality

$$\sum_{l=2}^{\infty}(l+k(l-1))|a_l| \le 1 \tag{12}$$

holds true, then $f \in k-\mathcal{ST}$.

Lemma 3. (see [5, 11]) A function $f \in \mathcal{A}$ of the form (1) is in $k-\mathcal{UCV}$ if it satisfies the condition

$$\sum_{l=2}^{\infty} l[l(k+1)-k]|a_l| \le 1. \tag{13}$$

Another sufficient condition for the class $k-\mathcal{UCV}$ is given in [7] as follows:
Lemma 4. (see [7, 11]) Let $f \in \mathcal{S}$ be of the form (1). If for some k $(0 \leq k < \infty)$, the inequality

$$\sum_{l=2}^{\infty} l(l-1)|a_l| \leq \frac{1}{k+2}, \tag{14}$$

holds true, then $f \in k-\mathcal{UCV}$. The number $\frac{1}{k+2}$ cannot be increased.
Lemma 5. (see [11]) Let $f \in \mathcal{A}$ be of the form (1). If the inequality

$$\sum_{l=2}^{\infty}[\beta+l-1]|a_l| \leq \beta \ (\beta > 0), \tag{15}$$

is satisfied, then $f \in \mathcal{S}^*_\beta$.
Lemma 6. (see [11]) Let $f \in \mathcal{A}$ be of the form (1). If

$$\sum_{l=2}^{\infty} l[\beta+l-1]|a_l| \leq \beta \quad (\beta > 0), \tag{16}$$

then $f \in C_\beta$.
Lemma 7. (see [1], Theorem 1 with p=1) Let the function $f(z)$ defined by (1) be in the class $\mathcal{R}^\lambda(A,B,\alpha)$, then

$$|a_l| \leq \frac{(B-A)(1-\alpha)cos\lambda}{l} \quad (l \geq 2). \tag{17}$$

3. Main Results

Unless otherwise stated, we assume throughout the sequel that $-1 \leq A < B \leq 1, |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1$.
Theorem 1. Let $k \geq 0$. If the inequality

$$\frac{1}{\binom{N}{n}}\left[M(k+1)A_1 - \binom{N-M}{n}\right] \leq \frac{sec\lambda}{(B-A)(1-\alpha)} - 1, \tag{18}$$

where

$$A_1 = \sum_{l=2}^{\infty}\binom{M-1}{l-2}\binom{N-M}{n-l+1} \tag{19}$$

is satisfied, then $\mathcal{J}(M,N,n)$ maps the class $\mathcal{R}^\lambda(A,B,\alpha)$ into $k-\mathcal{UCV}$.

Proof. Let the function f given by (1) be a member of $\mathcal{R}^\lambda(A,B,\alpha)$. By (10), we have

$$\mathcal{J}(M,N,n)f(z) = z + \frac{1}{\binom{N}{n}}\sum_{l=2}^{\infty}\binom{M}{l-1}\binom{N-M}{n-l+1}a_l z^l.$$

In view of Lemma 3, it is sufficient to show that

$$\frac{1}{\binom{N}{n}}\sum_{l=2}^{\infty} l[l(k+1)-k]\binom{M}{l-1}\binom{N-M}{n-l+1}|a_l| \leq 1.$$

By making use of Lemma 7, it is again sufficient to prove that

$$P_1 = \frac{1}{\binom{N}{n}}\sum_{l=2}^{\infty}[l(k+1)-k]\binom{M}{l-1}\binom{N-M}{n-l+1} \leq \frac{sec\lambda}{(B-A)(1-\alpha)}. \tag{20}$$

