

On Developable Surfaces with a Common Line of Curvature in Minkowski 3-Space E_1^3

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Abstract. In this paper we study the problem of constructing a spacelike and timelike developable surface family from a given non-null curve as a common line of curvature in Minkowski 3-space E_1^3 . The necessary and sufficient conditions when the resulting developable surface is a cylinder, cone or tangent surface are also analyzed. Some representative curves are chosen to construct the corresponding developable surfaces which possess these curves as lines of curvature. Finally, we have illustrated the convenience and efficiency of this approach by some representative examples.

1. INTRODUCTION

Line of curvature on a surface is an intrinsic geometric feature that plays an important role in a diversity of applications. Geometrically, a curve on a surface is classified as line of curvature if its tangents are always in the direction of the principal curvature. Additionally, the necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normal along this curve forms a developable surface. Many geometric operations are inherently related to lines of curvature. For instance, when a sheet is to be shaped by rolling, then it is fed into the rolls according to a principal direction and the rolls are adjusted according to the principal curvature. Line of curvature also has been widely applied in various industries, such as tent manufacturing, cutting and painting path, fiberglass tape windings in pipe manufacturing, textile manufacturing [1,2,7,9,13].

Generally, the research in line of curvature focused on how to find or compute them on the given surfaces, and there were a lot of papers working on such a problem [3–5]. In practical

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applications, the more interesting problem is how to construct those surfaces which possess the given spatial curve as a common line of curvature. This may be regarded as the reverse problem of the above-mentioned.

Cai-Yun et al. [3,4] studied such a problem. The basic idea is to regard the wanted surface as an extension from the given line of curvature, and represent it as a linear combination of the marching-scale functions $u(s, t)$, $v(s, t)$, $w(s, t)$ and the three vector functions $\mathbf{t}(s)$, $\mathbf{n}(s)$, $\mathbf{b}(s)$, which are the unit tangent, the principal normal and the binormal vector of the curve respectively. With the line of curvature and isoparametric constraints, they derived the necessary and sufficient conditions for the correct parametric representation of the surface family.

At the beginning of the 20th century, A. Einstein's theory opened a door to new geometries such as Lorentzian Geometry, which is simultaneously the geometry of special relativity, was established. Thereafter, researchers discovered a bridge between modern differential geometry and the mathematical physics of general relativity. Especially, E_1^4 (Minkowski space-time) has the very strong physical background. As the sub-space of E_1^4 , E_1^3 (the three-dimensional Minkowski space) has many properties which are different from the Euclidean 3-space E^3 . Some basic concepts, such as vector, frame, and the motion of point, have qualitative changes. Since the metric in E_1^3 is not positive definite metric, the distance function $\langle, \rangle$ can be positive, negative or zero, whereas the distance function in the Euclidean space can only be positive. Then, the study of surfaces in the Minkowski 3-space is more interesting than the Euclidean case.

According to the classifications of surfaces, they have different values for derivative of the vectors. So, the geometric interpretations can be more different.

In this paper, we derive the necessary and sufficient condition for a given spacelike or timelike curve to be both isoparametric and line of curvature on a timelike or spacelike developable ruled surface. Thus, we define family of spacelike or timelike developable surfaces with a common line of curvature. Also, we derive the conditions when the developable surface is cylinder, cone or tangent surface respectively. Meanwhile, some representative non-null curves are chosen to construct the corresponding timelike or spacelike developable surfaces which possess these curves as lines of curvature.

2. PRELIMINARIES

Let E_1^3 be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$\langle d\mathbf{x}, d\mathbf{x} \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{R}^3$. An arbitrary vector $\mathbf{x}$ of E_1^3 is said to be spacelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ or $\mathbf{x} = 0$, timelike if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ and lightlike or null if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ and $\mathbf{x} \neq 0$. A timelike or light-like vector in E_1^3 is said to be causal. For $\mathbf{x} \in E_1^3$ the norm is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$, then the vector $\mathbf{x}$ is called a spacelike unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$ and a timelike unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = -1$. Similarly, a regular curve in E_1^3 can locally be spacelike, timelike or

null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [6, 8, 10–12]. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of E_1^3 , the inner product is the real number $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$ and the vector product is defined by $\mathbf{x} \times \mathbf{y} = ((x_2 y_3 - x_3 y_2), (x_3 y_1 - x_1 y_3), -(x_1 y_2 - x_2 y_1))$. The angle between two vectors in Minkowski space is defined by [8].

