

Fuzzy Logistic Differential Equation with a Piecewise Constant Argument: Solution Representation, Stability, and Economic Applications

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Abstract. In this study, a fuzzy logistic differential equation with a piecewise constant argument is proposed to describe growth processes in which the state evolves continuously while saturation information is updated at discrete times. The model is formulated under granular differentiability, with uncertainty represented through the fuzzy initial state. Using horizontal membership functions, the fuzzy problem is reduced to a family of scalar differential equations with piecewise constant arguments, providing a basis for the solution representation and qualitative analysis. Existence, uniqueness, positivity, fuzzy reconstruction, stability, convergence, and oscillatory behavior are investigated, with rigorous proofs established for the main results. The proposed framework is then applied to technology adoption and market penetration under periodically revised information, and numerical examples are provided to illustrate the principal dynamical behaviors.

1. INTRODUCTION

The logistic law is one of the basic nonlinear mechanisms for growth under saturation. In its classical continuous form,

$$x'(\tau) = \Xi x(\tau) \left(1 - \frac{x(\tau)}{\mathcal{K}}\right), \quad \Xi > 0, \quad \mathcal{K} > 0, \quad (1.1)$$

where Ξ measures the intrinsic response intensity and $\mathcal{K}$ is the carrying capacity or saturation level. Beyond its historical role in population dynamics, logistic growth has a well-established economic interpretation. Griliches used logistic curves in the study of hybrid-corn diffusion [24], Mansfield derived a logistic-type law for the diffusion of industrial innovations [25], and Fisher and Pry employed logistic substitution curves to describe technological replacement [26]. Accordingly,

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when x denotes cumulative adoption, market penetration, installed capacity, or another bounded economic stock, $\mathcal{K}$ may be interpreted as the potential market or long-run saturation level.

Model (1.1) assumes that the saturation feedback is observed and updated continuously. In many economic environments, however, the state may evolve continuously while the information used for decision making is revised only at prescribed dates. Market-size estimates may be updated quarterly, production plans may be revised monthly, and adoption or installed-capacity statistics may be released periodically even though transactions, production, and adoption continue between reporting dates. Hybrid mechanisms of this type provide a natural motivation for differential equations with piecewise constant arguments (DEPCAs). The theory of DEPCAs was developed systematically by Cooke and Wiener [10], and periodic information updating also appears in economic models; for example, Cavalli and Naimzada [20] considered continuous production combined with learning performed only at discrete times.

Freezing the saturation signal during each unit sampling period leads to the logistic equation with piecewise constant argument

$$x'(\tau) = \Xi x(\tau) \left(1 - \frac{x(\lfloor \tau \rfloor)}{\mathcal{K}} \right), \quad \tau \geq 0, \quad (1.2)$$

where $\lfloor \tau \rfloor$ is the greatest integer not exceeding τ . Thus, on the interval $[n, n+1)$, the current state $x(\tau)$ changes continuously, whereas the saturation feedback is evaluated using the most recently available nodal value $x(n)$. At the next integer time, the feedback is updated and then held fixed over the following period. In this sense, (1.2) is not merely a discretization of the classical logistic equation but a continuous-time model with discrete information renewal. The normalized scalar form of (1.2) was studied by Carvalho and Cooke [11]; related logistic DEPCAs and population models have subsequently been investigated with respect to global attractivity, stability, semicycles, and bifurcation, among other properties; see [14–19].

A further modeling issue is uncertainty in the initial economic state. Initial adoption, effective installed capacity, or the current size of an emerging market may be known only imprecisely because of survey error, incomplete reporting, heterogeneous assessments, or ambiguous market definitions. Fuzzy numbers provide a nonprobabilistic framework for representing such imprecision. Since the introduction of fuzzy sets by Zadeh [1], several approaches to fuzzy arithmetic and fuzzy differential equations have been developed; see, for example, [2]. The horizontal membership function (HMF) proposed by Piegat and Landowski [3], together with the granular differentiability framework of Mazandarani, Pariz and Kamyad [5], is particularly suitable for the present nonlinear model because it preserves the dependence between repeated occurrences of the same fuzzy state. Granular methods have also been used in fuzzy dynamical and control problems [6]. Fuzzy logistic equations without piecewise constant arguments have been considered under other notions of fuzzy differentiability [7,8], while fuzzy differential equations with piecewise constant arguments have recently been studied in linear and fractional settings [9].

Motivated by these considerations, the present paper studies the fuzzy logistic model

$$\frac{d_{\text{gr}}}{d\tau} \vartheta(\tau) = \Xi \vartheta(\tau) \otimes_{\text{gr}} \left(1 \ominus_{\text{gr}} \frac{1}{\mathcal{K}} \vartheta(\lfloor \tau \rfloor) \right), \quad \vartheta(0) = \vartheta_0, \quad (1.3)$$

where $\vartheta(\tau)$ is fuzzy-valued and the derivative is understood in the granular sense. The parameters Ξ and $\mathcal{K}$ are kept positive and crisp, whereas the initial state ϑ_0 is fuzzy. This choice isolates uncertainty in the initial economic condition and keeps the meanings of the intrinsic response intensity and the saturation level unambiguous. The use of the same fuzzy state in both factors of (1.3) is handled through granular arithmetic, so the dependence structure is retained rather than replaced by independent interval evaluations.

The aim is to develop a rigorous solution and qualitative theory for (1.3) and to clarify its economic interpretation. The HMF formulation is used to connect the fuzzy problem with a family of scalar DEPCAs, after which existence, uniqueness, positivity, exact fuzzy reconstruction, stability, convergence, and oscillatory behavior are investigated. Particular attention is paid to distinguishing established ingredients from new results: standard facts from fuzzy analysis, DEPCAs theory, and discrete dynamics are cited at their point of use, while the fuzzy solution representation and the qualitative statements specific to the proposed model are proved explicitly. The resulting framework is then interpreted in the contexts of technology adoption and market penetration when saturation information is periodically revised.

The paper is organized as follows. Section 2 recalls the required notions from fuzzy analysis, DEPCAs, and discrete dynamics. Section 3 formulates the fuzzy logistic DEPCA and derives its solution representation. Section 4 develops the qualitative analysis. Section 5 discusses the economic interpretation of the proved adjustment regimes. Section 6 addresses modeling scope and limitations, and Section 7 concludes the paper.

2. THEORETICAL FRAMEWORK

This section records the fuzzy-analysis and discrete-dynamical material used later. The statements are standard and are included to fix notation. The presentation is adapted from the horizontal-membership-function and granular-calculus literature [3–5], together with standard references on fuzzy numbers [2], equations with piecewise constant arguments [10], and difference equations [22].

2.1. Fuzzy numbers, horizontal membership functions, and granular differentiability. Let $\mathbb{E}$ denote the space of fuzzy numbers on $\mathbb{R}$. For $u \in \mathbb{E}$, the ξ -level set is written

$$[u]^\xi = [u_\xi^-, u_\xi^+], \quad \xi \in [0, 1].$$

As usual, fuzzy numbers are assumed to be normal, convex, upper semicontinuous, and compactly supported; see [2].

Definition 2.1 (Horizontal membership function; [3, 4]). *For $u \in \mathbb{E}$, the horizontal membership function (HMF) is the real-valued map*

$$u^{\text{gr}} : [0, 1] \times [0, 1] \longrightarrow \mathbb{R},$$

$$(\xi, \alpha_u) \mapsto u^{\text{gr}}(\xi, \alpha_u) = u_{\xi}^{-} + (u_{\xi}^{+} - u_{\xi}^{-})\alpha_u, \quad (2.1)$$

where $\alpha_u \in [0, 1]$ is the relative-distance-measure (RDM) variable. One writes

$$\mathcal{P}(u) := u^{\text{gr}}(\xi, \alpha_u).$$

The HMF contains the same level-set information as the original fuzzy number. The inverse construction is recalled because it will be used explicitly in the reconstruction of the fuzzy solution.

