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APPROXIMATION THEOREMS FOR $q-$ ANALOUGE OF A LINEAR POSITIVE OPERATOR BY A. LUPAS

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Abstract. The purpose of the present paper is to introduce $q-$ analouge of a sequence of linear and positive operators which was introduced by A. Lupas [1]. First, we estimate moments of the operators and then prove a basic convergence theorem. Next, a local direct approximation theorem is established. Further, we study the rate of convergence and point-wise estimate using the Lipschitz type maximal function.

1. Introduction

At the International Dortmund Meeting held in Written (Germany, March, 1995), A. Lupas [1] introduced the following Linear positive operators:

$$L_n(f;x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f\left(\frac{k}{n}\right), x \geq 0. \tag{1}$$

with $f : [0,\infty] \to \mathbb{R}$. If we impose that $L_n e_1 = e_1$ we find that $a = 1/2$. Therefore operator (1) becomes

$$L_n(f;x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), x \geq 0,$$

where

$$(\alpha)_0 = 1,\ (\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1), k \geq 1.$$

The $q-$ analouge of the above operators is defined as:

$$L_{n,q}(f;x) = 2^{-[n]_q x} \sum_{k=0}^{\infty} \frac{([n]_q x)_k}{2^k [k]_q!} f\left(\frac{[k]_q}{[n]_q}\right), x \geq 0,$$

We denote $C_B[0,\infty)$ the space of real valued bounded continuous function f on the interval $[0,\infty)$, the norm on the space is defined as

$$\|f\| = \sup_{0 \leq x < \infty} |f(x)|.$$

Let $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. The Peetre's $K-$ functional is defined as

$$K_2(f,\delta) = \inf_{g \in W^2} \{\|f-g\| + \delta \|g''\|\},$$

where $\delta > 0$.
For $f \in C_B[0,\infty)$ a usual modulus of continuity is given by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{0\leq x<\infty} |f(x+h) - f(x))|.$$

The second order modulus of smoothness is given by

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{0\leq x<\infty} |f(x+2h) - 2f(x+h) + f(x))|.$$

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By [[3], p.177, Theorem 2.4] there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}).$$

In recent years, many results about the generalization of linear positive operators have been obtained by several mathematicians ([6]-[17]).

2. Moment estimates

Lemma 1. *The following relations hold:*

$$L_{n,q}(1;x) = 1, L_{n,q}(t;x) = x \text{ and } L_{n,q}(t^2;x) = qx^2 + \frac{1+q}{[n]}x.$$

Proof. We have

$$L_{n,q}(1;x) = 2^{-[n]_q x}\sum_{k=0}^{\infty}\frac{([n]_q x)_k}{2^k[k]_q!} = 1$$

Now,

$$\begin{aligned}
L_{n,q}(t;x) &= 2^{-[n]_q x}\sum_{k=0}^{\infty}\frac{([n]_q x)_k}{2^k[k]_q!}\frac{[k]_q}{[n]_q} \\
&= 2^{-[n]_q x}\sum_{k=0}^{\infty}\frac{([n]_q x)_k}{2^k[k-1]_q![n]_q} \\
&= \frac{2^{-[n]_q x-1}}{[n]_q}\sum_{k=1}^{\infty}\frac{[n]_q x([n]_q x+1)_{k-1}}{2^{k-1}[k-1]_q!} \\
&= 2^{-[n]_q x-1}x\sum_{k=1}^{\infty}\frac{([n]_q x+1)_{k-1}}{2^{k-1}[k-1]_q!} \\
&= 2^{-[n]_q x-1}x\sum_{k=0}^{\infty}\frac{([n]_q x+1)_k}{2^k[k]_q!} = x.
\end{aligned}$$

