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EXPONENTIAL STABILITY OF THE HEAT EQUATION WITH BOUNDARY TIME-VARYING DELAYS

MOUATAZ BILLAH MESMOULI[1], ABDELOUAHEB ARDJOUNI[1,2,*] AND AHCENE DJOUDI[1]

Abstract. In this paper, we consider the heat equation with a time-varying delays term in the boundary condition in a bounded domain of $\mathbb{R}^n$, the boundary Γ is a class C^2 such that $\Gamma = \Gamma_D \cup \Gamma_N$, with $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$, $\Gamma_D \neq \emptyset$ and $\Gamma_N \neq \emptyset$. Well-posedness of the problems is analyzed by using semigroup theory. The exponential stability of the problem is proved. This paper extends in n-dimensional the results of the heat equation obtained in [11].

1. Introduction

Time-delay often appears in many biological, electrical engineering systems and mechanical applications, and in many cases, delay is a source of instability [3]. In the case of distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system (see e.g. [1, 2, 8, 9, 10, 14]). The stability issue of systems with delay is, therefore, of theoretical and practical importance.

In present paper, we are interested in the effect of a time-varying delays in boundary stabilization of the heat equation in domains of $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with boundary Γ of class C^2. We assume that Γ is divided into two parts Γ_N and Γ_D; i.e., $\Gamma = \Gamma_D \cup \Gamma_N$ with $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$, $\Gamma_D \neq \emptyset$ and $\Gamma_N \neq \emptyset$.

In this domain Ω, we consider the initial boundary value problem

$$u_t(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,t) = 0 \text{ on } \Gamma_D \times (0,\infty), \tag{1.2}$$

$$\frac{\partial u}{\partial \nu}(x,t) = -\mu_1 u(x,t) - \mu_2 u(x, t-\tau(t)) \text{ on } \Gamma_N \times (0,\infty), \tag{1.3}$$

$$u(x,0) = u_0(x) \text{ in } \Omega, \tag{1.4}$$

$$u(x, t-\tau(0)) = f_0(x, t-\tau(0)) \text{ on } \Gamma_N \times (0, \tau(0)), \tag{1.5}$$

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, $\tau(t) > 0$, $\mu_1, \mu_2 \geq 0$ are fixed nonnegative real numbers, the initial datum (u_0, f_0) belongs to a suitable space.

On the functions $\tau(\cdot)$ we assume that there exists a positive constants $\overline{\tau}$, such that

$$0 < \tau_0 \leq \tau(t) \leq \overline{\tau}, \quad \forall t > 0, \tag{1.6}$$

Moreover, we assume

$$\tau'(t) < 1, \ \forall t > 0, \tag{1.7}$$

and

$$\tau \in W^{2,\infty}([0,T]), \ \forall t > 0. \tag{1.8}$$

Note that , if $t < \tau(t)$, then $u(x, t-\tau(t))$ is in the past and we need an initial value in the past. Moreover, by (1.7) and the mean value theorem, we have

$$\tau(t) - \tau(0) < t,$$

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which implies

$$t-\tau(t)>-\tau(0),$$

we thus obtain the initial condition (1.5).

The last boundary-value problem describes the propagation of heat in a homogeneous n-dimensional rod. Here a stands for the heat conduction coefficient, $u(x,t)$ is the value of the temperature field of the plant at time moment t and location x along the rod. In the sequel, the state dependence on time t and spatial variable x is suppressed whenever possible.

The above problem, with both $\mu_1,\mu_2>0$ and a time-varying delay, has been studied in one space dimension by Nicaise, Valein and Fridman [12]. In [12] an exponential stability result is given, under the condition

$$\mu_2<\sqrt{1-d}\mu_1, \tag{1.9}$$

where d is a constant such that

$$\tau'(t)\leq d<1,\ \forall t>0, \tag{1.10}$$

We are interested in giving an exponential stability result for such a problem. Let us denote by $\langle v,w\rangle$ the Euclidean inner product between two vectors $(v,w)\in\mathbb{R}^n$.

Under a suitable relation between the above coefficients we can give a well-posedness result and an exponential stability estimate for problem (1.1)–(1.5).