Proposition 2.1 (HMF reconstruction; [3, 5]). For $u \in \mathbb{E}$ and $\xi \in [0, 1]$,

$$[u]^{\xi} = \left[\inf_{\beta \geq \xi} \min_{\alpha_u \in [0, 1]} u^{\text{gr}}(\beta, \alpha_u), \quad \sup_{\beta \geq \xi} \max_{\alpha_u \in [0, 1]} u^{\text{gr}}(\beta, \alpha_u) \right]. \quad (2.2)$$

Granular arithmetic is carried out on HMF representatives and then interpreted again as a fuzzy number through the inverse HMF construction. If $u, v \in \mathbb{E}$ and $\otimes$ is an admissible real arithmetic operation, then

$$\mathcal{P}(u \otimes_{\text{gr}} v) := \mathcal{P}(u) \otimes \mathcal{P}(v), \quad (2.3)$$

with division used only when the denominator HMF does not vanish. Repeated appearances of the same fuzzy quantity are evaluated with the same RDM variable. This dependency rule is important in the nonlinear logistic term; see [3–5].

For later reference, the granular operations satisfy the familiar identities below [5]. For $u, v, w \in \mathbb{E}$,

$$u \ominus_{\text{gr}} v = (-1) \otimes_{\text{gr}} (v \ominus_{\text{gr}} u), \quad (2.4)$$

$$v \ominus_{\text{gr}} v = \widehat{0}, \quad (2.5)$$

$$u \otimes_{\text{gr}} (v \oplus_{\text{gr}} w) = (u \otimes_{\text{gr}} v) \oplus_{\text{gr}} (u \otimes_{\text{gr}} w). \quad (2.6)$$

Definition 2.2 (Granular order; [5]). Let $u, v \in \mathbb{E}$. One writes $u \leq_{\text{gr}} v$ if

$$u^{\text{gr}}(\xi, \alpha) \leq v^{\text{gr}}(\xi, \alpha) \quad \text{for every } (\xi, \alpha) \in [0, 1]^2.$$

The strict relation $u <_{\text{gr}} v$ is defined analogously. Equality in the granular sense is equivalent to equality of the corresponding HMFs for all common values of the membership and RDM variables.

Every $c \in \mathbb{R}$ is identified, when necessary, with the degenerate fuzzy number $\widehat{c} \in \mathbb{E}$ satisfying

$$[\widehat{c}]^{\xi} = [c, c], \quad \xi \in [0, 1].$$

For simplicity of notation, real constants appearing in granular operations are written without hats whenever no ambiguity arises.

Definition 2.3 (Fuzzy-valued function). A fuzzy-valued function is a mapping $\vartheta : J \subseteq \mathbb{R} \rightarrow \mathbb{E}$. Its HMF is written

$$\mathcal{P}(\vartheta(\tau)) = \vartheta^{\text{gr}}(\xi, \tau, \alpha_{\vartheta}).$$

If the entire trajectory is generated from one fuzzy initial value ϑ_0 , the same pair of HMF parameters is used throughout the trajectory. It is convenient to put

$$\theta = (\xi, \alpha_0) \in [0, 1]^2, \quad z_0(\theta) := \vartheta_0^{\text{gr}}(\xi, \alpha_0),$$

and to denote the corresponding real HMF trajectory by $z(\tau; \theta)$.

Definition 2.4 (Granular metric; [5]). For $u, v \in \mathbb{E}$, define

$$\|u \ominus_{\text{gr}} v\|_{\text{gr}} := \sup_{\xi \in [0, 1]} \max_{\alpha_u, \alpha_v \in [0, 1]} |u^{\text{gr}}(\xi, \alpha_u) - v^{\text{gr}}(\xi, \alpha_v)|. \quad (2.7)$$

This distance is called the granular metric. The metric space $(\mathbb{E}, \|\cdot\|_{\text{gr}})$ is complete; see [5].

When one argument is crisp, (2.7) simplifies substantially. If $c \in \mathbb{R}$ is identified with its degenerate fuzzy representative, then

$$\|u \ominus_{\text{gr}} c\|_{\text{gr}} = \sup_{\xi \in [0, 1]} \max_{\alpha_u \in [0, 1]} |u^{\text{gr}}(\xi, \alpha_u) - c|. \quad (2.8)$$

Hence convergence to a crisp equilibrium is equivalent to uniform convergence of all HMF representatives to that real value.

Definition 2.5 (Granular differentiability; [5]). A fuzzy-valued function $\vartheta : J \rightarrow \mathbb{E}$ is granular differentiable at $\tau_0 \in J$ if the granular difference quotient has a limit in $\mathbb{E}$ as the increment tends to zero. Its granular derivative is denoted by $d_{\text{gr}}\vartheta(\tau_0)/d\tau$.

The following characterization is the main device used in the sequel.

Proposition 2.2 (HMF characterization of granular differentiability; [5]). A fuzzy-valued function is granular differentiable at τ_0 if and only if its HMF is differentiable with respect to τ at τ_0 . In that case,

$$\mathcal{P}\left(\frac{d_{\text{gr}}\vartheta(\tau_0)}{d\tau}\right) = \frac{\partial}{\partial \tau} \vartheta^{\text{gr}}(\xi, \tau_0, \alpha_{\vartheta}). \quad (2.9)$$

For the numerical examples, the notation $\langle a, b, c \rangle$, $a \leq b \leq c$, denotes a triangular fuzzy number. Its ξ -cut is

$$[\langle a, b, c \rangle]^\xi = [a + (b - a)\xi, c - (c - b)\xi]. \quad (2.10)$$

This is standard fuzzy-number notation; see [2].

2.2. Piecewise constant arguments and the Ricker map. For $\tau \geq 0$, the greatest-integer function is constant on

$$I_n = [n, n + 1), \quad n \in \mathbb{N}_0.$$

Consequently, an equation of the form

$$x'(\tau) = F(x(\tau), x(\lfloor \tau \rfloor)) \quad (2.11)$$

can be treated successively on the intervals I_n once the nodal value $x(n)$ is known. This step-method viewpoint is fundamental in the theory of differential equations with piecewise constant arguments; see Cooke and Wiener [10]. The particular nonlinear equation

$$x'(t) = \mu x(t)(1 - x([t]))$$

was studied by Carvalho and Cooke [11, pp. 359–367]. Logistic DEPCAs with more general delayed nodal feedback, together with the induced difference equations for the nodal sequence, were subsequently studied in [14, 16–19].

The discrete map arising from the present model is a scaled Ricker map. Define

$$\mathcal{R}_{\Xi, \mathcal{K}}(x) := x \exp\left[\Xi\left(1 - \frac{x}{\mathcal{K}}\right)\right], \quad x \geq 0, \quad (2.12)$$

and, for the within-period evolution,

$$G_s(x) := x \exp\left[\Xi s\left(1 - \frac{x}{\mathcal{K}}\right)\right], \quad 0 \leq s \leq 1. \quad (2.13)$$

After $u = x/\mathcal{K}$, (2.12) becomes

$$u_{n+1} = u_n e^{\Xi(1-u_n)}, \quad (2.14)$$

which is the standard normalized form of the Ricker stock–recruitment map. Ricker’s original density-dependent recruitment law was introduced in [21]. For the logistic equation with discontinuous/piecewise constant feedback, the associated interval map is already present in Carvalho and Cooke [11]; see also Seifert [12] and later analyses such as [13]. Uesugi, Muroya and Ishiwata [16] explicitly derive the nodal difference equation for a broader logistic DEPCA class.

Straight differentiation yields

$$\mathcal{R}'_{\Xi, \mathcal{K}}(x) = \exp\left[\Xi\left(1 - \frac{x}{\mathcal{K}}\right)\right]\left(1 - \frac{\Xi x}{\mathcal{K}}\right), \quad (2.15)$$

$$\frac{\partial G_s}{\partial x}(x) = \exp\left[\Xi s\left(1 - \frac{x}{\mathcal{K}}\right)\right]\left(1 - \frac{\Xi s x}{\mathcal{K}}\right). \quad (2.16)$$

The fixed points of (2.12) are 0 and $\mathcal{K}$, and

$$\mathcal{R}'_{\Xi, \mathcal{K}}(\mathcal{K}) = 1 - \Xi. \quad (2.17)$$

Thus the standard one-dimensional fixed-point criterion gives local asymptotic stability of the positive fixed point for $0 < \Xi < 2$ and instability for $\Xi > 2$; see, for example, the general difference-equation theory in [22]. The critical and post-bifurcation dynamics belong to the classical Ricker-map literature and are not rederived here; see [12, 13, 23]. In the present paper these discrete facts are used only as cited background; the fuzzy continuous-time results are proved separately.

