

T-Spherical Fuzzy Graphs Framework for Wi-Fi Networks and Cybersecurity Based on Normal Dominating Mechanisms

Waheed Ahmad Khan^{1,*}, Misal Ahmad¹, Muhammad Asshad^{2,*}, Omar Almomani³,
Taher M. Ghazal^{3,4,5}

¹*Division of Science and Technology, Department of Mathematics, University of Education Lahore, Attock Campus, Attock 43600, Pakistan*

²*Faculty of Computer Studies, Arab open university oman*

³*Department of Networks and Cybersecurity, Hourani Center for Applied Scientific Research, Al-Ahliyya Amman University, Amman, Jordan*

⁴*Faculty of Computing and IT, Sohar University, Oman*

⁵*Center for Cyber Security, Faculty of Information Science and Technology, Universiti Kebangsaan Malaysia (UKM), 43600 Bangi, Selangor, Malaysia*

*Corresponding authors: muhammad.a@aou.edu.om, sirwak2003@yahoo.com

Abstract. Intelligent communication and cybersecurity networks require mathematical models capable of representing complex uncertainty while ensuring reliable network control. Existing domination models for fuzzy graph structures cannot adequately capture the richer uncertainty represented by T-spherical fuzzy information. In this study, we introduce the concept of normal domination in T-spherical fuzzy graphs (T-SFGs), where uncertainty is characterized by membership, neutrality and non-membership degrees satisfying the generalized T-spherical constraint. Fundamental concepts of normal domination are established together with their theoretical properties. The proposed framework provides a unified model for intelligent network analysis by integrating T-spherical fuzzy information with domination theory. Applications to Wi-Fi gateway selection and cybersecurity threat assessment demonstrate its practical effectiveness while comparative analysis confirms that the proposed T-SFG framework offers improved modeling flexibility, enhanced decision-making accuracy and broader applicability than existing fuzzy graph-based domination models.

Received: Aug. 30, 2026.

2020 *Mathematics Subject Classification.* 05C69; 03E72.

Key words and phrases. normal domination in T-SFGs; Wi-Fi communication networks; cybersecurity threat assessment; intelligent network analysis.

1. INTRODUCTION

Graph theory has become one of the most influential mathematical tools for representing and analyzing interconnected systems encountered in engineering, computer science, operations research, optimization and intelligent decision-making. Due to its capability of modeling complex relationships among interacting entities, numerous graph-theoretic concepts have been developed to address practical problems arising in communication networks, transportation systems, social interactions and information processing. Among these concepts, domination and independence have received considerable attention because of their ability to characterize network coverage, resource allocation, monitoring strategies and control mechanisms [1]. Consequently, domination theory has evolved into an active research area with a wide range of theoretical developments and practical applications [2]. Nevertheless, many real-world networks involve uncertain, vague or incomplete information that cannot be adequately represented by classical graph models. To overcome this limitation, fuzzy graph theory has provided an effective framework for incorporating uncertainty into graph structures.

The foundation of uncertainty modeling was established by Zadeh [3] through the introduction of fuzzy sets (FSs), where each element is characterized by a membership degree within the interval $[0, 1]$. After this, FSs have been successfully applied to numerous scientific and engineering problems involving imprecise information. To enhance their descriptive capability, several generalizations have subsequently been proposed. One of the earliest and most significant extensions was introduced by Atanassov [4] in the form of intuitionistic fuzzy sets (IFSs) in which each element is described by both membership (μ) and non-membership (η) degrees satisfying the constraint $0 < \mu + \eta < 1$. Although IFSs provide a richer representation of uncertainty than classical FSs however its constraint may become inadequate in situations where the combined membership and non-membership assessments naturally exceed one. To overcome this limitation, Yager [5] proposed Pythagorean fuzzy sets (PyFSs) which relax the admissible domain by replacing the linear constraint with the condition $0 \leq \mu_p^2(x) + \eta_p^2(x) \leq 1$. This modification considerably enlarges the feasible space for expressing uncertainty and making PyFSs more flexible than IFSs in many decision-making applications. Therefore, PyFSs have demonstrated superior adaptability in representing uncertain information encountered in practical systems. Although PyFSs significantly improve uncertainty representation but some real-life scenarios require an additional degree to express neutral or abstaining opinions. A typical example is a voting process in which individuals may vote in favor, vote against, abstain or refuse to participate. Such situations cannot be fully described using only membership and non-membership grades. To address this issue, Cuong [6] introduced picture fuzzy sets (PFSs), where every element is characterized by four possible responses: positive, neutral, negative and refusal. The addition of the neutral component substantially broadens the applicability of fuzzy modeling to problems involving human judgments and multi-state evaluations. Nevertheless, the admissible region of PFSs remains limited under certain highly uncertain environments. To further enhance modeling flexibility, the term

spherical fuzzy sets (SFSs) was proposed in [7]. It generalized PFSs by defining membership (μ), neutrality (v) and non-membership (η) values under the constraint $0 \leq \mu^2 + v^2 + \eta^2 \leq 1$. The spherical framework provides a considerably larger feasible domain and hence it provides a more expressive representations of uncertainty than FSs, IFSs and PFSs. Due to its more adaptability, SFSs have attracted substantial research interest in areas including intelligent decision-making, optimization and data analysis. For instance, Ashraf et al. [8] investigated their application to decision-making problems. Afterward, a more general framework was later introduced by Mahmood et al. [9] through the concept of T-spherical fuzzy sets (T-SFSs). Instead of fixing the power to two, T-SFSs employ the generalized condition $0 \leq \mu^n + v^n + \eta^n \leq 1$, where $n \geq 1$ is an integer. This generalization significantly enlarges the permissible uncertainty domain and provides greater flexibility for modeling complex systems characterized by ambiguous, incomplete or conflicting information. Consequently, T-SFSs have employed in transportation optimization [10], decision-making [11], information systems for sports culture analysis [12] and multi-criteria group decision-making through distance measures [13]. The development of FSs models leading to T-SFSs is summarized in Fig. 1. On the other hand, the concept of fuzzy graphs (FGs) was pioneered

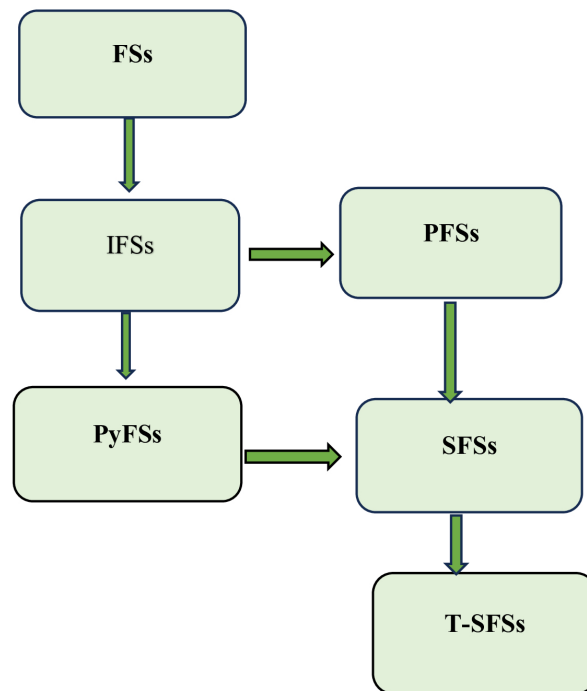


FIGURE 1. FSs extensions

by Rosenfeld [14] that extended graph theory to environments involving uncertain and imprecise relationships. Since then, FGs have evolved into an important mathematical framework for modeling systems in which the interactions among entities cannot be represented adequately by crisp graphs. Consequently, numerous classical graph-theoretic notions have been generalized to the

fuzzy setting which lead to effective solutions for problems involving uncertain relational structures. Consequently, FGs have found widespread applications in areas including communication networks, artificial intelligence, decision-making, pattern recognition and intelligent information systems. The continuous development of fuzzy graph theory has led to several generalized graph models with enhanced uncertainty-handling capabilities. Among these are intuitionistic fuzzy graphs (IFGs) [15], picture fuzzy graphs (PFGs) [16], interval-valued Pythagorean fuzzy graphs (IVPFGs) [17], and many other variants. Naz et al. [18] introduced Pythagorean fuzzy graphs (PyFGs) as a natural extension of IFGs and demonstrated their effectiveness in decision-making problems. Subsequently, several graph-theoretic concepts were established within interval-valued Pythagorean fuzzy graphs together with their application to socioeconomic analysis [19]. To further enlarge the modeling capability, spherical fuzzy graphs (SFGs) which generalize PFGs were proposed in [20]. Their structural properties have been investigated extensively including balanced SFGs [21] and various graph indices together with decision-making applications [22]. Further extension of PFG termed T-spherical fuzzy graphs (T-SFGs) was explored in [23]. Compared with previously developed fuzzy graph models, T-SFGs provide a significantly larger feasible domain for representing uncertain information, making them particularly suitable for modeling highly complex systems. Consequently, recent studies have focused on various theoretical developments and practical applications of T-SFGs. For example, shortest path algorithms have been developed within the T-SFG framework [24], several graph product operations have been established [25] and communication network applications have also been successfully investigated [26].

Domination is one of the most extensively investigated topics in graph theory because of its remarkable ability to model monitoring, coverage, resource allocation, communication and optimization problems in complex networks. Due to its broad applicability, numerous domination variants together with their theoretical properties and practical applications have been reported in the literature [27]. The incorporation of domination concepts into fuzzy graph theory was first established in [28] which opened new directions for analyzing networks characterized by uncertainty and imprecise relationships. Since then, considerable research has been devoted to developing new domination parameters and investigating their structural properties within fuzzy graphs [29–31]. These concepts have subsequently been generalized to several advanced fuzzy graph models including intuitionistic fuzzy graphs [32], intuitionistic fuzzy directed graphs based on effective arcs [33], intuitionistic fuzzy directed graphs with strong arcs [34], Pythagorean fuzzy graphs [35], picture fuzzy graphs [36], interval-valued Pythagorean fuzzy graphs [37] and spherical fuzzy graphs [38]. A comprehensive overview of recent developments in domination theory for fuzzy graphs has been presented in [39], highlighting the rapid growth of this research area. Furthermore, Banitalebi et al. [40] investigated several classes of dominating sets in PyFGs and demonstrated their applicability to decision-making problems. More recently, Jan et al. [41] extended these investigations to T-SFGs by introducing the notions of domination and double

domination as a result it broaden the applicability of domination theory to a more expressive fuzzy graph framework.

The abbreviations and notational conventions used throughout this article are summarized in Table 1.

Symbols	Description	Symbols	Description
FS	Fuzzy Set	$N(a)$	Neighbourhood of an element
IFS	Intuitionistic Fuzzy Set	$N^*(a)$	Normal neighbourhood of an element
$PyFS$	Pythagorean Fuzzy Set	$TNDN$	Total Normal Dominating Number
PFS	Picture Fuzzy Set	$mTNDN$	Minimal Total Normal Dominating Set
SFS	Spherical Fuzzy Set	$TNDS$	Total Normal Dominating Set
FG	Fuzzy Graph	$MABIS$	Maximal Abnormal Independent Set
IFG	Intuitionistic Fuzzy Graph	N_a	Normal Arc
$PYFG$	Pythagorean Fuzzy Graph	$NVCS$	Normal Vertex Covering Set
PFG	Picture Fuzzy Graph	$ABIS$	Abnormal Independent Set
SFG	Spherical Fuzzy Graph	$MNODS$	Maximal Normal Dominating Set
$T-SFG$	T-Spherical Fuzzy Graph	$mNODS$	Minimal Normal Dominating Set
$T-SFS$	T-Spherical Fuzzy Set	$NODS$	Normal Dominating Set
$PyFR$	Pythagorean Fuzzy Relation	$T-SFR$	T-Spherical Fuzzy Relation
DS	Dominating Set	MDS	Minimal Dominating Set
DN	Dominating/Domination Number	TDN	Total Dominating Number
TDS	Total Dominating Set	$PYFI$	Pythagorean Fuzzy Index

TABLE 1. Abbreviations used in this article.

Research Gap and Motivations. Although domination has been extensively investigated in FGs, IFGs, PyFGs and several other fuzzy graph models, the notion of normal domination has not yet been established within the framework of T-spherical fuzzy graphs (T-SFGs). Furthermore, existing domination studies primarily emphasize theoretical graph structures and provide limited support for intelligent communication systems operating under highly uncertain environments. In particular, the potential of normal domination for intelligent Wi-Fi gateway optimization and cybersecurity threat analysis remains unexplored.

Motivated by these limitations, we develop a comprehensive normal domination framework for T-SFGs. Since T-SFS provide a richer representation of uncertainty through membership, abstinence and non-membership grades under a flexible power parameter and hence they are particularly suitable for modeling complex intelligent systems characterized by uncertain interactions, dynamic network behavior and multiple decision factors.