2. Well-posedness of the problem

Using semigroup theory we can give the well-posedness of problem (1.1)-(1.5). Let us stand

$$z(x,\rho,t)=u(x,t-\tau(t)\rho),\ x\in\Gamma_N,\ \rho\in(0,1),\ t>0.$$

Then, the problem (1.1)-(1.5) is equivalent to

$$u_t(x,t)-\Delta u(x,t)=0 \text{ in } \Omega\times(0,\infty), \tag{2.1}$$

$$\tau(t)z_t(x,\rho,t)+(1-\tau'(t)\rho)z_\rho(x,\rho,t)=0 \text{ in } \Gamma_N\times(0,1)\times(0,\infty), \tag{2.2}$$

$$u(x,t)=0 \text{ on } \Gamma_D\times(0,\infty), \tag{2.3}$$

$$\frac{\partial u}{\partial\nu}(x,t)=-\mu_1u(x,t)-\mu_2z(x,1,t) \text{ on } \Gamma_N\times(0,\infty), \tag{2.4}$$

$$z(x,0,t)=u(x,t),\ x\in\Gamma_N,\ t>0, \tag{2.5}$$

$$u(x,0)=u_0(x) \text{ in } \Omega, \tag{2.6}$$

$$z(x,\rho,0)=f_0(x,-\tau(0)\rho),\ x\in\Gamma_N,\ \rho\in(0,1). \tag{2.7}$$

If we denote by

$$U:=(u,z)^T,$$

then

$$U'=\begin{pmatrix}u_t\\ z_t\end{pmatrix}=\begin{pmatrix}\Delta u\\ \frac{(\tau'(t)\rho-1)}{\tau(t)}z_\rho\end{pmatrix}.$$

Therefore, problem (2.1)–(2.7) can be rewritten as

$$\begin{cases}U'=\mathcal{A}(t)U,\\ U(0)=(u_0,f_0(\cdot,-\cdot\tau(0)))^T,\end{cases} \tag{2.8}$$

in the Hilbert space $\mathcal{H}$ defined by

$$\mathcal{H}:=L^2(\Omega)\times L^2(\Gamma_N\times(0,1)), \tag{2.9}$$

equipped with the standard inner product

$$\left\langle\begin{pmatrix}u\\ z\end{pmatrix},\begin{pmatrix}\tilde{u}\\ \tilde{z}\end{pmatrix}\right\rangle_{\mathcal{H}}:=\int_\Omega u(x)\tilde{u}(x)dx+\int_{\Gamma_N}\int_0^1 z(x,\rho)\tilde{z}(x,\rho)d\rho d\Gamma.$$

The time varying operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\begin{pmatrix} u \\ z \end{pmatrix} := \begin{pmatrix} \Delta u \\ \frac{(\tau'(t)\rho-1)}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}(t)) \quad : \quad &= \Big\{ (u,z)^T \in \big(E(\Delta, L^2(\Omega)) \cap V\big) \times L^2\left(\Gamma_N, H^1(0,1)\right) : \\ &\frac{\partial u}{\partial \nu} = -\mu_1 u - \mu_2 z(\cdot,1) \text{ on } \Gamma_N, \; u = z(\cdot,0) \text{ on } \Gamma_N \Big\}, \end{aligned}$$

where,

$$V = H^1_{\Gamma_D} = \left\{ u \in H^1(\Omega),\; u = 0 \text{ on } \Gamma_D \right\},$$

and

$$E(\Delta, L^2(\Omega)) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}.$$

Recall that for a function $u \in E(\Delta, L^2(\Omega))$, $\frac{\partial u}{\partial \nu}$ belongs to $H^{-1/2}(\Gamma_N)$ and the next Green formula is valid (see section 1.5 of [4])

$$\int_\Omega \nabla u \nabla \varphi dx = -\int_\Omega \Delta u \varphi dx + \langle \frac{\partial u}{\partial \nu}, \varphi \rangle_{\Gamma_N}, \; \forall \varphi \in H^1_{\Gamma_D}(\Omega), \tag{2.10}$$

where $\langle \cdot ; \cdot \rangle_{\Gamma_N}$ means the duality pairing between $H^{-1/2}(\Gamma_N)$ and $H^{1/2}(\Gamma_N)$.

Observe that the domain of $\mathcal{A}(t)$ is independent of the time t, i.e.,

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)),\; t > 0. \tag{2.11}$$

Note further that for $(u,z)^T \in \mathcal{D}(\mathcal{A}(t))$, $\partial u/\partial \nu$ belongs to $L^2(\Gamma_N)$, since $z(x,1)$ is in $L^2(\Gamma_N)$.

A general theory for equations of type (2.8) has been developed using semigroup theory [6, 7, 13]. The simplest way to prove existence and uniqueness results is to show that the triplet $\{\mathcal{A}, \mathcal{H}, \mathcal{D}(\mathcal{A}(0))\}$, with $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,T]\}$, for some fixed $T > 0$, forms a CD-system (or constant domain system, see [6, 7]). More precisely, we can obtain a well-posedness result using semigroup arguments by Kato [5, 6, 13]. The following result is proved in [5, Theorem 1.9].