Definition

i) *Spacelike angle*: Let $\mathbf{x}$ and $\mathbf{y}$ be spacelike vectors in E_1^3 that span a spacelike vector subspace; then we have $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|$, and hence, there is a unique real number $\theta \geq 0$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$. This number is called the spacelike angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

ii) *Central angle*: Let $\mathbf{x}$ and $\mathbf{y}$ be spacelike vectors in E_1^3 that span a timelike vector subspace; then we have $|\langle \mathbf{x}, \mathbf{y} \rangle| > \|\mathbf{x}\| \|\mathbf{y}\|$, and hence, there is a unique real number $\theta \geq 0$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cosh \theta$. This number is called the central angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

iii) *Lorentzian timelike angle*: Let $\mathbf{x}$ be spacelike vector and $\mathbf{y}$ be timelike vector in E_1^3 . Then there is a unique real number $\theta \geq 0$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \sinh \theta$. This number is called the Lorentzian timelike angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Let $\alpha = \alpha(s)$ be a unit speed non-null curve in E_1^3 has a spacelike or timelike rectifying plane; by $\kappa(s)$ and $\tau(s)$ we denote the natural curvature and torsion of $\alpha = \alpha(s)$, respectively.

We assume $\alpha''(s) \neq 0$ for all $s \in [0, L]$, since this would give us a straight line. Consider the Frenet frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ associated with curve $\alpha = \alpha(s)$, then the Frenet formulae read [6, 8, 12]:

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\epsilon_0 \epsilon_1 \kappa & 0 & \tau \\ 0 & -\epsilon_1 \epsilon_2 \tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \quad (2.1)$$

where $\epsilon_0 = \langle \mathbf{t}, \mathbf{t} \rangle = \pm 1$, $\epsilon_1 = \langle \mathbf{n}, \mathbf{n} \rangle = \pm 1$, $\epsilon_2 = \langle \mathbf{b}, \mathbf{b} \rangle = \pm 1$, $\epsilon_0 \epsilon_1 \epsilon_2 = -1$. In this paper, $\alpha'(s)$ denotes the derivative of α with respect to arc-length parameter.

2.1. Ruled surface. A ruled surface is a surface swept out by a straight line L moving along a curve $\alpha = \alpha(s)$. The various position of the generating lines L are called the rulings of the surface which has a parametrization in the ruled form:

$$\mathbf{P}(s, u) = \alpha(s) + u\mathbf{L}(s); \quad u \in \mathbb{R}, \quad (2.2)$$

where $\alpha(s)$ is called the base curve. The surface $\mathbf{P}(s, u)$ is regular if $\mathbf{P}_s \times \mathbf{P}_u \neq 0$ for all points, where $\mathbf{P}_s$ and $\mathbf{P}_u$ are the partial derivatives of $\mathbf{P}(s, u)$ with respect to s and u , respectively. The unit normal vector of the ruled surface $\mathbf{P}(s, u)$ at each point is defined by

$$\mathbf{N}(s, u) = \frac{\mathbf{P}_s \times \mathbf{P}_u}{\|\mathbf{P}_s \times \mathbf{P}_u\|} = \frac{\alpha'(s) \times \mathbf{L}(s) + u\mathbf{L}'(s) \times \mathbf{L}(s)}{\|\alpha'(s) \times \mathbf{L}(s) + u\mathbf{L}'(s) \times \mathbf{L}(s)\|}. \quad (2.3)$$

The base curve is not unique, since any curve of the form:

$$\mathbf{C}(s) = \alpha(s) - \mu(s)\mathbf{L}(s), \quad (2.4)$$

may be used as its base curve, $\mu(s)$ is a smooth function. If there exists a common perpendicular to two neighboring rulings on $\mathbf{P}(s, u)$, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the striction curve.

In Eq. (2.4) if

$$\mu(s) = \frac{\langle \alpha'(s), \mathbf{L}'(s) \rangle}{\|\mathbf{L}'\|^2}, \quad (2.5)$$

then $\mathbf{C}(s)$ is called the striction curve on the ruled surface and it is unique. In the case of $\eta = 0$ the base curve is the striction curve. The surface $\mathbf{P}(s, u)$ is called a spacelike/timelike surface if the induced metric is a Riemannian/Lorentzian metric on each tangent plane [10–12]. This is equivalent to saying that the normal vector $\mathbf{N}$ is timelike/spacelike at each point, where

$$\mathbf{N}(s, u) = \frac{\mathbf{P}_s \times \mathbf{P}_u}{\|\mathbf{P}_s \times \mathbf{P}_u\|}. \quad (2.6)$$

3. SPACELIKE AND TIMELIKE DEVELOPABLE SURFACES

Although Frenet frame can easily be computed, its rotation about the tangent of a general spine curve often leads to undesirable twist in motion design or surface modeling. Moreover, Frenet frame is not continuously defined for a C^1 -curve and even for a C^2 -curve, Frenet frame becomes undefined at an inflection point (i.e., the curvature vanishes), thus causing unacceptable discontinuity when used for surface modeling [12,13]. A moving frame that does not rotate about the instantaneous tangent of the curve $\alpha(s)$ is called a rotation minimizing frame (RMF) of $\alpha(s)$. It can be shown that the RMF is defined continuously for any C^1 -curve. Because of its minimal-twist property and stable behavior in the presence of inflection points, the RMF is preferred to Frenet frame in many applications in computer graphics, including free-form deformation with curve constraints.

