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# EXISTENCE RESULTS FOR SOME NONLINEAR FUNCTIONAL-INTEGRAL EQUATIONS IN BANACH ALGEBRA WITH APPLICATIONS

LAKSHMI NARAYAN MISHRA[1,2,*] H. M. SRIVASTAVA[3,4] AND MAUSUMI SEN[1]

Abstract. In the present manuscript, we prove some results concerning the existence of solutions for some nonlinear functional-integral equations which contains various integral and functional equations that considered in nonlinear analysis and its applications. By utilizing the techniques of noncompactness measures, we operate the fixed point theorems such as Darbo's theorem in Banach algebra concerning the estimate on the solutions. The results obtained in this paper extend and improve essentially some known results in the recent literature. We also provide an example of nonlinear functional-integral equation to show the ability of our main result.

## 1. Introduction

Measures of noncompactness and fixed point theorems are the most valuable and effective implements in the framework of nonlinear analysis, which act as principal role for the solvability of linear and nonlinear integral equations. Recently, the theory of such integral equations is developed effectively and emerge in the fields of mathematical analysis, engineering, mathematical physics and nonlinear functional analysis (see [2, 1, 33, 4, 13, 23, 27, 26, 34, 35, 36, 37, 19, 8, 15, 18, 7] and some references therein). In connection with some of the integro-differential equations, the paper should be further motivated by somehow connecting the work with the works [25, 17, 3, 12, 16, 31, 32].

Maleknejad et al. [29, 30] examined the existence of solutions for the nonlinear functional-integral equations (for short NLFIE) of the form

$$x(t) = g(t, x(t)) + f\left(t, \int_0^t u(t, s, x(s))ds, x(\alpha(t))\right), \tag{1.1}$$

and

$$x(t) = f(t, x(\alpha(t)) \int_0^t u(t, s, x(s))ds, \tag{1.2}$$

respectively, by availing the Darbo fixed-point theorem with suitable combination of measure of noncompactness defined in [5]. Banaś and Sadarangani [11] as well as Maleknejad et al. [28] discussed the existence of solutions for NLFIE

$$f\left(t, \int_0^t v(t, s, x(s))ds, x(\alpha(t))\right) \cdot g\left(t, \int_0^a u(t, s, x(s))ds, x(\beta(t))\right). \tag{1.3}$$

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Banaś and Rzepka [9, 10] dealt the existence of solutions of NLFIE and nonlinear quadratic Volterra integral equation

$$x(t) = f(t, x(t)) \int_0^t u(t, s, x(s))ds, \tag{1.4}$$

$$x(t) = p(t) + f(t, x(t)) \int_0^t v(t, s, x(s))ds, \tag{1.5}$$

respectively. The popular nonlinear Volterra integral equation and Urysohn integral equation are given as follows

$$x(t) = a(t) + \int_0^t u(t, s, x(s))ds, \tag{1.6}$$

$$x(t) = b(t) + \int_0^a v(t, s, x(s))ds, \tag{1.7}$$

respectively. Dhage [20] discussed the following nonlinear integral equation

$$x(t) = a(t) \int_0^a v(t, s, x(s))ds + \left( \int_0^t u(t, s, x(s))ds \right) \cdot \left( \int_0^a v(t, s, x(s))ds \right). \tag{1.8}$$

Moreover, the familiar quadratic integral equation of Chandrasekhar type [14] has the form

$$x(t) = 1 + x(t) \int_0^a \frac{t}{t+s} \phi(s)x(s)ds, \tag{1.9}$$

which is applicable in the theories of radiative transfer, neutron transport and kinetic energy of gases (see [14, 22, 24]).

In this paper, we study the existence of solutions of NLFIE

$$\begin{aligned} x(t) = &\left( q(t) + f(t, x(t), x(\theta(t))) + F\left( t, x(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, x(b(t)) \right) \right) \\ &\times G\left( t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t)) \right), \end{aligned} \tag{1.10}$$

for $t \in [0, a]$.