Now

$$P_1 = \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} [(l-1)(k+1)+1] \binom{M}{l-1} \binom{N-M}{n-l+1}$$

$$= \frac{1}{\binom{N}{n}} \left[\sum_{l=2}^{\infty} (k+1) \frac{M!}{(l-2)!(M-l+1)!} \binom{N-M}{n-l+1} + \sum_{l=2}^{\infty} \binom{M}{l-1} \binom{N-M}{n-l+1} \right]$$

$$= \frac{M(k+1)}{\binom{N}{n}} \sum_{l=2}^{\infty} \binom{M-1}{l-2} \binom{N-M}{n-l+1} + \frac{1}{\binom{N}{n}} \left[\sum_{l=0}^{\infty} \binom{M}{l} \binom{N-M}{n-l} - \binom{N-M}{n} \right]$$

$$= \frac{M(k+1)}{\binom{N}{n}} A_1 - \frac{\binom{N-M}{n}}{\binom{N}{n}} + 1,$$

where A_1 is defined as in (19).
Thus, in view of (20), if the inequality (18) is satisfied, then $\mathcal{J}(M,N,n)(f) \in k-\mathcal{UCV}$ as asserted. The proof of Theorem 1 is complete. □

Theorem 2. If the inequality

$$\frac{M}{\binom{N}{n}} A_1 \leq \frac{sec\lambda}{(k+2)(B-A(1-\alpha)} \tag{21}$$

is satisfied, then $\mathcal{J}(M,N,n)$ maps the class $\mathcal{R}^\lambda(A,B,\alpha)$ into $k-\mathcal{UCV}$.

Proof. Let the function f given by (1) be a member of $\mathcal{R}^\lambda(A,B,\alpha)$. By virtue of Lemma 4, it is sufficient to show that

$$\frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} l(l-1) \binom{M}{l-1} \binom{N-M}{n-l+1} |a_l| \leq \frac{1}{k+2}$$

Using the coefficient estimate (17), it is again sufficient to show that

$$P_2 = \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} (l-1) \binom{M}{l-1} \binom{N-M}{n-l+1} \leq \frac{sec\lambda}{(k+2)(B-A)(1-\alpha)}. \tag{22}$$

Now,

$$P_2 = \frac{M}{\binom{N}{n}} \sum_{l=2}^{\infty} \binom{M-1}{l-2} \binom{N-M}{n-l+1} = \frac{M}{\binom{N}{n}} A_1.$$

In view of (22), if the condition (21) is satisfied, then $\mathcal{J}(M,N,n)(f) \in k-\mathcal{UCV}$ as asserted. This ends the proof of Theorem 2. □

Theorem 3. If the inequality

$$(1+k) - \frac{(1+k)}{\binom{N}{n}} \binom{N-M}{n} - \frac{k}{\binom{N}{n}(M+1)} B_1 \leq \frac{sec\lambda}{(B-A)(1-\alpha)}, \tag{23}$$

where

$$B_1 = \sum_{l=2}^{\infty} \binom{M+1}{l} \binom{N-M}{n-l+1} \tag{24}$$

is satisfied, then $\mathcal{J}(M,N,n)$ maps the class $\mathcal{R}^\lambda(A,B,\alpha)$ into $k-\mathcal{ST}$.

Proof. Let the function f given by (1) be a member of $\mathcal{R}^\lambda(A,B,\alpha)$. By virtue of Lemma 2, it is sufficient to show that

$$\frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} [l+k(l-1)] \binom{M}{l-1} \binom{N-M}{n-l+1} |a_l| \leq 1.$$

Using the coefficient estimate (17), it is again sufficient to show that

$$P_3 = \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} \frac{[l+k(l-1)]}{l} \binom{M}{l-1} \binom{N-M}{n-l+1} \leq \frac{sec\lambda}{(B-A)(1-\alpha)} \tag{25}$$

Now,

$$P_3 = \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} \left[1 + (1 - \frac{1}{l})k\right] \binom{M}{l-1} \binom{N-M}{n-l+1} = \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} \left[(1+k) - \frac{k}{l}\right] \binom{M}{l-1} \binom{N-M}{n-l+1}$$
$$= (1+k)\left[1 - \frac{\binom{N-M}{n}}{\binom{N}{n}}\right] - \frac{k}{(M+1)\binom{N}{n}} B_1.$$