3. FUZZY LOGISTIC MODEL: SOLUTION REPRESENTATION, WELL-POSEDNESS, AND RECONSTRUCTION

3.1. Model formulation. Throughout this section, assume that

Assumption 3.1. *The growth parameter and carrying capacity satisfy*

$$\Xi > 0, \quad \mathcal{K} > 0,$$

and the fuzzy initial state $\vartheta_0 \in \mathbb{E}$ is strictly positive in the sense that

$$m_0 := \inf_{\xi \in [0,1]} \min_{\alpha_0 \in [0,1]} \vartheta_0^{\text{gr}}(\xi, \alpha_0) > 0. \quad (3.1)$$

The fuzzy logistic differential equation with piecewise constant argument is

$$\begin{cases} \frac{d_{\text{gr}}}{d\tau} \vartheta(\tau) = \Xi \vartheta(\tau) \otimes_{\text{gr}} \left(1 \ominus_{\text{gr}} \frac{1}{\mathcal{K}} \vartheta(\lfloor \tau \rfloor) \right), & \tau \geq 0, \\ \vartheta(0) = \vartheta_0. \end{cases} \quad (3.2)$$

Here, the parameters Ξ and $\mathcal{K}$ remain crisp, while all uncertainty is carried by the state and the initial condition.

Definition 3.1. *A fuzzy-valued function $\vartheta : [0, \infty) \rightarrow \mathbb{E}$ is a solution of (3.2) if it is granular continuous on $[0, \infty)$, granular differentiable on every open interval $(n, n+1)$, satisfies (3.2) there, satisfies $\vartheta(0) = \vartheta_0$, and has matching one-sided values at every positive integer.*

Classical differentiability at the switching instants is not imposed. This is consistent with the standard DEPCAs solution concept: the state is continuous, whereas its derivative may jump when the frozen argument is updated [10].

Fix $\theta = (\xi, \alpha_0) \in [0, 1]^2$ and put

$$z(\tau; \theta) := \vartheta^{\text{gr}}(\xi, \tau, \alpha_0), \quad z_0(\theta) := \vartheta_0^{\text{gr}}(\xi, \alpha_0).$$

By Proposition 2.2, together with the granular arithmetic rule using the same RDM variable for repeated occurrences of the state, every HMF representative must satisfy

$$\begin{cases} \frac{\partial z(\tau; \theta)}{\partial \tau} = \Xi z(\tau; \theta) \left(1 - \frac{z(\lfloor \tau \rfloor; \theta)}{\mathcal{K}} \right), \\ z(0; \theta) = z_0(\theta) > 0. \end{cases} \quad (3.3)$$

Thus the fuzzy problem is represented by a family of scalar logistic DEPCA indexed by the HMF parameters.

3.2. Explicit local representation and nodal recursion. Before stating the fuzzy result, it is useful to isolate the underlying scalar calculation and to identify what is already known. For a crisp initial value $x_0 > 0$, Carvalho and Cooke [11] studied the normalized equation $x'(t) = \mu x(t)(1 - x(\lfloor t \rfloor))$. The step method of Cooke and Wiener [10] gives, on $n \leq \tau < n+1$,

$$x(\tau) = x_n \exp \left[\Xi \left(1 - \frac{x_n}{\mathcal{K}} \right) (\tau - n) \right], \quad x_n = x(n), \quad (3.4)$$

and continuity at $n+1$ yields

$$x_{n+1} = x_n \exp \left[\Xi \left(1 - \frac{x_n}{\mathcal{K}} \right) \right]. \quad (3.5)$$

For broader logistic DEPCAs, the same nodal reduction is explicitly recorded in [16]. Formulas (3.4)–(3.5) are therefore not claimed as new. For completeness, their derivation is immediate: on $[n, n+1)$, $x(\lfloor \tau \rfloor) = x_n$ is constant, so $x' = \Xi(1 - x_n/\mathcal{K})x$ is a linear equation with constant coefficient, whose solution is (3.4); taking the left limit at $n+1$ gives (3.5).

For the fuzzy model, define

$$z_n(\theta) := z(n; \theta), \quad n \in \mathbb{N}_0. \quad (3.6)$$

The following theorem gives the exact HMF representation.

Theorem 3.2 (Explicit HMF representation and fuzzy Ricker recursion). *Under Assumption 3.1, every HMF representative of a solution of (3.2) satisfies, for $n \in \mathbb{N}_0$,*

$$z(\tau; \theta) = z_n(\theta) \exp \left[\Xi \left(1 - \frac{z_n(\theta)}{\mathcal{K}} \right) (\tau - n) \right], \quad n \leq \tau < n+1, \quad (3.7)$$

where the nodal values obey

$$z_{n+1}(\theta) = z_n(\theta) \exp \left[\Xi \left(1 - \frac{z_n(\theta)}{\mathcal{K}} \right) \right] = \mathcal{R}_{\Xi, \mathcal{K}}(z_n(\theta)), \quad (3.8)$$

with $z_0(\theta) = \mathfrak{g}_0^{\text{gr}}(\xi, \alpha_0)$. Equivalently,

$$z_n(\theta) = \mathcal{R}_{\Xi, \mathcal{K}}^n(z_0(\theta)). \quad (3.9)$$

Proof. Fix $\theta = (\xi, \alpha_0)$ and $n \in \mathbb{N}_0$. By (3.3), the HMF representative satisfies a scalar DEPCA. On the interval $I_n = [n, n+1)$ one has $\lfloor \tau \rfloor = n$, hence

$$\frac{\partial z(\tau; \theta)}{\partial \tau} = a_n(\theta) z(\tau; \theta), \quad a_n(\theta) := \Xi \left(1 - \frac{z_n(\theta)}{\mathcal{K}} \right). \quad (3.10)$$

The quantity $a_n(\theta)$ is constant with respect to τ . Multiplying (3.10) by $e^{-a_n(\theta)(\tau-n)}$ gives

$$\frac{\partial}{\partial \tau} \left[e^{-a_n(\theta)(\tau-n)} z(\tau; \theta) \right] = 0.$$

Therefore

$$e^{-a_n(\theta)(\tau-n)} z(\tau; \theta) = z(n; \theta) = z_n(\theta),$$

which proves (3.7) without dividing by z .

By Definition 3.1, the state is continuous at $n+1$. Hence, taking $\tau \rightarrow (n+1)^-$ in (3.7),

$$z_{n+1}(\theta) = z_n(\theta) e^{a_n(\theta)},$$

which is exactly (3.8). Starting from the prescribed $z_0(\theta)$ and applying this identity successively determines $z_1(\theta), z_2(\theta), \dots$ uniquely. Repeated substitution yields (3.9). Since θ was arbitrary, the representation holds for every HMF component. $\square$

Remark 3.1 (Reduction to the classical logistic DEPCA). If ϑ_0 is crisp, say $\vartheta_0 = \widehat{x}_0$, then $z_0(\theta) \equiv x_0$ for all θ . Theorem 3.2 then collapses exactly to the crisp formulas (3.4)–(3.5) studied in the classical DEPCAs literature [11, 16]. Thus the new content is the HMF/granular propagation and fuzzy reconstruction, not the scalar Ricker reduction itself. In general, (3.9) does not simplify to an elementary closed form in n because the Ricker recursion is nonlinear. One may equivalently write

$$z_n(\theta) = z_0(\theta) \exp \left[n\Xi - \frac{\Xi}{\mathcal{K}} \sum_{j=0}^{n-1} z_j(\theta) \right], \quad (3.11)$$

but previous nodal values remain in the sum.

Theorem 3.3 (Existence and uniqueness). Under Assumption 3.1, problem (3.2) admits exactly one solution in the sense of Definition 3.1. Its HMF components are generated by (3.7)–(3.8), and the fuzzy state at each time is obtained by the inverse HMF reconstruction of Proposition 2.1.

Proof. The proof is divided into construction and uniqueness.

Step 1: scalar construction for each HMF parameter. Fix θ . On $[0, 1)$, equation (3.3) is the linear initial-value problem

$$z' = \Xi \left(1 - \frac{z_0(\theta)}{\mathcal{K}} \right) z, \quad z(0) = z_0(\theta),$$

so it has the unique solution (3.7) with $n = 0$. Its left limit at 1 uniquely fixes $z_1(\theta)$ through (3.8). Assume inductively that $z_n(\theta)$ is uniquely known. Then on $[n, n+1)$ the coefficient in (3.10) is a known real constant, and the ordinary linear initial-value problem has exactly one solution. Its endpoint uniquely determines $z_{n+1}(\theta)$. Induction therefore constructs one and only one scalar HMF trajectory for every θ and every finite time.

