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CHARACTERIZATION OF (δ,γ)-DINI-LIPSCHITZ FUNCTIONS IN TERMS OF THEIR JACOBI-DUNKL TRANSFORMS

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Abstract. In this paper, we are going to define a generalized Dini-Lipschitz class and give a characterization for functions belonging to by means of an asymptotic estimating growth of the norm of their Jacobi-Dunkl transforms.

1. Introduction and Preliminaries

Younis theorem 5.2 [3] characterizes the set of functions in $L^2(\mathbb{R})$ satisfying the Dini-Lipschitz condition by means of an asymptotic estimating growth of the norm of their Fourier transforms. i.e.

Theorem 1.1. [3] *Let $\delta \in (0,1)$, $\gamma > 0$ and $f \in L^2(\mathbb{R})$. Then the following conditions are equivalents:*

(1) $\|f(t+h)-f(t)\|_{L^2(\mathbb{R})} = O\left(h^\delta(\log\frac{1}{h})^{-\gamma}\right)$, *as* $h \to 0$;

(2) $\displaystyle\int_{|\lambda|\geq r} |\hat{f}(\lambda)|^2 d\lambda = O\left(r^{-2\delta}(\log r)^{-2\gamma}\right)$, *as* $r \to +\infty$.

where $\hat{f}$ is the Fourier transform of f .

In the following, Let $\alpha\,,\beta$ *and* ρ denote 3 reals such that $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$ *and* $\rho = \alpha+\beta+1$,

$$A_{\alpha,\beta}(x) = 2^\rho(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

In [1] we have established a characterization of functions f in $L^2(\mathbb{R}, A_{\alpha,\beta}(x)dx)$ satisfying a certain Lipschitz condition, namely the equivalence between the two following conditions:

(1) $||\Delta_h f|| = ||\tau_h f + \tau_{-h} f - 2f||_{L^2(\mathbb{R},A_{\alpha,\beta}(x)dx)} = O(h^\delta)$, as $h \to 0$;

(2) $\displaystyle\int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta})$, *as* $r \to \infty$.

where $\mathcal{F}_{\alpha,\beta}(f)$ stands for the Jacobi-Dunkl transform of f, and τ_h is the related generalized translation operator .

This result has been generalized in [2] by using the higher powers: $\Lambda^r_{\alpha,\beta}$ and $\Delta^k_h f = \Delta_h(\Delta^{k-1}_h f)$, $r \in \mathbb{N}$, $k \in \mathbb{N}^*$. In this way, we are going in section 3 to define a generalized Dini-Lipschitz class $DLip[2,(\delta,\gamma),k,r]$, $\delta \in (0,1)$, $\gamma > 0$, and give a characterization for functions belonging to by means of an asymptotic estimating

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growth of the norm of their Jacobi-Dunkl transforms, i.e. we show the equivalence of the two following conditions:

(1) $f \in DLip[2,(\delta,\gamma),k,r]$;

(2) $\displaystyle\int_s^{\infty} \lambda^{2r} \left|\mathcal{F}_{\alpha,\beta}(f)(\lambda)\right|^2 d\sigma_{\alpha,\beta}(\lambda) = O\left(s^{-2\delta}(\log s)^{-2\gamma}\right)$, *as* $s \to +\infty$.

In the following section we recapitulate some results related to the harmonic analysis associated with the Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$ (see [4, 5, 6, 7, 9]).

2. Notations and Preliminaries

Notations:

- $d\sigma_{\alpha,\beta}(\lambda) = \dfrac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|}\mathbb{I}_{\mathbb{R}\setminus]-\rho,\rho[}(\lambda)d\lambda$

  where, $C_{\alpha,\beta}(\mu) = \dfrac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}$, $\mu \in \mathbb{C}\setminus(i\mathbb{N})$.

  and $\mathbb{I}_\Omega$ is the characteristic function of Ω .
- $L^p(A_{\alpha,\beta})$ (*resp.* $L^p(\sigma_{\alpha,\beta})$, $p \in]0,+\infty[$, the space of measurable functions g on $\mathbb{R}$ such that

$$||g||_{L^p(A_{\alpha,\beta})} = \left(\int_{\mathbb{R}} |g(t)|^p A_{\alpha,\beta}(t)dt\right)^{1/p} < +\infty .$$

$$(resp.\ ||g||_{L^p(\sigma_{\alpha,\beta})} = \left(\int_{\mathbb{R}} |g(\lambda)|^p d\sigma_{\alpha,\beta}(\lambda)\right)^{1/p} < +\infty) .$$

