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COMMON FIXED POINT THEOREMS FOR G–CONTRACTION IN C^*–ALGEBRA–VALUED METRIC SPACES

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Abstract. In this paper we prove the common fixed point theorems for two mappings in complete C^*–valued metric space endowed with the graph $G = (V, E)$, which satisfies G-contractive condition. Also, we provide an example in support of our main result.

1. Introduction and Preliminaries

The Banach contraction principle [5] plays an important role in solving non linear problems. The Banach contraction principle says that: if (X, d) be a complete metric space and f is a self mapping on X with the condition that there exists $\lambda \in (0, 1)$ such that

$$d(fx, fy) \leq \lambda d(x, y) \quad \text{for all } x, y \in X,$$

then f has a unique fixed point in X. Since then a lot of publications are devoted to the study and solutions of many practical and theoretical problems by using this condition. Due to a numerous applications of the fixed point theory, from the last few decades this theory is a central topic of research. In this theory one of the approach is the common fixed point theorems. The concept of the common fixed point theorems was investigated by Jungck [1]. Many authors studied the fixed and common fixed point theorems for different spaces, like in cone metric spaces [8], non-commutative Banach spaces [22], fuzzy metric spaces [14] and uniform metric spaces [21]. For more information about this topic see ([1, 6, 7, 9, 17, 18, 23]).

On the other hand the concept of C^*–algebra is well developed. Here we recall some basic definitions, notations and results of C^*–algebra that may be found in [13]. A $*$-algebra $\mathcal{A}$ is a complex algebra with linear involution $*$ such that $x^{**} = x$ and $(xy)^* = y^*x^*$, for any $x, y \in \mathcal{A}$. If $*$-algebra together with complete sub multiplicative norm satisfying $||x^*|| = ||x||$ for all $x \in \mathcal{A}$, then $*$-algebra is said to be a Banach $*$-algebra. A C^*–algebra is a Banach $*$-algebra such that $||x^*x|| = ||x||^2$ for all $x \in \mathcal{A}$. An element of $\mathcal{A}$ is called positive element, if $\mathcal{A}_+ = \{x^* = x | x \in \mathcal{A}\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the spectrum of an element $x \in \mathcal{A}$, i.e., $\sigma(x) = \{\lambda \in \mathbb{C} : \lambda I - x \text{ is not invertible}\}$. There is a natural partial ordering on $\mathcal{A}_+$ given by $x \preceq y$ if and only if $x - y \in \mathcal{A}_+$. In [12] Z. Ma et al., introduced the notion of C^*-algebra valued metric space and proved fixed point theorems for C^*-algebra valued contractive mapping.

Many researchers tried to obtain some fixed point theorems of Banach type contraction endowed with the graph G, we recommend [2, 3, 4, 15, 16, 20]. Recently, T. Kamran et al., in [19] extended the results of Ma et al., which was given in[12], by using C^*–valued metric spaces and G-contraction principles.

Now we give some definitions of graph theory which is found in any text on graph theory, for example [11]. Following Jachymski [10], let Δ denote the diagonal of the $X \times X$ in a metric space (X, d), and consider a directed graph $G = (V(G), E(G)) = (V, E)$ the set in which V of its vertices and E of its edges, and $\Delta \subseteq E$. Assume that G has no parallel edges. We may treat G as a weighted graph by assigning to each edge the distance between its vertices.

In this paper we will continue to study common fixed points in the C^*–valued metric space endowed with the graph G under G–contractive condition.

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Definition 1.1. *Let X be a nonempty set, and the mapping $d : X \times X \to \mathcal{A}$ endowed with the graph $G = (V, E)$, if it satisfies the following conditions:*
(1) $d(x, y) \geq 0$ *for all* $x, y \in X$ *and* $d(x, y) = 0 \Leftrightarrow x = y$*;*
(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*
(3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$*.*
Then d is called a C^*–valued metric on X, and $(X, d, \mathcal{A})$ is called C^*–valued metric space.

Definition 1.2. *Suppose that $(X, d, \mathcal{A})$ is a C^*–valued metric space. Let $x \in (X, d, \mathcal{A})$ and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to be convergent, if for any $\epsilon > 0$ there exists a positive integer N such that*

$$||d(x_n, x)|| \leq \epsilon \quad \text{for all } n \geq N.$$

The sequence $\{x_n\}$ is said to be Cauchy, if for any $\epsilon > 0$ there exists a positive integer N such that

$$||d(x_n, x_m)|| \leq \epsilon \quad \text{for all } n, m \geq N.$$

If every Cauchy sequence is convergent in $(X, d, \mathcal{A})$, then $(X, d, \mathcal{A})$ is said to be complete C^*–valued metric space.

Example 1.3. *Let $X = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{R})$. Define $d : X \times X \to \mathcal{A}$ such that*

$$d(x, y) = \begin{pmatrix} |x - y| & 0 \\ 0 & \alpha|x - y| \end{pmatrix} \quad \text{for all } x, y \in \mathbb{R} \text{ and } \alpha \geq 0.$$

It is essay to verify that d is a C^*–algebra valued metric space and $(X, d, M_2(\mathbb{R}))$ is a complete C^*–algebra valued metric space.