Theorem 1. *Assume that*

(i) *$\mathcal{D}(\mathcal{A}(0))$ is a dense subset of $\mathcal{H}$,*
(ii) *$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0))$ for all $t > 0$,*
(iii) *for all $t \in [0,T]$, $\mathcal{A}(t)$ generates a strongly continuous semigroup on $\mathcal{H}$ and the family $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,T]\}$ is stable with stability constants C and m independent of t (i.e. the semigroup $(S_t(s))_{s\geq 0}$ generated by $\mathcal{A}(t)$ satisfies $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$, for all $u \in \mathcal{H}$ and $s \geq 0$),*
(iv) *$\partial_t \mathcal{A}$ belongs to $L^\infty_*([0,T], B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H}))$, the space of equivalent classes of essentially bounded, strongly measurable functions from $[0,T]$ into the set $B(\mathcal{D}(\mathcal{A}(0)), \mathcal{H})$ of bounded operators from $\mathcal{D}(\mathcal{A}(0))$ into $\mathcal{H}$.*

Then, problem (2.8) has a unique solution $U \in C([0,T], \mathcal{D}(\mathcal{A}(0))) \cap C^1([0,T], \mathcal{H})$ for any initial datum in $\mathcal{D}(\mathcal{A}(0))$.

Our goal is then to check the above assumptions for problem (2.8).

Lemma 1. *$D(\mathcal{A}(0))$ is dense in $\mathcal{H}$.*

Proof. Let $(f,h)^T \in \mathcal{H}$ be orthogonal to all elements of $\mathcal{D}(\mathcal{A}(0))$, that is,

$$0 = \left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} g \\ h \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega u(x)g(x)dx + \int_{\Gamma_N}\int_0^1 z(x,\rho)h(x,\rho)d\rho d\Gamma,$$

for all $(u,z)^T \in \mathcal{D}(\mathcal{A}(0))$. We first take $u = 0$ and $z \in \mathcal{D}(\Gamma_N \times (0,1))$. As $(0,z)^T \in D(\mathcal{A}(0))$, we obtain

$$\int_{\Gamma_N}\int_0^1 z(x,\rho)h(x,\rho)d\rho d\Gamma = 0.$$

Since $\mathcal{D}(\Gamma_N \times (0,1))$ is dense in $L^2(\Gamma_N \times (0,1)$, we deduce that $h = 0$.

In the same way, by taking $z = 0$ and $u \in \mathcal{D}(\Omega)$ we see that $g = 0$. □

Assuming (1.9) and (1.10) hold. Let ξ be a positive constant that satisfies

$$\frac{\mu_2}{\sqrt{1-d}} \leq \xi \leq 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}. \tag{2.12}$$

Note that this choice of ξ is possible from assumption (1.9).

We define on the Hilbert space $\mathcal{H}$ the time dependent inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle_t := \int_\Omega u(x)\,\tilde{u}(x)\,dx + \xi\tau(t)\int_{\Gamma_N}\int_0^1 z(x,\rho)\,\tilde{z}(x,\rho)\,d\rho d\Gamma. \tag{2.13}$$

Using this time dependent inner product and Theorem 1, we can deduce a well-posedness result.

Theorem 2. *For any initial datum $U_0 \in \mathcal{D}(\mathcal{A}(0))$ there exists a unique solution*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A}(0))) \cap C^1([0,+\infty), \mathcal{H}),$$

of system (2.8).

Proof. We first observe that

$$\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \ \forall t,s \in [0,T], \tag{2.14}$$

where $\phi = (u,z)^T$ and c is a positive constant. Indeed, for all $s,t \in [0,T]$, we have

$$\begin{aligned} \|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} &= \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right)\int_\Omega u^2 dx \\ &\quad + \xi\left(\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|}\right)\int_{\Gamma_N}\int_0^1 z^2(x,\rho)d\rho d\Gamma. \end{aligned}$$

We notice that $1 - e^{\frac{c}{\tau_0}|t-s|} \leq 0$. Moreover $\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|} \leq 0$ for some $c > 0$. Indeed, $\tau(t) = \tau(s) + \tau'(a)(t-s)$, where $a \in (s,t)$, and thus,

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\tau'(a)|}{\tau(s)}|t-s|.$$

By (1.8), τ' is bounded on $[0,T]$ and therefore, recalling also (1.7),

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0}|t-s| \leq e^{\frac{c}{\tau_0}|t-s|},$$

which proves (2.14).