- $\mathcal{D}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$ with compact support.
- $\mathcal{S}(\mathbb{R})$ the usual Schwartz space of C^∞-functions on $\mathbb{R}$ rapidly decreasing together with their derivatives, equipped with the topology of semi-norms $L_{m,n}$, $(m,n) \in \mathbb{N}^2$, where

$$L_{m,n}(f) = \sup_{x\in\mathbb{R},0\le k\le n}\left[(1+x^2)^m\left|\frac{d^k}{dx^k}f(x)\right|\right] < +\infty.$$

- $\mathcal{S}^1(\mathbb{R}) = \{(\cosh t)^{-2\rho} f;\ f \in \mathcal{S}(\mathbb{R})\}$.

  The topology of this space is given by the semi-norms $L^1_{m,n}$, $(m,n) \in \mathbb{N}^2$, where

$$L^1_{m,n}(f) = \sup_{x\in\mathbb{R},0\le k\le n}\left[(\cosh t)^{-2\rho}(1+x^2)^m\left|\frac{d^k}{dx^k}f(x)\right|\right] < +\infty.$$

- $\left(\mathcal{S}^1(\mathbb{R})\right)'$ the topological dual of $\mathcal{S}^1(\mathbb{R})$.

Now, we introduce the Jacobi-Dunkl Transform and its basic properties:

The Jacobi-Dunkl function with parameters (α,β) , $\alpha \ge \beta \ge -\frac{1}{2}, \alpha \ne -\frac{1}{2}$, is defined by :

$$(1)\qquad \forall x \in \mathbb{R},\quad \psi_\lambda^{(\alpha,\beta)}(x) = \begin{cases} \varphi_\mu^{(\alpha,\beta)}(x) - \dfrac{i}{\lambda}\dfrac{d}{dx}\varphi_\mu^{(\alpha,\beta)}(x) & ,\ \text{if } \lambda \in \mathbb{C}\setminus\{0\}; \\ 1 & ,\ \text{if } \lambda = 0. \end{cases}$$

with $\lambda^2=\mu^2+\rho^2,\ \rho=\alpha+\beta+1$ and $\varphi_\mu^{(\alpha,\beta)}$ is the Jacobi function given by:

$$\varphi_\mu^{(\alpha,\beta)}(x)=F\left(\frac{\rho+i\mu}{2},\frac{\rho-i\mu}{2};\alpha+1,-(\sinh x)^2\right), \tag{2}$$

where F is the Gaussian hypergeometric function given by

$$F(a,b,c,z)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m m!}z^m\ ,|z|<1,$$

$a,b,z\in\mathbb{C}$ *and* $c\notin-\mathbb{N}$;
$(a)_0=1,\ (a)_m=a(a+1)...(a+m-1)$. (see [4, 10, 11]).
$\psi_\lambda^{(\alpha,\beta)}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases}\Lambda_{\alpha,\beta}u=i\lambda u & ,\ \lambda\in\mathbb{C};\\ u(0)=1.\end{cases} \tag{3}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta}u(x)=\frac{du}{dx}(x)+\frac{A'_{\alpha,\beta}(x)}{A_{\alpha,\beta}(x)}\times\frac{u(x)-u(-x)}{2}\ ;\ i.e.$$

$$\Lambda_{\alpha,\beta}u(x)=\frac{du}{dx}(x)+[(2\alpha+1)\coth x+(2\beta+1)\tanh x]\times\frac{u(x)-u(-x)}{2}\,.$$

The function $\psi_\lambda^{(\alpha,\beta)}$ can be written in the form below (See [5]),

$$\psi_\lambda^{(\alpha,\beta)}(x)=\varphi_\mu^{(\alpha,\beta)}(x)+i\frac{\lambda}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{(\alpha+1,\beta+1)}(x)\ ,\ \forall x\in\mathbb{R}\ , \tag{4}$$

where $\lambda^2=\mu^2+\rho^2$, $\rho=\alpha+\beta+1$.