3.1. The directrix $\alpha(s)$ is spacelike. In relation to Frenet frame, along a unit speed spacelike curve $\alpha(s)$ with spacelike principal normal $\mathbf{n}(s)$ and timelike binormal $\mathbf{b}(s)$, let us choose unit vectors:

$$\begin{aligned} \mathbf{f}_2(s) &= \cosh \theta \mathbf{n}(s) + \sinh \theta \mathbf{b}(s), \\ \mathbf{f}_3(s) &= \sinh \theta \mathbf{n}(s) + \cosh \theta \mathbf{b}(s), \end{aligned} \quad (3.1)$$

with a certain Lorentzian timelike angle $\theta(s) \geq 0$. Then

$$\mathbf{t}(s) \times \mathbf{f}_2(s) = -\mathbf{f}_3(s), \mathbf{t}(s) \times \mathbf{f}_3(s) = \mathbf{f}_2(s), \mathbf{f}_2(s) \times \mathbf{f}_3(s) = \mathbf{t}(s). \quad (3.2)$$

The angle θ specifies the difference of the two frames; it can be computed from the integral formula

$$\theta(s) = - \int_{s_0}^s \tau ds + \theta_0, \quad (3.3)$$

where s_0 is the starting value of the arc length. The frame $\{\mathbf{t}(s), \mathbf{f}_2(s), \mathbf{f}_3(s)\}$ is called rotation minimizing frame (RMF) [12]. Consequently, Frenet frame and the RMF coincide if and only if $\alpha(s)$ is a planar, i.e. $\tau = 0$ (See Fig. 1). In order to justify the name rotation minimizing frame, we

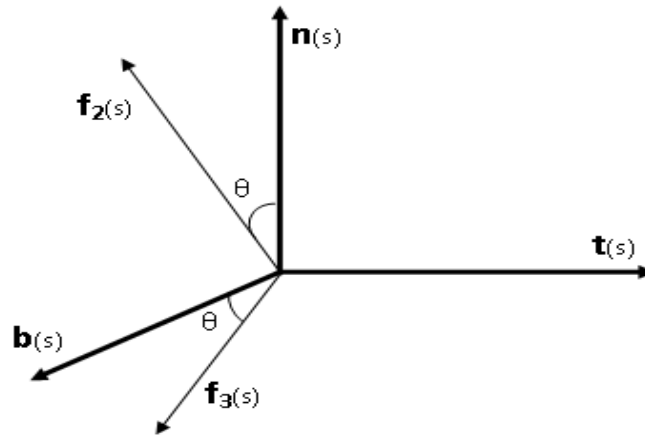


FIGURE 1. Rotation minimizing frame (RMF).

compute the derivatives of the frame vectors $\mathbf{f}_2(s)$ and $\mathbf{f}_3(s)$ as follow:

$$\begin{aligned}\mathbf{f}_2'(s) &= (\theta' + \tau)\mathbf{f}_3(s) + \kappa \cosh \theta \mathbf{t}(s), \\ \mathbf{f}_3'(s) &= (\theta' + \tau)\mathbf{f}_2(s) + \kappa \sinh \theta \mathbf{t}(s).\end{aligned}\quad (3.4)$$

In consequence of Eq. (3.1), both the coefficients of $\mathbf{f}_2$ and $\mathbf{f}_3$ vectors vanish. Hence, the $\mathbf{f}_3(s)$ -axis generates a developable ruled surface along the $\mathbf{t}$ -axis. That is the equation of this surface is given as follows

$$M: \mathbf{P}(s, u) = \alpha(s) + u\mathbf{f}_3(s). \quad (3.5)$$

Obviously, $\mathbf{P}(s, u_0) = \alpha(s)$, $0 \leq s \leq L$, that is the surface M interpolates the curve $\alpha(s)$. Generally, we choose $s_0 = 0$, and then $\theta_0 = \theta(0)$. Moreover, from Eqs. (3.4) and (3.5) the unit normal vector to the surface M at each point is

$$\mathbf{N}(s, u) = \frac{\mathbf{P}_s \times \mathbf{P}_u}{\|\mathbf{P}_s \times \mathbf{P}_u\|} = \mathbf{f}_2(s). \quad (3.6)$$