Step 2: reconstruction of a fuzzy value at each time. For fixed $\tau \geq 0$, define

$$L_\xi(\tau) := \inf_{\beta \geq \xi} \min_{\alpha_0 \in [0, 1]} z(\tau; \beta, \alpha_0), \quad (3.12)$$

$$U_\xi(\tau) := \sup_{\beta \geq \xi} \max_{\alpha_0 \in [0, 1]} z(\tau; \beta, \alpha_0). \quad (3.13)$$

For each fixed β , the initial HMF is affine and hence continuous in α_0 ; the maps $\mathcal{R}_{\Xi, \mathcal{K}}$ and G_s are continuous, so $z(\tau; \beta, \alpha_0)$ is continuous in α_0 . Consequently the inner minimum and maximum are attained. Moreover, the scalar construction produces finite values on every finite interval. As ξ increases, the admissible set $\{\beta : \beta \geq \xi\}$ decreases; hence $L_\xi(\tau)$ is nondecreasing and $U_\xi(\tau)$ is nonincreasing. Also $L_\xi(\tau) \leq U_\xi(\tau)$. Therefore

$$[L_\xi(\tau), U_\xi(\tau)], \quad 0 \leq \xi \leq 1,$$

is a nested family of nonempty compact intervals. By the standard level-set representation of fuzzy numbers [2] and, equivalently, the inverse-HMF construction (2.2), this family defines a fuzzy number, denoted by $\vartheta(\tau)$. At $\tau = 0$, Proposition 2.1 gives exactly $\vartheta(0) = \vartheta_0$.

Step 3: verification of the fuzzy differential equation. On each open interval $(n, n+1)$, every generating HMF component is differentiable and satisfies (3.3). The inverse-HMF arithmetic uses

these same dependent HMF components. Proposition 2.2 therefore converts the scalar identity back into the granular identity (3.2). Formula (3.8) gives matching left and right nodal values, so the reconstructed state is continuous at every integer. Thus the construction is a solution in the sense of Definition 3.1.

Step 4: uniqueness of the fuzzy solution. Suppose ϑ_1 and ϑ_2 are two fuzzy solutions with the same initial value. Fix any HMF parameter θ . Their HMF representatives satisfy the same scalar problem (3.3). Step 1 shows uniqueness on $[0, 1)$ and then successively on every $[n, n + 1)$. Hence the two HMF families coincide for all $\tau \geq 0$. Applying the reconstruction formula (2.2) gives identical level sets at every membership level, and therefore $\vartheta_1(\tau) = \vartheta_2(\tau)$ for all $\tau \geq 0$. $\square$

Example 3.1 (Detailed computation of the explicit solution). *Take*

$$\Xi = 0.8, \quad \mathcal{K} = 100, \quad \vartheta_0 = \langle 8, 10, 14 \rangle.$$

From (2.10),

$$[\vartheta_0]^\xi = [8 + 2\xi, 14 - 4\xi],$$

and an HMF of the initial fuzzy number is

$$z_0(\xi, \alpha_0) = 8 + 2\xi + 6(1 - \xi)\alpha_0. \quad (3.14)$$

For the lower support trajectory $z_0 = 8$, formula (3.7) gives on $0 \leq \tau < 1$

$$z(\tau) = 8e^{0.8(1-8/100)\tau} = 8e^{0.736\tau},$$

so

$$z_1 = 8e^{0.736} = 16.7005.$$

On $1 \leq \tau < 2$,

$$z(\tau) = 16.7005 \exp\left[0.8\left(1 - \frac{16.7005}{100}\right)(\tau - 1)\right],$$

and hence $z_2 = 32.5194$. Repetition of the same exact recursion gives the values in Table 1. The core trajectory and upper-support trajectory are obtained in exactly the same way.

TABLE 1. Nodal values from the exact recursion for Example 3.1.

n	$z_n(8)$	$z_n(10)$	$z_n(14)$
0	8.0000	10.0000	14.0000
1	16.7005	20.5443	27.8562
2	32.5194	38.7926	49.6107
3	55.7949	63.3003	74.2414
4	79.4655	84.9011	91.2306
5	93.6532	95.8015	97.8608

Figure 1 plots the exact continuous trajectories on the intervals $n = 0, \dots, 5$ and marks the nodal values.

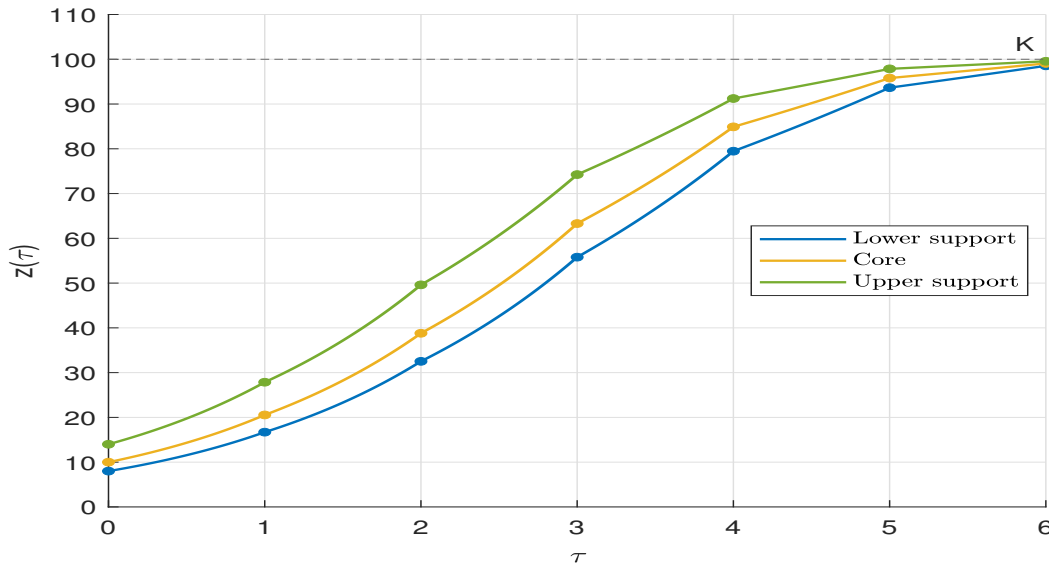


FIGURE 1. Exact within-period trajectories for the lower support, core, and upper support of $\langle 8, 10, 14 \rangle$ with $\Xi = 0.8$ and $\mathcal{K} = 100$.

Proposition 3.1 (Positivity). *Under Assumption 3.1, every HMF representative remains strictly positive:*

$$z(\tau; \theta) > 0, \quad \tau \geq 0, \quad (3.15)$$

and consequently every reconstructed level set of $\vartheta(\tau)$ is contained in $(0, \infty)$.

Proof. The initial HMF value is positive by (3.1). If $z_n(\theta) > 0$, then (3.8) is the product of $z_n(\theta)$ and a strictly positive exponential, so $z_{n+1}(\theta) > 0$. Induction gives positivity at all nodes, and (3.7) gives positivity between nodes. The reconstruction endpoints are infima and suprema of positive HMF values, and the uniform lower bound in Assumption 3.1, propagated over each finite time interval by the continuous scalar flow, keeps them positive. $\square$

3.3. Exact level-set reconstruction. The map $\mathcal{R}_{\Xi, \mathcal{K}}$ is not globally monotone. Therefore, outside a monotone regime, simply propagating the two endpoints of an initial interval need not give the exact image. The HMF formulation avoids this loss of dependency information.

For $x > 0$, let $\Phi_\tau(x)$ denote the unique scalar solution at time τ of the crisp problem corresponding to initial value x . If $\tau = n + s$, $n = \lfloor \tau \rfloor$ and $0 \leq s < 1$, then

$$\Phi_{n+s}(x) = G_s(\mathcal{R}_{\Xi, \mathcal{K}}^n(x)), \quad (3.16)$$

and

$$z(\tau; \theta) = \Phi_\tau(z_0(\theta)). \quad (3.17)$$

Theorem 3.4 (Exact level-set representation). *Let ϑ be the solution of (3.2). For every $\xi \in [0, 1]$ and $\tau \geq 0$,*

$$[\vartheta(\tau)]^\xi = \left[\inf_{\beta \geq \xi} \min_{\alpha_0 \in [0,1]} \Phi_\tau(\vartheta_0^{\text{gr}}(\beta, \alpha_0)), \sup_{\beta \geq \xi} \max_{\alpha_0 \in [0,1]} \Phi_\tau(\vartheta_0^{\text{gr}}(\beta, \alpha_0)) \right]. \quad (3.18)$$

Proof. By Theorem 3.2, the generating HMF component at time τ is precisely (3.17). Substituting this expression into the inverse-HMF reconstruction formula (2.2) gives (3.18). No monotonicity assumption is used. $\square$

The next lemma identifies a regime in which the extrema in (3.18) are attained at the ordinary endpoints of each initial level set.

