

Unconditional Two-Parameter Nonexpansive Mappings: Hausdorff Stability of Fixed Point Sets under Operator-Convergent Perturbations

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Abstract. We introduce a new unconditional two-parameter class of mappings, called unconditional (α, β) -nonexpansive mappings, satisfying $\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + \beta\|x - y\|$ for all $x, y \in C$, where $\alpha, \beta \geq 0$ and $2\alpha + \beta \leq 1$. This class contains the classical nonexpansive class as the special case $(\alpha, \beta) = (0, 1)$, and is not subsumed by the conditional frameworks of Pant and Shukla (2017) or Ullah et al. (2020), which impose a Suzuki-type pre-condition $(1/2)\|x - Tx\| \leq \|x - y\|$ before the contraction estimate is invoked. We analyse quantitative stability of the fixed-point set $F(T)$ under operator convergent perturbations $T_n \rightarrow T$ (i.e., $\eta_n := \sup_{x \in C} \|T_n x - Tx\| \rightarrow 0$) in two distinct regimes. In the interior regime $2\alpha + \beta < 1$, the fixed-point set is a singleton $\{x^*\}$ and we establish the explicit Hausdorff bound $d_H(F(T_n), F(T)) \leq \frac{1-\alpha}{1-2\alpha-\beta} \eta_n$, and prove that the constant $\frac{1-\alpha}{1-2\alpha-\beta}$ is sharp for every admissible (α, β) ; as a consequence, the fixed-point map $T \mapsto x^*(T)$ is Lipschitz on the operator space. In the boundary regime $2\alpha + \beta = 1$, the set of fixed points is closed and convex and is a subset of C ; we prove the Kuratowski upper limit inclusion $w\text{-}\text{Ls}(F(T_n)) \subseteq F(T)$ under the Opial property, and the full Kuratowski convergence $F(T_n) \xrightarrow{w} F(T)$ when $F(T)$ is a singleton, which occurs in particular under a strict form of the defining inequality. As an application, we derive explicit stability estimates for the solution set of parametric Hammerstein integral equations on $L^2([0, 1])$ under L^2 -convergent kernel perturbations.

1. INTRODUCTION

The study of fixed points for nonexpansive mappings is a central theme of modern functional analysis, with connections to variational inequalities, convex optimization, and nonlinear evolution equations. The foundational existence result is due independently to Browder [3], Kirk [7], and Göhde [6]: every nonexpansive self-mapping of a bounded, closed, convex subset of a uniformly convex Banach space [4] has a fixed point.

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1.1. Historical lineage of the mapping class. We trace the class introduced in the present paper through four generations of generalization.

Level 1 (Nonexpansive mappings). A mapping $T : C \rightarrow C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. This is the $\alpha = 0, \beta = 1$ special case of every class below.

Level 2 (Suzuki's condition (C) [12]). Suzuki replaced the universal requirement with a conditional one:

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|.$$

Every nonexpansive mapping satisfies condition (C), but the converse fails.

Level 3 (Generalized α -nonexpansive mappings (Pant-Shukla, 2017 [9]).) There exists $\alpha \in [0, 1)$ such that

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\| \quad (1.1)$$

for all $x, y \in C$. The conditional pre-condition $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ is a defining feature of the class; removing it and requiring the algebraic estimate for all pairs $x, y \in C$ yields precisely the unconditional class introduced in the present paper (Definition 3.1 with $\beta = 1 - 2\alpha$), which is distinct from the nonexpansive class whenever $\alpha > 0$. A related one-parameter class in the same conditional spirit was studied in [9].

Level 4 (Generalized (α, β) -nonexpansive mappings (Ullah-Ahmad-de la Sen, 2020 [13]).) There exist $\alpha, \beta \geq 0$ with $2\alpha + 2\beta \leq 1$ such that

$$\begin{aligned} \frac{1}{2}\|x - Gx\| \leq \|x - y\| &\implies \|Gx - Gy\| \\ &\leq \alpha(\|x - Gy\| + \|y - Gx\|) + \beta(\|x - Gx\| + \|y - Gy\|) \\ &\quad + (1 - 2\alpha - 2\beta)\|x - y\|. \end{aligned}$$

for all $x, y \in C$. The right-hand side contains five terms including self-distance terms $\|x - Gx\|$ and $\|y - Gy\|$, and the Suzuki-type pre-condition is retained.

Level 5 (Unconditional (α, β) -nonexpansive mappings (present paper).) We introduce a class that differs from all predecessors in a fundamental way: the contraction estimate holds unconditionally, i.e., for every pair $x, y \in C$ without any pre-condition. The right-hand side has three terms and contains no self-distance terms. This structural simplicity enables the sharp Hausdorff stability estimates in Theorem 4.1.

Patir, Goswami, and Mishra [10] introduced the $(B_{\gamma, \mu})$ condition, which has the same algebraic conclusion as our condition but under a Suzuki-type pre-condition. That framework is therefore a conditional variant of the one studied here, and their fixed-point theorems do not immediately apply to the unconditional setting.

Stability of fixed-point sets under perturbations of the underlying operators has been studied in several settings. Bruck [2] established structural properties of fixed-point sets in uniformly convex spaces; further results in the metric fixed-point setting are surveyed in [5]. To the best of our knowledge, we are not aware of any previous work establishing explicit quantitative

Hausdorff estimates for the fixed-point sets of unconditional (α, β) -nonexpansive mappings. The present paper provides such estimates (Theorem 4.1) together with a proof that the constant $C_{\alpha, \beta} = (1 - \alpha)/(1 - 2\alpha - \beta)$ is sharp (Theorem 4.2).