Therefore, M is spacelike developable ruled surface with the common spacelike line of curvature $\alpha(s)$. According to the above analysis, we obtain the following result:

Theorem 3.1. *In the Minkowski 3-space E_1^3 , for a given spacelike spatial curve $\alpha(s)$, there exists a spacelike developable surface that can be expressed by Eq. (3.5) possessing $\alpha(s)$ as a spacelike line of curvature.*

As it is customary in the literature, there are three types of developable surfaces, the given curve can be classified into three kinds correspondingly. In what follows, we will discuss the relationship between the given spacelike spatial curve $\alpha(s)$ and its isoparametric spacelike developable surface. In the light of above statements, we have:

$$\langle \mathbf{f}_3 \times \mathbf{f}_3', \alpha' \rangle = 0. \quad (3.7)$$

From Eq. (3.7), we note that there are two possible cases:

The first case: when

$$\mathbf{f}_3 \times \mathbf{f}_3' = 0 \Leftrightarrow \kappa \sinh \theta \mathbf{f}_2 = 0. \quad (3.8)$$

In this case, M is referred to as a spacelike cylindrical surface. Since $\mathbf{f}_2$ is a nonzero unit vector, then M is a spacelike cylindrical surface if and only if

$$\sinh \theta = 0 \Leftrightarrow \theta = 0. \quad (3.9)$$

The second case: by similar argument we can also have the following

$$\mathbf{f}_3 \times \mathbf{f}'_3 \neq 0. \quad (3.10)$$

This implies that M is a non-cylindrical surface. Therefore, the first derivative of the directrix is

$$\alpha'(s) = \mathbf{C}'(s) + \mu(s)\mathbf{f}'_3(s) + \mu'(s)\mathbf{f}_3(s). \quad (3.11)$$

Substituting Eq. (3.11) into Eq. (3.7) gives

$$\langle \mathbf{f}_3 \times \mathbf{f}'_3, \mathbf{C}' \rangle = 0. \quad (3.12)$$

Similarly, there are two possible cases which satisfy Eq. (3.12):

The first case is when the first derivative of the striction curve is $\mathbf{C}' = 0$. Geometrically, this condition implies that the striction curve degenerates to a point and the ruled surface becomes a cone; the striction point of a cone is commonly referred to as the vertex. Therefore, the surface represented by Eq. (3.5) is a cone if and only if there exists a fixed point $\mathbf{C}$ and a function $\mu(s)$ such that

$$\mu \kappa \sinh \theta = 1, \quad \mu' \sinh \theta = 0, \quad \mu' \cosh \theta = 0.$$

Since $\cosh \theta$ and $\sinh \theta$ are linearly independent, we get

$$\mu' = 0, \quad \mu \kappa \sinh \theta = 1,$$

which imply

$$\mu = \text{const.} = \frac{1}{\kappa \sinh \theta} \Leftrightarrow \kappa \sinh \theta = \kappa_0 \sinh \theta_0, \quad (3.13)$$

where $\theta_0 = \theta(0)$, and $\kappa_0 = \kappa(0)$.

The second case is when $\mathbf{C}' \neq 0$, i.e. $\kappa \sinh \theta \neq \kappa_0 \sinh \theta_0$. From Eq. (3.12), $\mathbf{C}'$ is perpendicular to $\mathbf{f}_3 \times \mathbf{f}'_3$, and therefore, $\mathbf{C}'$ is in the plane generated by $\mathbf{f}_3$ and $\mathbf{f}'_3$. The condition for $\mathbf{C}$ to be a striction curve is therefore equivalent to $\mathbf{C}'$ and $\mathbf{f}'_3$ are perpendicular to each other. Therefore, we may conclude that the ruling is parallel to the first derivative of the striction curve, which is also the tangent of the striction curve. This ruled surface is referred to as a tangent ruled surface. So, the surface represented by Eq. (3.5) is a spacelike tangent surface if and only if there exists a curve $\mathbf{C}(s)$ so that

$$\kappa \sinh \theta \neq \kappa_0 \sinh \theta_0, \quad (3.14)$$

where $\theta_0 = \theta(0)$, and $\kappa_0 = \kappa(0)$.