Therefore, in view of (25), if the inequality (23) is satisfied, then $\mathcal{J}(M,N,n)(f) \in k - \mathcal{ST}$ as asserted. This complete the proof of Theorem 3. □

Theorem 4. If $f \in \mathcal{R}^\lambda(A,B,\alpha)$ and the inequality

$$1 - \frac{\binom{N-M}{n}}{\binom{N}{n}} \leq \frac{1}{1+|B|}, \tag{26}$$

is satisfied, then $\mathcal{J}(M,N,n)(f) \in \mathcal{R}^\lambda(A,B,\alpha)$.

Proof. Let the function $f \in \mathcal{A}$ given by (1) be a member of $\mathcal{R}^\lambda(A,B,\alpha)$. By virtue of Lemma 1 and the coefficient inequality (17) it is sufficient to show that

$$P_4 = \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} \binom{M}{l-1} \binom{N-M}{n-l+1} \leq \frac{1}{1+|B|}. \tag{27}$$

Now P_4 is equivalently written as

$$P_4 = \sum_{l=1}^{\infty} \frac{\binom{M}{l}\binom{N-M}{n-l}}{\binom{N}{n}} = 1 - \frac{\binom{N-M}{n}}{\binom{N}{n}}$$

Thus, in view of (27), if the inequality (26) is satisfied, then $\mathcal{J}(M,N,n)(f) \in \mathcal{R}^\lambda(A,B,\alpha)$. The proof of Theorem 4 is complete. □

Theorem 5. Let $\beta > 0$, $f \in \mathcal{R}^\lambda(A,B,\alpha)$ and the inequality

$$\frac{\beta - 1}{(M+1)\binom{N}{n}} B_1 - \frac{\binom{N-M}{n}}{\binom{N}{n}} \leq \frac{\beta sec\lambda}{(B-A)(1-\alpha)} - 1, \tag{28}$$

is satisfied, then $\mathcal{J}(M,N,n)(f) \in \mathcal{S}^*_\beta$.

Proof. By making use of Lemma 5, it is sufficient to show that

$$\sum_{l=2}^{\infty} (\beta + l - 1) \frac{\binom{M}{l-1}\binom{N-M}{n-l+1}}{\binom{N}{n}} |a_l| \leq \beta.$$

Since $f \in \mathcal{R}^\lambda(A,B,\alpha)$, using the coefficient estimate (17), it is sufficient to show that

$$P_5 = \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} \left[\frac{\beta + l - 1}{l}\right] \binom{M}{l-1} \binom{N-M}{n-l+1} \leq \frac{\beta sec\lambda}{(B-A)(1-\alpha)}. \tag{29}$$

Now,

$$P_5 = \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} \left(\frac{\beta - 1}{l}\right) \binom{M}{l-1} \binom{N-M}{n-l+1} + \frac{1}{\binom{N}{n}} \sum_{l=2}^{\infty} \binom{M}{l-1} \binom{N-M}{n-l+1}$$
$$= \frac{\beta - 1}{(M+1)\binom{N}{n}} B_1 - \frac{\binom{N-M}{n}}{\binom{N}{n}} + 1.$$

Thus, in view of (29), if the inequality (28) is satisfied, then $\mathcal{J}(M,N,n)(f) \in \mathcal{S}^*_\beta$ as asserted. This proof the Theorem 5. □

Theorem 6. Let $\beta > 0$. If the inequality

$$\frac{1}{\binom{N}{n}}\left[MA_1 - \beta\binom{N-M}{n}\right] \le \beta\left[\frac{sec\lambda}{(B-A)(1-\alpha)} - 1\right] \tag{30}$$

is satisfied, then $\mathcal{J}(M,N,n)$ maps the class $\mathcal{R}^\lambda(A,B,\alpha)$ into $\mathcal{C}_\beta$.