Novelty of this work. The novelty of this work lies in integrating the expressive uncertainty modeling capability of T-SFGs with the structural control mechanism of normal domination into a unified mathematical framework. Unlike existing domination models developed for other fuzzy graph structures, the proposed framework incorporates normal domination in T-SFGs together with several new domination-related structures and establishes their theoretical interrelationships. Moreover, the proposed model extends beyond theoretical development by demonstrating its

practical applicability to intelligent Wi-Fi network optimization and cybersecurity threat analysis through computational algorithms and comparative evaluation.

Contributions. The main contributions of this study are outlined as follows.

- (1) A novel concept of normal domination based on normal arcs is introduced within the framework of T-SFGs.
- (2) Fundamental structural parameters of T-SFGs including graph order, graph size, edge strength, normal dominating sets, minimal normal dominating sets and normal domination numbers, are established and their theoretical properties are investigated.
- (3) Several classes of normal dominating sets and normal domination numbers are characterized through new theoretical results.
- (4) The notion of abnormal independent sets is introduced and its relationships with normal dominating sets are systematically established.
- (5) The concept of normal vertex covering is developed and its interactions with abnormal independent structures are investigated.
- (6) A computational framework together with dedicated algorithms is developed to apply the proposed model to intelligent Wi-Fi gateway optimization and cybersecurity threat analysis.
- (7) Comparative analysis demonstrates that the proposed framework provides greater modeling flexibility, precision and adaptability than existing fuzzy graph-based domination models.

Organization of this article. The article organized as: Section 2 contains some basic terminologies related to FSs and FGs. Initially, the concepts of order and size of T-SFGs are described in section 3. We then introduce the notion of normal domination in T-SFGs based on normal arcs and explore its important properties. Furthermore, the concepts of normal dominating sets ($NODS$ s), minimal normal dominating set ($mNODS$ s) etc are introduced with some significant results. Several important characterisation of normal domination in T-SFGs based on minimal and maximal normal dominating sets are also explored. The notion of abnormal independent set ($ABIS$) is initiated and several properties of T-SFGs are studied. Some interrelationship among the normal dominating set($NODS$) and abnormal independent set ($ABIS$) are also investigated. We also initiate the idea of a normal vertex covering set ($NVCS$) and explore its relationship with $ABIS$. Furthermore, the application of normal domination in T-SFG in the analysis of Wi-Fi system and cybersecurity threats are provided in section 4. Comparative study is conducted in section 5. Finally, section 6 contains some concluding remarks along with future aspects.

2. PRELIMINARIES

In this section, we provide several basic concepts related to generalized forms of fuzzy sets (FSs) and fuzzy graphs (FGs).

Definition 2.1. [3] A FS on X is $J = \{\langle x, \mu_J(x) | x \in X, \mu_J(x) \in [0, 1] \rangle\}$.

Definition 2.2. [4] An IFS on X is $I = \{\langle x, \mu_I(x), \eta_I(x) | x \in X \rangle\}$, where $\mu_I, \eta_I : X \rightarrow [0, 1]$ are the membership and non membership values of $x \in X$, satisfying $0 \leq \mu_I(x) + \eta_I(x) \leq 1$, for all x in X .

Definition 2.3. [5] A PyFS on X is $\mathcal{P} = \{x, \mu_{\mathcal{P}}(x), \eta_{\mathcal{P}}(x) | x \in X\}$, where $\mu_{\mathcal{P}}(x), \eta_{\mathcal{P}}(x) \in [0, 1]$ are the membership and non-membership degrees of x in X , with $0 \leq \mu_{\mathcal{P}}^2(x) + \eta_{\mathcal{P}}^2(x) \leq 1$. For all $x \in X$, $\delta_{\mathcal{P}}(x) = \sqrt{1 - (\mu_{\mathcal{P}}^2(x) + \eta_{\mathcal{P}}^2(x))}$ is pythagorean fuzzy index on $x \in X$.

Definition 2.4. [6] A PFS on X is $\mathfrak{p} = \{\langle x, \mu_{\mathfrak{p}}(x), v_{\mathfrak{p}}(x), \eta_{\mathfrak{p}}(x) | x \in X \rangle\}$, where $\mu_{\mathfrak{p}}, v_{\mathfrak{p}}, \eta_{\mathfrak{p}} : X \rightarrow [0, 1]$ are the membership, neutral and non-membership degrees of each $x \in X$, satisfying $0 \leq \mu_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(x) + \eta_{\mathfrak{p}}(x) \leq 1$, for all $x \in X$.

Definition 2.5. [7] A SFS on X is $\mathcal{S} = \{x, \mu_{\mathcal{S}}(x), v_{\mathcal{S}}(x), \eta_{\mathcal{S}}(x) | x \in X\}$, where $\mu_{\mathcal{S}}(x), v_{\mathcal{S}}(x), \eta_{\mathcal{S}}(x) \in [0, 1]$ are membership, neutral and non-membership degrees of x in X , satisfying $0 \leq \mu_{\mathcal{S}}^2(x) + v_{\mathcal{S}}^2(x) + \eta_{\mathcal{S}}^2(x) \leq 1$. For all $x \in X$, $\delta_{\mathcal{S}}(x) = \sqrt{1 - (\mu_{\mathcal{S}}^2(x) + v_{\mathcal{S}}^2(x) + \eta_{\mathcal{S}}^2(x))}$ is the refusal degree of x in X .

Definition 2.6. [9] A T-SFS $\mathcal{T}$ on X is $\mathcal{T} = \{x, \mu_{\mathcal{T}}(x), v_{\mathcal{T}}(x), \eta_{\mathcal{T}}(x) | x \in X\}$, where $\mu_{\mathcal{T}}(x), v_{\mathcal{T}}(x)$ and $\eta_{\mathcal{T}}(x) \in [0, 1]$ are the membership, neutral and non-membership degree of x in $\mathcal{T}$, satisfying $0 \leq \mu_{\mathcal{T}}^n(x) + v_{\mathcal{T}}^n(x) + \eta_{\mathcal{T}}^n(x) \leq 1$. For all x in X , $\delta_{\mathcal{T}}(x) = \sqrt[n]{1 - (\mu_{\mathcal{T}}^n(x) + v_{\mathcal{T}}^n(x) + \eta_{\mathcal{T}}^n(x))}$ denotes the refusal degree of x in X .

Definition 2.7. [14] A FG is a pair $G_F = (\mu, \tau)$ defined on $G^* = (A, B)$, where $\mu : A \rightarrow [0, 1]$ and $\tau : B \rightarrow [0, 1]$ satisfying $\tau(x, y) \leq (\mu(x) \wedge \mu(y))$, for $(x, y) \in B$.

Definition 2.8. [4] An IFG is a pair $G = (A, B)$, where

- (1) $A = \{a_1, a_2, \dots, a_n\}$ with $\mu_1, \eta_1 : V \rightarrow [0, 1]$ are the degrees of membership and non-membership of the element $a_i \in A$, satisfying $0 \leq \mu_1 + \eta_1 \leq 1$ for every $a_i \in A$.
- (2) $B \subseteq A \times A$, where $\mu_2, \eta_2 : A \times A \rightarrow [0, 1]$ with $\mu_2(a_i, a_j) \leq \min(\mu_1(a_i), \mu_1(a_j))$, $\gamma_2(a_i, a_j) \leq \max(\gamma_1(a_i), \gamma_1(a_j))$, satisfying $0 \leq \mu_2(a_i, a_j) + \gamma_2(a_i, a_j) \leq 1$ for every $(a_i, a_j) \in B$.

Definition 2.9. [18] A PyFG on a set X is $\mathcal{G} = (C, D)$, where C is a PyFS on X and D is a PyFR on X such that for each edge $(x, y) \in D$, $\mu_D(x, y) \leq \mu_C(x) \wedge \mu_C(y)$ and $\eta_D(x, y) \geq \eta_C(x) \vee \eta_C(y)$ satisfying $0 \leq \mu_D^2(x, y) + \eta_D^2(x, y) \leq 1$. For all $x, y \in X$, $\mu_D, \eta_D : X \times X \rightarrow [0, 1]$ are the membership and non-membership grades of D .

Definition 2.10. [16] A triplet $\mathcal{G} = (V, A, B)$ on $G^* = (V, E)$ is a PFG on $\mathcal{G}^*$, where $A = \{\mu_A(x), \eta_A(x), v_A(x) | x \in V\}$ is a PFS on V and $B = \{\mu_B(x, y), \eta_B(x, y), v_B(x, y) | (x, y) \in E\}$ is a PFS on $E \subseteq V \times V$ such that for each edge (x, y) , we have $\mu_B(x, y) \leq \{\mu_A(x) \wedge \mu_A(y)\}$, $\eta_B(x, y) \leq \{\eta_A(x) \wedge \eta_A(y)\}$ and $v_B(x, y) \geq \{v_A(x) \vee v_A(y)\}$ satisfying $0 \leq \mu_A(x) + \eta_A(x) + v_A(x) \leq 1$, for all $x \in V$ and $0 \leq \mu_B(x, y) + \eta_B(x, y) + v_B(x, y) \leq 1$, $\forall (x, y) \in V \times V$.

Definition 2.11. [20] Let X be a nonempty set of vertices. A spherical fuzzy graph (SFG) on X is represented by an ordered pair $G = (A, B)$, where A is a spherical fuzzy set (SFS) defined on X and B is

a spherical fuzzy relation (SFR) defined on $X \times X$. The spherical fuzzy information associated with every edge is determined by the corresponding values of its incident vertices. For every pair of vertices $a, b \in X$, the following conditions must be satisfied

$$\mu_B(a, b) \leq \mu_A(a) \wedge \mu_A(b)$$

$$v_B(a, b) \leq v_A(a) \wedge v_A(b)$$

$$\eta_B(a, b) \geq \eta_A(a) \vee \eta_A(b)$$

Here, $\mu_A(a)$, $v_A(a)$, and $\eta_A(a)$ denote the membership, neutrality and non-membership degrees of the vertex a , respectively. These three values are required to satisfy the spherical fuzzy admissibility condition

$$0 \leq \mu_A^2(a) + v_A^2(a) + \eta_A^2(a) \leq 1, \quad \forall a \in X.$$

Similarly, the spherical fuzzy values assigned to each edge $(a, b) \in X \times X$ must also satisfy

$$0 \leq \mu_B^2(a, b) + v_B^2(a, b) + \eta_B^2(a, b) \leq 1, \quad \forall (a, b) \in X \times X.$$

Definition 2.12. [23] Let X be a nonempty vertex set. A T-SFG on X is denoted by $G_{TSF} = (A, B)$, where A is a T-SFS defined on X and B is a T-spherical fuzzy relation (T-SFR) defined on $X \times X$. For every vertex $a \in X$, the associated membership, abstinance and non-membership degrees satisfy

$$\mu_A^n(a) + v_A^n(a) + \eta_A^n(a) \leq 1.$$

Moreover, for every pair of vertices $a, b \in X$, the edge values satisfy

$$\mu_B(a, b) \leq \mu_A(a) \wedge \mu_A(b)$$

$$v_B(a, b) \leq v_A(a) \wedge v_A(b)$$

$$\eta_B(a, b) \geq \eta_A(a) \vee \eta_A(b)$$

together with the admissibility condition

$$\mu_B^n(a, b) + v_B^n(a, b) + \eta_B^n(a, b) \leq 1, \quad \forall a, b \in X.$$

Hereafter, we recall some useful parameters of dominations related to FGs and PyFGs.

Definition 2.13. [28] Let $G^* = (C, D)$ be a fuzzy graph, where C and D represent the fuzzy vertex set and the fuzzy edge set, respectively. An edge $uv \in D$ is referred to as a strong edge whenever it satisfies

$$\rho(uv) = \rho(u) \wedge \rho(v)$$

for vertices $u, v \in C$. Under this condition, the vertex u is said to dominate the vertex v . A subset $C_1 \subseteq C$ is called a dominating set (DS) if every vertex outside C_1 is adjacent through a strong edge to at least one vertex contained in C_1 .

A dominating set is considered minimal whenever the removal of any vertex from C_1 causes it to lose its dominating property. The domination number of G^* , denoted by $\gamma(G^*)$, is defined as the minimum fuzzy cardinality among all dominating sets of G^* .

Definition 2.14. [28] Let G^* be a fuzzy graph without isolated vertices and C be a dominating set of G^* . A subset $C_1 \subseteq C$ is said to be a total dominating set (TDS) if every vertex of C is dominated by at least one vertex in C_1 . The total domination number, denoted by $\gamma_t(G^*)$, is defined as the minimum fuzzy cardinality over all total dominating sets of G^* .

Definition 2.15. [34] The neighbourhood of $x \in A$ i.e., $N(x)$ is described as $N(x) = \{y \in A \mid (x, y) \text{ is a strong arc}\}$.

Definition 2.16. [35] An arc f in G is referred to as normal arc, if $(\mu(f))^2 + (\eta(f))^2 = 1$ and so $(\mu(f)) = S\{f\} = \sqrt{1 - (\eta(f))^2}$.