3.2. The directrix $\alpha(s)$ is timelike. Here, the directrix $\alpha(s)$ is a timelike curve and the principal normal must be spacelike. As stated in the above case, we can get

$$\begin{aligned} \mathbf{f}_2(s) &= \cos \theta \mathbf{n}(s) + \sin \theta \mathbf{b}(s), \\ \mathbf{f}_3(s) &= -\sin \theta \mathbf{n}(s) + \cos \theta \mathbf{b}(s), \end{aligned} \quad (3.15)$$

with a certain spacelike angle $\theta(s) \geq 0$. Then

$$\mathbf{t}(s) \times \mathbf{f}_2(s) = \mathbf{f}_3(s), \mathbf{t}(s) \times \mathbf{f}_3(s) = -\mathbf{f}_2(s), \mathbf{f}_2(s) \times \mathbf{f}_3(s) = -\mathbf{t}(s). \quad (3.16)$$

In the same manner, we have

$$\mathbf{N}(s, u) = \frac{\mathbf{P}_s \times \mathbf{P}_u}{\|\mathbf{P}_s \times \mathbf{P}_u\|} = \mathbf{f}_2(s), \quad (3.17)$$

which is a spacelike unit vector. As stated in the above case, we can get the corresponding Theorem for Theorem 3.1:

Theorem 3.2. *In the Minkowski 3-space E_1^3 , for a given timelike spatial curve $\alpha(s)$, there exists a timelike developable surface possessing $\alpha(s)$ as a timelike line of curvature given by*

$$M: \mathbf{P}(s, u) = \alpha(s) + u\mathbf{f}_3(s). \quad (3.18)$$

As a result, the following corollary can be given:

Corollary 3.1. *In Minkowski 3-space E_1^3 , there is no spacelike developable ruled surface possessing $\alpha(s)$ as a timelike line of curvature.*

The relations between the given non-null line of curvature and its isoparametric developable surface are given as follows; we omit the details here:

1- M is a timelike cylindrical surface if and only if

$$\sin \theta = 0 \Leftrightarrow \theta = 0 \text{ or } \theta = \pi. \quad (3.19)$$

2- M is a timelike cone if and only if

$$\mu = \text{const.} = \frac{-1}{\kappa \sin \theta} \Leftrightarrow \kappa \sin \theta = \kappa_0 \sin \theta_0, \quad (3.20)$$

where $\theta_0 = \theta(0)$, and $\kappa_0 = \kappa(0)$.

3- M is a timelike tangent developable surface if and only if

$$\kappa \sin \theta \neq \kappa_0 \sin \theta_0. \quad (3.21)$$

4. EXAMPLES

In what follows some representative examples are illustrated to verify the method.

Example 4.1. *Suppose we are given a parametric spacelike helix*

$$\alpha(s) = (a \sinh \frac{s}{c}, \frac{bs}{c}, a \cosh \frac{s}{c}), \quad -1 \leq s \leq 1,$$

where $a, b, c \in \mathbb{R}$, and $a^2 + b^2 = c^2$. After simple computation, we have

$$\left. \begin{aligned} \mathbf{t}(s) &= \left(\frac{a}{c} \cosh \frac{s}{c}, \frac{b}{c}, \frac{a}{c} \sinh \frac{s}{c} \right), \\ \mathbf{n}(s) &= \left(\sinh \frac{s}{c}, 0, \cosh \frac{s}{c} \right), \\ \mathbf{b}(s) &= \left(\frac{b}{c} \cosh \frac{s}{c}, -\frac{a}{c}, \frac{b}{c} \sinh \frac{s}{c} \right) \\ \kappa &= \frac{a}{c^2}, \tau = \frac{b}{c^2}. \end{aligned} \right\}$$

and $\kappa = \frac{a}{c^2}$, $\tau = \frac{b}{c^2}$, then $\theta(s) = -\frac{b}{c^2}s + \theta_0$. Thus, the spacelike tangent developable surface family with a common spacelike line of curvature is given by

$$\mathbf{P}(s, u) = \left(a \sinh \frac{s}{c}, \frac{bs}{c}, a \cosh \frac{s}{c} \right) + u(0, \sinh \theta, \cosh \theta) \begin{pmatrix} \frac{a}{c} \cosh \frac{s}{c} & \frac{b}{c} & \frac{a}{c} \sinh \frac{s}{c} \\ \sinh \frac{s}{c} & 0 & \cosh \frac{s}{c} \\ \frac{b}{c} \cosh \frac{s}{c} & -\frac{a}{c} & \frac{b}{c} \sinh \frac{s}{c} \end{pmatrix}.$$

If we take $a = 2$, $b = 1$, $\theta_0 = 0$, and $-1 \leq u \leq 1$, then we immediately obtain a member of this family (See Fig. 2a). Fig. 2b shows another member in the family with $a = 2$, $b = 1$, $\theta_0 = .5$ and $-1 \leq u \leq 1$.