1.2. Contributions and structure. The main contributions are:

- i. We introduce the class of unconditional (α, β) -nonexpansive mappings (Definition 3.1), which is new in the literature.
- ii. We derive the Hausdorff bound with explicit constant $C_{\alpha, \beta} = (1 - \alpha)/(1 - 2\alpha - \beta)$ in the interior regime $2\alpha + \beta < 1$ (Theorem 4.1), prove that $C_{\alpha, \beta}$ is the best possible constant (Theorem 4.2), and show that the fixed-point map $T \mapsto x^*(T)$ is $C_{\alpha, \beta}$ -Lipschitz on the operator space (Corollary 4.2).
- iii. We prove the Kuratowski upper limit inclusion $w\text{-Ls}(F(T_n)) \subseteq F(T)$ in the boundary regime $2\alpha + \beta = 1$ under the Opial property (Theorem 5.1), and establish full Kuratowski convergence $F(T_n) \xrightarrow{w} F(T)$ when $F(T)$ is a singleton (Theorem 5.2); a sufficient condition for singleton $F(T)$ is the strict unconditional inequality (Proposition 5.1). When T is additionally demi-compact, weak convergence upgrades to norm convergence of every fixed-point sequence (Corollary 5.1).
- iv. We apply the theory to parametric Hammerstein integral equations on $L^2([0, 1])$ (Theorem 7.1).

Section 2 presents background material. Section 3 introduces the new class and its basic properties. Section 4 contains the interior-regime Hausdorff theorem. Section 5 treats the boundary regime. Section 6 derives one-sided excess estimates. Section 7 presents the Hammerstein application. Finally, Section 8 concludes the paper with a summary of the main results and a discussion of several directions for future research.

2. PRELIMINARIES

Throughout, $(X, \|\cdot\|)$ is a Banach space and C a nonempty, bounded, closed, convex subset.

2.1. Notation and Background. A Banach space X is said to satisfy the Opial property [8] if, whenever $x_n \rightharpoonup x$,

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\| \quad \text{for all } y \neq x.$$

Hilbert spaces and ℓ^p spaces, $1 < p < \infty$, satisfy this property. In contrast, Opial showed that this property fails in $L^p[0, 2\pi]$ for $1 < p < 2$ and $2 < p < \infty$.

For nonempty subsets $A, B \subset X$, the one-sided excess is

$$e(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|$$

and the Hausdorff distance is

$$d_H(A, B) = \max(e(A, B), e(B, A)).$$

A sequence (A_n) Kuratowski-converges weakly to A , written

$$A_n \xrightarrow{w} A,$$

if

$$\text{w-Ls}(A_n) \subseteq A \quad \text{and} \quad A \subseteq \text{w-Li}(A_n).$$

For $T, T_n : C \rightarrow C$, define

$$\eta_n := \sup_{x \in C} \|T_n x - Tx\|.$$

We say (T_n) converges uniformly to T on C if $\eta_n \rightarrow 0$.

2.2. Demiclosedness. $T : C \rightarrow X$ is demiclosed at zero if $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \rightarrow 0$ imply $Tx = x$.

Lemma 2.1 (Demiclosedness of $I - T$). *Let X satisfy the Opial property, $C \subset X$ weakly compact, and $T : C \rightarrow C$ unconditional (α, β) -nonexpansive with $2\alpha + \beta = 1$. Then $I - T$ is demiclosed at zero.*

Proof. Set

$$a_n := \|x_n - Tx\|, \quad b_n := \|x_n - Tx_n\| \rightarrow 0, \quad c_n := \|x_n - x\|.$$

Applying (3.1) to the pair (x_n, x) and using $\|Tx_n - x\| \leq b_n + c_n$:

$$\|Tx_n - Tx\| \leq \alpha(b_n + c_n) + \alpha a_n + \beta c_n.$$

Since $a_n \leq b_n + \|Tx_n - Tx\|$, substituting and rearranging yields

$$(1 - \alpha)a_n \leq (1 + \alpha)b_n + (\alpha + \beta)c_n.$$

Taking $\liminf$ and using $b_n \rightarrow 0$:

$$(1 - \alpha) \liminf_{n \rightarrow \infty} \|x_n - Tx\| \leq (\alpha + \beta) \liminf_{n \rightarrow \infty} \|x_n - x\|.$$

Since $2\alpha + \beta = 1$ gives $\alpha + \beta = 1 - \alpha$ and $\alpha \leq 1/2 < 1$:

$$\liminf_{n \rightarrow \infty} \|x_n - Tx\| \leq \liminf_{n \rightarrow \infty} \|x_n - x\|. \quad (2.1)$$

If $Tx \neq x$, the Opial property gives

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - Tx\|,$$

contradicting (2.1). Hence $Tx = x$. □

3. UNCONDITIONAL (α, β) -NONEXPANSIVE MAPPINGS

3.1. Definition and basic characteristics.

Definition 3.1 (Unconditional (α, β) -nonexpansive mapping). *Let $C \subset X$ and $\alpha, \beta \geq 0$ with $2\alpha + \beta \leq 1$. A mapping $T : C \rightarrow C$ is unconditional (α, β) -nonexpansive if*

$$\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + \beta\|x - y\|, \quad \forall x, y \in C. \quad (3.1)$$

Remark 3.1. (i) *The parameter set*

$$\{(\alpha, \beta) : \alpha, \beta \geq 0, 2\alpha + \beta \leq 1\}$$

is a triangle with vertices $(0,0)$, $(1/2,0)$, $(0,1)$. We call $2\alpha + \beta < 1$ the interior regime and $2\alpha + \beta = 1$ the boundary regime.

- (ii) The case $(\alpha, \beta) = (0, 1)$ reduces to the classical nonexpansive condition. Hence every nonexpansive mapping is unconditional $(0, 1)$ -nonexpansive.
- (iii) Condition (3.1) is unconditional: it holds for all $x, y \in C$ with no pre-condition. This distinguishes it fundamentally from (1.1) and from the class of [13], both of which require

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\|$$

before the estimate applies. These classes are mathematically distinct.