Definition 2.17. [35] Let $G = (A, B)$ be a Pythagorean fuzzy graph (PyFG).

- (1) A vertex $a \in A$ is said to normally dominate another vertex $b \in A$ whenever the edge joining a and b is a normal arc.
- (2) A subset $C \subseteq A$ is called a normal dominating set (NODS) of G if every vertex in $A \setminus C$ is normally dominated by at least one vertex belonging to C .

Definition 2.18. [35] Let $G = (A, B)$ be a Pythagorean fuzzy graph (PyFG). Then

- (1) A NODS C is called a minimal normal dominating set abbreviated as $mNODS$, if no proper subset of C is itself a NODS.
- (2) The lower normal domination number of G is defined as the minimum cardinality of all minimal NODS of G .
- (3) The upper normal domination number of G is defined as the maximum cardinality among all maximal normal dominating sets (MNODS) of G .

Definition 2.19. [35] Let $G = (A, B)$ be a PyFG. Two vertices $h, k \in A$ are said to be abnormally independent, if the edge joining h and k is not a normal arc.

3. NORMAL DOMINATIONS IN T-SPHERICAL FUZZY GRAPHS (T-SFGs)

In this section, we initiate the notion of normal dominations in T-SFGs and explore some interesting results relevant to them. Initially, bunch of useful terms including vertex and edge cardinalities, several types of strengths of edges and connectedness, normal arc, normal neighborhood, normal dominating set, normal vertex covering are introduced. Moreover, the concepts of total normal domination number is investigated.

Definition 3.1. Let G_{TSF} is a T-SFG. Then

- (1) The vertex cardinality of G_{TSF} is

$$|A| = \sum_{a_i \in A} \left(\frac{\mu(a_i) + \sqrt[n]{1 - (v(a_i))^n - (\eta(a_i))^n}}{2} \right)$$

It is also called the order of G_{TSF} .

(2) The edge cardinality of $\mathbb{G}_{TSF}$ is

$$|B| = \sum_{(a,b) \in B} \left(\frac{\mu(a,b) + \sqrt[n]{1 - (v(a,b))^n - (\eta(a,b))^n}}{2} \right)$$

It is also called the size of $\mathbb{G}_{TSF}$.

(3) The cardinality of $\mathbb{G}_{TSF}$ is $|\mathbb{G}_{TSF}| = |A| + |B|$.

Definition 3.2. Let $\mathbb{G}_{TSF}$ be a T-SFG and $a, b \in A$. Then

- (1) μ -strength of connectedness of (a, b) is $\mu_B^\infty = \sup\{\mu_B^k(a, b) | k = 1, 2, \dots, l\}$, where $\mu_B^k(a, b) = \min\{\mu_B(a, c_1), \mu_B(c_1, c_2), \mu_B(c_2, c_3), \dots, \mu_B(c_{k-1}, b) | a, c_1, \dots, c_{k-1}, b \in A, k = 1, 2, \dots, l\}$.
- (2) v -strength of connectedness of (a, b) is $v_B^\infty = \sup\{v_B^k(a, b) | k = 1, 2, \dots, l\}$, where $v_B^k(a, b) = \min\{v_B(a, c_1), v_B(c_1, c_2), v_B(c_2, c_3), \dots, v_B(c_{k-1}, b) | a, c_1, \dots, c_{k-1}, b \in A, k = 1, 2, \dots, l\}$.
- (3) η -strength of connectedness of (a, b) is $\eta_B^\infty = \min\{\eta_B^k(a, b) | k = 1, 2, \dots, l\}$, where $\eta_B^k(a, b) = \max\{\eta_B(a, c_1), \eta_B(c_1, c_2), \eta_B(c_2, c_3), \dots, \eta_B(c_{k-1}, b) | a, c_1, \dots, c_{k-1}, b \in A, k = 1, 2, \dots, l\}$.

Definition 3.3. An arc (x, y) is a strong, if $\mu(x, y) \geq \mu^\infty(x, y)$, $v(x, y) \geq v^\infty(x, y)$ and $\eta(x, y) \geq \eta^\infty(x, y)$.

Definition 3.4. A path in a T-SFG $\mathbb{G}_{TSF}$ is a sequence of distinct nodes $a_1, a_2, \dots, a_n$ satisfying one of the followings.

$$\begin{aligned} \mu_B(a_1, a_2) = 0, v_B(a_1, a_2) = 0, \eta_B(a_1, a_2) &> 0 \\ \mu_B(a_1, a_2) = 0, v_B(a_1, a_2) > 0, \eta_B(a_1, a_2) &= 0 \\ \mu_B(a_1, a_2) > 0, v_B(a_1, a_2) = 0, \eta_B(a_1, a_2) &= 0 \\ \mu_B(a_1, a_2) = 0, v_B(a_1, a_2) > 0, \eta_B(a_1, a_2) &> 0 \\ \mu_B(a_1, a_2) > 0, v_B(a_1, a_2) = 0, \eta_B(a_1, a_2) &> 0 \\ \mu_B(a_1, a_2) > 0, v_B(a_1, a_2) > 0, \eta_B(a_1, a_2) &= 0 \\ \mu_B(a_1, a_2) > 0, v_B(a_1, a_2) > 0, \eta_B(a_1, a_2) &> 0. \end{aligned}$$

Definition 3.5. An arc f in $\mathbb{G}_{TSF}$ denoted by $(\mathcal{N}_a)$ is referred as normal arc, if $(\mu(f))^n + (v(f))^n + (\eta(f))^n = 1$ and so $(\mu(f)) = S\{f\} = \sqrt[n]{1 - (\eta(f))^n - (v(f))^n}$.

Definition 3.6. The normal neighborhood of $a \in A$ (a set of vertices of underlying graph) is abbreviated as $N^*(a)$ and described as $N^*(a) = \{b \in A | ab \text{ is } \mathcal{N}_a \text{ in } \mathbb{G}_{TSF}\}$.

Definition 3.7. Let $\mathbb{G}_{TSF}$ be a T-SFG. Then

- (1) In $\mathbb{G}_{TSF}$, a normally dominates b and b normally dominates a , if there is a $\mathcal{N}_a$ between them.
- (2) $\mathbb{K} \subset A$ is referred as a normal dominating set ($NODS$) of $\mathbb{G}_{TSF}$, if for any $a \in A$ but not in $\mathbb{K}$, there is $k \in \mathbb{K}$, where k normally dominates a .
- (3) If there is no proper subset of a $NODS$ $\mathbb{K}$ of $\mathbb{G}_{TSF}$ is a $NODS$, then $\mathbb{K}$ is a minimal $NODS$ ($mNODS$).
- (4) Minimum node cardinality among all $mNODS$ s of $\mathbb{G}_{TSF}$ is a lower normal DN of $\mathbb{G}_{TSF}$ and is marked by $Y_A(\mathbb{G}_{TSF})$.

- (5) Maximum node cardinality among all $mNODS$ s of $\mathbb{G}_{TSF}$ is an upper normal DN of $\mathbb{G}_{TSF}$ and is represented by $\Gamma_A(\mathbb{G}_{TSF})$.
- (6) The normal DN of T-SFG is represented as $N^*D_A(\mathbb{G}_{TSF})$ and is described as

$$N^*D_A(\mathbb{G}_{TSF}) = \frac{Y_A(\mathbb{G}_{TSF}) + \Gamma_A(\mathbb{G}_{TSF})}{2}.$$

Example 3.1. Let $\mathbb{G}_{TSF}$ be a T-SFG shown in Fig. 2. Then $H = \{c\}$ is a $mNODS$ in $\mathbb{G}_{TSF}$. By doing a simple computations, we get $Y_A(\mathbb{G}) = \Gamma_A(\mathbb{G}) = N^*D_A(\mathbb{G}) = 0.799 \approx 0.8$.

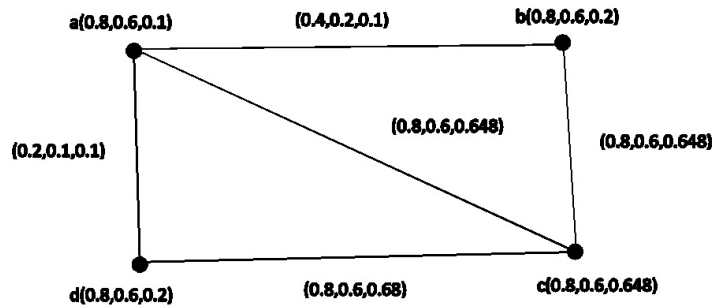


FIGURE 2. T- SFG

- Remark 3.1.** (1) If we add an arc f to B denoted as $B_f = B \cup \{f\}$ and then the T-SFG $\mathbb{G}_{TSF}$ is represented by $(\mathbb{G}_{TSF})_f = (A_f, B_f)$.
- (2) If arc f in T-SFG $(\mathbb{G}_{TSF})_f$ is a normal arc, then we represent it by $(\mathbb{G}_{TSF})_f^* = (A_f^*, B_f^*)$ instead of $(\mathbb{G}_{TSF})_f = (A_f, B_f)$.

Definition 3.8. A vertex $a \in A$ of a T-SFG $(\mathbb{G}_{TSF})$ is termed an abnormal isolated node, if it is not normally dominating any other vertex of $(\mathbb{G}_{TSF})$ i.e., $N^*(a) = \emptyset$.

- Remark 3.2.** (1) Let a be a node of a T-SFG $\mathbb{G}_{TSF} = (A, B)$. If $N^*(a) = \emptyset$, then every $NODS$ of $\mathbb{G}_{TSF}$ contains a .
- (2) For any $a, b \in A$. If a node a normal dominates b , then b normally dominates a and thus normal domination obeys symmetric relation on A .
- (3) For any $a \in A$, $N(a)$ is precisely the set of all vertices in A which are normally dominated by a .
- (4) If $(\mu_B(a, b))^n + (\eta_B(a, b))^n + (v_B(a, b))^n < 1$ for all $a, b \in A$ then the only $NODS$ of $\mathbb{G}_{TSF}$ is A .

Theorem 3.1. Let $f = w_1w_2$ is an additional N_a in $(\mathbb{G}_{TSF})_f$. Then

- (1) $N^*D_A(\mathbb{G}_{TSF}^n) \leq N^*D_A(\mathbb{G}_{TSF})$

$$(2) \ 0 \leq N^*D_A(\mathbb{G}_{TSF}) - N^*D_A(\mathbb{G}_{TSF}^n) \leq$$

$$\max\left\{\left(\frac{\mu(w_1) + \sqrt[n]{1 - (\eta(w_1))^n - (v(w_1))^n}}{2}\right), \left(\frac{\mu(w_2) + \sqrt[n]{1 - (\eta(w_2))^n - (v(w_2))^n}}{2}\right)\right\}.$$

Proof. (1) Let $\mathbb{K}$ be a $mNODS$ of $\mathbb{G}_{TSF}$ and $f = w_1w_2$ in $(\mathbb{G}_{TSF})_f$ an additional N_a in $(\mathbb{G}_{TSF})_f$. If w_1 or w_2 is considered as abnormal isolated node, then $\mathbb{H} - \{w_1\}$ or $H - \{w_2\}$ is a $mNODS(\mathbb{G}_{TSF}^n)_f$. Otherwise, H is a $mNODS(\mathbb{G}_{TSF}^n)_f$ in $(\mathbb{G}_{TSF}^n)_f$. Then

$$N^*\Gamma_A(\mathbb{G}_{TSF}^n)_f \leq N^*\Gamma_A(\mathbb{G}_{TSF})$$

$$N^*\Gamma_A(\mathbb{G}_{TSF}^n)_f \leq N^*\Gamma_A(\mathbb{G}_{TSF})$$

$$N^*D_A(\mathbb{G}_{TSF}^n)_f \leq N^*D_A(\mathbb{G}_{TSF}).$$

(2) From the proof of (1), we have

$$0 \leq Y_A(\mathbb{G}_{TSF}) - Y_A(\mathbb{G}_{TSF}^n) \leq \max\left\{\left(\frac{\mu(w_1) + \sqrt[n]{1 - (\eta(w_1))^n - (v(w_1))^n}}{2}\right), \left(\frac{\mu(w_2) + \sqrt[n]{1 - (\eta(w_2))^n - (v(w_2))^n}}{2}\right)\right\}$$

and

$$0 \leq \Gamma_A^*(\mathbb{G}_{TSF}) - \Gamma_A^*(\mathbb{G}_{TSF}^n) \leq \max\left\{\left(\frac{\mu(w_1) + \sqrt[n]{1 - (\eta(w_1))^n - (v(w_1))^n}}{2}\right), \left(\frac{\mu(w_2) + \sqrt[n]{1 - (\eta(w_2))^n - (v(w_2))^n}}{2}\right)\right\}.$$

Thus

$$0 \leq N^*D_A(\mathbb{G}_{TSF}) - N^*D_A(\mathbb{G}_{TSF}^n) \leq \max\left\{\left(\frac{\mu(w_1) + \sqrt[n]{1 - (\eta(w_1))^n - (v(w_1))^n}}{2}\right), \left(\frac{\mu(w_2) + \sqrt[n]{1 - (\eta(w_2))^n - (v(w_2))^n}}{2}\right)\right\}.$$

□

Definition 3.9. Let $\mathbb{G}_{TSF}$ be a T-SFG. Then

- (1) two nodes $a, b \in A$ are called abnormally independent, if there is no N_a between them.
- (2) $\mathbb{K} \subset A$ is referred as an abnormal independent set ($\mathcal{ABIS}$) of $\mathbb{G}_{TSF}$ whenever for any $a, b \in \mathbb{K}$, $(\mu_B(ab))^n + (\eta_B(ab))^n + (v_B(ab))^n < 1$.
- (3) an $\mathcal{ABIS}$ $\mathbb{K}$ in $\mathbb{G}_{TSF}$ is a maximal ($\mathcal{MABIS}$) whenever for any vertex $r \in A \setminus \mathbb{K}$, the set $\mathbb{K} \cup \{r\}$ is not $\mathcal{ABIS}$.
- (4) minimum node cardinality among all $\mathcal{MABIS}$ s is said to be a lower abnormal independent number of $\mathbb{G}_{TSF}$. It is denoted by $(\hat{a}\hat{b}\hat{i})_A(\mathbb{G}_{TSF})$.
- (5) maximum node cardinality among all $\mathcal{MABIS}$ s referred as an upper abnormal independent number of $\mathbb{G}_{TSF}$, denoted by $(\mathcal{ABI})_A(\mathbb{G}_{TSF})$.