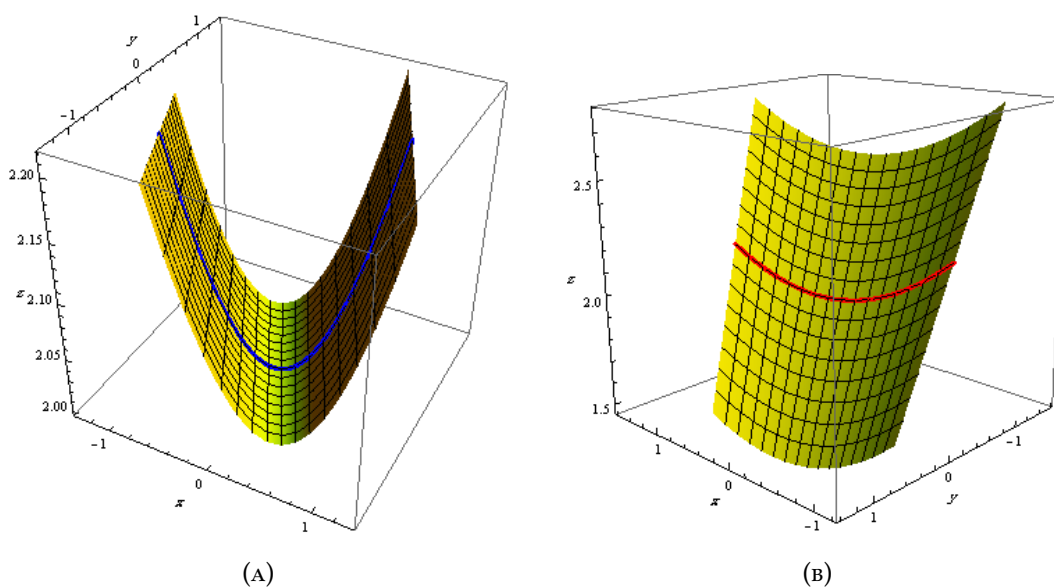


FIGURE 2. Spacelike tangent developable surfaces with a common spacelike line of curvature.

Example 4.2. Suppose we are given a parametric spacelike curve

$$\alpha(s) = (\sinh s, 0, \cosh s), \quad -1 \leq s \leq 1.$$

After simple computation, we have

$$\begin{aligned} \mathbf{t}(s) &= (\cosh s, 0, \sinh s), \quad \mathbf{n}(s) = (\sinh s, 0, \cosh s), \quad \mathbf{b}(s) = (0, -1, 0), \\ \kappa &= 1, \quad \tau = 0, \end{aligned}$$

and $\kappa = 1$, $\tau = 0$, then $\theta(s) = \theta_0$ is a constant. Likewise, the spacelike developable family with a common spacelike line of curvature is given by

$$\mathbf{P}(s, u) = (\sinh s, 0, \cosh s) + u(0, \sinh \theta_0, \cosh \theta_0) \begin{pmatrix} \cosh s & 0 & \sinh s \\ \sinh s & 0 & \cosh s \\ 0 & -1 & 0 \end{pmatrix}. \quad (4.1)$$

If $\theta_0 = 0$, and $-1 \leq u \leq 1$ the developable surface is a spacelike cylinder (See Fig. 3a). Take $\theta_0 = 1$, and $-1 \leq u \leq 1$, for example, the corresponding surface is a spacelike cone as shown in Fig. 3b.