Proof. In view of Lemma 6, it is sufficient to show that

$$\frac{1}{\binom{N}{n}}\sum_{l=2}^{\infty} l[\beta+l-1]\binom{M}{l-1}\binom{N-M}{n-l+1}|a_l| \le \beta.$$

Using coefficient inequality (17), it is enough to show that

$$P_6 = \frac{1}{\binom{N}{n}}\sum_{l=2}^{\infty}[\beta+l-1]\binom{M}{l-1}\binom{N-M}{n-l+1} \le \frac{\beta sec\lambda}{(B-A)(1-\alpha)}. \tag{31}$$

Now the expression P_4 can be equivalently written as

$$\begin{aligned} P_6 &= \frac{\beta}{\binom{N}{n}}\sum_{l=2}^{\infty}\binom{M}{l-1}\binom{N-M}{n-l+1} + \sum_{l=2}^{\infty}\frac{M!}{(l-2)!(M-l+1)!}\binom{N-M}{n-l+1} \\ &= \beta - \beta\frac{\binom{N-n}{n}}{\binom{N}{n}} + \frac{M}{\binom{N}{n}}\sum_{l=2}^{\infty}\binom{M-1}{l-2}\binom{N-M}{n-l+1} \\ &= \frac{M}{\binom{N}{n}}A_1 - \frac{\beta\binom{N-M}{n}}{\binom{N}{n}} + \beta. \end{aligned}$$

Thus, in view of (31) if the inequality (30) is satisfied, then $\mathcal{J}(M,N,n)(f) \in \mathcal{C}_\beta$ as desired. The proof of Theorem 6 is thus completed. □

References

[1] M. K. Aouf, On certain subclass of analytic p-valent functions of order alpha, Rend. Mat., 7(8) (1988), 89-104.

[2] M. K. Aouf, A. O. Mostafa and H. M. Zayed, Some constraints of hypergeometric functions to belong to certain subclasses of analytic functions, J. Egyptian Math. Soc., 24(3) (2016), 361-366.

[3] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Vol 259, Springer-Verlag, New York, 1983.

[4] A. W. Goodman, On uniformly convex functions, Ann. Polon. Math, 56(1)(1991), 87-92.

[5] S. Kanas and A. Wisnioska, Conic regions and k-uniform convexity, J. Comput. Appl. Math., 105(1999), 327-336.

[6] S. Kanas and A. Wisnioska, Conic domains and starlike functions, Rev. Roumaine Math. Pures Appl., 45(4) (2000), 647-657.

[7] W. Ma and D. Minda, Uniformly convex functions, Ann. Polon. Math., 57(2) (1992), 165-175.

[8] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, in: Monographs and Text Books in Pure and Applied Mathematics, 225, Marcel Dekker, New York, 2000.

[9] A. K. Mishra and T. Panigrahi, Class-mapping properties of the Hohlov operator, Bull. Korean Math. Soc., 48(1) (2011), 51-65.

[10] S. Ponnusamy and F. Ronning, Starlikeness properties for convolution involving hypergeometric series, Ann. Univ. Mariae Curie-Sklodowska I, II 1(16) (1998), 141-155.

[11] S. Porwal and S. Kumar, Confluent hypergeometric distribution and its applications on certain classes of univalent functions, Afr. Mat., (2016), doi:10.1007/s13370-016-0422-3.

[12] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc., 118(1) (1993), 189-196.

[13] A. K. Sharma, S. Porwal and K. K. Dixit , Class-mapping properties of convolution involving certain univalent function associated with hypergeometric function, Electronic J. Math. Anal. Appl., 1(2) (2013), 326-333.

Department of Mathematics, School of Applied Sciences, KIIT University, Bhubaneswar-751024, Orissa, India

*Corresponding author: trailokyap6@gmail.com