Next,

$$\begin{aligned}
L_{n,q}(t^2;x) &= 2^{-[n]_q x}\sum_{k=0}^{\infty}\frac{([n]_q x)_k}{2^k[k]_q!}\frac{[k]_q^2}{[n]_q^2} \\
&= 2^{-[n]_q x}\sum_{k=0}^{\infty}\frac{[n]_q x([n]_q x+1)_{k-1}}{2^k[k]_q[k-1]_q!}\frac{[k]_q^2}{[n]_q^2} \\
&= 2^{-[n]_q x-1}x\sum_{k=1}^{\infty}\frac{([n]_q x+1)_{k-1}}{2^{k-1}[k-1]_q!}\frac{[k]_q}{[n]_q} \\
&= \frac{2^{-[n]_q x-1}x}{[n]_q}\sum_{k=1}^{\infty}\frac{([n]_q x+1)_{k-1}[k]_q}{2^{k-1}[k-1]_q!} \\
&= \frac{2^{-[n]_q x-1}x}{[n]_q}\sum_{k=0}^{\infty}\frac{([n]_q x+1)_k[k+1]_q}{2^k[k]_q!} \\
&= \frac{2^{-[n]_q x-1}x}{[n]_q}\sum_{k=0}^{\infty}\frac{([n]_q x+1)_k(1+q[k]_q)}{2^k[k]_q!} \\
&= \frac{2^{-[n]_q x-1}x}{[n]_q}\sum_{k=0}^{\infty}\frac{([n]_q x+1)_k}{2^k[k]_q!} \\
&+ \frac{2^{-[n]_q x-1}x}{[n]_q}\sum_{k=0}^{\infty}\frac{([n]_q x+1)_k q[k]_q}{2^k[k]_q!} \\
&= I_1 + I_2, \text{say}.
\end{aligned}$$

We find that $I_1 = \frac{x}{[n]_q}$.
Now,

$$\begin{aligned} I_2 &= \frac{2^{-[n]_q x-1}x}{[n]_q}\sum_{k=0}^{\infty}\frac{([n]_q x+1)_k q[k]_q}{2^k[k]_q!} \\ &= \frac{2^{-[n]_q x-2}qx}{[n]_q}\sum_{k=1}^{\infty}\frac{([n]_q x+1)([n]_q x+2)_{k-1}}{2^{k-1}[k-1]_q!} \\ &= \frac{2^{-[n]_q x-2}qx([n]_q x+1)}{[n]_q}\sum_{k=1}^{\infty}\frac{([n]_q x+2)_{k-1}}{2^{k-1}[k-1]_q!} \\ &= \frac{2^{-[n]_q x-2}qx([n]_q x+1)}{[n]_q}\sum_{k=0}^{\infty}\frac{([n]_q x+2)_k}{2^k[k]_q!} = \frac{qx([n]_q x+1)}{[n]_q}. \end{aligned}$$

Hence, on combining I_1 and I_2, we get

$$L_{n,q}(t^2;x) = \frac{(1+q)x}{[n]_q} + qx^2.$$

□

Let us define mth order moment by $\psi_{n,m}(q;x) = L_{n,q}((t-x)^m;x)$.

Lemma 2. *Let $0 < q < 1$, then for $x \in [0,\infty)$ we have*

$$\psi_{n,1}(q;x) = 0 \text{ and } \psi_{n,2}(q;x) = \frac{x([2]-(1-q)[n]_q x)}{[n]_q}.$$

Proof. We have

$$\psi_{n,1}(q;x) = L_{n,q}(t-x;x) = 0.$$

Now,

$$\begin{aligned} \psi_{n,2}(q;x) &= L_{n,q}((t-x)^2;x) \\ &= L_{n,q}(t^2+x^2-2tx;x) \\ &= \frac{(1+q)x}{[n]_q} + (q-1)x^2. \end{aligned}$$

□

3. Basic Pointwise Convergence

The operators $L_{n,q}$ do not satisfy the conditions of the Bohman-Korovkin theorem in case $0 < q < 1$. To make this theorem applicable, we can choose a sequence (q_n) in place of the number q such that $q_n \to 1$ and $q_n^n \to 0$ as $n \to \infty$. With this modification we obtain the following Korovkin type result:

Theorem 1. *Let $f \in C_B[0,\infty)$ and q_n be a real sequence in $(0,1)$ such that $q_n \to 1$ and $q_n^n \to 0$ as $n \to \infty$. Then, for each $x \in [0,\infty)$ we have*

$$\lim_{n\to\infty} L_{n,q_n}(f;x) = f(x).$$

Proof. The proof is based on the well known Korovkin theorem regarding the convergence of a sequence of linear positive operators. So, it is enough to prove the conditions

$$\lim_{n\to\infty} L_{n,q_n}(t^m;x) = x^m, m = 0,1,2.$$

Now, using Lemma 1 we obtain

$$\lim_{n\to\infty} L_{n,q_n}(1;x) = 1,$$

$$\lim_{n\to\infty} L_{n,q_n}(t;x) = x$$

and

$$\begin{aligned}\lim_{n\to\infty} L_{n,q_n}(t;x) &= \lim_{n\to\infty} q_n x^2 + \frac{1+q_n}{[n]_{q_n}} x \\ &= x^2.\end{aligned}$$

