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GROWTH AND ZEROS OF MEROMORPHIC SOLUTIONS TO SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. The main purpose of this article is to investigate the growth of meromorphic solutions to homogeneous and non-homogeneous second order linear differential equations $f'' + Af' + Bf = F$, where $A(z)$, $B(z)$ and $F(z)$ are meromorphic functions with finite order having only finitely many poles. We show that, if there exist a positive constants $\sigma > 0$, $\alpha > 0$ such that $|A(z)| \geq e^{\alpha|z|^{\sigma}}$ as $|z| \to +\infty$, $z \in H$, where $\overline{dens}\{|z| : z \in H\} > 0$ and $\rho = \max\{\rho(B), \rho(F)\} < \sigma$, then every transcendental meromorphic solution f has an infinite order. Further, we give some estimates of their hyper-order, exponent and hyper-exponent of convergence of distinct zeros.

1. Introduction and statement of results

We will assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna theory of meromorphic functions (see [11], [14], [16]). In addition, for a meromorphic function f in the complex plane $\mathbb{C}$, we will use the notations $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponent of convergence of the zeros and the distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f.

In order to estimate the rate of growth of meromorphic function of infinite order more precisely, we recall the following definition.

Definition 1.1 ([13, 16]). Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. If f is an entire function, then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 1.2 ([7]). Let f be a meromorphic function. Then the hyper-exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r, \frac{1}{f}\right)}{\log r},$$

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where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z:|z|\leq r\}$. Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f)=\limsup_{r\to+\infty}\frac{\log\log\overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z:|z|\leq r\}$.

The linear measure of a set $E\subset(0,+\infty)$ is defined as $m(E)=\int_0^{+\infty}\chi_E(t)\,dt$. The logarithmic measure of a set $E\subset(1,+\infty)$ is defined by $lm(E)=\int_1^{+\infty}\frac{\chi_E(t)}{t}dt$, where $\chi_E(t)$ is the characteristic function of the set E. The upper density of a set $E\subset(0,+\infty)$ is defined by

$$\overline{dens}E=\limsup_{r\longrightarrow+\infty}\frac{m(E\cap[0,r])}{r}.$$

Consider the second-order linear differential equation

$$f''+A(z)f'+B(z)f=F, \tag{1.1}$$

where $A(z)$, $B(z)$ and $F(z)$ are meromorphic functions of finite order having only finitely many poles. Several authors have investigated the growth of solutions of the corresponding homogeneous equation

$$f''+A(z)f'+B(z)f=0. \tag{1.2}$$

From the works of Gundersen (see [10]) and Hellerstein et al. (see [12]), we know that if $A(z)$ and $B(z)$ are entire functions with $\rho(A)<\rho(B)$, or $A(z)$ is a polynomial, and $B(z)$ is transcendental, or $\rho(A)<\rho(B)\leq\frac{1}{2}$, then every solution $f\not\equiv0$ of (1.2) is of infinite order. For entire solutions of infinite order more precise estimates for their rate of growth would be an important achievement. Kwon (see [13]) and Chen and Yang (see [7]) have investigated the hyper-order $\rho_2(f)$ of solutions of (1.2), and obtained the following results.

Theorem A ([13]). *Let H be a set of complex numbers satisfying $\overline{dens}\{|z|:z\in H\}>0$, and let $A(z)$ and $B(z)$ be entire functions such that for real constants $\alpha(>0)$, $\beta(>0)$,*

$$|A(z)|\leq\exp\left\{o(1)|z|^{\beta}\right\}$$

and

$$|B(z)|\geq\exp\left\{(1+o(1))\alpha|z|^{\beta}\right\}$$

as $z\to+\infty$ for $z\in H$. Then every solution $f\not\equiv0$ of equation (1.2) has infinite order and $\rho_2(f)\geq\beta$.

Theorem B ([7]). *Let H be a set of complex numbers satisfying $\overline{dens}\{|z|:z\in H\}>0$, and let $A(z)$ and $B(z)$ be entire functions with $\rho(A)\leq\rho(B)=\rho<+\infty$ such that for real constant $C(>0)$ and for any given $\varepsilon>0$,*

$$|A(z)|\leq\exp\left\{o(1)|z|^{\rho-\varepsilon}\right\}$$

and

$$|B(z)|\geq\exp\left\{(1+o(1))C|z|^{\rho-\varepsilon}\right\}$$

as $z\to\infty$ for $z\in H$. Then every solution $f\not\equiv0$ of equation (1.2) has infinite order and $\rho_2(f)=\rho(B)$.