It is worthwhile mentioning that up to now equations (1.1)-(1.9) are a particular case of equation (1.10). Moreover, NLFIE (1.10) also involve with the functional equation of the first order having the form $x(t) = f(t, x(t), x(\theta(t)))$. This paper investigates existence of solutions of NLFIE (1.10) under some relevant results of fixed point theorem for the product of two operators which satisfies the Darbo condition with suitable combination of a measure of noncompactness in the Banach algebra of continuous functions in the interval $[0, a]$. The existence results are interesting in themselves although their solutions are continuous and stable.

## 2. Definitions and preliminaries

This section is devoted to revise some data which will be required in our further circumstances. Let $E$ is a real Banach space with the norm $\| \cdot \|$ and zero element $\theta'$. Symbolically $B(x, r)$ represents the closed ball centered at $x$ and with radius $r$, as well as we indicates by $B_r$ the ball $B(\theta', r)$. The notation $\mathcal{M}_E$ appears for the family of all nonempty and bounded subsets of $E$ and notation $\mathcal{N}_E$ also appears for its subfamily consisting of all relatively compact subsets. Additionally, if $X (\neq \phi) \subset E$

then the symbols $\bar{X}, ConvX$ in consideration of the closure and convex closure of $X$, respectively. We exercise the definition on the concept of a measure of noncompactness [5] as follows.

**Definition 2.1.** *Let $X \in \mathcal{M}_E$ and*

$$\mu(X) = \inf\left\{\delta > 0 : X = \bigcup_{i=1}^{m} X_i \ \ with \ \ diam(X_i) \leq \delta, \ i = 1, 2, ...m\right\},$$

*where for a fixed number $t \in [0, a]$, we denote*

$$diam\, X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

*Clearly, $0 \leq \mu(X) < \infty$. $\mu(X)$ is called the Kuratowski measure of noncompactness.*

**Theorem 2.1.** *Let $X, Y \in \mathcal{M}_E$ and $\lambda \in \mathbb{R}$. Then*

(i) *$\mu(X) = 0$ if and only if $X \in \mathcal{N}_E$;*
(ii) *$X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$;*
(iii) *$\mu(\bar{X}) = \mu(ConvX) = \mu(X)$;*
(iv) *$\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$;*
(v) *$\mu(\lambda X) = |\lambda|\mu(X)$, where $\lambda X = \{\lambda x : x \in X\}$;*
(vi) *$\mu(X + Y) \leq \mu(X) + \mu(Y)$, where $X + Y = \{x + y : x \in X, y \in Y\}$;*
(vii) *$|\mu(X) - \mu(Y)| \leq 2d_h(X, Y)$, where $d_h(X, Y)$ denotes the Hausdorff metric of $X$ and $Y$, i.e.*

$$d_h(X, Y) = \max\left\{\sup_{y\in Y} d(y, X), \sup_{x\in X} d(x, Y)\right\},$$

*where $d(.,.)$ is the distance from an element of $E$ to a set of $E$.*

Furthermore, every function $\mu : \mathcal{M}_E \to [0, \infty)$, satisfying conditions (i)-(vi) of Theorem 2.1, will be called a regular measure of noncompactness in the Banach space $E$ (cf. [9]).

Now let us theorize that $\Omega$ is a nonempty subset of a Banach space $E$ and $S : \Omega \to E$ is a continuous operator, which transforms bounded subsets of $\Omega$ to bounded ones. Additionally, let $\mu$ be a regular measure of noncompactness in $E$.

**Definition 2.2.** *(see [5]) The continuous operator $S$ satisfies the Darbo condition with a constant $K^{'}$ with respect to measure $\mu$ provided*

$$\mu(SX) \leq K^{'}\mu(X)$$

*for each $X \in \mathcal{M}_E$ such that $X \subset \Omega$.*
*If $K^{'} < 1$, then $S$ is called a contraction with respect to $\mu$.*

In the continuation, consider the space $C[0, a]$ is consisting of all real functions defined and continuous on the interval $[0, a]$. The space $C[0, a]$ is equipped with standard norm

$$\|x\| = \sup\{|x(t)| : t \in [0, a]\}.$$

Evidently, the space $C[0, a]$ has also the structure of Banach algebra.
Taking into our considerations, we will utilize a regular measure of noncompactness defined in [6] (cf. also [5]).
Let us fix a set $X \in \mathcal{M}_{C[0,a]}$. For $x \in X$ and for a given $\epsilon > 0$ denote by $w(x, \epsilon)$ the modulus of continuity of $x$, i.e.,

$$w(x, \epsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, a]; |t - s| \leq \epsilon\}.$$