Definition 3.10. The abnormal independent number of G_{TSF} , denoted by $\mathcal{ABI}(G_{TSF})$ is defined as

$$\mathcal{ABI}(G_{TSF}) = \frac{(\hat{abi})_A(G_{TSF}) + (\mathcal{ABI})_A(G_{TSF})}{2}.$$

Example 3.2. Let $G_{TSF} = (A, B)$ be a T-SFG, where $A = \{a, b, c, d, e, f\}$, $B = \{ab, bc, ad, cd, de, fe\}$ be the set of nodes and edges, respectively depicted in Fig. 3.

Here $n = 3$, clearly (a, d) and (e, f) are the normal edges. $I_1 = \{a, b, c, f\}$ and $I_2 = \{a, b, c, f\}$ are $\mathcal{MABIS}$ of G_{TSF} . Hence $(\hat{abi})_A(G_{TSF}) = 2.9$, $(\mathcal{ABI})_A(G_{TSF}) = 3.13$ and $\mathcal{ABI}(G_{TSF}) = 3.015$.

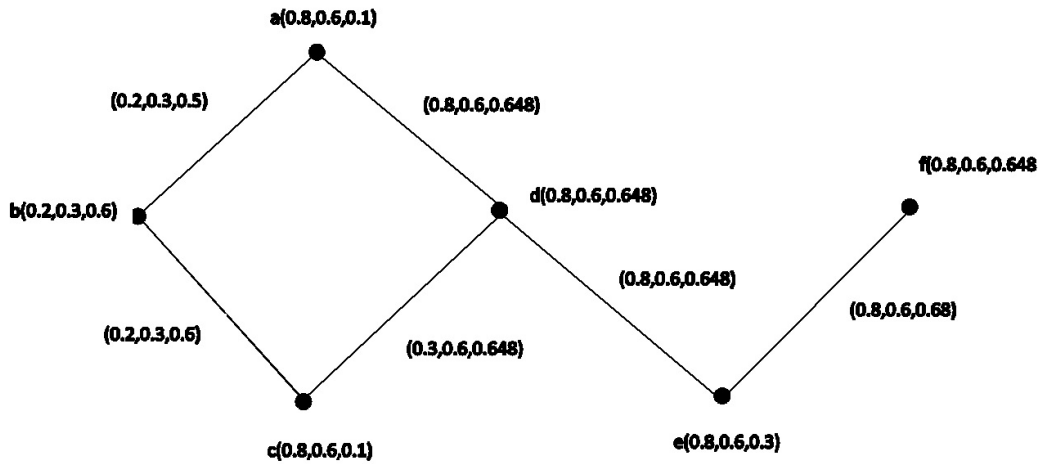


FIGURE 3. T-SFG

Definition 3.11. Let $e = (a, b)$ be a normal edge of a graph G . The vertices a and b are each said to be incident with the edge e . Equivalently, the edge e is incident with both endpoints a and b .

Definition 3.12. Let $G_{TSF} = (A, B)$ be a T-SFG and $e = (a, b)$ be its normal edge. A vertex b and the edge e are said to satisfy the covering relation, if b is incident with e .

Definition 3.13. A collection of vertices $S \subseteq A$ in T-SFG $G_{TSF} = (A, B)$ is said to form a normal vertex covering set ($\mathcal{NVCS}$) whenever every normal edge in the graph is represented by at least one of its incident vertices in S . The minimum cardinality of such a collection is called the normal vertex covering number and is abbreviated as $\lambda(G_{TSF})$.

Proposition 3.1. Let e be an additional N_a in $(G_{TSF})_e^*$. Then $\mathcal{ABI}(G_{TSF})_e^n \leq \mathcal{ABI}(G_{TSF})$.

Proof. It is obvious. □

Theorem 3.2. A $NODS$ S_n of G_{TSF} is a $mNODS$ if and only if for any node $a \in S_n$, one of the following statements holds.

- (1) a doesn't normally dominates any node of S_n

(2) there is a node $b \in S_n$, where $N_n(b) \cap S_n = \{a\}$.

Proof. Consider that S_n is a $mNODS$ of G_{TSF} . Following that, for any node $a \in S_n$, $S_n - \{a\}$ is not a $NODS$ and so there is $b \in A \setminus S_n - \{a\}$ that is not normally dominated by any other node in $S_n - \{a\}$. If $b = a$, then b is not a normally neighbor by any node in S_n . If $b \neq a$, then b is not normally dominated by $S_n - \{a\}$. However, S_n normally dominates b . Hence $N_n(b) \cap S_n = \{a\}$.

Conversely, assume that S_n is an $NODS$ and for any node $a \in S_n$, one of the statements holds. Suppose that S_n is not a $mNODS$. Therefore there exists a node $a \in S_n$, where $S_n - \{a\}$ be an $NODS$. Thus at least another node in $S_n - \{a\}$ normally dominates the node a and hence (1) is not valid. Also, $S_n - \{a\}$ is a $NODS$. Consequently, there is at least one node in $S_n - \{a\}$ such that it normally dominates any node of $A - S_n$. Hence (2) is not true, a contradiction. Thus S_n is a $mNODS$. $\square$

Theorem 3.3. If a subset $D \subseteq A$ in a T-SFG $G_{TSF} = (A, B)$ is an abnormally independent set of G_{TSF} , then $A - D$ is a normal vertex covering set of G_{TSF} .

Proof. By Definition 3.9, if D is an $\mathcal{ABIS}$ of G_{TSF} , then no two vertices of D are adjacent and every vertex of D is incident with at least one vertex of $A - D$. Hence $A - D$ is a normal vertex covering set of G_{TSF} . $\square$

Theorem 3.4. For any T-SFG $G_{TSF} = (A, B)$ without abnormal isolated nodes, $\lambda(G_{TSF}) + (\mathcal{ABI})_A(G_{TSF}) = |G_{TSF}|$.

Proof. Let us consider that S be an $\mathcal{ABIS}$ of G_{TSF} and K is a normal vertex covering set of G_{TSF} with $|S| = (\mathcal{ABI})_A(G_{TSF})$ and $|K| = \lambda(G_{TSF})$. Then by Theorem 3.3, if S is an $\mathcal{ABIS}$ then $A - S$ is normal vertex covering set of G_{TSF} . Hence $|K| \leq |A - S|$. It implies

$$\lambda(G_{TSF}) \leq |G_{TSF}| - (\mathcal{ABI})_A(G_{TSF})$$

and

$$\lambda(G_{TSF}) + (\mathcal{ABI})_A(G_{TSF}) \leq |G_{TSF}| \dots (1).$$

Also, by Theorem 3.3, if $A - K$ is an $\mathcal{ABIS}$ of G_{TSF} , then $A - (A - K) = K$ is a normal vertex cover set. Hence

$$|S| \geq |A - K| \Rightarrow (\mathcal{ABI})_A(G_{TSF}) \geq |G_{TSF}| - \lambda(G_{TSF})$$

and

$$(\mathcal{ABI})_A(G_{TSF}) + \lambda(G_{TSF}) \geq |G_{TSF}| \dots (2).$$

From (1) and (2), we obtain $\lambda(G_{TSF}) + (\mathcal{ABI})_A(G_{TSF}) = |G_{TSF}|$. $\square$

Theorem 3.5. Let G_{TSF} be a T-SFG and $\mathbb{D}_n$ is a $mNODS$. If there is no abnormal isolated nodes in G_{TSF} , then $A - \mathbb{D}_n$ is an $NODS$ of G_{TSF} .

Proof. Since G_{TSF} has no abnormal isolated nodes. Then for any node u of $\mathbb{D}_n$ there exists a node $v \in N(u)$. From Theorem 3.2, $v \in A - \mathbb{D}_n$. Thus each element of $\mathbb{D}_n$ is normally dominated by some member of $A - \mathbb{D}_n$ and $A - \mathbb{D}_n$ is a $NODS$ of G_{TSF} . $\square$

Theorem 3.6. *An $\mathcal{ABIS}$ of G_{TSF} is a $\mathcal{MABIS}$ if and only if it is a $\mathcal{NODS}$ of G_{TSF} .*

Proof. Assume that $\mathbb{F}$ is a $\mathcal{MABIS}$ of G_{TSF} . Since $\mathbb{F}$ is maximal with respect to the property of being an $\mathcal{ABIS}$, no vertex from $A \setminus \mathbb{F}$ can be added to $\mathbb{F}$ without violating the $\mathcal{ABIS}$ property. Consequently, for every vertex $c \in A \setminus \mathbb{F}$, there exists a vertex $v \in \mathbb{F}$ such that v normally dominates c . Otherwise, if no vertex of $\mathbb{F}$ normally dominated c , then the set $\mathbb{F} \cup \{c\}$ would remain an $\mathcal{ABIS}$, contradicting the maximality of $\mathbb{F}$. Hence, each vertex lying outside $\mathbb{F}$ is normally dominated by a vertex of $\mathbb{F}$, proving that $\mathbb{F}$ is a $\mathcal{NODS}$.

Conversely, suppose that $\mathbb{F}$ is both an $\mathcal{ABIS}$ and a $\mathcal{NODS}$ of G_{TSF} . Assume, to the contrary, that $\mathbb{F}$ is not a $\mathcal{MABIS}$. Then there exists a vertex $c \in A \setminus \mathbb{F}$ such that $\mathbb{F} \cup \{c\}$ is also an $\mathcal{ABIS}$. Since $\mathbb{F}$ is a $\mathcal{NODS}$, every vertex outside $\mathbb{F}$, including c , must be normally dominated by some vertex in $\mathbb{F}$. However, this normal domination prevents the inclusion of c while preserving the $\mathcal{ABIS}$ property, contradicting the assumption that $\mathbb{F} \cup \{c\}$ is an $\mathcal{ABIS}$. Therefore, no such vertex exists and hence $\mathbb{F}$ must be a $\mathcal{MABIS}$. Thus, an $\mathcal{ABIS}$ of G_{TSF} is a $\mathcal{MABIS}$ if and only if it is a $\mathcal{NODS}$. $\square$

Theorem 3.7. *Any $\mathcal{MABIS}$ of G_{TSF} is a $m\mathcal{NODS}$.*

Proof. Let $\mathbb{S}$ be a $\mathcal{MABIS}$ of G_{TSF} . Then $\mathbb{S}$ is a $\mathcal{NODS}$, by Theorem 3.6. Also, the set $\mathbb{S} - \{c\}$ is not a $\mathcal{NODS}$, for any vertex $c \in \mathbb{S}$. Hence $\mathbb{S}$ must be an $m\mathcal{NODS}$. $\square$

Definition 3.14. Let $G_{TSF} = (A, B)$ be a T-SFG without abnormal isolated vertex. Then the collection A_1 is total normal dominating set ($\mathcal{TND S}$), if for each node $a \in A$ there is a node $b \in A_1$ with a is not equal to b and a normally dominates b .

Definition 3.15. (1) By a minimal total normal dominating set ($m\mathcal{TND S}$) in T-SFG, we mean a $\mathcal{TND S}$ A_1 of a T-SFG with no proper subset of A_1 is a $\mathcal{TND S}$ except A_1 .

(2) The minimum cardinality of a $m\mathcal{TND S}$ is a lower total normal dominating number of G_{TSF} , abbreviated as $L_{tnd}(G_{TSF})$.