This completes the proof. □

4. Direct results

Theorem 2. *Let $f \in C_B[0,\infty)$ and $q \in (0,1)$. Then, for each $x \in [0,\infty)$ and $n \in \mathbb{N}$ there exists an absolute constant $C > 0$ such that*

$$|L_{n,q}(f;x) - f(x)| \leqslant C\omega_2\left(f, \sqrt{\frac{x([2]-(1-q)[n]_q x)}{[n]_q}}\right).$$

Proof. Let $g \in W^2$ and $x, t \in [0,\infty)$. Using Taylor's expansion we can write

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-v)g''(v)dv.$$

On application of Lemma 2 we obtain

$$L_{n,q}\big(g(t);x) - g(x)\big) = L_{n,q}\left(\int_x^t (t-v)g''(v)dv; x\right).$$

Now, we have $\left|\int_x^t (t-v)g''(v)dv\right| \leq (t-x)^2\|g''\|$. Therefore

$$|L_{n,q}(g(t);x) - g(x)| \leq L_{n,q}\left((t-x)^2;x\right)\|g''\| = \frac{x([2]-(1-q)[n]_q x)}{[n]_q}\|g''\|.$$

By Lemma 1, we have

$$|L_{n,q}(f;x)| \leq 2^{-[n]_q x}\sum_{k=0}^{\infty}\frac{([n]_q x)_k}{2^k [k]_q!}\left|f\left(\frac{[k]_q}{[n]_q}\right)\right| \leq \|f\|.$$

Thus

$$\begin{aligned}|L_{n,q}(f;x) - f(x)| &\leq |L_{n,q}(f-g;x) - (f-g)(x)| + |L_{n,q}(g;x) - g(x)| \\ &\leq 2\|f-g\| + \tfrac{x([2]-(1-q)[n]_q x)}{[n]_q}\|g''\|.\end{aligned}$$

At last, taking the infimum over all $g \in W^2$ and on application of the inequality $K_2(f,\delta) \leq C\omega_2(f,\delta^{1/2}), \delta > 0$, we get the required result. This completes the proof of the theorem.

□

5. Pointwise Estimates

In this section, we obtain some pointwise estimates of the rate of convergence of the $q-$ Baskakov-Durrmeyer operators. First, we discuss the relationship between the local smoothness of f and the local approximation.

Theorem 3. *Let $0 < \alpha \leq 1$ and E be any bounded subset of the interval $[0,\infty)$. If $f \in C_B[0,\infty) \cap Lip_M(\alpha)$ then we have*

$$|L_{n,q}(f;x) - f(x)| \leq M\{\psi_{n,2}^{\frac{\alpha}{2}}(q;x) + 2(d(x,E))^\alpha\}, x \in [0,\infty),$$

where M is a constant depending on α and f, $d(x,E)$ is the distance between x and E defined as $d(x,E) = \inf\{|t-x|; t \in E\}$ and $\psi_{n,2}(q;x) = L_{n,q}((t-x)^2;x)$.

Proof. From the property of infimum, it follows that there exists a point $t_0 \in \bar{E}$ such that $d(x,E) = |t_0 - x|$.

In view of the triangle inequality we have

$$|f(t) - f(x)| \leq |f(t) - f(t_0)| + |f(t_0) - f(x)|.$$

Using the definition of $Lip_M(\alpha)$, we get

$$\begin{aligned} |L_{n,q}(f;x) - f(x)| &\leq L_{n,q}(|f(t) - f(t_0)|;x) + L_{n,q}(|f(x) - f(t_0)|;x) \\ &\leq M\{L_{n,q}(|t - t_0|^{\alpha};x) + |x - t_0|^{\alpha}\} \\ &\leq M\{L_{n,q}(|t - x|^{\alpha};x) + 2|x - t_0|^{\alpha}\}. \end{aligned}$$

Choosing $p_1 = \frac{2}{\alpha}$ and $p_2 = \frac{2}{2-\alpha}$, we get $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then, Hölder's inequality yields

$$\begin{aligned} |L_{n,q}(f;x) - f(x)| &\leq M\{(L_{n,q}(|t - x|^{\alpha p_1};x))^{1/p_1}[L_{n,q}(1^{p_2};x)]^{1/p_2} + 2(d(x,E))^{\alpha}\} \\ &\leq M\{(L_{n,q}((t - x)^2;x))^{\alpha/2} + 2(d(x,E))^{\alpha}\} \\ &= M\{\psi_{n,2}^{\alpha/2}(q;x) + 2(d(x,E))^{\alpha}\}. \end{aligned}$$