- (iv) A strict variant of (3.1), in which the inequality is strict whenever $Tx \neq Ty$, is used in the boundary regime to derive uniqueness of fixed points and hence full Kuratowski convergence; see Proposition 5.1.

Proposition 3.1. Let $\alpha, \beta \geq 0$, $2\alpha + \beta \leq 1$.

- (i) Every nonexpansive $T : C \rightarrow C$ is unconditional $(0, 1)$ -nonexpansive.
- (ii) If $2\alpha + \beta < 1$, then $F(T)$ is a singleton $\{x^*\}$, and every orbit satisfies

$$\|Tx - x^*\| \leq q\|x - x^*\| \quad \text{for all } x \in C,$$

where

$$q = \frac{\alpha + \beta}{1 - \alpha} < 1.$$

Proof. (i) Immediate from $(\alpha, \beta) = (0, 1)$.

(ii) Using

$$\|Tx - y\| \leq \|Tx - Ty\| + \|Ty - y\|$$

and

$$\|Ty - x\| \leq \|Ty - Tx\| + \|Tx - x\|$$

in (3.1):

$$(1 - 2\alpha)\|Tx - Ty\| \leq \alpha(\|Tx - x\| + \|Ty - y\|) + \beta\|x - y\|.$$

On $F(T)$ the self-distance terms vanish, yielding

$$\|Ty - Tx\| \leq \frac{\beta}{1 - 2\alpha}\|x - y\|.$$

With $2\alpha + \beta < 1$, we have

$$\frac{\beta}{1 - 2\alpha} < 1,$$

so the Banach contraction principle [1] gives a unique fixed point x^* .

For any $x \in C$, apply (3.1) with $y = x^*$ (so $Tx^* = x^*$):

$$\|Tx - x^*\| = \|Tx - Tx^*\| \leq \alpha\|Tx - x^*\| + \alpha\|x^* - x\| + \beta\|x - x^*\|.$$

Rearranging:

$$(1 - \alpha)\|Tx - x^*\| \leq (\alpha + \beta)\|x - x^*\|,$$

i.e.,

$$\|Tx - x^*\| \leq q\|x - x^*\|$$

with

$$q := \frac{\alpha + \beta}{1 - \alpha}.$$

Since $2\alpha + \beta < 1$ gives $\alpha + \beta < 1 - \alpha$, we have $q < 1$ (note $\alpha \leq 1/2$ in the interior regime, so $1 - \alpha > 0$). Iterating gives

$$\|T^n x - x^*\| \leq q^n \|x - x^*\| \rightarrow 0.$$

□

Motivated by the lineage in Section 1.1 and the observation that all existing two-parameter generalizations [9, 10, 13] retain a conditional pre-condition, we introduce the unconditional class of Definition 3.1. The removal of the pre-condition is not merely cosmetic: it is what enables the uniform-on- C operator estimates underpinning Theorem 4.1.

3.2. Examples and fixed-point existence.

Example 3.1 (Banach contraction). *Let $T : C \rightarrow C$ satisfy*

$$\|Tx - Ty\| \leq L\|x - y\|$$

for some $L \in [0, 1)$. Then T is unconditional $(0, L)$ -nonexpansive ($\alpha = 0$, $\beta = L$), with $2\alpha + \beta = L < 1$ (interior regime).

Example 3.2 (Affine map in $\mathbb{R}$). *Let $C = [0, 2]$ and $T(x) = \frac{1}{4}x + \frac{1}{2}$. Then $T : C \rightarrow C$,*

$$\|Tx - Ty\| = \frac{1}{4}|x - y|,$$

so T is unconditional $(0, \frac{1}{4})$ -nonexpansive. The unique fixed point is $x^* = \frac{2}{3}$.

Example 3.3 (Strict contraction in Hilbert space). *Let H be a Hilbert space, $C \subset H$ nonempty bounded closed convex, and $T : C \rightarrow C$ an unconditional (α, β) -nonexpansive mapping with $2\alpha + \beta < 1$. By Proposition 3.1(ii), T is a strict contraction and has a unique fixed point. The orbit*

$$\|T^n x - x^*\| \leq q^n \|x - x^*\| \rightarrow 0$$

geometrically.

Lemma 3.1 (Asymptotic monotonicity). *Let $T : C \rightarrow C$ be unconditional (α, β) -nonexpansive with $2\alpha + \beta = 1$. Fix $x_0 \in C$, set $x_n = T^n x_0$, and define*

$$\varphi(y) = \limsup_{n \rightarrow \infty} \|x_n - y\|, \quad y \in X.$$

Then $\varphi(Tz) \leq \varphi(z)$ for every $z \in C$.

Proof. Fix $z \in C$. Set $s_n = \|x_{n+1} - Tz\|$ and $t_n = \|x_n - z\|$. Applying (3.1) to the pair (x_n, z) with $\beta = 1 - 2\alpha$:

$$\begin{aligned} s_n &= \|Tx_n - Tz\| \\ &\leq \alpha\|Tx_n - z\| + \alpha\|Tz - x_n\| + (1 - 2\alpha)\|x_n - z\| \\ &= \alpha t_{n+1} + \alpha s_{n-1} + (1 - 2\alpha)t_n, \end{aligned}$$

where $\|Tx_n - z\| = \|x_{n+1} - z\| = t_{n+1}$ and $\|Tz - x_n\| = \|x_n - Tz\| = s_{n-1}$.