Now we calculate $\langle \mathcal{A}(t)U, U\rangle_t$ for a fixed t. Take $U = (u,z)^T \in \mathcal{D}(\mathcal{A}(t))$. Then,

$$\begin{aligned} \langle \mathcal{A}(t)U, U\rangle_t &= \left\langle \begin{pmatrix} \Delta u \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ z \end{pmatrix} \right\rangle_t \\ &= \int_\Omega u(x)\Delta u(x)dx - \xi\int_{\Gamma_N}\int_0^1 (1 - \tau'(t)\rho)\,z_\rho(x,\rho)\,z(x,\rho)\,d\rho d\Gamma. \end{aligned}$$

So, by Green's formula,

$$\begin{aligned} \langle \mathcal{A}(t)U, U\rangle_t &= \int_{\Gamma_N} \frac{\partial u(x)}{\partial \nu} u(x)d\Gamma - \int_\Omega |\nabla u(x)|^2 dx \\ &\quad - \xi\int_{\Gamma_N}\int_0^1 (1 - \tau'(t)\rho)z_\rho(x,\rho)z(x,\rho)d\rho d\Gamma. \end{aligned} \tag{2.15}$$

Integrating by parts in ρ, we obtain

$$\begin{aligned} &\int_{\Gamma_N}\int_0^1 z_\rho(x,\rho)z(x,\rho)(1 - \tau'(t)\rho)\,d\rho d\Gamma \\ &= \int_{\Gamma_N}\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho} z^2(x,\rho)(1 - \tau'(t)\rho)d\rho d\Gamma \\ &= \frac{\tau'(t)}{2}\int_{\Gamma_N}\int_0^1 z^2(x,\rho)d\rho d\Gamma + \frac{1}{2}\int_{\Gamma_N}\{z^2(x,1)\,(1 - \tau'(t)) - z^2(x,0)\}d\Gamma. \end{aligned} \tag{2.16}$$

Therefore, from (2.15) and (2.16),

$$\begin{aligned}
&\langle \mathcal{A}(t)U,U\rangle_t \\
&= \int_{\Gamma_N} \frac{\partial u(x)}{\partial \nu} u(x)d\Gamma - \int_{\Omega} |\nabla u(x)|^2 dx \\
&- \frac{\xi}{2}\int_{\Gamma_N} \{z^2(x,1)(1-\tau'(t)) - z^2(x,0)\} d\Gamma - \frac{\xi\tau'(t)}{2}\int_{\Gamma_N}\int_0^1 z^2(x,\rho)d\rho d\Gamma \\
&= -\int_{\Gamma_N} [\mu_1 u(x) + \mu_2 z(x,1)] u(x) d\Gamma - \int_{\Omega} |\nabla u(x)|^2 dx \\
&+ \frac{\xi}{2}\int_{\Gamma_N} u^2(x)d\Gamma - \frac{\xi}{2}\int_{\Gamma_N}\{z^2(x,1)(1-\tau'(t))\, d\Gamma - \frac{\xi\tau'(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma \\
&= -\left(\mu_1 - \frac{\xi}{2}\right)\int_{\Gamma_N} u^2(x)d\Gamma - \mu_2\int_{\Gamma_N} z(x,1)u(x)d\Gamma - \int_{\Omega}|\nabla u(x)|^2 dx \\
&- \frac{\xi}{2}\int_{\Gamma_N}\{z^2(x,1)(1-\tau'(t))\, d\Gamma - \frac{\xi\tau'(t)}{2}\int_{\Gamma_N}\int_0^1 z^2(x,\rho)d\rho d\Gamma,
\end{aligned}$$

from which, using Cauchy-Schwarz's, Poincaré's inequality and (1.10), it follows that

$$\begin{aligned}
\langle \mathcal{A}(t)U,U\rangle_t \leq &\left(-\mu_1 + \frac{\xi}{2} + \frac{\mu_2}{2\sqrt{1-d}} - \frac{1}{C_p}\right)\int_{\Gamma_N} u^2(x)d\Gamma \\
&+ \left(\frac{\mu_2\sqrt{1-d}}{2} - \frac{\xi}{2}(1-d)\right)\int_{\Gamma_N} z^2(x,1)d\Gamma + \kappa(t)\langle U,U\rangle_t,
\end{aligned} \tag{2.17}$$

where

$$\kappa(t) = \frac{(\tau'^2(t)+1)^{\frac{1}{2}}}{2\tau(t)}. \tag{2.18}$$

Now, observe that from (2.12),

$$-\mu_1 + \frac{\xi}{2} + \frac{\mu_2}{2\sqrt{1-d}} \leq 0, \;\; \frac{\mu_2\sqrt{1-d}}{2} - \frac{\xi}{2}(1-d) \leq 0.$$

Then

$$\langle \mathcal{A}(t)U,U\rangle_t - \kappa(t)\langle U,U\rangle_t \leq 0, \tag{2.19}$$

which means that the operator $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ is dissipative.