The Jacobi-Dunkl transform of a function $f\in L^1(A_{\alpha,\beta})$ is defined by :

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda)=\int_{\mathbb{R}}f(x)\psi_{-\lambda}^{(\alpha,\beta)}(x)A_{\alpha,\beta}(x)dx,\ \forall\lambda\in\mathbb{R}\ ; \tag{5}$$

The inverse Jacobi-Dunkl transform of a function $h\in L^1(\sigma_{\alpha,\beta})$ is:

$$\mathcal{F}_{\alpha,\beta}^{-1}(h)(t)=\int_{\mathbb{R}}h(\lambda)\psi_\lambda^{(\alpha,\beta)}(t)d\sigma_{\alpha,\beta}(\lambda)\,. \tag{6}$$

$\mathcal{F}_{\alpha,\beta}$ is a topological isomorphism from $\mathcal{S}^1(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$, and extends uniquely to a unitary isomorphism from $L^2(A_{\alpha,\beta})$ onto $L^2(\sigma_{\alpha,\beta})$. The Plancherel formula is given by

$$\|f\|_{L^2(A_{\alpha,\beta})}=\|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2(\sigma_{\alpha,\beta})}\ . \tag{7}$$

For $f\in\mathcal{S}^1(\mathbb{R})$ we have the following inversion formula

$$f(x)=\int_{\mathbb{R}}\mathcal{F}_{\alpha,\beta}(f)(\lambda)\psi_\lambda^{(\alpha,\beta)}(x)d\sigma_{\alpha,\beta}(\lambda)\,,\ \forall x\in\mathbb{R}\,, \tag{8}$$

and the relation

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}f)(\lambda)=i\lambda\mathcal{F}_{\alpha,\beta}(f)(\lambda)\,. \tag{9}$$

Let $f \in L^2(A_{\alpha,\beta})$. For all $x \in \mathbb{R}$ the operator of Jacobi-Dunkl translation τ_x is defined by:

$$\tau_x f(y) = \int_{\mathbb{R}} f(z) d\nu^{\alpha,\beta}_{x,y}(z)\ ,\ \forall\ y \in \mathbb{R}\ . \tag{10}$$

where $\nu^{\alpha,\beta}_{x,y}$, $x, y \in \mathbb{R}$ are the signed measures given by

$$d\nu^{\alpha,\beta}_{x,y}(z) = \begin{cases} K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & ,\ \text{if}\ \ x,y \in \mathbb{R}^*; \\ \delta_x & ,\ \text{if}\ \ y = 0; \\ \delta_y & ,\ \text{if}\ \ x = 0. \end{cases} \tag{11}$$

Here, δ_x is the Dirac measure at x. And

$$\begin{aligned} K_{\alpha,\beta}(x,y,z) = \quad & M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha}\mathbb{I}_{I_{x,y}} \times \int_0^\pi \rho_\theta(x,y,z) \\ & \times (g_\theta(x,y,z))_+^{\alpha-\beta-1}\sin^{2\beta}\theta d\theta. \end{aligned}$$

$$\begin{aligned} I_{x,y} &= [-|x|-|y|, -||x|-|y||] \cup [||x|+|y||, |x|+|y|]\ , \\ \rho_\theta(x,y,z) &= 1 - \sigma^\theta_{x,y,z} + \sigma^\theta_{z,x,y} + \sigma^\theta_{z,y,x} \end{aligned}$$

$$\sigma^\theta_{x,y,z} = \begin{cases} \dfrac{\cosh(x)+\cosh(y)-\cosh(z)\cos(\theta)}{\sinh(x)\sinh(y)} & ,\ \text{if}\ \ xy \neq 0; \\ 0 & ,\ \text{if}\ \ xy = 0. \end{cases}$$

for all $x, y, z \in \mathbb{R}$, $\theta \in [0,\pi]$.

$$g_\theta(x,y,z) = 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2\cosh x \cosh y \cosh z \cos\theta\ .$$

$$t_+ = \begin{cases} t & ,\ \text{if}\ \ t > 0; \\ 0 & ,\ \text{if}\ \ t \leq 0. \end{cases}$$

and

$$M_{\alpha,\beta} = \begin{cases} \dfrac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} & ,\ \text{if}\ \ \alpha > \beta; \\ 0 & ,\ \text{if}\ \ \alpha = \beta. \end{cases}$$