Let

$$S = \limsup_{n \rightarrow \infty} s_n = \varphi(Tz) \quad \text{and} \quad r = \limsup_{n \rightarrow \infty} t_n = \varphi(z).$$

Taking lim sup across the recurrence, and using the index-shift identity

$$\limsup_{n \rightarrow \infty} a_{n+k} = \limsup_{n \rightarrow \infty} a_n,$$

so that

$$\limsup_{n \rightarrow \infty} t_{n+1} = r \quad \text{and} \quad \limsup_{n \rightarrow \infty} s_{n-1} = S,$$

we obtain

$$S \leq \alpha r + \alpha S + (1 - 2\alpha)r = \alpha S + (1 - \alpha)r.$$

Hence

$$(1 - \alpha)S \leq (1 - \alpha)r,$$

and since $0 \leq \alpha < 1$ we obtain $\varphi(Tz) \leq \varphi(z)$. □

Theorem 3.1 (Fixed Point Existence). *Let X be a uniformly convex Banach space, $C \subset X$ nonempty, bounded, closed, convex, and $T : C \rightarrow C$ unconditional (α, β) -nonexpansive with $2\alpha + \beta \leq 1$. Then $F(T) \neq \emptyset$.*

Proof. Interior regime ($2\alpha + \beta < 1$): The Banach contraction principle [1] via Proposition 3.1 (ii).

Boundary regime ($2\alpha + \beta = 1$): Fix $x_0 \in C$, set $x_n = T^n x_0 \in C$, and define

$$\varphi(y) = r(y, \{x_n\}) = \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

Existence and uniqueness of the asymptotic center. Since C is weakly compact, Lemma 9.1(a) of [5] gives $A(C, \{x_n\}) \neq \emptyset$. Since X is uniformly convex, $\varepsilon_0(X) = 0$ [5, Definitions 5.2–5.3, p. 52–5], and Theorem 9.2 of [5] yields

$$\text{diam } A(C, \{x_n\}) \leq \varepsilon_0(X)r(C, \{x_n\}) = 0.$$

Hence $A(C, \{x_n\}) = \{z^*\}$ is the unique minimizer of φ over C .

The minimizer is a fixed point. By Lemma 3.1, $\varphi(Tz^*) \leq \varphi(z^*)$. Since $Tz^* \in C$, minimality of z^* gives $\varphi(Tz^*) \geq \varphi(z^*)$. Therefore $\varphi(Tz^*) = \varphi(z^*)$, so $Tz^* \in A(C, \{x_n\}) = \{z^*\}$, i.e. $Tz^* = z^*$. □

4. HAUSDORFF STABILITY IN THE INTERIOR REGIME

Standing assumptions for this section. $T : C \rightarrow C$ is unconditional (α, β) -nonexpansive with $2\alpha + \beta < 1$; $F(T) = \{x^*\}$. Each $T_n : C \rightarrow C$ is unconditional (α, β) -nonexpansive (same parameters), and

$$\eta_n = \sup_{x \in C} \|T_n x - Tx\| \rightarrow 0.$$

$F(T_n) = \{x_n^*\}$ exists by Theorem 3.1.

Lemma 4.1 (Fixed-point comparison).

$$\|x_n^* - x^*\| \leq \frac{1 - \alpha}{1 - 2\alpha - \beta} \eta_n.$$

Proof. Using $T_n x_n^* = x_n^*$, $Tx^* = x^*$, and the triangle inequality:

$$\begin{aligned} \|x_n^* - x^*\| &= \|T_n x_n^* - Tx^*\| \\ &\leq \|T_n x_n^* - Tx_n^*\| + \|Tx_n^* - Tx^*\| \\ &\leq \eta_n + \|Tx_n^* - Tx^*\|. \end{aligned}$$

Applying (3.1) to T at x_n^* and x^* , with $Tx^* = x^*$:

$$\begin{aligned} \|Tx_n^* - Tx^*\| &\leq \alpha \|Tx_n^* - x^*\| + \alpha \|Tx^* - x_n^*\| + \beta \|x_n^* - x^*\| \\ &= \alpha \|Tx_n^* - x^*\| + \alpha \|x_n^* - x^*\| + \beta \|x_n^* - x^*\|. \end{aligned}$$

Also, $\|Tx_n^* - x^*\| = \|Tx_n^* - Tx^*\|$, so applying the same bound:

$$(1 - \alpha) \|Tx_n^* - x^*\| \leq (\alpha + \beta) \|x_n^* - x^*\|.$$

Substituting:

$$\begin{aligned} \|Tx_n^* - Tx^*\| &\leq \frac{\alpha(\alpha + \beta)}{1 - \alpha} \|x_n^* - x^*\| + (\alpha + \beta) \|x_n^* - x^*\| \\ &= \frac{\alpha + \beta}{1 - \alpha} \|x_n^* - x^*\|. \end{aligned}$$

Combining with the first estimate:

$$\|x_n^* - x^*\| \leq \eta_n + \frac{\alpha + \beta}{1 - \alpha} \|x_n^* - x^*\|.$$

Hence

$$\frac{1 - 2\alpha - \beta}{1 - \alpha} \|x_n^* - x^*\| \leq \eta_n,$$

giving the result. □

Theorem 4.1 (Hausdorff stability, interior regime).

$$d_H(F(T_n), F(T)) \leq \frac{1 - \alpha}{1 - 2\alpha - \beta} \eta_n \rightarrow 0.$$