These results were improved by Belaïdi in [2, 3] by considering more general conditions to higher order linear differential equations with entire coefficients. Recently in [8] Chen extended the previous results by studying the zeros and the growth of meromorphic solutions of equation (1.1) when $A(z)$, $B(z)$, $F(z)$ are meromorphic functions.

There exists a natural question: How about the growth of (1.1) when $A(z)$, $B(z)$ and $F(z)$ are meromorphic functions of finite order having only finitely many poles and the dominant coefficient is $A(z)$ instead of $B(z)$?

In this paper, we answer the above question and obtain the following results.

Theorem 1.1 *Let $H \subset [0,+\infty)$ be a set with a positive upper density, and let $A(z)$, $B(z)$ and $F(z)$ be meromorphic functions of finite order having only finitely many poles. Suppose there exist positive constants $\sigma > 0$, $\alpha > 0$ such that $|A(z)| \geq e^{\alpha r^{\sigma}}$ as $|z| = r \in H$, $r \to +\infty$, and $\rho = \max\{\rho(B), \rho(F)\} < \sigma$. Then every transcendental meromorphic solution f of equation* (1.1) *satisfies*

$$\rho(f) = +\infty \quad \text{and} \quad \rho_2(f) \leq \rho(A).$$

Furthermore, if $F(z) \not\equiv 0$ then every transcendental meromorphic solution f of equation (1.1) *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq \rho(A).$$

Remark 1.1 It is clear that $\rho(A) = \beta \geq \sigma$ in Theorem 1.1. Indeed, suppose that $\rho(A) = \beta < \sigma$. Then, by using Lemma 2.2 of this paper, there exists a set $E_2 \subset (1,+\infty)$ that has finite linear measure such that when $|z| = r \notin [0,1] \cup E_2$, $r \longrightarrow +\infty$, we have for any given ε $(0 < \varepsilon < \sigma - \beta)$

$$|A(z)| \leq e^{r^{\beta+\varepsilon}}. \tag{1.3}$$

On the other hand, by the hypotheses of Theorem 1.1, there exist positive constants $\sigma > 0$, $\alpha > 0$ such that

$$|A(z)| \geq e^{\alpha r^{\sigma}} \tag{1.4}$$

as $|z| = r \in H$, $r \to +\infty$, where H is a set with $m(H) = \infty$. From (1.3) and (1.4), we obtain for $|z| = r \in H \setminus [0,1] \cup E_1$, $r \to +\infty$

$$e^{\alpha r^{\sigma}} \leq |A(z)| \leq e^{r^{\beta+\varepsilon}}$$

and by ε $(0 < \varepsilon < \sigma - \beta)$ this is a contradiction as $r \to +\infty$. Hence $\rho(A) = \beta \geq \sigma$.

Corollary 1.1 *Let $A(z)$, $B(z)$, $F(z)$ be meromorphic functions of finite order having only finitely many poles such that $\rho = \max\{\rho(B), \rho(F)\} < \rho(A) = \sigma < \frac{1}{2}$. Then every transcendental meromorphic solution f of equation* (1.1) *satisfies*

$$\rho(f) = +\infty \text{ and } \rho_2(f) \leq \rho(A) = \sigma.$$

Furthermore, if $F(z) \not\equiv 0$ then every transcendental meromorphic solution f of equation (1.1) *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty \text{ and } \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq \rho(A) = \sigma.$$

2. Lemmas for the proofs of Theorems

Our results depend mainly on the following lemmas.