Further, put

$$w(X, \epsilon) = \sup\{w(x, \epsilon) : x \in X\},$$
$$w_0(X) = \lim_{\epsilon\to 0} w(X, \epsilon).$$

The function $w_0(X)$ is a regular measure of noncompactness in the space $C[0, a]$, which can be shown in [6].
For our purposes we will require the following lemma and theorem [21, 6].

**Lemma 2.1.** *Let $D$ be a bounded, closed and convex subset of $E$. If operator $S : D \to D$ is a strict set contraction, then $S$ has a fixed point in $D$.*

**Theorem 2.2.** *Let us suppose that $\Omega$ is a nonempty, bounded, convex and closed subset of $C[0,a]$ and the operators $P$ and $T$ transform continuously the set $\Omega$ into $C[0,a]$, just like that $P(\Omega)$ and $T(\Omega)$ are bounded. Furthermore, let the operator $S = P \cdot T$ transform $\Omega$ into itself. If the each operators $P$ and $T$ satisfies the Darbo condition on the set $\Omega$ with the constants $K_1$ and $K_2$, respectively, then the operator $S$ satisfies the Darbo condition on $\Omega$ with the constant*

$$\|P(\Omega)\|K_2 + \|T(\Omega)\|K_1.$$

**Remark 2.1.** *In Theorem 2.2, if $\|P(\Omega)\|K_2 + \|T(\Omega)\|K_1 < 1$, then $S$ is a contraction with respect to the measure $w_0$ and has at least one fixed point in the set $\Omega$.*

Now we will identify solutions of the integral equation (1.10).

## 3. Main result

In this section, we will study the solvability of NLFIE (1.10) for $x \in C[0,a]$, under the following hypotheses.

$(A_1)$ The function $q : [0,a] \to \mathbb{R}$ is continuous and bounded with $k = \sup_{t\in[0,a]} |q(t)|$.

$(A_2)$ The functions $f : [0,a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $F, G : [0,a] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exists nonnegative constants $l, m$ such that

$$|f(t,0,0)| \le l,$$
$$|F(t,0,0,0)| \le m,$$
$$|G(t,0,0,0)| \le m.$$

$(A_3)$ There exists the continuous functions $a_j : [0,a] \to [0,a]$, for $j = 1, 2, ...8$ such that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le a_1(t)|x_1 - y_1| + a_2(t)|x_2 - y_2|,$$
$$|F(t,x_1,y_1,x_2) - F(t,x_3,y_2,x_4)| \le a_3(t)|x_1 - x_3| + a_4(t)|y_1 - y_2| + a_5(t)|x_2 - x_4|,$$
$$|G(t,x_1,y_1,x_2) - G(t,x_3,y_2,x_4)| \le a_6(t)|x_1 - x_3| + a_7(t)|y_1 - y_2| + a_8(t)|x_2 - x_4|,$$

for all $t \in [0,a]$ and $x_1, x_2, x_3, x_4, y_1, y_2 \in \mathbb{R}$.

$(A_4)$ The functions $u = u(t,s,x(\acute{a}(s)))$ and $v = v(t,s,x(c(s)))$ act continuously from the set $[0,a] \times [0,a] \times \mathbb{R}$ into $\mathbb{R}$. Moreover, the functions $\theta, \acute{a}, b, c$ and $d$ transform continuously the interval $[0,a]$ into itself.

$(A_5)$ There exists a nonnegative constant $K$ such that

$$K = \max_j \{a_j(t) : t \in [0,a]\},$$

for $j = 1, 2, ...8$.

$(A_6)$ (Sublinear condition) There exists the constants $\xi$ and $\eta$ such that

$$|u(t,s,x(\acute{a}(s)))| \le \xi + \eta|x|,$$
$$|v(t,s,x(c(s)))| \le \xi + \eta|x|,$$

for all $t, s \in [0,a]$ and $x \in \mathbb{R}$.