Moreover,

$$\kappa'(t) = \frac{\tau''(t)\tau'(t)}{2\tau(t)(\tau'^2(t)+1)^{\frac{1}{2}}} - \frac{\tau'(t)(\tau'^2(t)+1)^{\frac{1}{2}}}{2\tau(t)^2},$$

is bounded on $[0,T]$ for all $T>0$ (by (1.6) and (1.7)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2} z_\rho \end{pmatrix},$$

with $\frac{\tau''(t)\tau(t)\rho-\tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2}$ bounded on $[0,T]$. Thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L_*^\infty([0,T], B(D(\mathcal{A}(0)),\mathcal{H})), \tag{2.20}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0,T]$ into $B(D(\mathcal{A}(0)),\mathcal{H})$.

Now, we show that $\lambda I - \mathcal{A}(t)$ is surjective for fixed $t>0$ and $\lambda>0$. Given $(g,h)^T \in \mathcal{H}$, we seek $U=(u,z)^T \in \mathcal{D}(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t))\begin{pmatrix} u \\ z\end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix},$$

that is verifying

$$\left\{ \begin{array}{l} \lambda u-\Delta u=g, \\ \lambda z+\frac{1-\tau'(t)\rho}{\tau(t)}z_{\rho}=h. \end{array} \right. \tag{2.21}$$

Suppose that we have found u with the appropriate regularity. We can then determine z, indeed z satisfies the differential equation,

$$\lambda z(x,\rho)+\frac{1-\tau'\left(t\right)\rho}{\tau\left(t\right)}z_{\rho}(x,\rho)=h(x,\rho),\ \text{for } x\in\Gamma,\rho\in\left(0,1\right),$$

and the boundary condition

$$z(x,0)=u(x),\ \text{for } x\in\Gamma_N. \tag{2.22}$$

Therefore z is explicitly given by

$$z(x,\rho)=u(x)e^{-\lambda\rho\tau(t)}+\tau(t)e^{-\lambda\rho\tau(t)}\int_0^{\rho}h(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma,$$

if $\tau'(t)=0$, and

$$\begin{aligned} z(x,\rho)&=u(x)e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)} \\ &\quad+e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho)}\int_0^{\rho}\frac{h(x,\sigma)\tau(t)}{1-\tau'(t)\sigma}e^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\sigma)}d\sigma, \end{aligned}$$

otherwise. This means that once u is found with the appropriate properties, we can find z.

In particular, if $\tau'(t)=0$,

$$z(x,1)=u(x)e^{-\lambda\tau(t)}+z_0(x),\ x\in\Gamma_N, \tag{2.23}$$

with $z_0\in L^2(\Gamma_N)$ defined by

$$z_0(x)=\tau(t)e^{-\lambda\tau(t)}\int_0^1 h(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma,\ x\in\Gamma_N, \tag{2.24}$$

and, if $\tau'(t)\neq 0$,

$$z(x,1)=u(x)e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))}+z_0(x),\ x\in\Gamma_N, \tag{2.25}$$

with $z_0\in L^2(\Gamma_N)$ defined by

$$z_0(x)=e^{\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t))}\int_0^1\frac{h(x,\sigma)\tau(t)}{1-\tau'(t)\sigma}e^{-\lambda\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\sigma)}d\sigma, \tag{2.26}$$

for $x\in\Gamma_N$. Then, we have to find u. In view of the equation

$$\lambda u-\Delta u=g. \tag{2.27}$$

Multiplying this identity by a test function ϕ and integrating in space

$$\int_{\Omega}\left(\lambda u\phi-\Delta u\phi\right)dx=\int_{\Omega}g\phi dx,\ \forall\phi\in H^1_{\Gamma_D}, \tag{2.28}$$

using Green's formula, we obtain

$$\begin{aligned} \int_{\Omega}\left(\lambda u\phi-\Delta u\phi\right)dx &= \int_{\Omega}\left(\lambda u\phi+\nabla u\nabla\phi\right)dx-\int_{\Gamma_N}\frac{\partial u}{\partial\nu}\phi d\Gamma \\ &= \int_{\Omega}\left(\lambda u\phi+\nabla u\nabla\phi\right)dx+\int_{\Gamma_N}\left(\mu_1 u+\mu_2 z\left(x,1\right)\right)\phi d\Gamma. \end{aligned}$$

By (2.23), we obtain

$$\begin{aligned} \int_{\Omega}\left(\lambda u\phi-\Delta u\phi\right)dx &= \int_{\Omega}\left(\lambda u\phi+\nabla u\nabla\phi\right)dx \\ &\quad+\int_{\Gamma_N}\left(\mu_1 u+\mu_2\left(ue^{-\lambda\tau(t)}+z_0\right)\right)\phi d\Gamma, \end{aligned}$$

if $\tau'(t) = 0$, and by (2.25)

$$\int_{\Omega} (\lambda u\phi - \Delta u\phi)\, dx = \int_{\Omega} (\lambda u\phi + \nabla u \nabla\phi)\, dx + \int_{\Gamma_N} \left(\mu_1 u + \mu_2 \left(u e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + z_0\right)\right) \phi d\Gamma,$$