Lemma 2.1 ([9]). *Let $f(z)$ be a transcendental meromorphic function of finite order ρ, and let $\varepsilon > 0$ be a given constant. Then, there exists a set $E_0 \subset (1,+\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E_0 \cup [0,1]$, and for all k, j, $0 \leq j < k$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

Similarly, there exists a set $E_1 \subset [0,2\pi)$ that has linear measure zero such that for all $z = re^{i\theta}$ with $|z|$ sufficiently large and $\theta \in [0,2\pi) \setminus E_1$, and for all k, j, $0 \leq j < k$, the inequality (2.1) *holds.*

Lemma 2.2 ([6]). *Let $f(z)$ be a meromorphic function of order $\rho(f) = \rho < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E_2 \subset (1,+\infty)$ that has finite linear measure and finite logarithmic measure such that when $|z| = r \notin [0,1] \cup E_2$, $r \longrightarrow +\infty$, we have*

$$|f(z)| \leq \exp\left\{r^{\rho+\varepsilon}\right\}.$$

Lemma 2.3 ([14]). *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with $a_n \neq 0$ ($n \geq 1$ is an integer) be a non constant polynomial. Then for every $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that for all z, $|z| = r > R$, we have*

$$(1-\varepsilon)|a_n| r^n \leq |P(z)| \leq (1+\varepsilon)|a_n| r^n.$$

Lemma 2.4 ([1]). *Suppose that $k \geq 2$ and $A_0, A_1, A_2, \cdots, A_{k-1}$ (for at least $A_s \not\equiv 0$, $s \in \{0, 1, \cdots, k-1\}$) are meromorphic functions that have finitely many poles.*
Let $\rho = \max\{\rho(A_j)\ (j = 0, 1, \cdots, k-1),\ \rho(F)\} < +\infty$ and let $f(z)$ be a meromorphic solution of infinite order of equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F.$$

Then $\rho_2(f) \leq \rho$.

Lemma 2.5 ([1]). *Let $f(z)$ be a meromorphic function having only finitely many poles, and suppose that*

$$G(z) := \frac{\log^+ \left|f^{(s)}(z)\right|}{|z|^\rho}, \quad (s \geq 1 \text{ is an integer})$$

is unbounded on some ray $\arg z = \theta$ with constant $\rho > 0$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ ($n = 1, 2, \cdots$) tending to infinity such that $G(z_n) \to \infty$ and

$$\left|\frac{f^{(j)}(z_n)}{f^{(s)}(z_n)}\right| \leq \frac{1}{(s-j)!}(1+o(1))|z_n|^{s-j} \quad (j = 0, \cdots, s-1) \quad \text{as} \quad n \to +\infty.$$

Lemma 2.6 ([15]). *Let $f(z)$ be an entire function with $\rho(f) < +\infty$. Suppose that there exists a set $E_3 \subset [0, 2\pi]$ which has linear measure zero, such that $\log^+ \left|f\left(re^{i\theta}\right)\right| \leq Mr^\sigma$ for any ray $\arg(z) = \theta \in [0, 2\pi] \backslash E_3$, where M is a positive constant depending on θ, while σ is a positive constant independent of θ. Then $\rho(f) \leq \sigma$.*

Lemma 2.7 ([4]). *Let $f(z)$ be an entire function of order ρ where $0 < \rho(f) = \rho < \frac{1}{2}$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H \subset [0, +\infty)$ with $\overline{dens} H \geq 1 - 2\rho$ such that for all z satisfying $|z| = r \in H$, we have*

$$|f(z)| \geq \exp\left\{r^{\rho-\varepsilon}\right\}.$$

Lemma 2.8 ([5]). *Let $A_j\ (j = 0, 1, \cdots, k-1)$, $F \not\equiv 0$ be finite order meromorphic functions. If $f(z)$ is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then f satisfies $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

3. Proof of Theorem 1.1

Assume that f is a transcendental ($f' \not\equiv 0$) meromorphic solution of (1.1) with $\rho(f) < \sigma$. It follows from (1.1) that

$$-\frac{f''}{f'} - B(z)\frac{f}{f'} + \frac{F(z)}{f'} = A(z). \tag{3.1}$$

Since $\rho = \max\{\rho(B),\ \rho(F)\} < \sigma$, then the order of growth of the left side of equation (3.1) is $\rho_1 = \max\{\rho(B),\ \rho(F), \rho(f)\} < \sigma$, hence $\rho(A) \leq \rho_1$. By Lemma 2.2, for any given ε $(0 < \varepsilon < \sigma - \rho_1)$, there exists a set $E_2 \subset (1, +\infty)$ with a finite linear measure and finite logarithmic measure such that