We have

$$\mathcal{F}_{\alpha,\beta}(\tau_h f)(\lambda) = \psi^{\alpha,\beta}_\lambda(h).\mathcal{F}_{\alpha,\beta}(f)(\lambda) \quad ; \quad h,\lambda \in \mathbb{R}\ . \tag{12}$$

Let $g \in L^2(\sigma_{\alpha,\beta})$. Then the distribution $T_{g\sigma_{\alpha,\beta}}$ defined by

$$\langle T_{g\sigma_{\alpha,\beta}}, \varphi\rangle = \int_{\mathbb{R}} g(\lambda)\varphi(\lambda)d\sigma_{\alpha,\beta}(\lambda)\ , \quad \varphi \in \mathcal{D}(\mathbb{R})\ , \tag{13}$$

belongs to $\mathcal{S}'(\mathbb{R})$.

Let $f \in L^2(A_{\alpha,\beta})$. Then the distribution $T_{fA_{\alpha,\beta}}$ defined by

$$\langle T_{fA_{\alpha,\beta}}, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)A_{\alpha,\beta}(x)dx\ , \quad \varphi \in \mathcal{S}^1(\mathbb{R})\ , \tag{14}$$

belongs to $\left(\mathcal{S}^1(\mathbb{R})\right)'$.

Via the correspondance $f \mapsto T_{fA_{\alpha,\beta}}$, we identify $L^2(A_{\alpha,\beta})$ as a subspace of $\left(\mathcal{S}^1(\mathbb{R})\right)'$.

The jacobi-dunkl transform of a distribution $T \in \left(\mathcal{S}^1(\mathbb{R})\right)'$ is defined by:

$$\langle \mathcal{F}_{\alpha,\beta}(T), \varphi \rangle = \langle T, \mathcal{F}^{-1}_{\alpha,\beta}(\check{\varphi}) \rangle \,, \ \varphi \in \mathcal{S}(\mathbb{R}) \,, \tag{15}$$

where $\check{\varphi}$ is given by $\check{\varphi}(x) = \varphi(-x)$.
It is clear that $\mathcal{F}_{\alpha,\beta}(T) \in \mathcal{S}'(\mathbb{R})$.

The jacobi-dunkl transform of a distribution defined by $f \in L^2(A_{\alpha,\beta})$ is given by the distribution $T_{\mathcal{F}_{\alpha,\beta}(f)\sigma_{\alpha,\beta}}$; i.e.

$$\mathcal{F}_{\alpha,\beta}(T_{fA_{\alpha,\beta}}) = T_{\mathcal{F}_{\alpha,\beta}(f)\sigma_{\alpha,\beta}} \,. \tag{16}$$

We identify the tempered distribution given by $\mathcal{F}_{\alpha,\beta}(f)$ and the function $\mathcal{F}_{\alpha,\beta}(f)$.
Let $T \in \left(\mathcal{S}^1(\mathbb{R})\right)'$ and consider the distribution $\Lambda_{\alpha,\beta}T$ defined by

$$\langle \Lambda_{\alpha,\beta}(T), \varphi \rangle = -\langle T, \Lambda_{\alpha,\beta}(\varphi) \rangle \,, \text{ for all } \varphi \in \mathcal{S}^1(\mathbb{R}) \,. \tag{17}$$

(Note that $\mathcal{S}^1(\mathbb{R})$ is unvariant under $\Lambda_{\alpha,\beta}$).
By using (9) it is easy to see that

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}(T)) = i\lambda \mathcal{F}_{\alpha,\beta}(T) \,. \tag{18}$$

For $f \in L^2(A_{\alpha,\beta})$, we define the finite differences of first and higher order as follows:

$$\begin{aligned} \Delta_h^1 f &= \Delta_h f = \tau_h f + \tau_{-h} f - 2f = (\tau_h + \tau_{-h} - 2E)f; \\ \Delta_h^k f &= \Delta_h(\Delta_h^{k-1}) f = (\tau_h + \tau_{-h} - 2E)^k f \,, \quad k = 2, 3, ...; \end{aligned}$$

where E is the unit operator in $L^2(A_{\alpha,\beta})$.