$(A_7)$ $4\sigma\tau < 1$, for $\sigma = 4K + Ka\eta$ and $\tau = k + l + Ka\xi + m$.

Now we can formulate the main result of this paper.

**Theorem 3.1.** *Under the assumptions $(A_1) - (A_7)$, NLFIE (1.10) has at least one solution in the Banach algebra $C = C[0,a]$.*

*Proof.* To prove this result using Theorem 2.2, we consider the operators $P$ and $T$ on the Banach algebra $C[0,a]$ in the following way:

$$(Px)(t) = q(t) + f(t,x(t),x(\theta(t))) + F\left(t, x(t), \int_0^t u(t,s,x(\acute{a}(s)))ds, x(b(t))\right),$$

$$(Tx)(t) = G\left(t, x(t), \int_0^a v(t,s,x(c(s)))ds, x(d(t))\right),$$

for $t \in [0, a]$.
Now, taking into account the assumptions $(A_1), (A_2)$ and $(A_4)$, it is clear that $P$ and $T$ transforms the Banach algebra $C[0, a]$ into itself.
Now, the operator $S$ defined on the algebra $C[0, a]$ as follows

$$Sx = (Px) \cdot (Tx).$$

Definitely, $S$ transform $C[0, a]$ into itself.
Next, let us fix $x \in C[0, a]$, then using our imposed assumptions for $t \in [0, a]$, we obtain

$$\begin{aligned}
&|(Sx)(t)| = |(Px)(t)| \times |(Tx)(t)| \\
&= \left| q(t) + f(t, x(t), x(\theta(t))) + F\left(t, x(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, x(b(t))\right)\right| \\
&\quad \times \left| G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right)\right| \\
&\leq \Bigg\{ k + |f(t, x(t), x(\theta(t))) - f(t, 0, 0)| + |f(t, 0, 0)| \\
&\quad + \left| F\left(t, x(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, x(b(t))\right) - F(t, 0, 0, 0)\right| + |F(t, 0, 0, 0)| \Bigg\} \\
&\quad \times \left\{ \left| G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right) - G(t, 0, 0, 0)\right| + |G(t, 0, 0, 0)| \right\} \\
&\leq \left( k + a_1(t)|x(t)| + a_2(t)|x(\theta(t))| + l + a_3(t)|x(t)| + a_4(t)\int_0^t |u(t, s, x(\acute{a}(s)))|ds + a_5(t)|x(b(t))| + m \right) \\
&\quad \times \left( a_6(t)|x(t)| + a_7(t)\int_0^a |v(t, s, x(c(s)))|ds + a_8(t)|x(d(t))| + m \right) \\
&\leq \{k + 4K\|x\| + l + Ka(\xi + \eta\|x\|) + m\} \cdot \{2K\|x\| + Ka(\xi + \eta\|x\|) + m\} \\
&\leq \{(4K + Ka\eta)\|x\| + k + l + Ka\xi + m\}^2.
\end{aligned}$$

Let $\sigma = 4K + Ka\eta$ and $\tau = k + l + ka\xi + m$, then from the above estimate, it follows that

$$\|Px\| \leq \sigma\|x\| + \tau, \tag{3.1}$$

$$\|Tx\| \leq \sigma\|x\| + \tau, \tag{3.2}$$

$$\|Sx\| \leq (\sigma\|x\| + \tau)^2, \tag{3.3}$$

for $x \in C[0, a]$.
From estimate (3.3), we conclude that the operator $S$ maps the ball $B_r \subset C[0, a]$ into itself for $r_1 \leq r \leq r_2$, where

$$r_1 = \frac{1 - 2\sigma\tau - \sqrt{1 - 4\sigma\tau}}{2\sigma^2},$$

$$r_2 = \frac{1 - 2\sigma\tau + \sqrt{1 - 4\sigma\tau}}{2\sigma^2}.$$

In the following, we will assume that $r = r_1$.
Moreover, let us observe that from estimates (3.1) and (3.2), we obtain

$$\|PB_r\| \leq \sigma r + \tau, \tag{3.4}$$

$$\|TB_r\| \leq \sigma r + \tau. \tag{3.5}$$

Now, we have to prove that the operator $P$ is continuous on the ball $B_r$. To do this, fix $\epsilon > 0$ and take arbitrary $x, y \in B_r$ such that $\|x - y\| \leq \epsilon$. Then for $t \in [0, a]$, we have