$$|A(z)| \leq e^{r^{\rho_1+\varepsilon}} \tag{3.2}$$

holds for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, $r \to +\infty$. From hypotheses of Theorem 1.1, there exist a set H with $\overline{dens} H > 0$ and positive constants $\sigma > 0$, $\alpha > 0$ such that

$$|A(z)| \geq e^{\alpha r^\sigma} \tag{3.3}$$

holds for all z satisfying $|z| = r \in H$, $r \to +\infty$. By (3.2) and (3.3), we conclude that

$$e^{\alpha r^\sigma} \leq e^{r^{\rho_1+\varepsilon}}$$

that is,

$$e^{(1-o(1))\alpha r^{\sigma}} \leq 1$$

for all z satisfying $|z| = r \in H \setminus [0,1] \cup E_2$, $r \to +\infty$, this contradicts the fact $e^{(1-o(1))\alpha r^{\sigma}} \to +\infty$. Consequently, any transcendental meromorphic solution f of (1.1) is of $\rho(f) \geq \sigma$.

Now, we prove that $\rho(f) = +\infty$. Let f be a transcendental meromorphic solution of (1.1). We assume that f is of finite order and $\rho(f) = \delta$. Then, we have $\rho(f) = \delta \geq \sigma$. It follows from (1.1) that

$$|A| \leq \left|\frac{f''}{f'}\right| + |B|\left|\frac{f}{f'}\right| + \left|\frac{F}{f'}\right|. \tag{3.4}$$

By Lemma 2.1, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\theta \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| = r \geq R_0$, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \leq r^{2\delta}. \tag{3.5}$$

We now proceed to show that

$$G(z) = \frac{\log^+ |f'(z)|}{|z|^{\rho+\varepsilon}}$$

is bounded on the ray $\arg z = \theta$. Supposing that this is not the case, then by Lemma 2.5, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \cdots)$ tending to infinity such that

$$\left|\frac{f(z_m)}{f'(z_m)}\right| \leq (1 + o(1))|z_m| \quad \text{as} \quad m \to +\infty \tag{3.6}$$

and

$$\frac{\log^+ |f'(z_m)|}{|z_m|^{\rho+\varepsilon}} \to \infty. \tag{3.7}$$

From (3.7) for any positive constant number $M > 0$, we have

$$|f'(z_m)| > e^{M|z_m|^{\rho+\varepsilon}} \quad \text{as} \quad m \to +\infty. \tag{3.8}$$

Since $F(z)$ is a meromorphic function with only finitely many poles, then by Hadamard factorization theorem, we can write $F(z) = \frac{H(z)}{\pi(z)}$ where $\pi(z)$ is a polynomial, $H(z)$ is an entire function with $\rho(H) = \rho(F)$. From (3.8), for m sufficiently large ($r_m \to +\infty$), we have

$$\left|\frac{F(z_m)}{f'(z_m)}\right| = \left|\frac{H(z_m)}{\pi(z_m)\ f'(z_m)}\right| \leq \left|\frac{H(z_m)}{cr_m^s e^{M|z_m|^{\rho+\varepsilon}}}\right| \leq \frac{|H(z_m)|}{e^{M|z_m|^{\rho+\varepsilon}}},$$

where $c > 0$ is a constant and $s = \deg \pi \geq 1$ is an integer. Since $\rho(H) = \rho(F) \leq \rho$, then we have

$$\left|\frac{H(z_m)}{\pi(z_m) f'(z_m)}\right| \leq \frac{|H(z_m)|}{e^{M|z_m|^{\rho+\varepsilon}}} \to 0 \ \text{ as } m \to +\infty. \tag{3.9}$$