(3) The maximum cardinality of a $m\mathcal{TND S}$ is the upper total normal dominating number of G_{TSF} , denoted by $U_{tnd}(G_{TSF})$.

Definition 3.16. A total normal domination number ($\mathcal{TND N}$) of G_{TSF} , abbreviated as $\zeta_{tn}(G_{TSF})$ is the $\zeta_{tnd}(G_{TSF}) = \frac{L_{tnd}(G_{TSF}) + U_{tnd}(G_{TSF})}{2}$.

Theorem 3.8. Let G_{TSF} be a T-SFG. Then, $L_{tnd}(G_{TSF}) = |G_{TSF}|$ if and only if every vertex of G_{TSF} has exactly one neighbor.

Proof. Assume that every vertex of G_{TSF} has exactly one neighbor. Then, removing any vertex from the vertex set A leaves its unique adjacent vertex without a normal dominator. Consequently, no proper subset of A satisfies the conditions of a $\mathcal{TND S}$. Hence, A is the only $\mathcal{TND S}$ of G_{TSF} , and therefore $L_{tnd}(G_{TSF}) = |A| = |G_{TSF}|$. Conversely, suppose that $L_{tnd}(G_{TSF}) = |G_{TSF}|$. Assume, to the contrary, that there exists a vertex $a \in A$ having at least two distinct neighbors, say u and

x . Then the set $A \setminus \{a\}$ still normally dominates every vertex of $\mathbb{G}_{TSF}$, since both u and x remain adjacent to vertices in $A \setminus \{a\}$, while all other domination relations are unaffected. Thus, $A \setminus \{a\}$ is a $\mathcal{TND S}$ of cardinality $|A| - 1$, implying $L_{tnd}(\mathbb{G}_{TSF}) \leq |A| - 1 < |\mathbb{G}_{TSF}|$ which contradicts the assumption. Therefore, no vertex can have more than one neighbor. Hence, every vertex of $\mathbb{G}_{TSF}$ has exactly one neighbor. Hence $L_{tnd}(\mathbb{G}_{TSF}) = |\mathbb{G}_{TSF}|$ if and only if every vertex of $\mathbb{G}_{TSF}$ has exactly one neighbor. $\square$

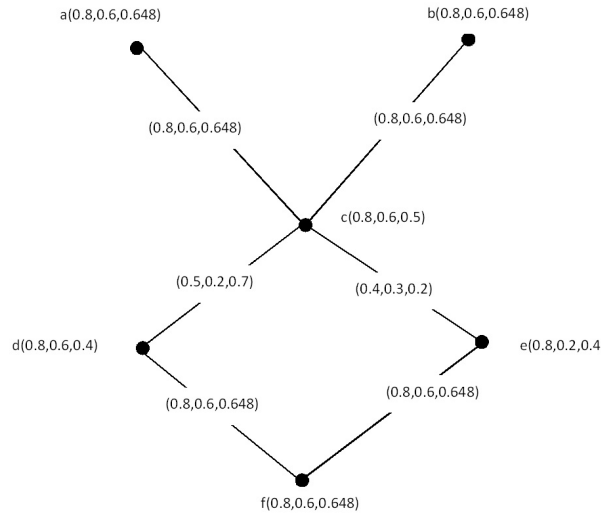


FIGURE 4. T-Spherical Fuzzy Graph

Example 3.3. From Fig. 4, the arcs (a, c) , (a, b) , (d, f) and (f, e) are the normal arcs. The $m\mathcal{TND S}$ are $D_1 = \{c, f\}$, $D_2 = \{a, b, f\}$, $D_3 = \{c, d, e\}$ and $D_4 = \{a, b, d, e\}$. For $n = 3$, the cardinalities of $m\mathcal{TND S}$ are $|D_1| = 1.61$, $|D_2| = 2.54$, $|D_3| = 1.60$ and $|D_4| = 3.34$. Hence $L_{tnd}(\mathbb{G}_{TSF}) = 1.60$, $U_{tnd}(\mathbb{G}_{TSF}) = 3.34$ and $\zeta_{tnd}(\mathbb{G}_{TSF}) = 2.47$.

4. APPLICATIONS OF NORMAL DOMINATION IN T-SPHERICAL FUZZY GRAPHS

This section comprises of two subsections. In section 4.1, we construct a model based on domination in T-SFGs, accompanied by an illustrative numerical example, Flowchart and Algorithm. In section 4.2, we further extend the model for cybersecurity threats analysis.

The complete methodology adopted in this section is highlighted in Fig. 5 which provides the sequence of modeling, computation and application stages. Moreover, Table 2 highlights the major stages of the proposed application frameworks along with their objectives and corresponding outputs. However, the overall application framework of this section is illustrated in Fig. 6. As shown in the infographic, the proposed normal domination methodology is first applied to Wi-Fi communication networks, then extended to cybersecurity threat analysis and finally evaluated through a comparative analysis with existing graph models.

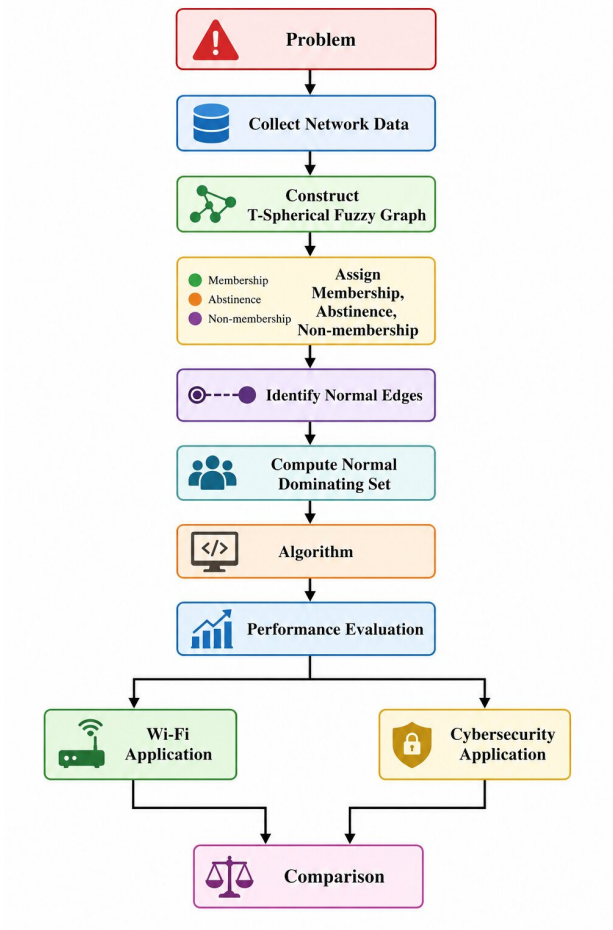


FIGURE 5. Flowchart diagram of Applications

TABLE 2. Workflow of the proposed application framework.

Step	Methodology	Objective	Output
1	Problem formulation	Define engineering problem	Network requirements
2	Construct T-SFG	Model uncertainty	T-SFG
3	Compute normal edges	Identify reliable connections	Normal graph
4	Determine NODS	Select dominant nodes	Gateway/monitoring nodes
5	Algorithm execution	Efficient computation	Normal dominating set
6	Performance analysis	Evaluate effectiveness	Improved coverage
7	Engineering application	Wi-Fi and cybersecurity	Optimized network

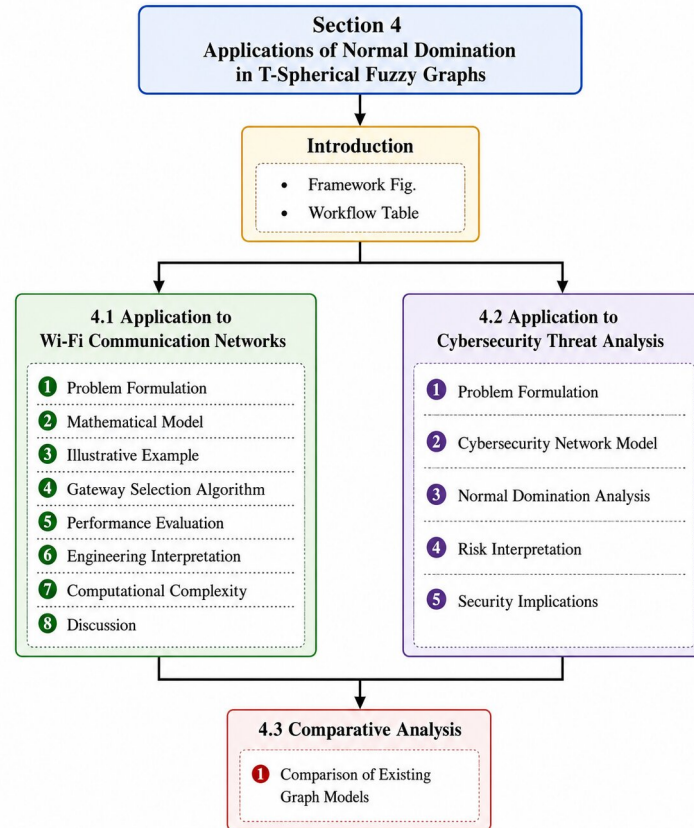


FIGURE 6. Overall Section 4 Infographic Representation.

4.1. Application to Wi-Fi Communication Networks. Graphs have a wide range of applications across various scientific disciplines, including computer science, physics, mathematics, chemistry, biology and the social sciences. In the study of Wi-Fi systems, it is observed that connections are just like the social relationships among devices within a network. Just as social networks link individuals within a group or organization, Wi-Fi networks interconnect various devices. These systems inherently involve uncertainties. Thus, the classical graphs may fail to handle uncertainties in such systems. Consequently, modelling these issues through FGs provide us precise results. With the passage of time, as we encounter increasingly complex and uncertain scenarios, numerous extended forms of fuzzy graphs (FGs) have been introduced to effectively address these situations. It has been observed that T-SFGs being the most extended form of fuzzy graphs (FGs) have a greater capacity to accommodate various types of uncertainties compared to other forms such as IFGs, PyFGs and PFGs. Here, we present and analyze a model for a Wi-Fi network through normal dominations within the framework of T-SFGs.

4.1.1. Problem Statement. In practical Wi-Fi communication systems, connecting every access point directly to the central controller results in excessive communication overhead, increased bandwidth consumption and higher deployment costs. Therefore, an important optimization problem

is to determine the minimum number of gateway devices capable of providing complete communication coverage while maintaining reliable connectivity under uncertain network conditions. Since communication quality among devices is inherently uncertain due to signal fluctuations, interference and environmental effects, conventional graph models cannot adequately represent such uncertainty. The proposed T-SFG normal domination framework addresses this problem by identifying an optimal set of gateway devices that normally dominate the entire communication network while preserving communication reliability.

Consider a situation where a Wi-Fi system consists of numerous connected devices such as routers, computers, servers and IoT devices, each is represented by a node in a T-SFG. Moreover, in a Wi-Fi system, each device connected to the network is also considered as a node in a T-spherical fuzzy graph(T-SFG). It means that every routers, smartphone, laptop, tablet, IoT device or any other gadget accessing the Wi-Fi network is represented by a node in a T-SFG. A Wi-Fi system have a several connections in companies, houses, schools, offices etc. The connections between Wi-Fi and devices are represented by edges of T-SFG. Due to various factors, Wi-Fi network signals often experience fluctuations. Hence we categorize the connections into three types: strong connection, weak connection and no connection (no signal). Each edge in the T-SFG represents the strength of the connection to Wi-Fi, corresponding to one of these categories. If there is no connection between a device and the Wi-Fi network, then no edge exists between them in the graph. Practically, it has observed that the strength of connection of each device is not stable everywhere. Moreover, it is quite understandable that if the Wi-Fi system connects with certain routers and get all devices connected to it with the help of router, then the Wi-Fi system needs not be to connected each device in a single mode (category). Thus, it is important to categorize the connection error, strength of connections etc. According to our approach, we categorize the connections in Wi-Fi system through dominations in T-SFG. In this way, it become easier to check the strength of connections by analyzing the *NODS* of routers.

4.1.2. Mathematical Model. The Wi-Fi communication network is modeled as a T-SFG $G_{TSF} = (A, B)$, where each vertex represents a wireless access point or IoT device and each edge represents a communication link between two devices. The corresponding T-spherical fuzzy membership, abstinence and non-membership degrees quantify the communication reliability, uncertainty and communication failure, respectively.

Within this framework

- each vertex represents an IoT device or wireless access point
- each normal edge represents a highly reliable communication link
- a normal dominating set represents the selected gateway devices responsible for supervising neighboring devices
- the normal domination number determines the minimum number of gateways required to maintain complete network coverage.

Consequently, the gateway selection problem is transformed into determining a minimum normal dominating set of the proposed T-SFG.

We demonstrate the practical applicability of our model through a numerical example supported by Algorithm 1 and Flowchart (given in Fig. 7).