Lemma 2.1. *The following inequalities are valids for Jacobi functions* $\varphi_\mu^{\alpha,\beta}(h)$

(1) $|\varphi_\mu^{(\alpha,\beta)}(h)| \leq 1$;
(2) $|1 - \varphi_\mu^{(\alpha,\beta)}(h)| \leq h^2\lambda^2$; *where* $\lambda^2 = \mu^2 + \rho^2$.

Proof. (See [12], Lemmas 3.1-3.2) □

For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind defined by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(n+\alpha+1)} \quad , \ z \in \mathbb{C}.$$

We see that $\lim\limits_{z\to 0} \dfrac{j_\alpha(z) - 1}{z^2} \neq 0$, by consequence, there exists $c_1 > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Rightarrow \quad |j_\alpha(z) - 1| \geq c_1 |z|^2 \,. \tag{19}$$

Lemma 2.2. *Let* $\alpha \geq \beta \geq \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$. *Then for* $|\upsilon| \leq \rho$, *there exists a positive constant* c_2 *such that*

$$|1 - \varphi_{\mu+i\upsilon}^{(\alpha,\beta)}(t)| \geq c_2 |1 - j_\alpha(\mu t)| \,.$$

Proof. (See [8], Lemma 9) □

3. Main Results

We denote by $W^{2,k}_{\alpha,\beta}$, $k\in\mathbb{N}^*$, the Sobolev space constructed by the operator $\Lambda_{\alpha,\beta}$; i.e.

$$W^{2,k}_{\alpha,\beta}=\left\{f\in L^2(A_{\alpha,\beta});\ \Lambda^j_{\alpha,\beta}f\in L^2(A_{\alpha,\beta}),\ j=0,1,2,...,k\right\}\ ; \tag{20}$$

where, $\Lambda^0_{\alpha,\beta}f=f,\ \Lambda^1_{\alpha,\beta}f=\Lambda_{\alpha,\beta}f$, $\Lambda^r_{\alpha,\beta}f=\Lambda_{\alpha,\beta}(\Lambda^{r-1}_{\alpha,\beta}f),\ r=2,3,...$

Lemma 3.1. *Let* $f\in W^{2,k}_{\alpha,\beta}$, $k\in\mathbb{N}^*$. *Then*

$$\left\|\Delta^k_h\Lambda^r_{\alpha,\beta}f\right\|^2_{L^2(A_{\alpha,\beta})}=2^{2k}\int_{\mathbb{R}}\lambda^{2r}|1-\varphi_\mu(h)|^{2k}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma_{\alpha,\beta}(\lambda)\ ,$$

where $r=0,1,...,k$.

Proof. Using the eveness of φ_μ and formula (4) we get

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}(\tau_hf+\tau_{-h}f-2f)(\lambda&=(\psi^{(\alpha,\beta)}_\lambda(h)+\psi^{(\alpha,\beta)}_\lambda(-h)-2).\mathcal{F}_{\alpha,\beta}(f)(\lambda)\\&=2(\varphi^{(\alpha,\beta)}_\mu(h)-1).\mathcal{F}_{\alpha,\beta}(f)(\lambda).\end{aligned}$$

and

$$\mathcal{F}_{\alpha,\beta}(\Delta^k_hf)(\lambda)=2^k(\varphi^{(\alpha,\beta)}_\mu(h)-1)^k.\mathcal{F}_{\alpha,\beta}(f)(\lambda). \tag{21}$$

Furthermore, we obtain by the formula (18)

$$\mathcal{F}_{\alpha,\beta}(\Lambda^r_{\alpha,\beta}f)(\lambda)=(i\lambda)^r\mathcal{F}_{\alpha,\beta}(f)(\lambda)\,. \tag{22}$$

Using the formulas (21) and (22) we get

$$\mathcal{F}_{\alpha,\beta}(\Delta^k_h\Lambda^r_{\alpha,\beta}f)(\lambda)=2^k(i\lambda)^r.(\varphi^{(\alpha,\beta)}_\mu(h)-1)^k.\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

By the Plancherel formula (7), we have the result. □

Definition 3.2. *Let* $\delta\in(0,1),\gamma>0$ *and* $k\in\mathbb{N}^*$. *A function* $f\in W^{2,k}_{\alpha,\beta}$ *is said to be in the* (δ,γ)*-Dini-Lipschitz class, denoted by* $DLip[2,(\delta,\gamma),k,r]$ *, if*

$$\left\|\Delta^k_h\Lambda^r_{\alpha,\beta}f\right\|_{L^2(A_{\alpha,\beta})}=O\left(h^\delta(\log\frac{1}{h})^{-\gamma}\right)\ ,\quad as\ h\longrightarrow0,$$

where $r=0,1,...,k$.