Proof. $F(T) = \{x^*\}$, $F(T_n) = \{x_n^*\}$, so

$$d_H(F(T_n), F(T)) = \|x_n^* - x^*\| \leq \frac{1 - \alpha}{1 - 2\alpha - \beta} \eta_n$$

by Lemma 4.1. □

Corollary 4.1. *If $\eta_n \leq Mr^n$ for some $M > 0$, $r \in (0, 1)$, then*

$$d_H(F(T_n), F(T)) \leq \frac{M(1 - \alpha)}{1 - 2\alpha - \beta} r^n.$$

Corollary 4.2 (Lipschitz continuity of the fixed-point map). *Let $\mathcal{F}_{\alpha, \beta}(C)$ denote the family of all unconditional (α, β) -nonexpansive self-maps of C with $2\alpha + \beta < 1$, equipped with the metric*

$$d(S, T) := \sup_{x \in C} \|Sx - Tx\|.$$

The fixed-point map

$$x^* : \mathcal{F}_{\alpha, \beta}(C) \rightarrow C, \quad T \mapsto x^*(T),$$

is $C_{\alpha, \beta}$ -Lipschitz:

$$\|x^*(S) - x^*(T)\| \leq \frac{1 - \alpha}{1 - 2\alpha - \beta} d(S, T).$$

Proof. Apply Theorem 4.1 with $T_n := S$ and $\eta_n := d(S, T)$. □

Theorem 4.2 (Sharpness of $C_{\alpha, \beta}$). *The constant*

$$C_{\alpha, \beta} = \frac{1 - \alpha}{1 - 2\alpha - \beta}$$

in Theorem 4.1 is sharp for every (α, β) with $\alpha, \beta \geq 0$ and $2\alpha + \beta < 1$: there exist unconditional (α, β) -nonexpansive operators T, T_ε on a convex compact $C \subset \mathbb{R}$ such that

$$\frac{d_H(F(T_\varepsilon), F(T))}{\eta_\varepsilon} = C_{\alpha, \beta}$$

for every $\varepsilon > 0$.

Proof. Set

$$q := \frac{\alpha + \beta}{1 - \alpha} \in [0, 1)$$

and $C := [0, 1] \subset \mathbb{R}$. Define $T, T_\varepsilon : C \rightarrow C$ by

$$Tx := qx, \quad T_\varepsilon x := qx + \varepsilon,$$

for $\varepsilon \in (0, 1 - q]$; both map C into C since $T(1) = q \leq 1$ and $T_\varepsilon(1) = q + \varepsilon \leq 1$.

Verification of condition (3.1) for T and T_ε . For any $x, y \in C$ and any map $S \in \{T, T_\varepsilon\}$ with $Sx - Sy = q(x - y)$, the reverse triangle inequality gives

$$\begin{aligned}
|Sx - y| + |Sy - x| &\geq |(Sx - y) - (Sy - x)| \\
&= |Sx - Sy + (x - y)| \\
&= |q(x - y) + (x - y)| \\
&= (q + 1)|x - y|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha|Sx - y| + \alpha|Sy - x| + \beta|x - y| &\geq \alpha(q + 1)|x - y| + \beta|x - y| \\
&= [\alpha(q + 1) + \beta]|x - y| \\
&= q|x - y| = |Sx - Sy|,
\end{aligned}$$

where the last equality uses

$$\alpha(q + 1) + \beta = \alpha q + \alpha + \beta = q$$

(since $q(1 - \alpha) = \alpha + \beta$). Both T and T_ε thus satisfy (3.1).

Sharpness. The unique fixed points are $x^*(T) = 0$ and

$$x^*(T_\varepsilon) = \frac{\varepsilon}{1 - q}.$$

The perturbation size is

$$\eta_\varepsilon = \sup_{x \in C} |T_\varepsilon x - Tx| = \varepsilon.$$

Hence

$$\frac{d_H(F(T_\varepsilon), F(T))}{\eta_\varepsilon} = \frac{\varepsilon / (1 - q)}{\varepsilon} = \frac{1}{1 - q} = \frac{1 - \alpha}{1 - 2\alpha - \beta} = C_{\alpha, \beta}.$$

□

Lemma 4.2 (Orbit contraction). *For any $x \in C$,*

$$\|T^n x - x^*\| \leq q^n \|x - x^*\|,$$

where

$$q = \frac{\alpha + \beta}{1 - \alpha} < 1.$$

Proof.

$$\|Tx - x^*\| = \|Tx - Tx^*\| \leq \frac{\alpha + \beta}{1 - \alpha} \|x - x^*\|$$

by the bound in the proof of Proposition 3.1 (ii). Iterate.

□

Lemma 4.3 (Approximation under perturbation). *For any $x \in C$ and $k \geq 1$:*

$$\|T_n^k x - x_n^*\| \leq q^k \|x - x_n^*\|.$$

Proof. Apply Lemma 4.2 to T_n .

□

5. KURATOWSKI CONVERGENCE IN THE BOUNDARY REGIME

Standing assumptions for this section. X satisfies the Opial property; $C \subset X$ is weakly compact, closed, convex. $T, T_n : C \rightarrow C$ are unconditional (α, β) -nonexpansive with $2\alpha + \beta = 1$ and $\eta_n \rightarrow 0$.

Theorem 5.1 (Kuratowski weak upper limit).

$$\text{w-Ls}(F(T_n)) \subseteq F(T).$$

Proof. Let $x_{n_k}^* \in F(T_{n_k})$ with $x_{n_k}^* \rightharpoonup x$. Then

$$\|x_{n_k}^* - Tx_{n_k}^*\| = \|T_{n_k}x_{n_k}^* - Tx_{n_k}^*\| \leq \eta_{n_k} \rightarrow 0.$$

By Lemma 2.1, $Tx = x$, so $x \in F(T)$. □

Remark 5.1. *The full Kuratowski convergence*

$$F(T_n) \xrightarrow{w} F(T)$$

does not hold in general. Indeed, take $X = \ell^2$, $C = \{x \in \ell^2 : \|x\| \leq 1\}$, $T = \text{id}$ (unconditional $(0, 1)$ -nonexpansive), and

$$T_n(x_1, x_2, \dots) = (x_1, (1 - 1/n)x_2, (1 - 1/n)x_3, \dots)$$

(also unconditional $(0, 1)$ -nonexpansive, $\eta_n = 1/n \rightarrow 0$). Then

$$F(T) = C$$

while

$$F(T_n) = \{ae_1 : |a| \leq 1\};$$

the point $e_2 \in F(T)$ belongs to no weak cluster set of $(F(T_n))$. The lower inclusion

$$F(T) \subseteq \text{w-Li}(F(T_n))$$

therefore requires additional structural assumptions; a sufficient condition ensuring full convergence is the uniqueness of the fixed point of T , established in Theorem 5.2 below.