4.1.3. Illustrative Example. Consider a Wi-Fi network demonstrated through T-SFG given in Fig. 8, where we assume a town of ten different devices represented by the ten vertices of T-SFG and arcs represent their connections. The values of vertices and arcs are in the form of T -SFN. These values are proposed by experts due to percentage of connection reliability, percentage of packet loss and disconnection probability. $A = \{a_i | i = 1, 2, \dots, 10\}$ is the set of all devices and the set of all arcs (connections) is represented by $b_i(1, 2, \dots, 10)$. By doing simple calculations, we get $NODS$ in T-SFG G_{TSF} i.e., $D = \{a_1, a_5, a_8\}$. Hence $A - D = \{a_2, a_3, a_4, a_6, a_7, a_9, a_{10}\}$. Thus it becomes clear that the Wi-Fi (main server) only need to connected to the devices (routers) a_1, a_5, a_8 to establish an efficient Wi-Fi connection system.

Extended Explanation of Example 4.1.3. To understand the scenario intuitively presented in Example 4.1.3, we imagine that the Wi-Fi server (Main Server) does not attempt to directly maintain the stable links with all ten devices simultaneously. However, it only ensures strong and reliable connections with the $NODS$ members. For instance, device a_1 not only maintains its own connection with the server but at the same time it also provides reliable paths to its neighbors a_2, a_3 and a_4 . Similarly, device a_5 covers nodes including a_6 and a_7 , and the device a_8 connects further to a_9 and a_{10} . Thus, although only three devices are directly linked to the server, the remaining seven devices indirectly receiving access through these dominating nodes. From a network management point of view: the server essentially designates a small set of devices to function as the local relays or gateways. These selected devices (the $NODS$) accommodate the variance in connection strengths and propagate the stable communication further into the network. Instead of spreading resources thin by attempting to connect uniformly to every device, the Wi-Fi system benefits from a structured connection plan where the $NODS$ ensures coverage of the entire network. In this way, the notion of a $NODS$ indicates the minimal number of direct server–device connections required and guaranteeing the connectivity to all devices in the system. It models how the server can manage bandwidth, reduce redundant communication and improve coverage by relying on a carefully chosen few devices (subset of devices) to maintain connectivity.

4.1.4. Gateway Selection Algorithm. Hereafter, we present a reproducible Algorithm 1 along with its execution Flowchart 7. **Algorithm 1. Normal domination in a T -spherical fuzzy graph for Wi-Fi networks** **Input:** T -spherical fuzzy graph $G_{TSF} = (V, E)$; vertex membership $\sigma(v_k) = (\mu_k, \eta_k, \nu_k)$; edge membership $\tau(v_p, v_q) = (\mu_{pq}, \eta_{pq}, \nu_{pq})$; T -spherical power $n \in \mathbb{N}$.

Output: Normal dominating set $D \subseteq V$.

- (1) Initialize $D \leftarrow \emptyset$ and $Covered \leftarrow \emptyset$.
- (2) For each $v \in V$, set $NormalNeighbors[v] \leftarrow \emptyset$.

(3) For each edge $(v_p, v_q) \in E$, compute

$$S_{pq} = \mu_{pq}^n + \eta_{pq}^n + \nu_{pq}^n.$$

(4) If $S_{pq} = 1$, then add v_q to $NormalNeighbors[v_p]$ and v_p to $NormalNeighbors[v_q]$.

(5) While $Covered \neq V$, do:

(a) Select a vertex $u \in V \setminus Covered$ with maximum $|NormalNeighbors[u] \setminus Covered|$.

(b) Add u to D .

(c) Update

$$Covered \leftarrow Covered \cup \{u\} \cup NormalNeighbors[u].$$

(6) Return the set D .

4.1.5. *Performance Analysis.* The obtained normal dominating set consists of only three gateway devices namely $D = \{a_1, a_5, a_8\}$ which are sufficient to supervise the entire communication network. Table 3 summarizes the performance improvement achieved through the proposed optimization framework.

TABLE 3. Performance comparison before and after gateway optimization.

Performance Metric	Before Optimization	After Optimization
Gateway devices	10	3
Communication coverage	100%	100%
Communication redundancy	High	Low
Bandwidth utilization	Moderate	Improved
Network management cost	High	Reduced
Scalability	Moderate	High

4.1.6. *Engineering Benefits.* The proposed normal domination within the framework of T-SFG provides several practical advantages for intelligent Wi-Fi systems.

- It minimizes the number of gateway devices while maintaining complete communication coverage.
- It reduces communication redundancy by eliminating unnecessary direct connections.
- It improves bandwidth utilization through efficient gateway deployment.
- It decreases network maintenance and operational costs.
- It provides a scalable framework suitable for large IoT communication infrastructures operating under uncertain environments.

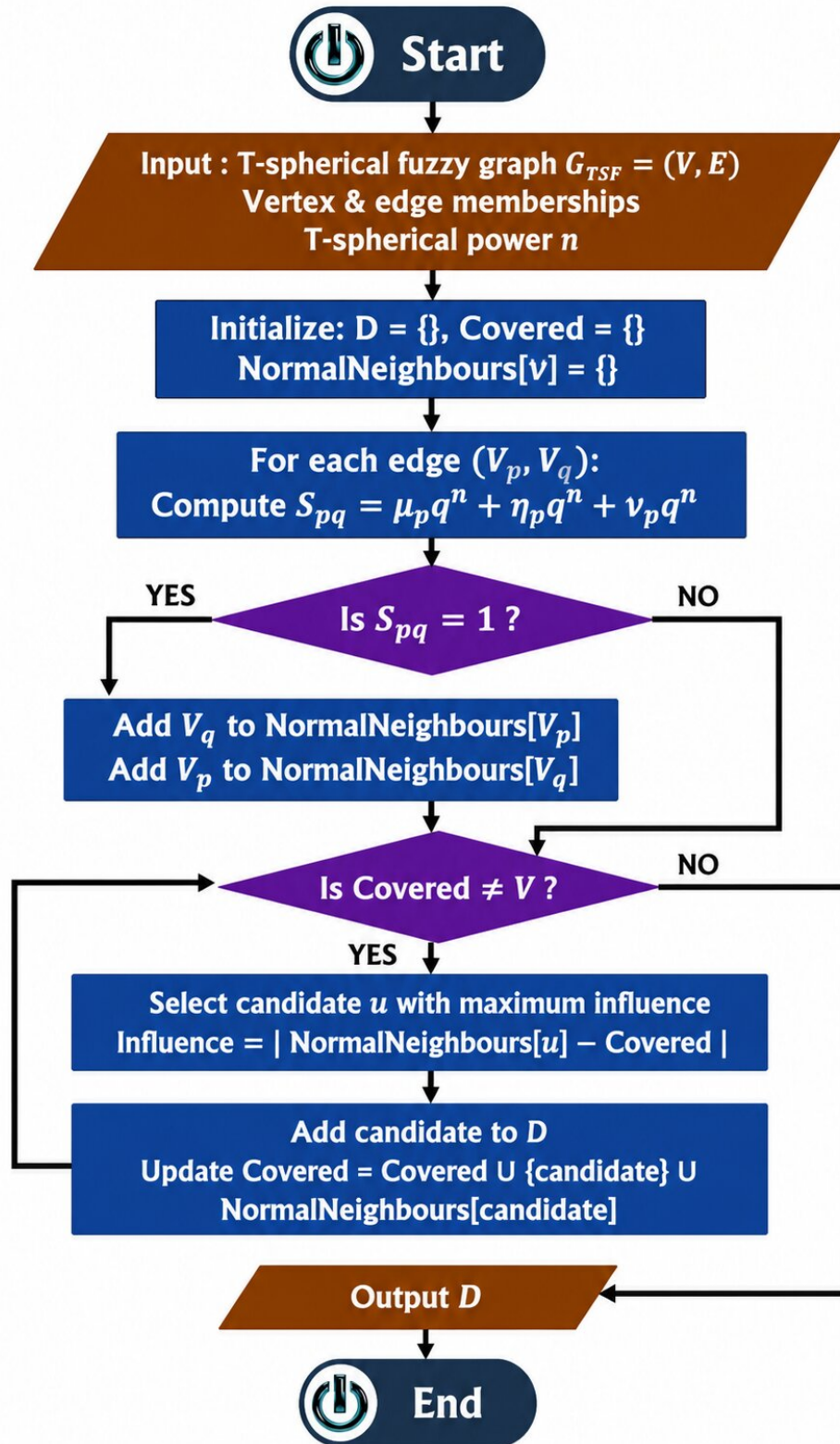


FIGURE 7. Flowchart illustrating the hierarchy of Algorithm 1 for computing the normal dominating set in a T-SFG.

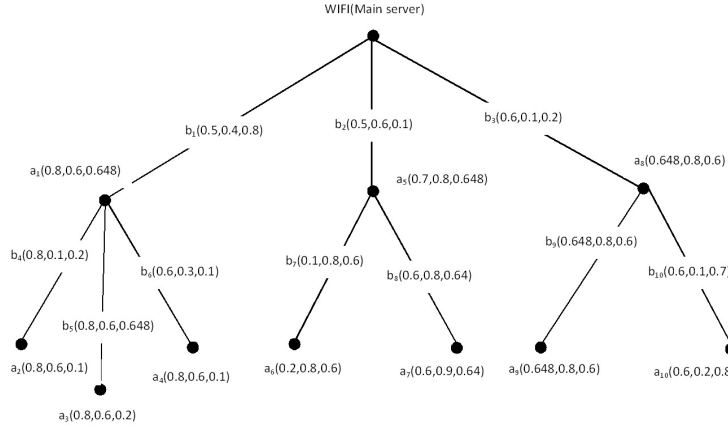


FIGURE 8. T-SFG representing Wi-Fi networks

4.1.7. *Computational Complexity.* Algorithm 1 first identifies all normal neighbors by examining each communication link which requires $O(|B|)$ operations. Subsequently, the gateway selection procedure iteratively evaluates the uncovered vertices, requiring at most $O(|A|^2)$ operations in the worst case. Therefore, the overall computational complexity of the proposed algorithm is

$$O(|A|^2 + |B|)$$

where $|A|$ and $|B|$ denote the numbers of vertices and communication links, respectively. This polynomial-time complexity makes the proposed approach computationally efficient for medium- and large-scale Wi-Fi communication networks.

4.1.8. *Discussions.* The obtained solution demonstrates that only three gateway devices are sufficient to supervise all communication nodes within the considered Wi-Fi network. Consequently, unnecessary direct communication with the central controller is eliminated and hence it reducing communication overhead and improving bandwidth utilization. Furthermore, the T-spherical fuzzy representation effectively captures uncertainty associated with communication reliability. Thus, the proposed framework is more capable of selecting realistic gateway devices than conventional graph-theoretic approaches.

4.2. **Application to Cybersecurity Threat Analysis.** Using the same computational framework presented in Fig. 6, the proposed methodology is extended to cybersecurity threat analysis in this section.

In cybersecurity, detecting and handling risks involves a high degree of uncertainties due to vague intelligence, evolving threats and incomplete data. Traditional graph models often fail in accurate modeling such uncertain circumstances. The proposed T-SFGS model provides us an extended framework for dealing complex, vague relationships between system components, vulnerability propagation paths etc.

Each node in the T-SFG can represent a component in a networked system (e.g., servers, databases, IoT devices), while edges represent potential threat paths. The T-spherical fuzzy values assigned to vertices and edges depict the confidence level, risk exposure and ambiguity of interactions based on security intelligence (e.g., likelihood of attack success, patch uncertainty, detection reliability).

For instance, consider a network with known vulnerabilities where interruption (intrusion) detection systems provide uncertain threat alerts. Through normal dominating set in T-SFG, security analysts can detect critical nodes that are dominating the threat space and they can prioritize monitoring, mitigation and patching. Additionally, the model is also useful to simulate risk propagation paths which helps the analyst to predict how vulnerabilities may spread under varying degrees of threat certainty. Consequently, he can enhance the responsiveness of cybersecurity defense strategies.

This application demonstrates the broad utilization of the proposed model beyond Wi-Fi networks which highlights its capability to support decision-making under uncertainty in critical environments like cybersecurity.

4.2.1. Problem Formulation. Modern cybersecurity infrastructures consist of numerous interconnected devices whose interactions are uncertain due to dynamic attacks, incomplete information and varying threat levels. Monitoring every component continuously is computationally expensive and economically impractical. Therefore, an important optimization problem is to determine the minimum number of monitoring nodes capable of supervising the entire network while maintaining effective security coverage. The proposed normal domination framework models such optimization problems using T-SFG and identifies an optimal set of monitoring nodes under uncertain threat conditions.