By Lemma 2.2, for any given ε $(0 < \varepsilon < \sigma - \rho)$, there exists a set $E_2 \subset (1, +\infty)$ with a finite linear measure and a finite logarithmic measure such that

$$|B(z)| \leq e^{r^{\rho+\varepsilon}} \tag{3.10}$$

holds for all z satisfying $|z| = r \notin [0,1] \cup E_2$, $r \to +\infty$. Also by the hypotheses of Theorem 1.1, there exists a set H with $\overline{dens}H > 0$, such that for all z satisfying $|z| = r \in H$, $r \to +\infty$, we have

$$|A(z)| \geq e^{\alpha r^{\sigma}}. \tag{3.11}$$

Using (3.5), (3.6), (3.9), (3.10) and (3.11), we conclude from (3.4) that for all $z_m = r_m e^{i\theta}$ satisfying $\theta \in [0, 2\pi) \setminus E_1$ and $r_m \in H \setminus [0,1] \cup E_2$, $r_m \to +\infty$, we have

$$e^{\alpha r_m^{\sigma}} \leq r_m^{2\delta} + e^{r_m^{\rho+\varepsilon}} r_m (1 + o(1)) + o(1) \leq 3r_m^{2\delta+1} e^{r_m^{\rho+\varepsilon}},$$

that is,

$$e^{\alpha(1-o(1))r_m^{\sigma}} \leq 3r_m^{2\delta+1}$$

which is a contradiction for m is large enough. Therefore, $\frac{\log^+|f'(z)|}{|z|^{\rho+\varepsilon}}$ is bounded on the ray $\arg(z)=\theta$, then there exists a bounded constant $M_1>0$ such that

$$|f'(z)|\le e^{M_1|z|^{\rho+\varepsilon}}$$

on the ray $\arg(z)=\theta$. Then

$$|f(z)|\le(1+o(1))\,r\,|f'(z)|\le e^{M_1 r^{\rho+2\varepsilon}} \tag{3.12}$$

on the ray $\arg(z)=\theta$. Since A , B and F are meromorphic functions having only finitely many poles and the poles of f can only occur at the poles of A, B and F, then $f(z)$ must have only finitely many poles. Therefore, by Hadamard factorization theorem, we can write f as $f(z)=\frac{g(z)}{d(z)}$ where $d(z)$ is a polynomial and $g(z)$ is an entire function with $\rho(g)=\rho(f)\ge\sigma$. From (3.12), we have

$$\left|\frac{g(z)}{d(z)}\right|\le e^{M_1 r^{\rho+2\varepsilon}}$$

on the ray $\arg(z)=\theta$. Then

$$|g(z)|\le|d(z)|\,e^{M_1 r^{\rho+2\varepsilon}}\le Ar^k e^{M_1 r^{\rho+2\varepsilon}}$$

on the ray $\arg(z)=\theta$, where $A>0$ is a constant and $k=\deg d\ge 1$ is an integer. Hence

$$|g(z)|\le e^{M_1 r^{\rho+3\varepsilon}} \tag{3.13}$$

on the ray $\arg(z)=\theta$. Therefore, for any given $\theta\in[0,2\pi)\setminus E_1$, where $E_1\subset[0,2\pi)$ is a set of linear measure zero, we have (3.13) holds, for sufficiently large $|z|=r$. Then by Lemma 2.6, we get $\rho(g)\le\rho+3\varepsilon<\sigma$ for a small positive ε, a contradiction with $\rho(g)\ge\sigma$. Hence, every transcendental meromorphic solution f of (1.1) must be of infinite order. By Remark 1.1 we have $\rho(A)\ge\sigma$ and since $\rho=\max\{\rho(B),\ \rho(F)\}<\sigma$, then by using Lemma 2.4, we obtain

$$\rho_2(f)\le\rho(A). \tag{3.14}$$

Suppose that $F\not\equiv 0$. Then, by Lemma 2.8, we obtain

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty.$$

We know that if f has a zero at z_0 of order l $(l>2)$, and $A(z)$, $B(z)$ are analytic at z_0, then $F(z)$ must have a zero at z_0 of order $l-2$. Therefore, we get by $F\not\equiv 0$ that

$$N\left(r,\frac{1}{f}\right)\le 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+N(r,A)+N(r,B). \tag{3.15}$$

On the other hand, (1.1) may be rewritten as follows

$$\frac{1}{f}=\frac{1}{F}\left[\frac{f''}{f}+A\frac{f'}{f}+B\right].$$

So

$$m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{1}{F}\right)+m(r,A)+m(r,B)+\sum_{j=1}^{2}m\left(r,\frac{f^{(j)}}{f}\right)+O(1). \tag{3.16}$$