Lemma 3.1 (Monotone invariant regime). *Assume $0 < \Xi \leq 1$. Then:*

- (i) $\mathcal{R}_{\Xi, \mathcal{K}}$ maps $(0, \mathcal{K}]$ into $(0, \mathcal{K}]$;
- (ii) $\mathcal{R}_{\Xi, \mathcal{K}}$ is strictly increasing on $(0, \mathcal{K}]$;
- (iii) G_s is increasing on $(0, \mathcal{K}]$ for every $s \in [0, 1]$.

Proof. Let $x = \mathcal{K}u$ with $u \in (0, 1]$. Since $\log u \leq u - 1$,

$$\log \frac{\mathcal{R}_{\Xi, \mathcal{K}}(x)}{\mathcal{K}} = \log u + \Xi(1 - u) \leq (u - 1) + \Xi(1 - u) = -(1 - \Xi)(1 - u) \leq 0.$$

Thus $0 < \mathcal{R}_{\Xi, \mathcal{K}}(x) \leq \mathcal{K}$, proving (i). Formula (2.15) gives

$$\mathcal{R}'_{\Xi, \mathcal{K}}(x) = e^{\Xi(1-x/\mathcal{K})} \left(1 - \frac{\Xi x}{\mathcal{K}} \right) \geq 0$$

for $0 < x \leq \mathcal{K}$ and $0 < \Xi \leq 1$. The derivative is positive except possibly at the endpoint $(\Xi, x) = (1, \mathcal{K})$, so the map is strictly increasing on $(0, \mathcal{K}]$. This proves (ii). Finally, (2.16) gives

$$\frac{\partial G_s}{\partial x}(x) = e^{\Xi s(1-x/\mathcal{K})} \left(1 - \frac{\Xi s x}{\mathcal{K}} \right) \geq 0,$$

which proves (iii). $\square$

Corollary 3.1 (Endpoint propagation below the carrying capacity). *Assume that $0 < \Xi \leq 1$ and*

$$0 < \vartheta_{0, \xi}^- \leq \vartheta_{0, \xi}^+ \leq \mathcal{K}, \quad \xi \in [0, 1], \quad (3.19)$$

where

$$[\vartheta_0]^\xi = [\vartheta_{0, \xi}^-, \vartheta_{0, \xi}^+].$$

Then, for every $\tau \geq 0$ and $\xi \in [0, 1]$,

$$[\vartheta(\tau)]^\xi = [\Phi_\tau(\vartheta_{0, \xi}^-), \Phi_\tau(\vartheta_{0, \xi}^+)]. \quad (3.20)$$

Proof. Let $\tau = n + s$, where $n = \lfloor \tau \rfloor$ and $s \in [0, 1]$. By Lemma 3.1, $\mathcal{R}_{\Xi, \mathcal{K}}$ maps $(0, \mathcal{K}]$ into itself and is strictly increasing on this interval, while G_s is increasing on $(0, \mathcal{K}]$. Hence,

$$\Phi_\tau = G_s \circ \mathcal{R}_{\Xi, \mathcal{K}}^n$$

is increasing on $(0, \mathcal{K}]$.

For each $\beta \in [\xi, 1]$, the HMF representation of the initial fuzzy value satisfies

$$\{\vartheta_0^{\text{gr}}(\beta, \alpha_0) : \alpha_0 \in [0, 1]\} = [\vartheta_{0,\beta}^-, \vartheta_{0,\beta}^+].$$

Since Φ_τ is increasing, it follows that

$$\min_{\alpha_0 \in [0,1]} \Phi_\tau(\vartheta_0^{\text{gr}}(\beta, \alpha_0)) = \Phi_\tau(\vartheta_{0,\beta}^-)$$

and

$$\max_{\alpha_0 \in [0,1]} \Phi_\tau(\vartheta_0^{\text{gr}}(\beta, \alpha_0)) = \Phi_\tau(\vartheta_{0,\beta}^+).$$

Moreover, the nestedness of the level sets of ϑ_0 gives

$$\vartheta_{0,\xi}^- \leq \vartheta_{0,\beta}^-, \quad \vartheta_{0,\beta}^+ \leq \vartheta_{0,\xi}^+, \quad \beta \geq \xi.$$

Therefore, using again the monotonicity of Φ_τ ,

$$\inf_{\beta \geq \xi} \Phi_\tau(\vartheta_{0,\beta}^-) = \Phi_\tau(\vartheta_{0,\xi}^-),$$

and

$$\sup_{\beta \geq \xi} \Phi_\tau(\vartheta_{0,\beta}^+) = \Phi_\tau(\vartheta_{0,\xi}^+).$$

Substitution into (3.18) yields (3.20). □

Example 3.2 (Exact propagation of a nontrivial level set). *Continue Example 3.1. At membership level $\xi = 1/2$,*

$$[\vartheta_0]^{1/2} = [9, 12].$$

Since $\Xi = 0.8 \leq 1$ and $[9, 12] \subset (0, 100)$, Corollary 3.1 applies. Therefore

$$[\vartheta(n)]^{1/2} = [\Phi_n(9), \Phi_n(12)].$$

The first five updates are

n	lower endpoint	upper endpoint
0	9.0000	12.0000
1	18.6384	24.2619
2	35.7345	44.4699
3	59.7543	69.3421
4	82.4512	88.6163
5	94.8785	97.0654

This computation illustrates why the endpoint formula is useful in the monotone regime: no optimization over the HMF parameters is required. Outside such a regime, Theorem 3.4, rather than naive endpoint propagation, is the correct reconstruction formula.

4. QUALITATIVE BEHAVIOR RELEVANT TO MARKET ADJUSTMENT

4.1. Equilibrium and stability notions. The scalar HMF equation has the two constant solutions 0 and $\mathcal{K}$. Consequently the fuzzy model has the two degenerate equilibria $\widehat{0}$ and $\widehat{\mathcal{K}}$. Under Assumption 3.1, the economically relevant equilibrium is the positive saturation level $\widehat{\mathcal{K}}$.

Definition 4.1 (Granular stability about $\widehat{\mathcal{K}}$). *The equilibrium $\widehat{\mathcal{K}}$ is locally stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left\| \vartheta_0 \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}} < \delta \implies \left\| \vartheta(\tau) \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}} < \varepsilon \quad \text{for all } \tau \geq 0.$$

It is locally asymptotically stable if it is locally stable and, for all sufficiently small initial perturbations,

$$\lim_{\tau \rightarrow \infty} \left\| \vartheta(\tau) \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}} = 0.$$

Definition 4.2 (Componentwise damped oscillation). *A fuzzy solution is componentwise damped oscillatory about $\widehat{\mathcal{K}}$ if, for every HMF parameter θ with $z_0(\theta) \neq \mathcal{K}$, the nodal deviations $z_n(\theta) - \mathcal{K}$ eventually alternate in sign and*

$$\left\| \vartheta(\tau) \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}} \longrightarrow 0 \quad (\tau \rightarrow \infty).$$

If alternation starts at $n = 0$, the word “eventually” is omitted.

For the economic application, damped oscillation is more informative than oscillation alone: it describes repeated over- and under-adjustment while retaining convergence to the long-run market target. Accordingly, the paper does not develop a separate theorem for persistent oscillations.

4.2. Local asymptotic stability.

Theorem 4.1 (Local granular asymptotic stability). *Let $0 < \Xi < 2$. Then there exist $\delta > 0$, $q \in (0, 1)$, and $M \geq 1$ such that*

$$\left\| \vartheta_0 \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}} < \delta \tag{4.1}$$

implies

$$\left\| \vartheta(\tau) \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}} \leq M q^{\lfloor \tau \rfloor} \left\| \vartheta_0 \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}}, \quad \tau \geq 0. \tag{4.2}$$

Hence $\widehat{\mathcal{K}}$ is locally asymptotically stable in the granular metric.