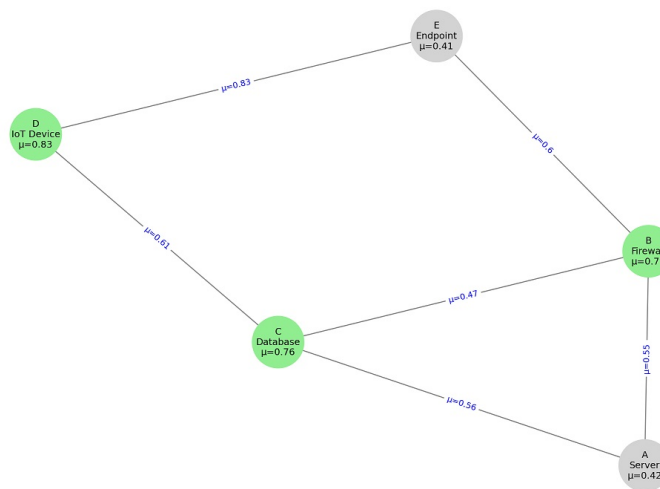


FIGURE 9. T-SFG for cybersecurity components

4.2.2. *Cybersecurity Network Model with Normal domination evaluation in T-SFGs (Cybersecurity Context)*. Let us consider T-SFG given in Fig. 9 representing a simplified cybersecurity framework. Each node in it is a system component while each edge denotes the risk propagation strength (μ) between the components. Moreover, corresponding values are depicted in Table 4. Moreover, nodes values (system components) and corresponding edge values (Risk links) are given in Table 5 and Table 6, respectively. Furthermore, by taking a membership threshold of $\mu \geq 0.6$ for strong nodes (trusted/dominant), we evaluate the domination under uncertainty as follows.

- (1) Strong nodes: ($\mu \geq 0.6$) are B (Firewall) with $\mu = 0.73$; C (Database) with $\mu = 0.76$ and D (IoT Device) with $\mu = 0.83$.
- (2) Computing *NODS*: We get B (Firewall) and D (IoT Device), it means that Node B (Firewall) and node D (IoT Device) together dominating all other nodes in the graph due to their strong membership. All components of the system are covered by this set.

Device	μ : Connection Reliability	v : Packet Loss/ Interruption Rate	η : Disconnection Probability
a_1	0.8	0.6	0.648
a_2	0.8	0.6	0.1
a_3	0.8	0.6	0.2
a_4	0.8	0.6	0.1
a_5	0.7	0.8	0.648
a_6	0.2	0.8	0.6
a_7	0.6	0.9	0.64
a_8	0.648	0.8	0.6
a_9	0.648	0.8	0.6
a_{10}	0.6	0.2	0.8
b_1	0.5	0.4	0.8
b_2	0.5	0.6	0.1
b_3	0.6	0.1	0.2
b_4	0.8	0.1	0.2
b_5	0.8	0.6	0.648
b_6	0.6	0.3	0.1
b_7	0.1	0.8	0.6
b_8	0.6	0.8	0.64
b_9	0.648	0.8	0.6
b_{10}	0.6	0.1	0.7

TABLE 4. Wi-Fi device connection parameters, including connection reliability, packet loss/interruption rate, and disconnection probability.

4.2.3. *Risk Interpretation*. Following the above calculations, we interpret the cybersecurity risk propagation as follows.

- (1) B and D are critical components and must be monitoring for the risk control.

Node	μ (Membership)	η (Non-membership)	ν (Neutrality)
A (Server)	0.42	0.44	0.79
B (Firewall)	0.73	0.03	0.68
C (Database)	0.76	0.40	0.51
D (IoT Device)	0.83	0.35	0.43
E (Endpoints)	0.41	0.39	0.82

TABLE 5. T-spherical fuzzy values of the cybersecurity network components.

Edge	μ (Membership)	η (Non-membership)	ν (Neutrality)
A-B	0.55	0.15	0.82
A-C	0.56	0.42	0.71
B-C	0.47	0.26	0.84
B-E	0.60	0.27	0.75
C-D	0.61	0.11	0.78
D-E	0.83	0.12	0.54

TABLE 6. T-spherical fuzzy values associated with the cybersecurity network risk links.

- (2) Full network coverage is possible by deploying security controls at these nodes under uncertain environment.
- (3) It also helps us to reduce resource allocation points while maintaining the system integrity.

4.2.4. Security Implications. The obtained normal dominating set identifies the minimum collection of monitoring nodes required to supervise the entire cybersecurity infrastructure. Consequently, the proposed framework significantly reduces monitoring complexity while preserving complete network surveillance. Moreover, by incorporating T-spherical fuzzy information, the model effectively represents uncertain threat propagation and varying communication reliability. Hence, the proposed model provides us a more realistic decision-support framework for intelligent cybersecurity systems.

4.2.5. Comparison Analysis of Proposed Models. To further evaluate the effectiveness of the proposed T-SFG-framework, a qualitative comparison with conventional FGs and SFGs is presented. Although these graph models are capable of representing uncertain relationships however they differ considerably in their ability to capture complex uncertainty, communication reliability and network optimization requirements. In particular, FGs consider only membership information whereas SFGs additionally incorporate abstinence information. The proposed T-SFG framework further enhances uncertainty representation by providing a more flexible modeling environment for intelligent decision-making in Wi-Fi communication systems. Table 7 compares these graph models with respect to several important characteristics such as uncertainty representation, communication reliability, gateway optimization capability and suitability for intelligent Wi-Fi and cybersecurity applications. Evidently, it has observed that the proposed T-SFG framework shows greater modeling flexibility and provides a more effective foundation for uncertainty-aware gateway selection than existing graph-based approaches.

TABLE 7. Comparison of graph models for Wi-Fi network optimization.

Property	FG	SFG	Proposed T-SFG
Represents uncertainty	Yes	Yes	Yes
Represents abstinence	No	Yes	Yes
Flexible uncertainty modeling	Limited	Moderate	High
Gateway optimization	Limited	Moderate	Excellent
Communication reliability	Moderate	Good	Excellent
Suitable for intelligent Wi-Fi	Moderate	Good	Excellent
Cybersecurity modeling	Limited	Moderate	Excellent

5. COMPARATIVE STUDY

In this section, we provide a comparative analysis of normal domination in T-SFG with PyFGs and PFGs. Refer to Example 4.1.3, a Wi-Fi network is dealt through T-SFG in which each node and edge are given values from T-SFSs. Inherently, T-SFSs has more capacity to express uncertainties precisely compared to PyFSs and PFSSs. Moreover, T-SFG in Fig. 8 can be considered as PyFG by setting neutral value sets to zero in Fig. 9. Similarly, this graph can also be considered as PFG, whenever $n = 1$. Thus, the data we deal within the frameworks of both PyFGs and PFGs is analyzed more effectively and automatically when handled through T-SFGs.

On the other hand, the problems solved by using T-SFGs cannot be handled through PyFGs and PFGs. For instance, PyFGs and PFGs are unable to deal the data depicted in Figs.[10& 11]. In fact, the PyFGs and PFGs can't handled the problem (application) we just talked about, as these have limited flexibilities. As PyFG structure does not give information about the neutral and refusal degrees. However, if we perform this experiment in the context of PFGs, then again PFGs has no vast domain compared to T-SFGs. As a result, we analyze that our proposed method in the context of T-SFG is an extension of the existing methods. Additionally, the superiority of our proposed model is further demonstrated in the Table 8 and Table 9.

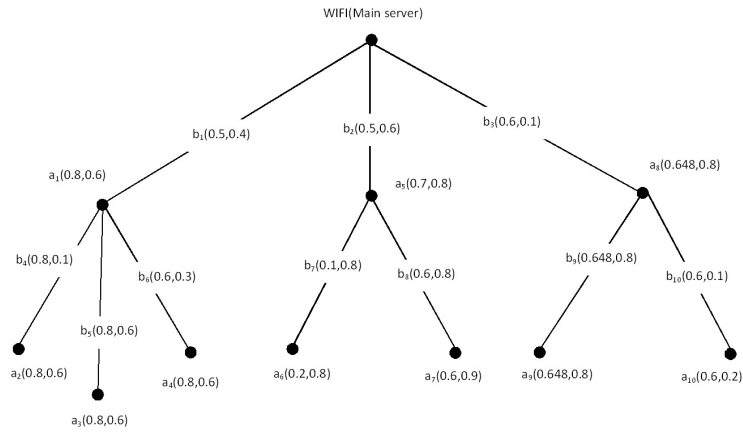


FIGURE 10. Wi-Fi depicted through PyFG

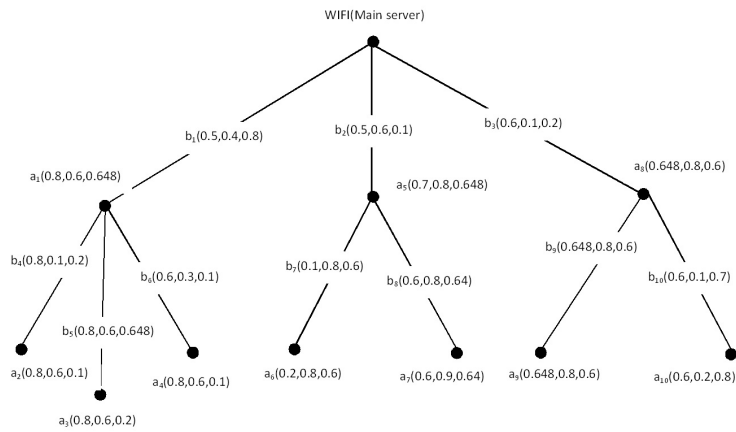


FIGURE 11. Wi-Fi network via PFG

Vertices/ Edges	Degrees	Sum of Degrees	Interpretation (Sum)	Sum of Cubes of De- grees	Interpretation (Cubes)
a_1	(0.8, 0.6, 0.648)	2.048	Fail to connect	1.000	Strongly con- nected
a_2	(0.8, 0.6, 0.1)	1.500	Fail to connect	0.700	Connected
a_3	(0.8, 0.6, 0.2)	1.600	Fail to connect	0.736	Connected
a_4	(0.8, 0.6, 0.1)	1.500	Fail to connect	0.700	Connected
a_5	(0.7, 0.8, 0.648)	2.148	Fail to connect	1.100	Fail to connect
a_6	(0.2, 0.8, 0.6)	1.600	Fail to connect	0.700	Connected
a_7	(0.6, 0.9, 0.64)	2.140	Fail to connect	1.200	Fail to connect
a_8	(0.648, 0.8, 0.6)	2.048	Fail to connect	1.000	Strongly con- nected
a_9	(0.648, 0.8, 0.6)	2.048	Fail to connect	1.000	Strongly con- nected
a_{10}	(0.6, 0.2, 0.8)	1.600	Fail to connect	0.700	Connected
b_1	(0.5, 0.4, 0.8)	1.700	Fail to connect	0.700	Connected
b_2	(0.5, 0.6, 0.1)	1.200	Fail to connect	0.342	Connected
b_3	(0.6, 0.1, 0.2)	0.900	Connected	0.225	Connected
b_4	(0.8, 0.1, 0.2)	1.100	Fail to connect	1.000	Strongly con- nected
b_5	(0.8, 0.6, 0.648)	2.048	Fail to connect	1.000	Strongly con- nected
b_6	(0.6, 0.3, 0.1)	1.000	Strongly con- nected	0.244	Connected
b_7	(0.1, 0.8, 0.6)	1.500	Fail to connect	0.700	Connected
b_8	(0.6, 0.8, 0.64)	2.040	Fail to connect	1.000	Strongly con- nected
b_9	(0.648, 0.8, 0.6)	2.048	Fail to connect	1.000	Strongly con- nected
b_{10}	(0.6, 0.1, 0.7)	1.400	Fail to connect	0.560	Connected

TABLE 8. Comparative analysis of Wi-Fi network connectivity using Picture Fuzzy Graphs (PFGs) and T-spherical Fuzzy Graphs (T-SFGs).

Object	Degrees	Sum of Degrees	Interpretation	Sum of Cubes of Degrees	Interpretation
a_1	(0.8, 0.6, 0.648)	1.419	Fail to connect	1.000	Strongly connected
a_2	(0.8, 0.6, 0.1)	1.010	Fail to connect	0.700	Connected
a_3	(0.8, 0.6, 0.2)	1.600	Fail to connect	0.736	Connected
a_4	(0.8, 0.6, 0.1)	1.010	Fail to connect	0.700	Connected
a_5	(0.7, 0.8, 0.648)	1.540	Fail to connect	1.100	Fail to connect
a_6	(0.2, 0.8, 0.6)	1.040	Fail to connect	0.700	Connected
a_7	(0.6, 0.9, 0.64)	1.570	Fail to connect	1.200	Fail to connect
a_8	(0.648, 0.8, 0.6)	1.410	Fail to connect	1.000	Strongly connected
a_9	(0.648, 0.8, 0.6)	1.410	Fail to connect	1.000	Strongly connected
a_{10}	(0.6, 0.2, 0.8)	1.040	Fail to connect	0.700	Connected
b_1	(0.5, 0.4, 0.8)	1.050	Fail to connect	0.700	Connected
b_2	(0.5, 0.6, 0.1)	0.620	Connected	0.342	Connected
b_3	(0.6, 0.1, 0.2)	0.900	Connected	0.225	Connected
b_4	(0.8, 0.1, 0.2)	0.410	Connected	1.000	Strongly connected
b_5	(0.8, 0.6, 0.648)	1.410	Fail to connect	1.000	Strongly connected
b_6	(0.6, 0.3, 0.1)	0.460	Connected	0.244	Connected
b_7	(0.1, 0.8, 0.6)	1.010	Fail to connect	0.700	Connected
b_8	(0.6, 0.8, 0.64)	1.409	Fail to connect	1.000	Strongly connected
b_9	(0.648, 0.8, 0.6)	1.410	Fail to connect	1.000	Strongly connected
b_{10}	(0.6, 0.1, 0.7)	0.860	Connected	0.560	Connected

TABLE 9. Comparative analysis of T-spherical fuzzy information using the proposed T-SFG model.