Definition 1.4. *Let $(X, d, \mathcal{A})$ be a C^*–valued metric space. A mapping $f : X \to X$ is said to be a C^*–algebra–valued contraction mapping on X if there exists an $a \in \mathcal{A}$ with $||a|| < 1$ such that*

$$d(fx, fy) \leq a^* d(x, y)a, \quad \text{for all } x,\ y \in X. \tag{1.1}$$

Theorem 1.5. [12] *Let $(X, d, \mathcal{A})$ be a complete C^*–algebra-valued metric space and f satisfies (1.1), then f has a unique fixed point in X.*

Property 1.6. [12]

(1) *For any $\{x_n\} \in X$ such that x_n converges to x with $(x_{n+1}, x_n) \in E$ for all $n \geq 1$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x, x_{n_k}) \in E$.*
(2) *For any $\{f^n x\} \in X$ such that $f^n x$ converges to $x \in X$ with $(f^{n+1}x, f^n x) \in E$ there exists a subsequence $\{f^{n_k} x\}$ and $n_0 \in \mathbb{N}$ such that $(x, f^{n_k} x) \in E$ for all $k \geq n_0$.*

2. Main Result

In this section, we prove common fixed point theorems for two mappings satisfying G–contractive condition in a complete C^*–valued metric space endowed with the graph $G = (V, E)$.

Definition 2.1. *Let $(X, d, \mathcal{A})$ be a C^*–valued metric space endowed with the graph $G = (V, E)$. The mappings $f, g : X \to X$ are said to be C^*–valued G–contractive on X, if there exists an $a \in \mathcal{A}$ with $||a|| < 1$ such that*

$$d(fx, gy) \leq a^* d(x, y)a, \quad \text{for all} \quad (x, y) \in E. \tag{2.1}$$

Theorem 2.2. *Let $(X, d, \mathcal{A})$ is a complete C^*–valued metric space endowed with the graph $G = (V, E)$. Suppose that the mappings $f, g : X \to X$ are C^*–valued G–contractive mappings on X satisfying the Property 1.6 (2) and the following conditions*
(1) *if $(x, y) \in E$ then $(fx, gy) \in E$,*
(2) *there exists $z_0 \in X$ such that $(z_0, fz_0), (z_0, gz_0) \in E$.*
Then f and g has a unique common fixed point in X.

Proof. Let $z_1 \in X$, and construct sequence $\{z_n\} \in X$, such that $z_{2n+1} = fz_{2n}$, $z_{2n+2} = gz_{2n+1}$, and $(z_{2n-1}, z_{2n}) \in E$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} d(z_{2n+1}, z_{2n+2}) &= d(gz_{2n+1}, fz_{2n}) \\ &\leq a^* d(z_{2n+1}, z_{2n}) a \\ &\leq (a^*)^2 d(z_{2n}, z_{2n-1})(a)^2 \\ &\cdot \\ &\cdot \\ &\cdot \\ &\leq (a^*)^{2n+1} d(z_1, z_0)(a)^{2n+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} d(z_{2n+1}, z_{2n}) &= d(fz_{2n}, gz_{2n-1}) \\ &\leq a^* d(z_{2n}, z_{2n-1}) a \\ &\cdot \\ &\cdot \\ &\cdot \\ &\leq (a^*)^{2n} d(z_1, z_0)(a)^{2n} \\ &= (a^*)^{2n} Q(a)^{2n}. \end{aligned}$$

Let us denote $d(z_1, z_0)$ by $Q \in \mathcal{A}$. Then for any $n \in \mathbb{N}$

$$\begin{aligned} d(z_{n+1}, z_n) &= (a^*)^n d(z_1, z_0)(a)^n \\ &= (a^*)^n Q(a)^n, \end{aligned}$$

then for any $q \in \mathbb{N}$ and applying the triangular inequality (3) for the C^*–valued metric spaces,

$$\begin{aligned} d(z_{n+q}, z_n) &= d(z_{n+q}, z_{n+q-1}) + d(z_{n+q-1}, z_{n+q-2}) + \cdots + d(z_{n+1}, z_n) \\ &\leq \sum_{j=n}^{n+q-1} (a^*)^j d(z_1, z_0)(a)^j \\ &= \sum_{j=n}^{n+q-1} (a^*)^j Q(a)^j \\ &= \sum_{j=n}^{n+q-1} (a^*)^j Q^{\frac{1}{2}} Q^{\frac{1}{2}} (a)^j \\ &= \sum_{j=n}^{n+q-1} (Q^{\frac{1}{2}} a^j)^* (Q^{\frac{1}{2}} a^j) \\ &= \sum_{j=n}^{n+q-1} |Q^{\frac{1}{2}} a^j|^2 \\ &\leq \sum_{j=n}^{n+q-1} || \, |Q^{\frac{1}{2}} a^j|^2 ||.I \\ &= ||Q^{\frac{1}{2}}||^2 \sum_{j=n}^{n+q-1} ||a^{2j}||. \end{aligned}$$

Since $||a|| < 1$, thus $d(z_{n+q}, z_n) \to 0$ as $n \to \infty$. Thus we conclude that the sequence $\{z_n\}$ is a Cauchy sequence, with respect to $\mathcal{A}$. Using the completeness of X, there exists an element $z_0 \in X = V$, such that $z_n \to z_0$ as $n \to \infty$.