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \leq |f(t, x(t), x(\theta(t))) - f(t, y(t), y(\theta(t)))| \\
&\quad + \left| F\left(t, x(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, x(b(t))\right) - F\left(t, y(t), \int_0^t u(t, s, y(\acute{a}(s)))ds, y(b(t))\right) \right| \\
&\leq a_1(t)|x(t) - y(t)| + a_2(t)|x(\theta(t)) - y(\theta(t))| \\
&\quad + \left| F\left(t, x(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, x(b(t))\right) - F\left(t, y(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, y(b(t))\right) \right| \\
&\quad + \left| F\left(t, y(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, y(b(t))\right) - F\left(t, y(t), \int_0^t u(t, s, y(\acute{a}(s)))ds, y(b(t))\right) \right| \\
&\leq a_1(t)|x(t) - y(t)| + a_2(t)|x(\theta(t)) - y(\theta(t))| + a_3(t)|x(t) - y(t)| + a_5(t)|x(\theta(t)) - y(\theta(t))| \\
&\quad + a_4(t) \int_0^t |u(t, s, x(\acute{a}(s))) - u(t, s, y(\acute{a}(s)))| ds \\
&\leq 4K\|x - y\| + Ka\ w(u, \epsilon) \\
&\leq 4K\epsilon + Ka\ w(u, \epsilon),
\end{aligned}$$

where

$$w(u, \epsilon) = \sup\{|u(t, s, x) - u(t, s, y)| : t, s \in [0, a]; x, y \in [-r, r]; \|x - y\| \leq \epsilon\}.$$

In view of uniformly continuous of the function $u = u(t, s, x)$ on the bounded subset $[0, a] \times [0, a] \times [-r, r]$, we have that $w(u, \epsilon) \to 0$ as $\epsilon \to 0$. Thus, from the above inequality the operator $P$ is continuous on $B_r$. Similarly, the operator $T$ is also continuous on $B_r$. Hence, we conclude that $S$ is continuous operator on $B_r$.

Next, we prove that the operators $P$ and $T$ satisfies the Darbo condition with respect to the measure $w_0$, defined in Section 2, in the ball $B_r$. To do this, we take a nonempty subset $X$ of $B_r$ and $x \in X$. Let $\epsilon > 0$ be fixed and $t_1, t_2 \in [0, a]$ with $t_2 - t_1 \leq \epsilon$ and we can assume that $t_1 \leq t_2$. Then, taking into account our assumptions, it follows

$$\begin{aligned}
|(Px)(t_2) - (Px)(t_1)| &\leq |q(t_2) - q(t_1)| + |f(t_2, x(t_2), x(\theta(t_2))) - f(t_1, x(t_1), x(\theta(t_1)))| \\
&\quad + \left| F\left(t_2, x(t_2), \int_0^{t_2} u(t_2, s, x(\acute{a}(s)))ds, x(b(t_2))\right)\right. \\
&\quad \left. - F\left(t_1, x(t_1), \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds, x(b(t_1))\right) \right| \\
&\leq w(q, \epsilon) + |f(t_2, x(t_2), x(\theta(t_2))) - f(t_2, x(t_1), x(\theta(t_1)))| + |f(t_2, x(t_1), x(\theta(t_1))) \\
&\quad - f(t_1, x(t_1), x(\theta(t_1)))| + \left| F\left(t_2, x(t_2), \int_0^{t_2} u(t_2, s, x(\acute{a}(s)))ds, x(b(t_2))\right)\right.
\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
& - F\left(t_2, x(t_1), \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds, x(b(t_1))\right)\Bigg| \\
& + \left|F\left(t_2, x(t_1), \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds, x(b(t_1))\right)\right. \\
& - F\left(t_1, x(t_1), \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds, x(b(t_1))\right)\Bigg| \\
\leq\ & w(q,\epsilon) + a_1(t)|x(t_2) - x(t_1)| + a_2(t)|x(\theta(t_2)) - x(\theta(t_1))| + w_f(\epsilon,.,.) + a_3(t)|x(t_2) - x(t_1)| \\
& + a_4(t)\left|\int_0^{t_2} u(t_2, s, x(\acute{a}(s)))ds - \int_0^{t_1} u(t_1, s, x(\acute{a}(s)))ds\right| + a_5(t)|x(b(t_2)) - x(b(t_1))| \\
& + w_F(\epsilon,.,.,.) \\
\leq\ & w(q,\epsilon) + 2Kw(x,\epsilon) + Kw(x, w(\theta,\epsilon)) + w_f(\epsilon,.,.) \\
& + K\left\{\int_0^{t_1} |u(t_2, s, x(\acute{a}(s))) - u(t_1, s, x(\acute{a}(s)))|ds + \int_{t_1}^{t_2} |u(t_2, s, x(\acute{a}(s)))|ds\right\} \\
& + Kw(x, w(b,\epsilon)) + w_F(\epsilon,.,.,.)
\end{aligned}
$$