Proposition 5.1 (Strict condition implies uniqueness). *If $T : C \rightarrow C$ is strictly unconditional (α, β) -nonexpansive with $2\alpha + \beta = 1$, meaning*

$$\|Tx - Ty\| < \alpha\|Tx - y\| + \alpha\|Ty - x\| + \beta\|x - y\|, \quad \text{where } Tx \neq Ty,$$

then $F(T)$ contains at most one element. If additionally, X is uniformly convex, then $F(T)$ is a singleton.

Proof. Suppose $x, y \in F(T)$ with $x \neq y$. Then $Tx = x \neq y = Ty$, so the strict inequality gives

$$\begin{aligned} \|x - y\| &= \|Tx - Ty\| \\ &< \alpha\|Tx - y\| + \alpha\|Ty - x\| + \beta\|x - y\| \\ &= (2\alpha + \beta)\|x - y\| = \|x - y\|, \end{aligned}$$

a contradiction. Hence $F(T)$ contains at most one element. Under uniform convexity, Theorem 3.1 gives $F(T) \neq \emptyset$, so $F(T)$ is a singleton. □

Theorem 5.2 (Full Kuratowski convergence under uniqueness). *Suppose the standing assumptions of this section hold and, additionally, X is uniformly convex and $F(T) = \{x^*\}$. Then*

$$\text{w-Li}(F(T_n)) = \text{w-Ls}(F(T_n)) = \{x^*\} = F(T),$$

i.e.

$$F(T_n) \xrightarrow{\text{w}} F(T).$$

Moreover, every sequence $(x_n^*)_{n \geq 1}$ with $x_n^* \in F(T_n)$ satisfies

$$x_n^* \rightharpoonup x^*.$$

Proof. The inclusion

$$\text{w-Ls}(F(T_n)) \subseteq F(T) = \{x^*\}$$

is Theorem 5.1. It remains to show

$$\{x^*\} \subseteq \text{w-Li}(F(T_n)).$$

Let (n_k) be any subsequence. Each T_{n_k} satisfies the hypotheses of Theorem 3.1 (X uniformly convex, $T_{n_k} : C \rightarrow C$ unconditional (α, β) -nonexpansive, $2\alpha + \beta = 1$), so

$$F(T_{n_k}) \neq \emptyset.$$

Pick any $y_{n_k} \in F(T_{n_k})$.

Since $y_{n_k} \in C$ and C is weakly sequentially compact, there is a further subsequence

$$y_{n_{k_j}} \rightharpoonup \bar{x}.$$

As

$$\|y_{n_{k_j}} - Ty_{n_{k_j}}\| = \|T_{n_{k_j}} y_{n_{k_j}} - Ty_{n_{k_j}}\| \leq \eta_{n_{k_j}} \rightarrow 0,$$

Lemma 2.1 yields $T\bar{x} = \bar{x}$, hence

$$\bar{x} \in F(T) = \{x^*\}.$$

Consequently, every weakly convergent subsequence of (y_{n_k}) has limit x^* . Since C is weakly sequentially compact, if $y_{n_k} \not\rightharpoonup x^*$, then some further subsequence would lie eventually outside a weak neighbourhood of x^* ; any convergent sub-subsequence thereof would then have a limit other than x^* , contradicting the above. Therefore

$$y_{n_k} \rightharpoonup x^*,$$

so

$$x^* \in \text{w-Li}(F(T_n)).$$

The statement for arbitrary (x_n^*) with $x_n^* \in F(T_n)$ follows by applying the argument to the full index sequence. $\square$

Corollary 5.1 (Norm convergence under demi-compactness). *Under the hypotheses of Theorem 5.2, suppose additionally that T is demi-compact at 0 [11]: every bounded sequence $\{z_n\} \subset C$ satisfying*

$$\|z_n - Tz_n\| \rightarrow 0$$

possesses a strongly convergent subsequence. Then $x_n^* \rightarrow x^*$ in norm for every sequence (x_n^*) with $x_n^* \in F(T_n)$.

Proof. Let $x_n^* \in F(T_n)$. Since $T_n x_n^* = x_n^*$,

$$\|x_n^* - Tx_n^*\| = \|T_n x_n^* - Tx_n^*\| \leq \eta_n \rightarrow 0.$$

Demi-compactness of T yields a subsequence

$$x_{n_k}^* \rightarrow \bar{x}$$

in norm, hence $x_{n_k}^* \rightarrow \bar{x}$. By Lemma 2.1, $T\bar{x} = \bar{x}$, so

$$\bar{x} \in F(T) = \{x^*\}.$$

Since every subsequence of (x_n^*) has a further subsequence converging in norm to x^* , we conclude

$$x_n^* \rightarrow x^*$$

in norm. □

Proposition 5.2 (Fixed-point set is closed convex). *If $T : C \rightarrow C$ is unconditional (α, β) -nonexpansive with $2\alpha + \beta \leq 1$, then $F(T)$ is closed. If additionally, X is strictly convex and $2\alpha + \beta = 1$, then $F(T)$ is convex.*

Proof. Closedness. Let $x_n \in F(T)$, $x_n \rightarrow x$. Apply (3.1) to the pair (x_n, x) . Since $Tx_n = x_n$, we have

$$\|Tx_n - x\| = \|x_n - x\|$$

and

$$\|Tx - x_n\| = \|Tx - Tx_n\| = \|Tx_n - Tx\|,$$

so

$$\|Tx_n - Tx\| \leq \alpha\|x_n - x\| + \alpha\|Tx_n - Tx\| + \beta\|x_n - x\|,$$

giving

$$(1 - \alpha)\|Tx_n - Tx\| \leq (\alpha + \beta)\|x_n - x\| \rightarrow 0.$$

Since $Tx_n = x_n \rightarrow x$ and $Tx_n \rightarrow Tx$, we get $Tx = x$.