This completes the proof of the theorem. □

Next, we obtain a local direct estimate of operators $L_{n,q}$ using the Lipschitz-type maximal function of order α introduced by Lenze [2] as

$$\tilde{\omega}_{\alpha}(f,x) = \sup_{t \neq x, t \in [0,\infty)} \frac{|f(t) - f(x)|}{|t - x|^{\alpha}}, \; x \in [0,\infty) \, and \, \alpha \in (0,1]. \tag{2}$$

Theorem 4. *Let $0 < \alpha \leq 1$ and $f \in C_B[0,\infty)$, then for all $x \in [0,\infty)$ we have*

$$|L_{n,q}(f;x) - f(x)| \leq \tilde{\omega}_{\alpha}(f,x)\psi_{n,2}^{\alpha/2}(q;x).$$

Proof. In view of (2), we get

$$|f(t) - f(x)| \leq \tilde{\omega}_{\alpha}(f,x)|t - x|^{\alpha}$$

and hence

$$|L_{n,q}(f;x) - f(x)| \leq L_{n,q}(|f(t) - f(x)|;x) \leq \tilde{\omega}_{\alpha}(f,x)L_{n,q}(|t - x|^{\alpha};x).$$

Now, using the Hölder's inequality with $p = \frac{2}{\alpha}$ and $\frac{1}{q} = 1 - \frac{1}{p}$, we obtain

$$|L_{n,q}(f;x) - f(x)| \leq \tilde{\omega}_{\alpha}(f,x)(L_{n,q}(|t - x|^2;x))^{\alpha/2} = \tilde{\omega}_{\alpha}(f,x)\psi_{n,2}^{\alpha/2}(x).$$

Thus, the proof is completed.

□

6. Weighted Approximation

In this section, we discuss about the weighted approximation theorem for the operators $L_{n,q}(f)$.

Let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty} \frac{|f(x)|}{1+x^2}$ is finite.

Theorem 5. *Let q_n be a sequence in $(0,1)$ such that $q_n \to 1$ and $q_n^n \to 0$, as $n \to \infty$. For each $C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty} \|L_{n,q_n}(f) - f\|_{x^2} = 0. \tag{3}$$

Proof. In order to proof (3) it is sufficient to show that ([5])

$$\lim_{n\to\infty} \|L_{n,q_n}(t^\nu;x) - x^\nu\|_{x^2} = 0,\ \nu = 0,1,2. \tag{4}$$

Since, $L_{n,q_n}(1;x) = 1$, (4) holds true for $\nu = 0$.

Now, by Lemma 1, we have

$$\begin{aligned}\|L_{n,q_n}(t;x) - x\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|L_{n,q_n}(t;x) - x|}{1+x^2}\\ &\to 0, \text{ as } n\to\infty.\end{aligned}$$

Therefore, (4) is true for $\nu = 1$.

Again, by Lemma 1, we may write

$$\begin{aligned}\|L_{n,q_n}(t^2;x) - x^2\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|L_{n,q_n}(t^2;x) - x^2|}{1+x^2}\\ &= \sup_{x\in[0,\infty)} \frac{\left|\frac{(1+q_n)x + q_n[n]_{q_n}x^2}{[n]_{q_n}} - x^2\right|}{1+x^2}\\ &\le \frac{1+q_n}{[n]_{q_n}} \sup_{x\in[0,\infty)} \frac{x}{1+x^2}\\ &+ (q_n - 1) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}\\ &= \frac{1+q_n}{[n]_{q_n}} + (q_n - 1).\end{aligned}$$

Hence, (4) follows for $\nu = 2$. This completes the proof of the theorem. □

Theorem 6. *Let $f \in C_{x^2}[0,\infty), q = q_n \in (0,1)$ such that $q_n \to 1$ and $q_n^n \to 0$ as $n\to\infty$ and ω_{a+1} be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, $a > 0$. Then, for every $n \ge 1$*

$$\|L_{n,q}(f) - f\|_{C[0,a]} \le \frac{12M_f(1+a^2)a}{[n]_q} + 2\omega_{a+1}\left(f, \sqrt{\frac{2a}{[n]_q}}\right).$$

Proof. For $x \in [0,a]$ and $t > a+1$. Since $t - x > 1$, we have

$$\begin{aligned}|f(t) - f(x)| &\le M_f(2 + x^2 + t^2)\\ &\le M_f(2 + 3x^2 + 2(t-x)^2)\\ &\le 3M_f(1 + x^2 + (t-x)^2)\\ &\le 6M_f(1+x^2)(t-x)^2\\ &\le 6M_f(1+a^2)(t-x)^2. \qquad (5)\end{aligned}$$

For $x \in [0,a]$ and $t \le a+1$, we have

$$|f(t) - f(x)| \le \omega_{a+1}(f, |t-x|) \le \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \tag{6}$$

where $\delta > 0$.