Theorem 3.3. *Let* $f\in W^{2,k}_{\alpha,\beta}$, $k\in\mathbb{N}^*$. *Then the following are equivalents:*

(1) $f\in DLip[2,(\delta,\gamma),k,r]$;

(2) $\displaystyle\int_s^\infty\lambda^{2r}\left|\mathcal{F}_{\alpha,\beta}(f)(\lambda)\right|^2d\sigma_{\alpha,\beta}(\lambda)=O\left(s^{-2\delta}(\log s)^{-2\gamma}\right)$, *as* $s\to+\infty$.

Proof. (1) ⇒ (2): Assume that $f\in DLip[2,(\delta,\gamma),k,r]$; then

$$\left\|\Delta^k_h\Lambda^r_{\alpha,\beta}f\right\|_{L^2(A_{\alpha,\beta})}=O\left(h^\delta(\log\frac{1}{h})^{-\gamma}\right)\quad as\ h\longrightarrow0.$$

by lemma 3.1, we have

$$\int_{\mathbb{R}}\lambda^{2r}|1-\varphi_\mu(h)|^{2k}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2d\sigma_{\alpha,\beta}(\lambda)=O\left(h^{2\delta}(\log\frac{1}{h})^{-2\gamma}\right)$$

If $|\lambda| \in [\frac{\eta}{2h}, \frac{\eta}{h}]$ then $|\mu h| \leq \eta$ (recall that $\lambda^2 = \mu^2 + \rho^2$). We get by (19): $|j_\alpha(\mu h) - 1| \geq c_1 \mu^2 h^2$. From $|\lambda| \geq \dfrac{\eta}{2h}$ we have, $\mu^2 h^2 \geq \dfrac{\eta^2}{4} - \rho^2 h^2$; then we can find a positive constant $c_3 = c_3(\eta, \alpha, \beta)$ such that $\mu^2 h^2 \geq c_3$ (take $h < 1$); thus, $|j_\alpha(\mu h) - 1| \geq c_1 c_3$. This inequality and lemma 2.2 implys that: $|1 - \varphi_\mu^{(\alpha,\beta)}(h)| \geq c_1 c_2 c_3 = C$.
Hence, $1 \leq \dfrac{1}{C^2} |1 - \varphi_\mu^{(\alpha,\beta)}(h)|^2$. Then,

$$\begin{aligned}\int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) &\leq \frac{1}{C^{2k}} \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{2r} |1 - \varphi_\mu^{(\alpha,\beta)}(h)|^{2k} \\ &\qquad \times |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) \\ &\leq \frac{1}{C^{2k}} \int_{\mathbb{R}} \lambda^{2r} |1 - \varphi_\mu^{(\alpha,\beta)}(h)|^{2k} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) \\ &= O\left(h^{2\delta} (\log \frac{1}{h})^{-2\gamma}\right).\end{aligned}$$

Then we have,

$$\int_{s \leq |\lambda| \leq 2s} \lambda^{2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) = O\left(s^{-2\delta} (\log s)^{-2\gamma}\right) \quad , \quad as\ s \to +\infty.$$