otherwise. Therefore, (2.28) can be rewritten as

$$\int_{\Omega} (\lambda u\phi + \nabla u \nabla\phi)\, dx + \int_{\Gamma_N} \left(\mu_1 u + \mu_2 \left(u e^{-\lambda\tau(t)} + z_0\right)\right) \phi d\Gamma = \int_{\Omega} g\phi dx, \tag{2.29}$$

if $\tau'(t) = 0$, and

$$\int_{\Omega} (\lambda u\phi + \nabla u \nabla\phi)\, dx + \int_{\Gamma_N} \left(\mu_1 u + \mu_2 \left(u e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + z_0\right)\right) \phi d\Gamma = \int_{\Omega} g\phi dx, \tag{2.30}$$

otherwise. As the left-hand side of (2.29) or (2.30) is coercive on $H^1_{\Gamma_D}(\Omega)$, the Lax-Milgram lemma guarantees the existence and uniqueness of a solution $u \in H^1_{\Gamma_D}(\Omega)$ of (2.29), (2.30).

If we consider $\phi \in \mathcal{D}(\Omega)$ in (2.29), (2.30), we have that u solves (2.27) in $\mathcal{D}'(\Omega)$ and thus $u \in E(\Delta, L^2(\Omega))$.

Using Green's formula (2.10) in (2.29) and using (2.27), we obtain, if $\tau'(t) = 0$

$$\int_{\Gamma_N} \left(\mu_1 + \mu_2 e^{-\lambda\tau(t)}\right) u\phi d\Gamma + \langle \frac{\partial u}{\partial\nu}, \phi\rangle_{\Gamma_N} = -\mu_2 \int_{\Gamma_N} z_0 \phi d\Gamma,$$

from which follows

$$\frac{\partial u}{\partial\nu} + \left(\mu_1 + \mu_2 e^{-\lambda\tau(t)}\right) u = -\mu_2 z_0 \text{ on } \Gamma_N,$$

which imply that

$$\frac{\partial u}{\partial\nu} = -\mu_1 u - \mu_2 z(\cdot, 1) \text{ on } \Gamma_N,$$

where we have used (2.23) and (2.27).

We find the same result if $\tau'(t) \neq 0$.

In conclusion, we have found $(u, z)^T \in \mathcal{D}(\mathcal{A})$, which verifies (2.21), and thus $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$. Again as $\kappa(t) > 0$, this proves that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t) \text{ is surjective,} \tag{2.31}$$

for any $\lambda > 0$ and $t > 0$.

Then, (2.14), (2.19) and (2.31) imply that the family $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t) : t \in [0, T]\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of t, by [6, Proposition 1.1]. Therefore, the assumptions (i)-(iv) of Theorem 1 are satisfied by (2.11), (2.14), (2.19), (2.31), (2.20) and Lemma 1, and thus, the problem

$$\begin{cases} \tilde{U}' = \tilde{\mathcal{A}}(t)\tilde{U}, \\ \tilde{U}(0) = U_0, \end{cases}$$

has a unique solution $\tilde{U} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H})$ for $U_0 \in D(\mathcal{A}(0))$. The requested solution of (2.8) is then given by

$$U(t) = e^{\beta(t)} \tilde{U}(t),$$

with $\beta(t) = \int_0^t \kappa(s) ds$, because

$$\begin{aligned} U' &e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{U}'(t) \\ &= \kappa(t) e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{\mathcal{A}}(t)\tilde{U}(t) \\ &= e^{\beta(t)}(\kappa(t)\tilde{U}(t) + \tilde{\mathcal{A}}(t)\tilde{U}(t)) \\ &= e^{\beta(t)}\mathcal{A}(t)\tilde{U}(t) = \mathcal{A}(t) e^{\beta(t)}\tilde{U}(t) \\ &= \mathcal{A}(t)U(t). \end{aligned}$$

This concludes the proof. □

3. The decay of the energy

Let us choose the following energy

$$E(t)=\frac{1}{2}\int_{\Omega}u^{2}(x,t)\,dx+\frac{\xi\tau(t)}{2}\int_{\Gamma_N}\int_{0}^{1}u^{2}(x,t-\tau(t)\rho)\,dpd\Gamma, \tag{3.1}$$

where ξ is a suitable positive constant.