$$
\begin{aligned}
w(Px,\epsilon) \leq\ & w(q,\epsilon) + 2Kw(x,\epsilon) + Kw(x, w(\theta,\epsilon)) + w_f(\epsilon,.,.) + K\{w_u(\epsilon,.,.)a + K'\epsilon\} \\
& + Kw(x, w(b,\epsilon)) + w_F(\epsilon,.,.,.)
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
w_f(\epsilon,.,.) &= \sup\{|f(t, x_1, x_2) - f(t', x_1, x_2)| : t, t' \in [0,a]; |t - t'| \leq \epsilon; x_1, x_2 \in [-r, r]\}, \\
w_u(\epsilon,.,.) &= \sup\{|u(t, s, x) - u(t', s, x)| : t, t' \in [0,a]; |t - t'| \leq \epsilon; x \in [-r, r]\}, \\
w_F(\epsilon,.,.,.) &= \sup\{|F(t, x_1, y_1, x_2) - F(t', x_1, y_1, x_2)| : t, t' \in [0,a]; |t - t'| \leq \epsilon; x_1, x_2 \in [-r, r]; \\
& \qquad y_1 \in [-K'a, K'a]\}, \\
K' &= \sup\{|u(t, s, x)| : t, s \in [0,a];\ x \in [-r, r]\}.
\end{aligned}
$$

Since, the functions $q = q(t)$, $f = f(t, x_1, x_2)$ and $F = F(t, x_1, y_1, x_2)$ are uniformly continuous on the set $[0,a]$, $[0,a]\times\mathbb{R}\times\mathbb{R}$ and $[0,a]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, respectively, and the function $u = u(t, s, x)$ is also uniformly continuous on the set $[0,a]\times[0,a]\times\mathbb{R}$. Hence, we deduce that $w(q,\epsilon) \to 0, w_f(\epsilon,.,.) \to 0, w_u(\epsilon,.,.) \to 0$ and $w_F(\epsilon,.,.,.) \to 0$ as $\epsilon \to 0$. Thus, from the above estimate (3.6) we conclude

$$
w_0(PX) \leq 4Kw_0(X). \tag{3.8}
$$

Similarly, we can show that

$$
w_0(TX) \leq 2Kw_0(X). \tag{3.9}
$$

Finally, from the estimates (3.4), (3.5), (3.7), (3.8) and keeping in mind Theorem 2.2, we conclude that the operator $S$ satisfies the Darbo condition on $B_r$ with respect to the measure $w_0$ with constant $4K(\sigma r + \tau) + 2K(\sigma r + \tau)$. Thus, we have

$$
\begin{aligned}
6K(\sigma r + \tau) &= 6K(\sigma r_1 + \tau) \\
&= 6K\left\{\sigma\left(\frac{(1 - 2\sigma\tau) - \sqrt{1 - 4\sigma\tau}}{2\sigma^2}\right) + \tau\right\} \\
&= \frac{3K}{\sigma}(1 - \sqrt{1 - 4\sigma\tau}).
\end{aligned}
$$

Taking into account the assumption $(A_7)$, since $1-\sqrt{1-4\sigma\tau}<1$ and $\dfrac{3K}{\sigma}=\dfrac{3K}{4K+Ka\eta}<1$. Therefore, the operator $S$ is a contraction on $B_r$ with respect to measure $w_0$. Thus, $S$ has at least one fixed point in the ball $B_r$, by applying Theorem 2.2 and Remark 2.1. Consequently, the NLFIE (1.10) has at least one solution in the ball $B_r$. □

## 4. An example

Now, we begin with an example of a NLFIE and to illustrate the existence of its solutions by using Theorem 3.1.