From (5) and (6), we can write

$$|f(t) - f(x)| \le 6M_f(1+a^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta) \tag{7}$$

For $x \in [0,a]$ and $t \ge 0$ and applying Schwarz inequality, we obtain

$$\begin{aligned}|L_{n,q}(f;x) - f(x)| &\le L_{n,q}(|f(t) - f(x)|;x)\\ &\le 6M_f(1+a^2)L_{n,q}((t-x)^2;x)\\ &+ \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}L_{n,q}((t-x)^2;x)^{\frac{1}{2}}\right).\end{aligned}$$

Hence, using Lemma 2, for every $q \in (0,1)$ and $x \in [0,a]$

$$\begin{aligned}|L_{n,q}(f;x)-f(x)| &\leq 6M_f(1+a^2)\frac{x([2]-(1-q)[n]_qx)}{[n]_q}\\ &+ C\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{x([2]-(1-q)[n]_qx)}{[n]_q}}\right.\\ &\leq \frac{12M_f(1+a^2)a}{[n]_q}\\ &+ \omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{2a}{[n]_q}}\right).\end{aligned}$$

Taking $\delta=\sqrt{\frac{2a}{[n]_q}}$, we get the required result.

This completes the proof of Theorem. □

Now, we prove a theorem to approximate all functions in $C_{x^2}[0,\infty)$. Such type of results are given in [4] for locally integrable functions.

Theorem 7. *Let $q=q_n\in(0,1)$ such that $q_n\to 1$ and $q_n^n\to 0$, as $n\to\infty$. For each $f\in C^*_{x^2}[0,\infty)$, and $\alpha>1$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|L_{n,q_n}(f;x)-f(x)|}{(1+x^2)^\alpha}=0.$$

Proof. For any fixed $x_0>0$,

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|L_{n,q_n}(f;x)-f(x)|}{(1+x^2)^\alpha} &\leq \sup_{x\leq x_0}\frac{|L_{n,q_n}(f;x)-f(x)|}{(1+x^2)^\alpha}+\sup_{x>x_0}\frac{|L_{n,q_n}(f;x)-f(x)|}{(1+x^2)^\alpha}\\ &\leq \|L_{n,q_n}(f)-f\|_{C[0,x_0]}+\|f\|_{x^2}\sup_{x\geq x_0}\frac{|L_{n,q_n}(1+t^2;x)|}{(1+x^2)^\alpha}\\ &\quad+\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^\alpha}.\end{aligned}\tag{8}$$

Since, $|f(x)|\leq M_f(1+x^2)$, we have

$$\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^\alpha}\leq\sup_{x\geq x_0}\frac{M_f}{(1+x^2)^{\alpha-1}}\leq\frac{M_f}{(1+x_0^2)^{\alpha-1}}.$$

Let $\epsilon>0$ be arbitrary. We can choose x_0 to be large that

$$\frac{M_f}{(1+x_0^2)^{\alpha-1}}<\frac{\epsilon}{3}\tag{9}$$

and in view of Lemma 1, we obtain

$$\begin{aligned}\|f\|_{x^2}\lim_{n\to\infty}\frac{|L_{n,q_n}(1+t^2;x)|}{(1+x^2)^\alpha} &= \frac{1+x^2}{(1+x^2)^\alpha}\|f\|_{x^2}\\ &= \frac{\|f\|_{x^2}}{(1+x^2)^{\alpha-1}}\\ &\leq \frac{\|f\|_{x^2}}{(1+x_0^2)^{\alpha-1}}\\ &< \frac{\epsilon}{3}.\end{aligned}\tag{10}$$

Using Theorem 6 we can see that the first term of the inequality (8) implies that

$$\|L_{n,q_n}(f;.)-f\|_{C[0,x_0]}<\frac{\epsilon}{3},\text{ as }n\to\infty.\tag{11}$$

Combining (8)-(11), we get the desired result.

□

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