Hence, by the lemma of logarithmic derivative [11], there exists a set E having finite linear measure such that for all $r\notin E$, we have

$$m\left(r,\frac{f^{(j)}}{f}\right)=O(\log(rT(r,f)))\quad(j=1,2). \tag{3.17}$$

By (3.15), (3.16) and (3.17), we obtain

$$T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)\le 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+N(r,A)+N(r,B)$$
$$+m\left(r,\frac{1}{F}\right)+m(r,A)+m(r,B)+\sum_{j=1}^{2}m\left(r,\frac{f^{(j)}}{f}\right)+O(1)\le 2\overline{N}\left(r,\frac{1}{f}\right)$$
$$+T(r,F)+T(r,A)+T(r,B)+C\log(rT(r,f)), \tag{3.18}$$

where C is a positive constant. Set $\beta = \rho(A) = \max\{\rho, \rho(A)\}$. Then, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$C \log(rT(r,f)) \leq \frac{1}{2} T(r,f), \quad T(r,A) \leq r^{\beta+\varepsilon}, \; T(r,B) \leq r^{\beta+\varepsilon}, \; T(r,F) \leq r^{\beta+\varepsilon}. \tag{3.19}$$

Then for $r \notin E$ and r sufficiently large, by using (3.18) and (3.19), we conclude that

$$T(r,f) \leq 2\overline{N}\left(r, \frac{1}{f}\right) + 3r^{\beta+\varepsilon} + \frac{1}{2} T(r,f),$$

that is,

$$T(r,f) \leq 4\overline{N}\left(r, \frac{1}{f}\right) + 6r^{\beta+\varepsilon}. \tag{3.20}$$

Hence, by (3.20), we get $\rho_2(f) \leq \overline{\lambda}_2(f)$. It follows that

$$\lambda_2(f) \geq \overline{\lambda}_2(f) \geq \rho_2(f). \tag{3.21}$$

We have $\overline{N}(r,f) \leq N(r,f) \leq T(r,f)$, then

$$\overline{\lambda}_2(f) \leq \lambda_2(f) \leq \rho_2(f). \tag{3.22}$$

Therefore, by (3.21) and (3.22), we obtain $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f)$. From (3.14), we get

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq \rho(A).$$

4. Proof of Corollary 1.1

Since A is a meromorphic function having only finitely many poles and $\rho(A) = \sigma$, then by Hadamard factorization theorem, we can write A to $A(z) = \frac{K(z)}{P(z)}$, where $K(z)$ is an entire function with $\rho(A) = \rho(K) = \sigma$ and $P(z)$ is a polynomial. Hence, by Lemma 2.7, for any ε $(0 < \varepsilon < \sigma)$, there exists a set $H \subset [0, +\infty)$ with $\overline{dens}H \geq 1 - 2\sigma > 0$ such that

$$|K(z)| \geq e^{r^{\sigma-\varepsilon}} \tag{4.1}$$

holds for all z, $|z| = r \in H$ and $r \to +\infty$. Also, by Lemma 2.3, there exist positive constants $c > 0$, $m \geq 1$ such that

$$|P(z)| \leq cr^m. \tag{4.2}$$

Hence from (4.1) and (4.2), we have

$$|A(z)| = \left|\frac{K(z)}{P(z)}\right| \geq \frac{e^{r^{\sigma-\varepsilon}}}{cr^m} \geq e^{r^{\sigma-2\varepsilon}}. \tag{4.3}$$

Since $\rho = \max\{\rho(B), \rho(F)\} < \sigma$, then for any given ε with $0 < 2\varepsilon < \sigma - \rho$, we have (4.3) and

$$\rho = \max\{\rho(B), \rho(F)\} < \sigma - 2\varepsilon. \tag{4.4}$$

By using Theorem 1.1 for equation (1.1), we find that every transcendental meromorphic solution f of equation (1.1) satisfies

$$\rho(f) = +\infty \quad \text{and} \quad \rho_2(f) \leq \rho(A) = \sigma. \tag{4.5}$$

Furthermore, by using (4.5) and the fact $F \not\equiv 0$, we conclude from Theorem 1.1 that every transcendental meromorphic solution f of equation (1.1) with $F \not\equiv 0$ satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq \rho(A) = \sigma.$$

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