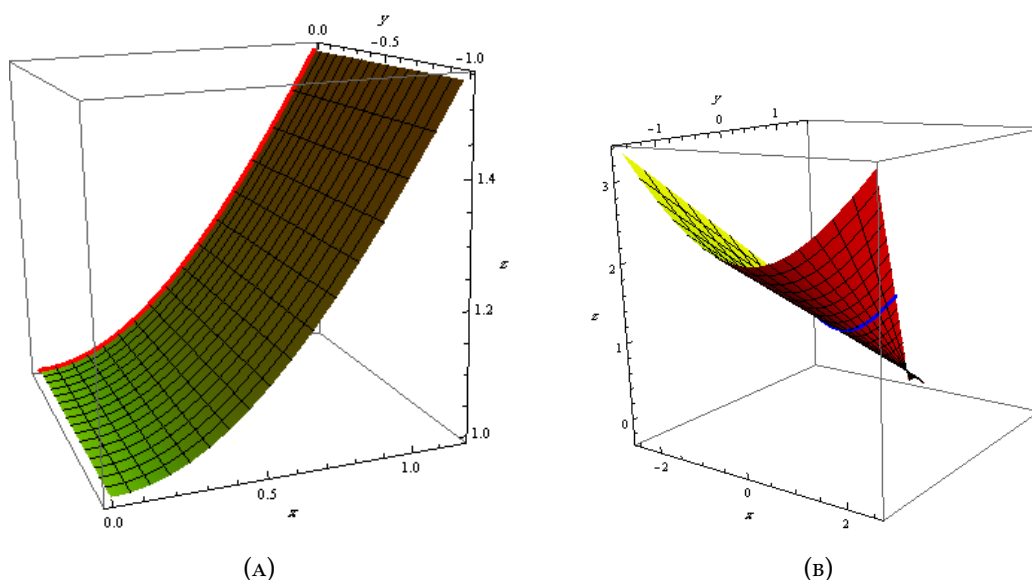


FIGURE 3. Spacelike developable surfaces with a common spacelike line of curvature.

Example 4.3. Suppose we are given a parametric timelike helix

$$\alpha(s) = (a \cosh \frac{s}{c}, b \frac{s}{c}, a \sinh \frac{s}{c}), \quad a > 0, \quad b \neq 0, \quad a^2 - b^2 = c^2, \quad -2 \leq s \leq 2.$$

We will construct a family of timelike developable surfaces sharing the helix $\alpha(s)$ as the timelike line of curvature. After simple computation, we have

$$\left. \begin{aligned} \mathbf{t}(s) &= (\frac{a}{c} \sinh \frac{s}{c}, \frac{b}{c}, \frac{a}{c} \cosh \frac{s}{c}), \\ \mathbf{n}(s) &= (\cosh \frac{s}{c}, 0, \sinh \frac{s}{c}), \\ \mathbf{b}(s) &= (\frac{b}{c} \sinh \frac{s}{c}, \frac{a}{c}, \frac{b}{c} \cosh \frac{s}{c}), \\ \kappa &= \frac{a}{c^2}, \quad \tau = \frac{b}{c^2}, \end{aligned} \right\}$$

and $\theta(s) = -\frac{b}{c^2}s + \theta_0$. Thus, the timelike tangent developable surface family with common timelike line of curvature is given by

$$\mathbf{P}(s, u) = \left(a \cosh \frac{s}{c}, \frac{bs}{c}, a \sinh \frac{s}{c}\right) + u(0, -\sin \theta, \cos \theta) \begin{pmatrix} \frac{a}{c} \sinh \frac{s}{c} & \frac{b}{c} & \frac{a}{c} \cosh \frac{s}{c} \\ \cosh \frac{s}{c} & 0 & \sinh \frac{s}{c} \\ \frac{b}{c} \sinh \frac{s}{c} & \frac{a}{c} & -\frac{b}{c} \cosh \frac{s}{c} \end{pmatrix},$$

If we take $a = 2$, $b = 1$, $\theta_0 = 0$, and $-1 \leq u \leq 1$, then we immediately obtain a member of this family (See Fig. 4a). Fig. 4b shows another member in the family with $a = 2$, $b = 1$, $\theta_0 = \frac{\pi}{4}$, and $-1 \leq u \leq 1$.