6. CONCLUSION

Graph theory finds numerous practical applications across various scientific fields such as computer science, physics, chemistry, operations research etc. However, many problems manipulated through graphs such as data analysis, networking etc involve uncertainties. On the other hand, FGs have emerged as a useful tool to tackle these challenges. Over the past few decades, FGs have demonstrated significant applications in modern science and technology. In this study, we have introduced novel concepts of normal domination in T-SFGs with application in Wi-Fi networks. Initially, some useful terms are described such as cardinality, order, strength of normal domination in T-SFG. After this, several significant characterizations of normal domination in T-SFGs (T-SFSs) based on minimal and maximal *NODS* are studied. We have also introduced the notion of *ABIS* and discussed some its significant characterizations. Some relationships among *NODS* and *ABIS* and between *NVCS* and *ABIS*. To show the effectiveness of this study, we provide application of newly established concepts in Wi-Fi system. Comparative study has also conducted to show the superiority of our presented model. One can discuss these concepts within the framework of q-rung orthopair picture fuzzy graphs.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] E.J. Cockayne, S.T. Hedetniemi, Towards a Theory of Domination in Graphs, *Networks* 7 (1977), 247–261. <https://doi.org/10.1002/net.3230070305>.
- [2] T.W. Haynes, S.T. Hedetniemi, M.A. Henning (eds), *Topics in Domination in Graphs*, Springer International Publishing, 2020. <https://doi.org/10.1007/978-3-030-51117-3>.
- [3] L.A. Zadeh, Fuzzy Sets, *Inf. Control* 8 (1965), 338–353. [https://doi.org/10.1016/S0019-9958\(65\)90241-X](https://doi.org/10.1016/S0019-9958(65)90241-X).
- [4] K.T. Atanassov, Intuitionistic Fuzzy Sets, *Fuzzy Sets Syst.* 20 (1986), 87–96. [https://doi.org/10.1016/S0165-0114\(86\)80034-3](https://doi.org/10.1016/S0165-0114(86)80034-3).
- [5] R.R. Yager, Pythagorean Fuzzy Subsets, in: 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), IEEE, 2013, pp. 57–61. <https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375>.
- [6] B.C. Cuong, Picture Fuzzy Sets, *J. Comput. Sci. Cybern.* 30 (2014), 409–420. <https://doi.org/10.15625/1813-9663/30/4/5032>.
- [7] F. Kutlu Gündoğdu, C. Kahraman, Spherical Fuzzy Sets and Spherical Fuzzy TOPSIS Method, *J. Intell. Fuzzy Syst.* 36 (2019), 337–352. <https://doi.org/10.3233/JIFS-181401>.
- [8] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, T. Mahmood, Spherical Fuzzy Sets and Their Applications in Multi-Attribute Decision Making Problems, *J. Intell. Fuzzy Syst.* 36 (2019), 2829–2844. <https://doi.org/10.3233/JIFS-172009>.
- [9] T. Mahmood, K. Ullah, Q. Khan, N. Jan, An Approach toward Decision-Making and Medical Diagnosis Problems Using the Concept of Spherical Fuzzy Sets, *Neural Comput. Appl.* 31 (2019), 7041–7053. <https://doi.org/10.1007/s00521-018-3521-2>.
- [10] Ö.F. Görçün, P. Chatterjee, Ž. Stević, H. Küçükönder, An Integrated Model for Road Freight Transport Firm Selection in Third-Party Logistics Using T-Spherical Fuzzy Sets, *Transp. Res. Part E Logist. Transp. Rev.* 186 (2024), 103542. <https://doi.org/10.1016/j.tre.2024.103542>.

- [11] M. Sarfraz, M. Azeem, Parametric Similarity Measurement of T-Spherical Fuzzy Sets for Enhanced Decision-Making, *Int. J. Knowl. Innov. Stud.* 2 (2024), 29–44. <https://doi.org/10.56578/ijkis020104>.
- [12] P. Wang, A Novel Decision Making Approach to Analyze Optimization of Key Factors in Sports Culture Dissemination Using T-Spherical Fuzzy Information, *IEEE Access* 13 (2025), 37737–37747. <https://doi.org/10.1109/ACCESS.2025.3543979>.
- [13] F. Karamaz, F. Karaaslan, Distance Measures of r, s, t -Spherical Fuzzy Sets and Their Applications in MCGDM Based on TOPSIS, *J. Supercomput.* 81 (2025), 173. <https://doi.org/10.1007/s11227-024-06560-5>.
- [14] A. Rosenfeld, Fuzzy Graphs, in: *Fuzzy Sets and their Applications to Cognitive and Decision Processes*, Elsevier, pp. 77–95, (1975). <https://doi.org/10.1016/B978-0-12-775260-0.50008-6>.
- [15] K.T. Atanassov, A. Shannon, A First Step to a Theory of the Intuitionistic Fuzzy Graphs, in: *Proceedings of the First Workshop on Fuzzy Based Expert Systems*, Sofia, pp. 59–61, 1994.
- [16] C. Zuo, A. Pal, A. Dey, New Concepts of Picture Fuzzy Graphs with Application, *Mathematics* 7 (2019), 470. <https://doi.org/10.3390/math7050470>.
- [17] X. Shi, S. Kosari, W.A. Khan, Some Novel Concepts of Interval-Valued Picture Fuzzy Graphs with Applications toward the Transmission Control Protocol and Social Networks, *Front. Phys.* 11 (2023), 1260785. <https://doi.org/10.3389/fphy.2023.1260785>.
- [18] S. Naz, S. Ashraf, M. Akram, A Novel Approach to Decision-Making with Pythagorean Fuzzy Information, *Mathematics* 6 (2018), 95. <https://doi.org/10.3390/math6060095>.
- [19] W.A. Khan, I.A. Shah, T.T. Nguyen, M.H. Pham, H.V. Pham, Some Concepts of Interval Valued Pythagorean Fuzzy Graphs and Analysis of Socioeconomic Conditions of Communities, *IEEE Access* 13 (2025), 55140–55155. <https://doi.org/10.1109/ACCESS.2025.3553351>.
- [20] M. Akram, D. Saleem, T. Al-Hawary, Spherical Fuzzy Graphs with Application to Decision-Making, *Math. Comput. Appl.* 25 (2020), 8. <https://doi.org/10.3390/mca25010008>.
- [21] B. Some, P. Das, A. Pal, Balance Spherical Fuzzy Graph and Their Applications, *TWMS J. Appl. Eng. Math.* 15 (2025), 728–747.
- [22] B. Some, A. Pal, Comparative Energy Analysis of Spherical Fuzzy Indices in Decision-Making Problems, *Int. J. Math. Ind.* 17 (2025), 2450027. <https://doi.org/10.1142/S2661335224500278>.
- [23] A. Guleria, R.K. Bajaj, T-Spherical Fuzzy Graphs: Operations and Applications in Various Selection Processes, *Arab. J. Sci. Eng.* 45 (2020), 2177–2193. <https://doi.org/10.1007/s13369-019-04107-y>.
- [24] L. Zedam, N. Jan, E. Rak, T. Mahmood, K. Ullah, An Approach towards Decision-Making and Shortest Path Problems Based on T-Spherical Fuzzy Information, *Int. J. Fuzzy Syst.* 22 (2020), 1521–1534. <https://doi.org/10.1007/s40815-020-00820-1>.
- [25] F. Karaaslan, F. Karamaz, Some New Product Operations of T-Spherical Fuzzy Graphs and an Application of T-Spherical Fuzzy Graphs in MCGDM, *Artif. Intell. Rev.* 56 (2023), 13663–13689. <https://doi.org/10.1007/s10462-023-10477-5>.
- [26] S.U. Khan, F. Hussain, T. Senapati, S. Hussain, Z. Ali, et al., Analysis of Computer Communication Networks Based on Evaluation of Domination and Double Domination for Interval-Valued T-Spherical Fuzzy Graphs and Their Applications in Decision-Making Problems, *Eng. Appl. Artif. Intell.* 139 (2025), 109650. <https://doi.org/10.1016/j.engappai.2024.109650>.
- [27] T.W. Haynes, S. Hedetniemi, P. Slater, *Fundamentals of Domination in Graphs*, CRC Press, 2013. <https://doi.org/10.1201/9781482246582>.
- [28] A. Somasundaram, S. Somasundaram, Domination in Fuzzy Graphs – I, *Pattern Recognit. Lett.* 19 (1998), 787–791. [https://doi.org/10.1016/S0167-8655\(98\)00064-6](https://doi.org/10.1016/S0167-8655(98)00064-6).
- [29] A. Nagoorgani, V.T. Chandrasekaran, Domination in Fuzzy Graph, *Adv. Fuzzy Sets Syst.* 1 (2006), 17–26.
- [30] A. Nagoor Gani, K. Prasanna Devi, 2-Domination in Fuzzy Graphs, *Int. J. Fuzzy Math. Arch.* 9 (2015), 119–124.

- [31] A. Nagoor Gani, K. Prasanna Devi, M. Pal, Reduction of Domination Parameter in Fuzzy Graphs, *Glob. J. Pure Appl. Math.* 13 (2017), 3307–3315.
- [32] R. Parvathi, G. Thamizhendhi, Domination in Intuitionistic Fuzzy Graphs, *Notes Intuitionistic Fuzzy Sets* 16 (2010), 39–49.
- [33] W.A. Khan, K. Ali, A. Fida, M. Asif, H.V. Pham, et al., Some Novel Concepts of Intuitionistic Fuzzy Directed Graphs with Application in Selecting a Suitable Place for Opening Restaurant, *Heliyon* 10 (2024), e33950. <https://doi.org/10.1016/j.heliyon.2024.e33950>.
- [34] H. Guan, W.A. Khan, A. Fida, K. Ali, J. Shafi, et al., Dominations in Intuitionistic Fuzzy Directed Graphs with Applications towards Influential Graphs, *Mathematics* 12 (2024), 872. <https://doi.org/10.3390/math12060872>.
- [35] S. Banitalebi, R.A. Borzooei, Domination in Pythagorean Fuzzy Graphs, *Granul. Comput.* 8 (2023), 959–966. <https://doi.org/10.1007/s41066-023-00362-5>.
- [36] R. Ismail, S.U. Khan, S. Al Ghour, E.H.A. Al-Sabri, M.M.S. Mohammed, et al., A Complete Breakdown of Politics Coverage Using the Concept of Domination and Double Domination in Picture Fuzzy Graph, *Symmetry* 15 (2023), 1044. <https://doi.org/10.3390/sym15051044>.
- [37] W.A. Khan, A. Zahid, H. Rashmanlou, Novel Concepts of Strong Paired Domination in Interval-Valued Picture Fuzzy Graphs and Its Applications Towards Selection Criteria, *J. Appl. Math. Comput.* 70 (2024), 6165–6193. <https://doi.org/10.1007/s12190-024-02212-2>.
- [38] W.A. Khan, M. Asif, A. Ahmed, S. Abbas, M. Yousaf, Decision Making Based on Strong Paired Domination in Spherical Fuzzy Graphs, *Arab. J. Sci. Eng.* 50 (2025), 19187–19199. <https://doi.org/10.1007/s13369-025-10132-x>.
- [39] A. Rana, A Survey on the Domination of Fuzzy Graphs, *Discret. Math. Algorithms Appl.* 13 (2021), 2130001. <https://doi.org/10.1142/S1793830921300010>.
- [40] S. Banitalebi, S.S. Ahn, Y.B. Jun, R.A. Borzooei, Normal m -Domination and Inverse m -Domination in Pythagorean Fuzzy Graphs with Application in Decision Making, *J. Intell. Fuzzy Syst.* 43 (2022), 5053–5062. <https://doi.org/10.3233/JIFS-220319>.
- [41] N. Jan, T. Mahmood, L. Zedam, L. Abdullah, K. Ullah, Analysis of Double Domination by Using the Concept of Spherical Fuzzy Information with Application, *J. Ambient Intell. Humaniz. Comput.* 14 (2023), 1147–1162. <https://doi.org/10.1007/s12652-021-03370-y>.