On the other hand, using the triangular inequality, we get

$$\begin{aligned} d(z_0, fz_0) &= d(z_0, z_{2n+1}) + d(z_{2n+1}, fz_0) \\ &= d(z_0, z_{2n+1}) + d(gz_{2n}, fz_0) \\ &\leq d(z_0, z_{2n+1}) + a^* d(z_{2n}, z_0)a. \end{aligned}$$

Thus if $n \to \infty$, then $d(z_0, fz_0) \to 0$ i.e. $fz_0 = z_0$. Similarly we can prove that $gz_0 = z_0$. Now we will show the uniqueness of common fixed points in X. For this we assume that there is another point $z^* \in X = V$, such that$(z_0, z^*) \in E$. Consider

$$d(z_0, z^*) = d(fz_0, gz_0) \leq a^* d(z_0, z^*)a.$$

Since $||a|| < 1$, then the above inequality yields that

$$0 \leq ||d(z_0, z^*)|| \leq ||a||^2 ||d(z_0, z^*)|| < ||d(z_0, z^*)||.$$

Which is a contradiction. Thus, $||d(z_0, z^*)|| = 0$ which implies that $d(z_0, z^*) = 0$ i.e. $z_0 = z^*$. Thus the proof is complete.

Corollary 2.3. *Suppose that $(X, d, \mathcal{A})$ is a C^*–valued metric space endowed with the graph G, and suppose that the mappings $f, g : X \to X$ are G–contractive, satisfying*

$$||d(fx, gy)|| \leq ||a|| ||d(x, y)||, \text{ for all } (x, y) \in E,$$

where $a \in \mathcal{A}$ with $||a|| < 1$. Then f and g have a unique common fixed point in X.

Corollary 2.4. *Let $(X, d, \mathcal{A})$ is a C^*–valued metric space endowed with the graph G, and suppose that the mapping $f : X \to X$ is G–contractive, satisfying*

$$||d(f^m x, f^n y)|| \leq a^* d(x, y)a, \quad \text{for all } (x, y) \in E,$$

where $a \in \mathcal{A}$ with $||a|| < 1$ and m, n are positive integers. Then f has a unique fixed point in X.

Remark 2.5. *In Theorem 2.2, if $g = f$, then we have*

$$d(fx, fy) \leq a^* d(x, y)a, \quad \text{for all } (x, y) \in E. \tag{2.2}$$

In this case we have the following corollary, which can also be found in [12].

Corollary 2.6. *Let $(X, d, \mathcal{A})$ be a complete C^*–valued metric space, and consider the mapping $f : X \to X$ such that it satisfies (2.2), then f has a unique fixed point in X.*

Example 2.7. *Consider, $\mathcal{A} = M_{2\times 2}(\mathbb{R})$, of all 2×2 matrices with the usual operation of addition, scalar multiplication, and matrix multiplication. Thus $\mathcal{A}$ becomes C^*–algebra. Let us define $d : \mathbb{R}\times\mathbb{R} \to \mathcal{A}$ by*

$$d(x, y) = \begin{pmatrix} |x - y| & 0 \\ 0 & |x - y| \end{pmatrix}.$$

It is essay to check that d satisfies all the conditions of Definition 1.1. Therefore $(\mathbb{R}, \mathcal{A}, d)$ is C^*–valued metric space. Define $f, g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{x^2}{4} \quad \text{and} \quad g(x) = \frac{x^2}{3},$$

and consider the graph $G = (V, E)$, where $V = \mathbb{R}$ and

$$E = \Big\{\Big(\frac{1}{4^m}, \frac{1}{3^{2m+1}}\Big); m = 1, 2, \dots\Big\} \cup \Big\{\Big(\frac{1}{4^m}, 0\Big); m = 1, 2, \dots\Big\} \cup \{(x, x); x \in \mathbb{R}\}.$$

Note that, for each $m \in \mathbb{N}$,

$$\Big(f(\frac{1}{4^m}), g(\frac{1}{3^{2m+1}})\Big) = \Big(\frac{1}{4^{2n+1}}, \frac{1}{3^{4n+3}}\Big) \in E,$$

and

$$\Big(f(\frac{1}{4^m}), g(0)\Big) = \Big(\frac{1}{4^{2m+1}}, 0\Big) \in E.$$

Also, $(fx, gx) = (\frac{x^2}{4}, \frac{x^2}{3})$*, for each* $x \in \mathbb{R}$*, which is again in* E*. Moreover, by taking* $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$*, we have* $||A|| < 1$*, so all the conditions of Theorem 2.2 are satisfied and thus the common fixed point of* f *and* g *is 0.*

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