Convexity. Let $u, v \in F(T)$, $w = (1 - t)u + tv$. From (3.1) with $2\alpha + \beta = 1$ and $Tu = u$, $Tv = v$:

$$(1 - \alpha)\|Tw - u\| \leq (\alpha + \beta)\|w - u\| = (1 - \alpha)t\|u - v\|,$$

so

$$\|Tw - u\| \leq t\|u - v\|.$$

Similarly,

$$\|Tw - v\| \leq (1 - t)\|u - v\|.$$

In a strictly convex space these force $Tw = w$. □

6. ONE-SIDED EXCESS ESTIMATES

Theorem 6.1 (Backward excess). *In the interior regime,*

$$e(F(T), F(T_n)) \leq \frac{1 - \alpha}{1 - 2\alpha - \beta} \eta_n.$$

Proof.

$$e(F(T), F(T_n)) = \|x^* - x_n^*\|;$$

apply Lemma 4.1. □

Theorem 6.2 (Forward excess). *In the interior regime, if T is compact then*

$$e(F(T_n), F(T)) \leq \frac{1 - \alpha}{1 - 2\alpha - \beta} \eta_n.$$

Proof. For $x_n^* \in F(T_n)$:

$$\|x_n^* - Tx_n^*\| \leq \eta_n.$$

Compactness of T gives a subsequence

$$Tx_{n_k}^* \rightarrow y^*,$$

so

$$x_{n_k}^* \rightarrow y^*$$

and $Ty^* = y^*$, i.e., $y^* = x^*$. The quantitative bound is Lemma 4.1. □

Corollary 6.1. *Under the hypotheses of Theorem 6.2,*

$$d_H(F(T_n), F(T)) \leq \frac{1 - \alpha}{1 - 2\alpha - \beta} \eta_n.$$

7. APPLICATION TO PARAMETRIC HAMMERSTEIN INTEGRAL EQUATIONS

7.1. Problem formulation. Consider the parametric Hammerstein equation on $L^2([0, 1])$:

$$u(s) = f(s) + \lambda \int_0^1 k(s, t)g(t, u(t)) dt, \quad s \in [0, 1], \quad \lambda \in \mathbb{R}, \quad (7.1)$$

where $f \in L^2([0, 1])$, $k \in L^2([0, 1]^2)$, and $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$|g(t, u) - g(t, v)| \leq L|u - v| \quad \text{a.e.}$$

and $g(\cdot, 0) \in L^2([0, 1])$. The integral operator is

$$(T_\lambda u)(s) := f(s) + \lambda \int_0^1 k(s, t)g(t, u(t)) dt.$$

Fixed points of T_λ are solutions of (7.1).

7.2. Nonexpansiveness and kernel perturbations.

Proposition 7.1 (Invariant ball). *Suppose*

$$\beta_0 := |\lambda|L\|k\|_{L^2} < 1.$$

Define

$$R_0 := \frac{\|f\|_{L^2} + |\lambda|\|k\|_{L^2}\|g(\cdot, 0)\|_{L^2}}{1 - \beta_0}.$$

Then for every $R \geq R_0$, the closed ball

$$C_R := \{u \in L^2([0, 1]) : \|u\|_{L^2} \leq R\}$$

satisfies

$$T_\lambda(C_R) \subseteq C_R.$$

Proof. For $u \in C_R$, the Lipschitz bound on g gives

$$|g(t, u(t))| \leq |g(t, 0)| + L|u(t)| \quad \text{a.e.,}$$

so by the triangle inequality and Cauchy–Schwarz:

$$\begin{aligned} \|T_\lambda u\|_{L^2} &\leq \|f\|_{L^2} + |\lambda|\|k\|_{L^2}\|g(\cdot, u(\cdot))\|_{L^2} \\ &\leq \|f\|_{L^2} + |\lambda|\|k\|_{L^2}(\|g(\cdot, 0)\|_{L^2} + LR) \\ &= R_0(1 - \beta_0) + \beta_0 R \leq R, \end{aligned}$$

where the last inequality uses $R \geq R_0$ and $\beta_0 < 1$. □

Proposition 7.2. *Let $\beta_0 := |\lambda|L\|k\|_{L^2} \leq 1$ and $C \subset L^2([0, 1])$ nonempty, bounded, closed, convex, T_λ -invariant. Then $T_\lambda : C \rightarrow C$ is nonexpansive; specifically, unconditional $(0, \beta_0)$ -nonexpansive.*

Proof. By Cauchy–Schwarz:

$$\|T_\lambda u - T_\lambda v\|_{L^2} \leq |\lambda|L\|k\|_{L^2}\|u - v\|_{L^2} = \beta_0\|u - v\|_{L^2}.$$

This is (3.1) with $\alpha = 0, \beta = \beta_0$. □

Remark 7.1. *Proposition 7.2 shows that T_λ is nonexpansive with $(\alpha, \beta) = (0, \beta_0)$. The correct interpretation is $\beta = \beta_0 \leq 1$ (boundary if $\beta_0 = 1$, interior if $\beta_0 < 1$). There is no equivalence between nonexpansiveness and condition (3.1) at a special parameter value; they coincide at $\alpha = 0, \beta = 1$.*

Let $k_n \in L^2([0, 1]^2)$ with

$$\|k_n - k\|_{L^2} \rightarrow 0,$$

and define $T_{\lambda,n}$ by replacing k with k_n .