Proof. From (2.17),

$$|\mathcal{R}'_{\Xi, \mathcal{K}}(\mathcal{K})| = |1 - \Xi| < 1.$$

By continuity of $\mathcal{R}'_{\Xi, \mathcal{K}}$, choose $\delta \in (0, \mathcal{K})$ so that, on

$$I_\delta = [\mathcal{K} - \delta, \mathcal{K} + \delta],$$

$$q := \sup_{x \in I_\delta} |\mathcal{R}'_{\Xi, \mathcal{K}}(x)| < 1. \tag{4.3}$$

Since $\mathcal{R}_{\Xi, \mathcal{K}}(\mathcal{K}) = \mathcal{K}$, the mean value theorem gives

$$|\mathcal{R}_{\Xi, \mathcal{K}}(x) - \mathcal{K}| \leq q|x - \mathcal{K}|, \quad x \in I_\delta. \tag{4.4}$$

In particular, $\mathcal{R}_{\Xi, \mathcal{K}}(I_\delta) \subset I_\delta$.

Condition (4.1) and (2.8) imply $|z_0(\theta) - \mathcal{K}| < \delta$ for every θ . Applying (4.4) to the nodal recursion and using induction yields

$$|z_n(\theta) - \mathcal{K}| \leq q^n |z_0(\theta) - \mathcal{K}|. \quad (4.5)$$

Now write $\tau = n + s$, where $n = \lfloor \tau \rfloor$ and $0 \leq s < 1$. Theorem 3.2 gives

$$z(\tau; \theta) = G_s(z_n(\theta)).$$

The derivative $\partial G_s(x)/\partial x$ is continuous on the compact set $I_\delta \times [0, 1]$, so

$$M := \max \left\{ 1, \max_{x \in I_\delta, 0 \leq s \leq 1} \left| \frac{\partial G_s}{\partial x}(x) \right| \right\} < \infty. \quad (4.6)$$

Since $G_s(\mathcal{K}) = \mathcal{K}$, another mean value estimate gives

$$|z(\tau; \theta) - \mathcal{K}| \leq M |z_n(\theta) - \mathcal{K}| \leq M q^n |z_0(\theta) - \mathcal{K}|.$$

Taking the supremum over the HMF parameters and using (2.8) proves (4.2). Because $q^{\lfloor \tau \rfloor} \rightarrow 0$, asymptotic convergence follows, and the same estimate yields local stability. $\square$

Remark 4.1 (Classical limit of the stability result). *For a degenerate initial fuzzy number, Theorem 4.1 reduces to the classical local stability condition for the positive fixed point of the Ricker map, namely $|1 - \Xi| < 1$; see [22] and the Ricker/EPCA literature [11, 12]. The new statement is the uniform transfer of the nodal contraction to all HMF components and to the full continuous-time fuzzy trajectory.*

Example 4.1 (Local convergence from both sides of the target). *Let $\Xi = 0.8$, $\mathcal{K} = 100$, and $\vartheta_0 = \langle 96, 98, 101 \rangle$. The support is a small neighborhood of 100. For instance, on $I = [95, 105]$, direct evaluation of (2.15) gives*

$$\sup_{x \in I} |\mathcal{R}'_{\Xi, \mathcal{K}}(x)| < 0.25 < 1,$$

so the contraction neighborhood required by Theorem 4.1 can be chosen to contain the entire initial support. The representative nodal sequences starting from 96, 98, and 101 are

n	$z_n(96)$	$z_n(98)$	$z_n(101)$
0	96.0000	98.0000	101.0000
1	99.1217	99.5806	100.1952
2	99.8206	99.9153	100.0389
3	99.9640	99.9830	100.0078
4	99.9928	99.9966	100.0016
5	99.9986	99.9993	100.0003

Thus perturbations both below and above the saturation level contract rapidly toward 100, in agreement with Theorem 4.1.

4.3. Global convergence below capacity for $0 < \Xi \leq 1$.

Theorem 4.2 (Global convergence from below and uncertainty collapse). *Assume $0 < \Xi \leq 1$ and*

$$0 < m_0 \leq z_0(\theta) \leq \mathcal{K} \quad \text{for all } \theta \in [0, 1]^2. \quad (4.7)$$

Then:

- (i) every HMF trajectory $z(\tau; \theta)$ is nondecreasing and remains in $(0, \mathcal{K}]$;
- (ii) $z(\tau; \theta) \rightarrow \mathcal{K}$ for every θ ;
- (iii) the convergence is uniform in the HMF parameters,

$$\lim_{\tau \rightarrow \infty} \left\| \vartheta(\tau) \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}} = 0; \quad (4.8)$$

- (iv) every level-set width tends to zero,

$$\lim_{\tau \rightarrow \infty} \text{diam}[\vartheta(\tau)]^\xi = 0, \quad \xi \in [0, 1]. \quad (4.9)$$

Proof. Fix θ . If $0 < z_n(\theta) < \mathcal{K}$, then (3.8) gives

$$\frac{z_{n+1}(\theta)}{z_n(\theta)} = \exp \left[\Xi \left(1 - \frac{z_n(\theta)}{\mathcal{K}} \right) \right] > 1,$$

so $z_{n+1}(\theta) > z_n(\theta)$. Lemma 3.1 gives $z_{n+1}(\theta) \leq \mathcal{K}$. Hence the nodal sequence is increasing and bounded above. Let its limit be $L > 0$. By continuity of $\mathcal{R}_{\Xi, \mathcal{K}}$,

$$L = \mathcal{R}_{\Xi, \mathcal{K}}(L) = L e^{\Xi(1-L/\mathcal{K})}.$$

Because $L > 0$, this identity implies $L = \mathcal{K}$. If $z_n(\theta) = \mathcal{K}$ at some step, all later nodal values remain equal to $\mathcal{K}$.

On $[n, n+1)$, the coefficient $\Xi(1 - z_n/\mathcal{K})$ in (3.7) is nonnegative, so the continuous trajectory increases from z_n to z_{n+1} . This proves (i) and (ii).

For uniform convergence, let $x_*(\tau) = \Phi_\tau(m_0)$. By Corollary 3.1 and monotonicity of the scalar flow,

$$x_*(\tau) \leq z(\tau; \theta) \leq \mathcal{K} \quad \text{for every } \theta.$$

The scalar argument above gives $x_*(\tau) \rightarrow \mathcal{K}$. Hence

$$0 \leq \mathcal{K} - z(\tau; \theta) \leq \mathcal{K} - x_*(\tau) \rightarrow 0$$

uniformly in θ . Formula (2.8) gives (iii). Finally, every endpoint of every reconstructed level set lies between the minimum and maximum HMF values and therefore converges uniformly to $\mathcal{K}$; the difference of the two endpoints tends to zero. This proves (iv). $\square$

Remark 4.2 (Crisp special case and relation to earlier EPCA results). *If ϑ_0 is degenerate, Theorem 4.2 becomes a scalar convergence statement for the logistic EPCA. Convergence and global-attractivity questions for crisp logistic equations with piecewise constant arguments have been treated much more generally in [11, 14, 16]. The restriction $0 < \Xi \leq 1$ here is not presented as the sharpest crisp stability range; it is imposed because it gives an invariant order-preserving regime in which exact fuzzy endpoint propagation and uniform uncertainty-collapse arguments are immediate.*

Remark 4.3 (Meaning of uncertainty collapse). *Conclusion (4.9) concerns only uncertainty inherited from the initial state. It does not assert that real economic uncertainty disappears. If Ξ , $\mathcal{K}$, or future shocks are uncertain, a model with persistent sources of fuzziness is required.*

Example 4.2 (Global convergence of the fuzzy band). *Example 3.1 satisfies all assumptions of Theorem 4.2. The support endpoints evolve from $[8, 14]$ to approximately $[93.6532, 97.8608]$ by $n = 5$ and continue toward the common limit 100. At membership level $1/2$, Example 3.2 gives width 3 initially and approximately 2.1869 at $n = 5$. The width need not decrease monotonically at the first few updates; Theorem 4.2 asserts asymptotic collapse, not monotone contraction of every level-set width. Figure 1 displays representative continuous-time trajectories. Thus the same example simultaneously verifies the exact representation, monotone endpoint propagation, and asymptotic loss of initial-state ambiguity predicted by the theorem.*

4.4. Damped overshooting for $1 < \Xi < 2$.

Theorem 4.3 (Local componentwise damped oscillation). *Let $1 < \Xi < 2$. There exists $\delta > 0$ such that, if*

$$\left\| \vartheta_0 \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}} < \delta, \quad (4.10)$$

then, for every HMF parameter θ with $z_0(\theta) \neq \mathcal{K}$,

$$(z_{n+1}(\theta) - \mathcal{K})(z_n(\theta) - \mathcal{K}) < 0, \quad n \in \mathbb{N}_0, \quad (4.11)$$

and

$$\lim_{\tau \rightarrow \infty} \left\| \vartheta(\tau) \ominus_{\text{gr}} \widehat{\mathcal{K}} \right\|_{\text{gr}} = 0.$$

Hence every non-equilibrium HMF component alternates about the saturation level with decreasing amplitude; the fuzzy solution is componentwise damped oscillatory.