Or equivalently

$$\int_{s \leq |\lambda| \leq 2s} \lambda^{2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) \leq K_1 O\left(s^{-2\delta} (\log s)^{-2\gamma}\right) \quad , \quad as\ s \to +\infty,$$

where K_1 is some positive constant . It follows that,

$$\begin{aligned}\int_{|\lambda| \geq s} \lambda^{2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) &= \sum_{i=0}^{\infty} \int_{2^i s \leq |\lambda| \leq 2^{i+1} s} \lambda^{2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) \\ &\leq K_1 \sum_{i=0}^{\infty} (2^i s)^{-2\delta} (\log 2^i s)^{-2\gamma} \\ &\leq K_1 \left(\sum_{i=0}^{\infty} (2^i)^{-2\delta} \right) \left(s^{-2\delta} (\log s)^{-2\gamma}\right) \\ &\leq K \left(s^{-2\delta} (\log s)^{-2\gamma}\right).\end{aligned}$$

where $K = \dfrac{K_1}{1 - 2^{-2\delta}}$. This proves that:

$$\int_{|\lambda| \geq s} \lambda^{2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) = O\left(s^{-2\delta} (\log s)^{-2\gamma}\right) \quad , \ as\ s \to +\infty.$$

(2) $\Rightarrow$ (1) : Suppose now that

$$\int_{|\lambda| \geq s} \lambda^{2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) = O\left(s^{-2\delta} (\log s)^{-2\gamma}\right) \quad , \ as\ s \to +\infty.$$

we have to show that:

$$\int_{\mathbb{R}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) = O\left(h^{2\delta}(\log\frac{1}{h})^{-2\gamma}\right) \quad , \ as\ h\to 0.$$

Write:

$$\int_{\mathbb{R}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) = I_1+I_2,$$

where:

$$\begin{aligned} I_1 &= \int_{|\lambda|\le\frac{1}{h}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)\,; \\ I_2 &= \int_{|\lambda|>\frac{1}{h}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda). \end{aligned}$$

Estimate I_1 and I_2. From (1) of lemma 2.1 we can write,

$$\begin{aligned} I_2 &\le 4^k\int_{|\lambda|>\frac{1}{h}} \lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)\,, \quad (s=\frac{1}{h}) \\ &= O\left(h^{2\delta}(\log\frac{1}{h})^{-2\gamma}\right). \end{aligned}$$

Using the inequalities (1) and (2) of lemma 2.1 we get

$$\begin{aligned} I_1 &= \int_{|\lambda|\le\frac{1}{h}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) \\ &\le 2^{2k-1}\int_{|\lambda|\le\frac{1}{h}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|.|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) \\ &\le 2^{2k-1}h^2\int_{|\lambda|\le\frac{1}{h}} \lambda^{2r}.\lambda^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda). \end{aligned}$$

Consider the function

$$\psi(s)=\int_s^\infty \lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda).$$

Since $\psi(s)=O\left(s^{-2\delta}(\log s)^{-2\gamma}\right)$, an integration by parts gives:

$$\begin{aligned} 2^{2k-1}h^2\int_0^{\frac{1}{h}} \lambda^{2r}.\lambda^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) &= 2^{2k-1}h^2\int_0^{\frac{1}{h}}\left(-s^2\psi'(s)\right)ds \\ &= 2^{2k-1}h^2\left(-\frac{1}{h^2}\psi(\frac{1}{h})+2\int_0^{\frac{1}{h}} s\psi(s)ds\right) \\ &\le 2^{2k-1}h^2\int_0^{\frac{1}{h}} s\psi(s)ds \\ &\le C_1.h^2\int_0^{\frac{1}{h}} s^{1-2\delta}(\log s)^{-2\gamma}ds \\ &\le C_2.h^{2\delta}(\log\frac{1}{h})^{-2\gamma}. \end{aligned}$$

Hence,

$$I_1 = O\left(h^{2\delta}(\log\frac{1}{h})^{-2\gamma}\right).$$

Finally we get

$$\begin{aligned} I_1 + I_2 &= O\left(h^{2\delta}(\log\frac{1}{h})^{-2\gamma}\right) + O\left(h^{2\delta}(\log\frac{1}{h})^{-2\gamma}\right) \\ &= O\left(h^{2\delta}(\log\frac{1}{h})^{-2\gamma}\right) \end{aligned}$$

Which completes the proof of the theorem.

□

Corollary 3.4. *Let $f \in W^{2,k}_{\alpha,\beta}$ such that $f \in DLip[2,(\delta,\gamma),k,r]$. Then*

$$\int_s^{\infty} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 \, d\sigma_{\alpha,\beta}(\lambda) = O\left(s^{-2(\delta+r)}(\log s)^{-2\gamma}\right) \quad , \ as \ s \to +\infty.$$

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