Proposition 1. *Let (1.9) and (1.10) be satisfied. Then for all regular solution of problem (2.8), the energy is decreasing and satisfies*

$$E'(t)\leq -C\left(\int_{\Gamma_N}u^{2}(x,t)\,d\Gamma+\int_{\Gamma_N}u^{2}(x,t-\tau(t))\,d\Gamma\right). \tag{3.2}$$

Proof. Differentiating (3.1), we get

$$\begin{aligned}E'(t)=&\int_{\Omega}uu_t dx+\frac{\xi\tau'(t)}{2}\int_{\Gamma_N}\int_{0}^{1}u^{2}(x,t-\tau(t)\rho)\,dpd\Gamma\\&+\xi\tau(t)\int_{\Gamma_N}\int_{0}^{1}(1-\tau'(t)\rho)\,u(x,t-\tau(t)\rho)\,u_t(x,t-\tau(t)\rho)\,dpd\Gamma,\end{aligned}$$

then

$$\begin{aligned}E'(t)=&\int_{\Omega}u\Delta u dx+\frac{\xi\tau'(t)}{2}\int_{\Gamma_N}\int_{0}^{1}u^{2}(x,t-\tau(t)\rho)\,dpd\Gamma\\&+\xi\tau(t)\int_{\Gamma_N}\int_{0}^{1}(1-\tau'(t)\rho)\,u(x,t-\tau(t)\rho)\,u_t(x,t-\tau(t)\rho)\,dpd\Gamma.\end{aligned}$$

By Green's formula and integrating by parts in ρ, we obtain

$$\begin{aligned}E'(t)=&-\int_{\Omega}|\nabla u|^{2}dx+\int_{\Gamma_N}u\frac{\partial u}{\partial\nu}d\Gamma\\&-\frac{\xi}{2}\int_{\Gamma_N}u^{2}(x,t-\tau(t))\,(1-\tau'(t))\,d\Gamma+\frac{\xi}{2}\int_{\Gamma_N}u^{2}(x,t)\,d\Gamma,\end{aligned}$$

and by (1.3), we obtain

$$\begin{aligned}E'(t)=&-\int_{\Omega}|\nabla u|^{2}dx-\int_{\Gamma_N}\left[\mu_1u^{2}(x,t)+\mu_2u(x,t)\,u(x,t-\tau(t))\right]d\Gamma\\&-\frac{\xi}{2}\int_{\Gamma_N}u^{2}(x,t-\tau(t))\,(1-\tau'(t))\,d\Gamma+\frac{\xi}{2}\int_{\Gamma_N}u^{2}(x,t)\,d\Gamma.\end{aligned}$$

By Cauchy-Schwarz's and Poincaré's inequality, we get,

$$\begin{aligned}E'(t)\leq&\left(-\frac{1}{C_p}-\mu_1+\frac{\xi}{2}+\frac{\mu_2}{2\sqrt{1-d}}\right)\int_{\Gamma_N}u^{2}(x,t)\,d\Gamma\\&-\left(\frac{\xi(1-d)}{2}+\frac{\mu_2\sqrt{1-d}}{2}\right)\int_{\Gamma_N}u^{2}(x,t-\tau(t))\,d\Gamma.\end{aligned}$$

Since the condition (2.12), we deduce that

$$-\frac{1}{C_p}-\mu_1+\frac{\xi}{2}+\frac{\mu_2}{2\sqrt{1-d}}\leq 0.$$

which concludes the proof. □

4. Exponential stability

In this section, we will give an exponential stability result for the problem (1.1)–(1.5) by using the following Lyapunov functional

$$\mathcal{E}(t) = E(t) + \gamma \widehat{E}(t), \tag{4.1}$$

where $\gamma > 0$ is a parameter that will be fixed small enough later on, E is the standard energy defined by (3.1) and $\widehat{E}$ is defined by

$$\widehat{E}(t) = \xi\tau(t)\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}u^2(x,t-\tau(t)\rho)\,d\rho d\Gamma. \tag{4.2}$$

Note that, the functional $\widehat{E}$ is equivalent to the energy E, that is there exist two positive constant d_1, d_2 such that

$$d_1 E(t) \le \mathcal{E}(t) \le d_2 E(t). \tag{4.3}$$

Theorem 3. *Assume (1.6) and (1.7). Then, there exist positive constants C_1, C_2 such that for any solution of problem (1.1)–(1.5),*

$$E(t) \le C_1 E(0) e^{-C_2 t}, \ \forall t \ge 0.$$

Proof. First, we differentiate $\widehat{E}(t)$ to have

$$\begin{aligned}\frac{d}{dt}\widehat{E}(t) &= \frac{\tau'(t)}{\tau(t)}\widehat{E}(t)\\ &\quad + \xi\tau(t)\int_{\Gamma_N}\int_0^1 (-2\tau'(t)\rho)\, e^{-2\tau(t)\rho}u^2(x,t-\tau(t)\rho)\,d\rho d\Gamma + J,\end{aligned}$$

where

$$J = 2\xi\tau(t)\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}(1-\tau'(t)\rho)\,u_t(x,t-\tau(t)\rho)\,u(x,t-\tau(t)\rho)\,d\rho d\Gamma.$$