**Example 4.1.** *Consider the following NLFIE:*

$$x(t)=\left[te^{-(t+3)}+\frac{t}{7(1+t)}\arctan|x(t)|+\frac{t}{16}\ln(1+|x(1-t)|)+\frac{1}{12}\int_0^t\left\{\frac{\cos(x(1-s))}{3}\right.\right.$$

$$\left.\left.+2t\arctan\left(\frac{|x(1-s)|}{1+|x(1-s)|}\right)\right\}ds\right]\times\left[\frac{1}{17}\int_0^1\left\{\frac{t\sin x(\sqrt{s})}{3}+(1+t)\ln(1+|x(\sqrt{s})|)\right\}ds\right], \tag{4.1}$$

*where $t\in[0,1]$.*

*Observe that equation (4.1) is a particular case of equation (1.10). Let us take $q:[0,1]\to\mathbb{R}; f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}; F,G:[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $u,v:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ and comparing (4.1) with equation (1.10), we get*

$$q(t)=te^{-(t+3)},f(t,x_1,x_2)=\frac{t}{7(1+t)}\arctan|x_1|+\frac{t}{16}\ln(1+|x_2|),$$

$$F(t,x_1,y_1,x_2)=\frac{1}{12}y_1,G(t,x_1,y_1,x_2)=\frac{1}{17}y_1,$$

$$u(t,s,x)=\frac{\cos x}{3}+2t\arctan\left(\frac{|x|}{1+|x|}\right),v(t,s,x)=\frac{t\sin x}{3}+(1+t)\ln(1+|x|),$$

*then we can easily test that the assumptions of Theorem 3.1 are satisfied. In fact, we have that the function $q(t)$ is continuous and bounded on $[0,1]$ with $k=e^{-4}=0.0183156\ldots$ . Thus, the assumption $(A_1)$ is satisfied. Moreover, these functions are continuous and satisfies the assumption $(A_3)$ with $a_1=\frac{1}{14},a_2=\frac{1}{16},a_3=a_5=a_6=a_8=0,a_4=\frac{1}{12},a_7=\frac{1}{17}$.*
*In this case, we have*

$$K=\max\left\{\frac{1}{14},\frac{1}{16},0,\frac{1}{12},\frac{1}{17}\right\}=\frac{1}{12}.$$

*Further,*

$$|f(t,0,0)|=0,|F(t,0,0,0)|=0,|G(t,0,0,0)|=0,$$

$$|u(t,s,x)|\le\frac{1}{3}+2|x|,|v(t,s,x)|\le\frac{1}{3}+2|x|.$$

*It is observed that $l=m=0,\xi=\frac{1}{3},\eta=2$ and $a=1$.*
*Finally, we see that*

$$4\sigma\tau=4(4K+Ka\eta)(k+l+Ka\xi+m)<1.$$

*Hence, all the assumptions from $(A_1)$ to $(A_7)$ are satisfied. Now, on the basis of result obtained in Theorem 3.1, we deduce that NLFIE (4.1) has at least one solution in Banach algebra $C[0,1]$.*

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[1]Department of Mathematics, National Institute of Technology, Silchar 788 010, Cachar, Assam, India

[2]L. 1627 Awadh Puri Colony, Phase III, Beniganj, Opposite Industrial Training Institute (I.T.I.), Ayodhya Main Road, Faizabad 224 001, Uttar Pradesh, India

[3]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada

[4]China Medical University, Taichung 40402, Taiwan, Republic of China

[*]Corresponding author: lakshminarayanmishra04@gmail.com