Proposition 7.3.

$$\eta_n := \sup_{u \in C} \|T_{\lambda,n}u - T_\lambda u\|_{L^2} \leq |\lambda|G_R\|k_n - k\|_{L^2} \rightarrow 0,$$

where

$$G_R := \sup_{u \in C} \|g(\cdot, u(\cdot))\|_{L^2} < \infty.$$

Proof. Since C is bounded in L^2 with radius

$$R := \sup_{u \in C} \|u\|_{L^2} < \infty$$

and

$$|g(t, u)| \leq |g(t, 0)| + L|u| \quad \text{a.e.},$$

we have

$$\|g(\cdot, u(\cdot))\|_{L^2} \leq \|g(\cdot, 0)\|_{L^2} + LR$$

for all $u \in C$, so

$$G_R \leq \|g(\cdot, 0)\|_{L^2} + LR < \infty.$$

By Cauchy–Schwarz:

$$\begin{aligned} \|T_{\lambda,n}u - T_\lambda u\|_{L^2}^2 &= \lambda^2 \int_0^1 \left(\int_0^1 [k_n - k](s, t) g(t, u(t)) dt \right)^2 ds \\ &\leq \lambda^2 \|k_n - k\|_{L^2}^2 \|g(\cdot, u)\|_{L^2}^2 \\ &\leq \lambda^2 G_R^2 \|k_n - k\|_{L^2}^2. \end{aligned}$$

□

7.3. Stability of solution sets.

Theorem 7.1 (Hammerstein solution stability). *Suppose*

$$\beta_0 := |\lambda|L\|k\|_{L^2} < 1$$

(interior regime) and

$$\|k_n - k\|_{L^2} \rightarrow 0.$$

Then

$$\|u_n^* - u^*\|_{L^2} \leq \frac{|\lambda|G_R}{1 - \beta_0} \|k_n - k\|_{L^2}.$$

Proof. Choose $R \geq R_0$ as in Proposition 7.1; then C_R is a nonempty, bounded, closed, convex, T_λ -invariant subset of $L^2([0, 1])$. Since

$$\|k_n - k\|_{L^2} \rightarrow 0,$$

for all sufficiently large n we have

$$|\lambda|L\|k_n\|_{L^2} < 1$$

and C_R is also $T_{\lambda,n}$ -invariant (the same bound applies with k_n in place of k , using $\|k_n\|_{L^2} \rightarrow \|k\|_{L^2}$.)

By Propositions 7.2 and 7.3, T_λ and $T_{\lambda,n}$ are unconditional $(0, \cdot)$ -nonexpansive on C_R in the interior regime. Applying Theorem 4.1 with $\alpha = 0, \beta = \beta_0$:

$$\begin{aligned} \|u_n^* - u^*\|_{L^2} &\leq \frac{1}{1 - \beta_0} \eta_n \\ &\leq \frac{|\lambda|G_R}{1 - \beta_0} \|k_n - k\|_{L^2}. \end{aligned}$$

□

Corollary 7.1. *If*

$$\|k_n - k\|_{L^2} \leq Dr^n$$

for some $D > 0$, $r \in (0, 1)$, then

$$\|u_n^* - u^*\|_{L^2} \leq \frac{D|\lambda|G_R}{1 - \beta_0} r^n.$$

8. CONCLUSIONS

We have introduced the class of unconditional (α, β) -nonexpansive mappings and derived explicit Hausdorff stability bounds for fixed-point sets under operator-convergent perturbations. The class is genuinely new: it is not subsumed by the conditional frameworks of [9] or [13], which both require a Suzuki-type pre-condition.

The main results are:

- (i) sharp Hausdorff bound

$$d_H(F(T_n), F(T)) \leq C_{\alpha, \beta} \eta_n, \quad C_{\alpha, \beta} = \frac{1 - \alpha}{1 - 2\alpha - \beta},$$

proved sharp (Theorems 4.1 and 4.2) and the fixed-point map $T \mapsto x^*(T)$ shown $C_{\alpha, \beta}$ -Lipschitz (Corollary 4.2), in the interior regime;

- (ii) Kuratowski upper limit inclusion $w\text{-}Ls(F(T_n)) \subseteq F(T)$ at the boundary (Theorem 5.1);
- (iii) full Kuratowski convergence $F(T_n) \xrightarrow{w} F(T)$ and, under demi-compactness, norm convergence of fixed-point sequences, when $F(T)$ is a singleton (Theorem 5.2 and Corollary 5.1);
- (iv) explicit L^2 -stability for Hammerstein solution sets (Theorem 7.1).

Natural directions for future research include the following.

- (1) **Multivalued unconditional mappings.** Extending the present framework to multivalued unconditional (α, β) -nonexpansive mappings and establishing the corresponding Hausdorff stability and Kuratowski convergence results for fixed-point sets remains open.
- (2) **Lower Kuratowski limit without uniqueness.** Theorem 5.2 requires $F(T)$ to be a singleton. Characterising the weak lower Kuratowski limit $w\text{-}Li(F(T_n))$ when $F(T)$ is not a singleton is an open problem that would complete the picture in the boundary regime.
- (3) **Variable parameter families.** The results of the present paper assume a common parameter pair (α, β) . Another natural direction is to extend the theory to sequences (or families) of unconditional (α_n, β_n) -nonexpansive mappings with varying parameters, under suitable uniform assumptions (for example, $\sup_n C_{\alpha_n, \beta_n} < \infty$). An interesting question is whether quantitative Hausdorff stability and Kuratowski convergence continue to hold in this more general setting.

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