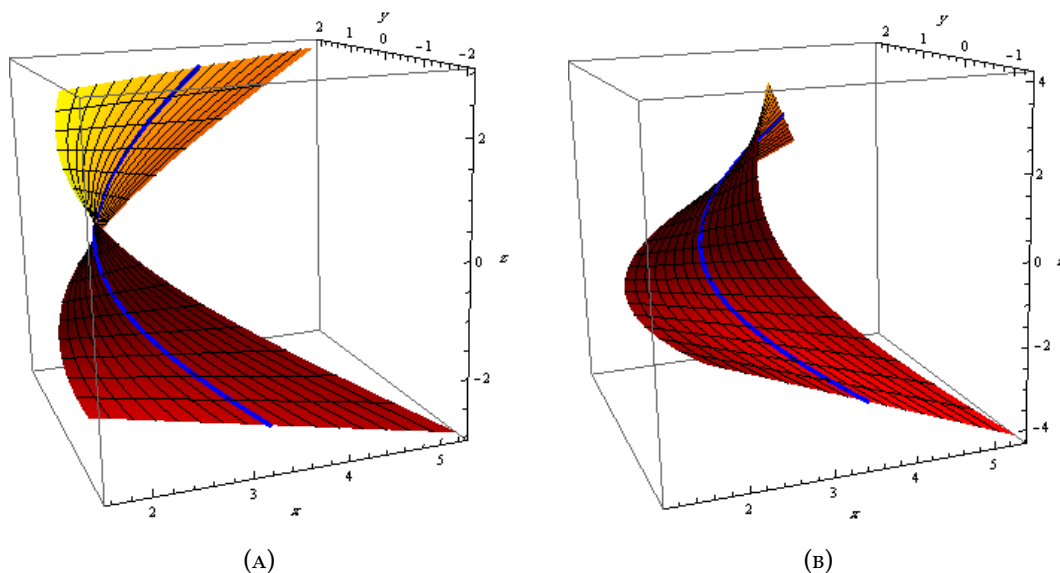


FIGURE 4. Timelike tangent developable surfaces with a common timelike line of curvature.

Example 4.4. Suppose we are given a parametric timelike circle

$$\alpha(s) = (0, \cosh s, \sinh s), \quad -1 \leq s \leq 1.$$

After simple computation, we have

$$\begin{aligned} \mathbf{t}(s) &= (0, \sinh s, \cosh s), \quad \mathbf{n}(s) = (0, \cosh s, \sinh s), \quad \mathbf{b}(s) = (-1, 0, 0), \\ \kappa &= 1, \quad \tau = 0, \end{aligned}$$

and $\theta(s) = \theta_0$. Thus, the timelike developable family with common timelike line of curvature is given by

$$\mathbf{P}(s, u) = (0, \cosh s, \sinh s) + u(0, -\sin \theta_0, \cos \theta_0) \begin{pmatrix} 0 & \sinh s & \cosh s \\ 0 & \cosh s & \sinh s \\ -1 & 0 & 0 \end{pmatrix}.$$

If $\theta_0 = 0$ or π , the developable surface is a timelike cylinder. Take $\theta_0 = 0$, for example, the corresponding surface is shown in Fig. 5a. If we choose $\theta_0 = \frac{\pi}{3}$, then the corresponding surface is a timelike cone as shown in Fig. 5b.

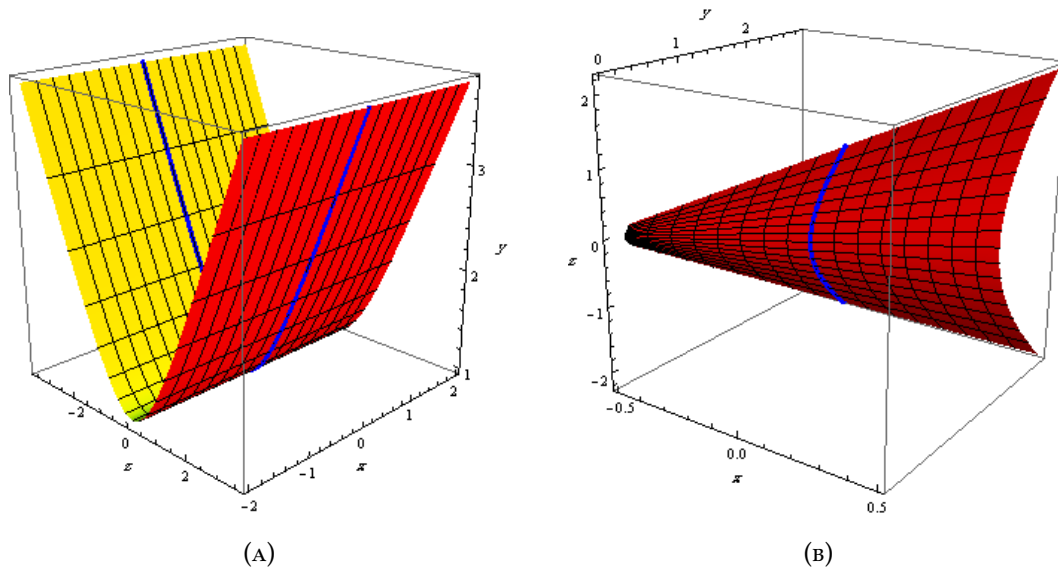


FIGURE 5. Timelike developable surfaces with a common timelike line of curvature.

5. CONCLUSION

In this study, we have presented a method for finding a spacelike and timelike developable surface family whose members all share a given non-null line of curvature. By representing the spacelike and timelike developable surface by the combination of the given spacelike and timelike curve and the RMF, we have derived the necessary and sufficient conditions for the given non-null curve to be the line of curvature for the parametric spacelike and timelike developable surface. The necessary and sufficient conditions when the resulting developable timelike surface is a cylinder, cone or tangent surface are also analyzed. The results, in addition to being of theoretical interest, have application in geometric modeling and the manufacturing of products. Finally, some representative examples are given and plotted.

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