Moreover, by noticing one more time that

$$z(x,\rho,t) = u(x,t-\tau(t)\rho), \ x \in \Gamma_N, \ \rho \in (0,1), \ t > 0,$$

and by integrating by parts in ρ, we have

$$\begin{aligned}J &= -\xi\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}(1-\tau'(t)\rho)\,\frac{\partial}{\partial\rho}(z(x,\rho,t))^2\,d\rho d\Gamma\\ &= \xi\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}\left[-2\tau(t)(1-\tau'(t)\rho)-\tau'(t)\right]z^2(x,\rho,t)\,d\rho d\Gamma\\ &\quad -\xi\int_{\Gamma_N} e^{-2\tau(t)}(1-\tau'(t))\,z^2(x,1,t)\,d\Gamma + \xi\int_{\Gamma_N} z^2(x,0,t)\,d\Gamma\\ &= \xi\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}\left[-2\tau(t)(1-\tau'(t)\rho)-\tau'(t)\right]u^2(x,t-\tau(t)\rho)\,d\rho d\Gamma\\ &\quad -\xi\int_{\Gamma_N} e^{-2\tau(t)}(1-\tau'(t))\,u^2(x,t-\tau(t))\,d\Gamma + \xi\int_{\Gamma_N} u^2(x,t)\,d\Gamma.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{d}{dt}\widehat{E}(t) &= \frac{\tau'(t)}{\tau(t)}\widehat{E}(t) + \xi\int_{\Gamma_N}\int_0^1 e^{-2\tau(t)\rho}\left[-2\tau(t)-\tau'(t)\right]u^2(x,t-\tau(t)\rho)\,d\rho d\Gamma\\ &\quad -\xi\int_{\Gamma_N} e^{-2\tau(t)}(1-\tau'(t))\,u^2(x,t-\tau(t))\,d\Gamma + \xi\int_{\Gamma_N} u^2(x,t)\,d\Gamma\\ &= -2\widehat{E}(t) - \xi\int_{\Gamma_N} e^{-2\tau(t)}(1-\tau'(t))\,u^2(x,t-\tau(t))\,d\Gamma + \xi\int_{\Gamma_N} u^2(x,t)\,d\Gamma.\end{aligned}$$

As $\tau'(t) < 1$, we obtain

$$\frac{d}{dt}\widehat{E}(t) \leq -2\widehat{E}(t) + \xi \int_{\Gamma_N} u^2(x,t)\, d\Gamma. \tag{4.4}$$

Consequently, gathering (3.2), (4.1) and (4.4), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}E(t) + \gamma \frac{d}{dt}\widehat{E}(t) \\ &\leq -2\gamma\widehat{E}(t) + \gamma\xi \int_{\Gamma_N} u^2(x,t)\, d\Gamma \\ &\quad - C\int_{\Gamma_N} \left(u^2(x,t) + u^2(x,t-\tau(t))\right) d\Gamma. \end{aligned}$$

Then, for γ sufficiently small, we can estimate

$$\frac{d}{dt}\mathcal{E}(t) \leq -2\gamma\widehat{E}(t) - C\int_{\Gamma_N} \left(u^2(x,t) + u^2(x,t-\tau(t))\right) d\Gamma. \tag{4.5}$$

Now, observe that by assumption (1.6) on $\tau(t)$, we can deduce

$$\begin{aligned} \widehat{E}(t) &\geq \xi\tau(t)\int_{\Gamma_N}\int_0^1 e^{-2\overline{\tau}\rho} u^2(x,t-\tau(t)\rho)\, dpd\Gamma \\ &\geq \frac{k\xi\tau(t)}{2}\int_{\Gamma_N}\int_0^1 u^2(x,t-\tau(t)\rho)\, dpd\Gamma, \end{aligned} \tag{4.6}$$

for some positive constant k. Therefore, from (4.5) and (4.6),

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &\leq -2\gamma\widehat{E}(t) - C\int_{\Gamma_N} \left(u^2(x,t) + u^2(x,t-\tau(t))\right) d\Gamma \\ &\leq -kE(t) \leq -K\mathcal{E}(t). \end{aligned}$$

for suitable positive constants k, K; where we used also the first inequality in (4.3). This clearly implies

$$\mathcal{E}(t) \leq e^{-Kt}\mathcal{E}(0),$$

and so, using (4.3),

$$E(t) \leq C_1 e^{-C_2 t} E(0),$$

for suitable constants $C_1, C_2 > 0$. □

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[1] Applied Mathematics Lab, Faculty of Sciences, Department of Mathematics, Univ Annaba, P.O. Box 12, Annaba 23000, Algeria

[2] Department of Mathematics and Informatics, Univ Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria

*Corresponding author: abd_ardjouni@yahoo.fr