Proof. Because $1 < \Xi < 2$, one has $\mathcal{K}/\Xi < \mathcal{K}$ and $|1 - \Xi| < 1$. By continuity of $\mathcal{R}'_{\Xi, \mathcal{K}}$, choose $\delta > 0$ sufficiently small that

$$I_\delta = [\mathcal{K} - \delta, \mathcal{K} + \delta] \subset (\mathcal{K}/\Xi, \infty)$$

and

$$q := \sup_{x \in I_\delta} |\mathcal{R}'_{\Xi, \mathcal{K}}(x)| < 1.$$

On I_δ , formula (2.15) gives $1 - \Xi x/\mathcal{K} < 0$, hence $\mathcal{R}'_{\Xi, \mathcal{K}}(x) < 0$ throughout this neighborhood. As in Theorem 4.1, the mean value theorem gives

$$|\mathcal{R}_{\Xi, \mathcal{K}}(x) - \mathcal{K}| \leq q|x - \mathcal{K}|,$$

so $\mathcal{R}_{\Xi, \mathcal{K}}(I_\delta) \subset I_\delta$ and all nodal HMF values remain in I_δ .

Since $\mathcal{R}_{\Xi, \mathcal{K}}$ is strictly decreasing there and $\mathcal{R}_{\Xi, \mathcal{K}}(\mathcal{K}) = \mathcal{K}$,

$$x < \mathcal{K} \implies \mathcal{R}_{\Xi, \mathcal{K}}(x) > \mathcal{K}, \quad x > \mathcal{K} \implies \mathcal{R}_{\Xi, \mathcal{K}}(x) < \mathcal{K}.$$

This proves the strict alternation (4.11). The contraction estimate also gives $z_n(\theta) \rightarrow \mathcal{K}$ geometrically. Finally, Theorem 4.1 transfers nodal convergence uniformly to the complete continuous-time fuzzy

solution, yielding convergence in the granular metric. Alternation together with convergence is precisely componentwise damped oscillation. $\square$

Remark 4.4 (Why no separate oscillation theorem is needed). *Theorem 4.3 already implies oscillation of every non-equilibrium HMF nodal sequence in the stated neighborhood, while adding the economically important fact that the oscillation decays toward the target. For a market-saturation application, this is more informative than a theorem asserting sign changes alone. Persistent cycles and instability for larger values of Ξ are classical features of the Ricker map [12, 13, 23] and are not pursued as separate fuzzy economic regimes here.*

Example 4.3 (Damped over-adjustment). *Let*

$$\Xi = 1.6, \quad \mathcal{K} = 100, \quad \vartheta_0 = \langle 95, 97, 99 \rangle.$$

The initial fuzzy support lies close to $\mathcal{K}$. In fact, with $I = [94, 106]$, formula (2.15) gives

$$\mathcal{R}'_{\Xi, \mathcal{K}}(x) < 0, \quad \sup_{x \in I} |\mathcal{R}'_{\Xi, \mathcal{K}}(x)| < 0.633 < 1,$$

and $[95, 99] \subset I$. Thus the hypotheses used in the proof of Theorem 4.3 are verified quantitatively for this example. Representative nodal values are Every listed trajectory alternates around 100 and its deviation

TABLE 2. Alternating nodal values for Example 4.3.

n	$z_n(95)$	$z_n(97)$	$z_n(99)$
0	95.0000	97.0000	99.0000
1	102.9123	101.7696	100.5967
2	98.2269	98.9286	99.6408
3	101.0534	100.6391	100.2151
4	99.3645	99.6152	99.8708
5	100.3800	100.2304	100.0775

decreases. Figure 2 gives the exact continuous trajectories on the intervals $n = 0, \dots, 5$.

5. ECONOMIC INTERPRETATION AND ILLUSTRATIVE APPLICATION

5.1. Periodically revised market-penetration model. The state $\vartheta(\tau)$ can be interpreted as an imprecisely known cumulative adoption stock, installed-capacity index, or market-penetration measure. Logistic curves have a long history in the economics of innovation diffusion and technological substitution [24–26]. The present model adds a different mechanism: the state changes continuously, but the saturation signal is revised only at discrete reporting dates. This type of separation between continuous activity and periodic information updating has an economic precedent in piecewise-constant-argument modeling; see Cavalli and Naimzada [20].

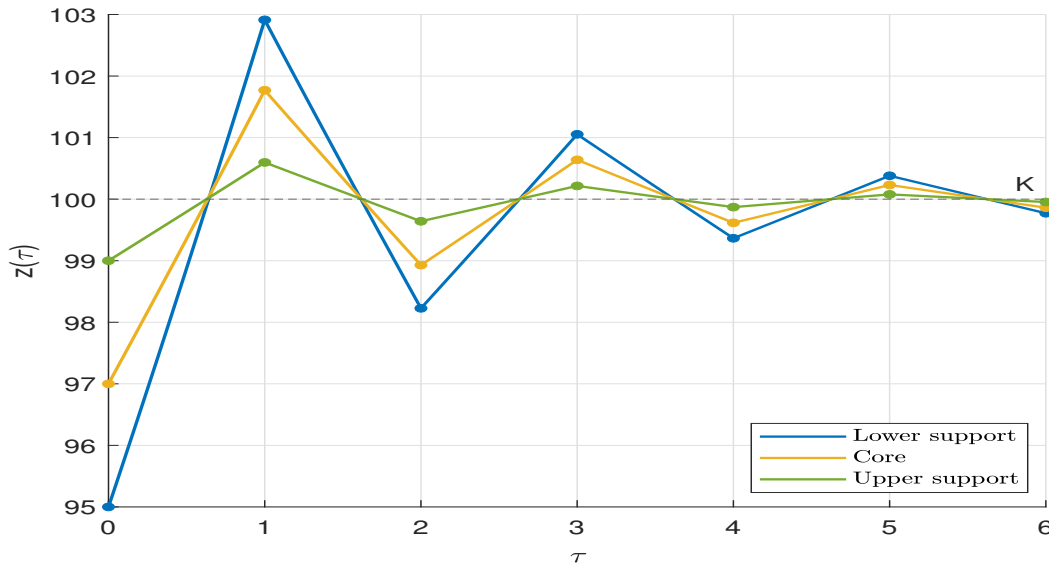


FIGURE 2. Damped oscillation for $\Xi = 1.6$, $\mathcal{K} = 100$, and representative values from $\langle 95, 97, 99 \rangle$.

On a reporting interval $n \leq \tau < n + 1$, an HMF representative satisfies

$$\frac{\partial z}{\partial \tau} = \Xi z(\tau) \left(1 - \frac{z_n}{\mathcal{K}} \right). \quad (5.1)$$

The current stock $z(\tau)$ therefore responds to the last observed saturation state z_n . If $z_n < \mathcal{K}$, the reported market remains below potential and the within-period growth coefficient is positive. If $z_n > \mathcal{K}$, the last report indicates overshooting and the coefficient is negative. The feedback is not revised until the next reporting date.

The parameters have the following stylized interpretations:

- $\mathcal{K}$ is the long-run market potential or target installed capacity;
- Ξ is the response intensity over one reporting period; if the physical growth rate is r and reports arrive every h time units, the effective parameter is $\Xi = rh$;
- ϑ_0 represents ambiguity in the initial market state, for example uncertainty caused by incomplete surveys, heterogeneous expert assessments, or early-stage reporting error.

The model is intentionally parsimonious: Ξ and $\mathcal{K}$ are crisp and constant, so the paper isolates how initial fuzzy ambiguity propagates under periodically updated saturation feedback.

5.2. Two economically relevant adjustment regimes. The main theoretical regimes have direct interpretations.

For $0 < \Xi \leq 1$, Theorem 4.2 shows that any positive fuzzy initial state lying below $\mathcal{K}$ converges monotonically to the market potential. Example 3.1 may be read as a market with potential $\mathcal{K} = 100$ and an initial fuzzy assessment $\langle 8, 10, 14 \rangle$. The last available saturation signal is conservative enough that no nodal overshoot occurs; the uncertainty band contracts because all admissible initial states are attracted to the same target.

For $1 < \Xi < 2$, a sufficiently small perturbation of the target remains stable but the latest reported deviation is over-corrected. Example 4.3, with $\Xi = 1.6$, represents this situation: an observation below 100 induces sufficiently strong within-period expansion that the next report lies above 100, after which the sign of the adjustment reverses. Theorem 4.3 proves that, locally, these overcorrections decay. This regime is economically meaningful when decisions rely on stale information but the adjustment intensity is not strong enough to destroy stability.

The paper deliberately does not develop the post-instability Ricker regimes as an economic application. Cycles and complex dynamics for larger response parameters are mathematically well known in the scalar Ricker literature [12, 13, 23], but interpreting them as technology-adoption dynamics would require additional behavioral and empirical justification. For the present purpose, monotone saturation and damped overshooting are the two regimes most directly aligned with convergence to a long-run market potential.

5.3. Exact computational procedure. No numerical ODE solver is needed between reporting dates. Given $\Xi, \mathcal{K}$, and the initial HMF, the exact computation is:

Algorithm 5.1 (HMF propagation). *For each desired parameter pair $\theta = (\xi, \alpha_0)$:*

- (1) Set $z_0(\theta) = \mathfrak{g}_0^{\text{gr}}(\xi, \alpha_0)$.
- (2) Generate the nodal values from

$$z_{n+1}(\theta) = \mathcal{R}_{\Xi, \mathcal{K}}(z_n(\theta)).$$

- (3) For $\tau = n + s, 0 \leq s < 1$, evaluate

$$z(\tau; \theta) = G_s(z_n(\theta)).$$

- (4) Reconstruct the fuzzy level sets using Theorem 3.4. In the monotone regime of Corollary 3.1, propagate only the two endpoints of each initial level set.

6. DISCUSSION

The model studied in this paper sits at the intersection of three established ideas: logistic saturation, piecewise constant feedback, and fuzzy uncertainty. The separate ingredients are classical, but their combination changes the qualitative picture in a useful way.

First, replacing $x(\tau)$ by $x(\lfloor \tau \rfloor)$ only in the saturation factor is not a numerical Euler discretization of the logistic differential equation. The resulting equation is itself a hybrid continuous-time model. Between integers, the current state $x(\tau)$ still evolves continuously, while the density or saturation signal is held at the last nodal value. The exact nodal map happens to be a Ricker map because the within-period equation is linear once the frozen feedback is fixed. This distinction matters in applications: the model describes periodic information or control updating, not merely a computational approximation.

Second, the Ricker reduction provides a clear explanation for behavior that cannot occur in the ordinary logistic ODE. In (1.1), the positive equilibrium is asymptotically stable for every $\Xi > 0$ and

positive solutions approach it without repeated overshooting. In the piecewise-feedback model, the one-period map has multiplier $1 - \Xi$ at $\mathcal{K}$. Hence the response intensity relative to the reporting interval matters. For $1 < \Xi < 2$, the equilibrium is locally stable but approached through alternating overcorrections. The loss of local stability beyond the classical Ricker threshold is well known in the scalar theory and is mentioned only as background, because the economically relevant focus here is convergence to the target.

Third, fuzziness and dynamical instability should not be conflated. Fuzzy width represents uncertainty about the state, whereas oscillation or instability concerns the deterministic evolution of each HMF representative. Theorem 4.2 shows that in a stable monotone regime an initially fuzzy state can converge to a crisp equilibrium even though no probabilistic learning mechanism is built into the model. This occurs because every admissible initial state is attracted to the same fixed point. Conversely, when the underlying map is unstable, a narrow fuzzy initial set may expand or become dynamically complicated. The fuzzy representation therefore reveals the sensitivity of the economic trajectory to imprecise initial information.

Fourth, the use of granular differentiability is not cosmetic. The nonlinear recursion repeatedly reuses the same fuzzy state. HMF/RDM representation offers a way to propagate this dependency through ordinary real functions. The exact level-set formula (3.18) makes explicit that global monotonicity cannot be assumed for the Ricker map. In particular, when Ξ or the state is large enough that $\mathcal{R}'_{\Xi, \mathcal{K}}$ changes sign, a naive endpoint calculation can fail. The monotone endpoint formula in Corollary 3.1 is therefore stated only under conditions that are proved to make it valid.

The model also has clear limitations. The parameters Ξ and $\mathcal{K}$ are treated as crisp and time independent. In economic settings the market potential may itself be uncertain and may change with prices, income, regulation, or competing technologies. Fuzzifying $\mathcal{K}$ or Ξ is possible in principle, but the nonlinear expression then contains several dependent fuzzy quantities and the stability analysis requires additional care. Likewise, external forcing and stochastic shocks are omitted. The current results should therefore be read as a structural benchmark for uncertainty in the initial state rather than a complete forecasting model.

Another limitation concerns the choice of the floor function. The unit interval is only a normalization. If information is updated every $h > 0$ time units, the model can be written

$$\frac{dx(\tau)}{d\tau} = rx(\tau) \left(1 - \frac{x(h \lfloor \tau/h \rfloor)}{\mathcal{K}} \right), \quad (6.1)$$

and the nodal map becomes

$$x_{n+1} = x_n \exp \left[rh \left(1 - \frac{x_n}{\mathcal{K}} \right) \right].$$

Hence the effective dimensionless parameter is $\Xi = rh$. The threshold $\Xi = 2$ may therefore be interpreted either as excessive intrinsic responsiveness r , an excessively long information-update interval h , or their combination. This observation is especially relevant economically: even a moderate adjustment rate can become destabilizing if market information is updated too infrequently.

The economic application in Section 5 is illustrative rather than empirical. It demonstrates how the mathematical results translate into two transparent patterns: monotone saturation and alternating reported overshoot with decreasing amplitude. An empirical study would require estimation of Ξ and $\mathcal{K}$, construction of a fuzzy initial state from data or expert assessments, and model comparison against ordinary logistic, Bass, Gompertz, or alternative diffusion specifications. The classical empirical literature [24–26] provides precedents for logistic diffusion, while the boundedly rational monopolist model in [20] provides an economic precedent for continuous activity coupled with periodic information updating. Neither strand, by itself, empirically validates the particular fuzzy logistic mechanism proposed here.

Several mathematical extensions are natural. One is to replace $\lfloor \tau \rfloor$ by a delayed or alternating piecewise constant argument, thereby separating reporting delay from within-period time. A second is to allow fuzzy $\mathcal{K}$ and Ξ while preserving dependency information in the HMF representation. A third is to include harvesting, advertising, policy intervention, or external innovation terms. Finally, the critical and post-bifurcation regimes could be studied in fuzzy terms by combining the extensive Ricker-map literature with fuzzy set propagation. Those extensions are beyond the scope of the present article because the priority here is to establish a mathematically transparent baseline model with complete proofs.

7. CONCLUSION

A fuzzy logistic differential equation with the piecewise constant argument $\lfloor \tau \rfloor$ has been formulated under granular differentiability. The HMF representation converts the fuzzy problem into a family of scalar DEPCAs. The scalar local formula and the induced Ricker nodal map are classical ingredients and are cited accordingly; the fuzzy HMF representation, reconstruction, and continuous-time qualitative statements are established directly in the paper.

The analysis proves existence and uniqueness, positivity, exact level-set reconstruction, and an endpoint propagation formula in a monotone regime. The positive crisp equilibrium is locally asymptotically stable in the granular metric for $0 < \Xi < 2$. For $0 < \Xi \leq 1$, positive fuzzy data below the carrying capacity converge globally to the crisp saturation level and their level-set widths vanish. For $1 < \Xi < 2$, sufficiently small perturbations exhibit componentwise damped oscillation: the nodal HMF components alternate around the target while the complete fuzzy trajectory converges to it. Detailed examples and reproducible MATLAB scripts illustrate the exact solution formula and the two stable adjustment regimes.

In the economic interpretation, ϑ represents an imprecisely known adoption or market-penetration stock, $\mathcal{K}$ is the market potential, and Ξ measures responsiveness over one information-update interval. Conservative updating produces monotone saturation, whereas stronger but still stable updating produces decaying overcorrection. The model is best regarded as a structural benchmark for markets in which activity is continuous but the saturation signal is revised periodically.

Extensions with fuzzy parameters, external shocks, or more general piecewise constant arguments are natural directions for further work.

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