

Integral Type Contraction in the Setting of ω_t -Distance Mappings with an Application**Tariq Qawasmeh****Department of Mathematics, College of Arts and Sciences, Amman Arab University, Amman 11953,
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Abstract. The purpose of this paper is to introduce a new family of contractive conditions of integral type formulated through an ω_t -distance mapping defined on a b -metric space. To this end we employ two auxiliary classes of functions: the class Φ of nonnegative Lebesgue integrable mappings whose integral over every nondegenerate interval is positive and finite, and the class Γ of continuous non-decreasing comparison mappings whose iterates converge pointwise to zero. Coupling these two classes with the class Θ of altering distance functions, we define the notion of an (ω_t, Θ) integral contraction and we prove that every such self-mapping of a complete b -metric space possesses a unique fixed point, provided that either the mapping is continuous or a Kada type infimum condition is fulfilled. Two corollaries, corresponding to the linear comparison function $\gamma(a) = \eta a$ with $\eta \in [0, 1)$ and to the power type comparison function $\gamma(a) = (1 + a)^r - 1$ with $r \in (0, 1)$, are deduced. Since neither the symmetry of the ω_t -distance nor the continuity of the b -metric is required, our results extend and unify several known theorems obtained in metric, b -metric and Ω -distance settings. A nontrivial example illustrating the sharpness of the hypotheses is supplied, and the abstract theory is applied to establish the existence and uniqueness of the solution of a class of Volterra integral equations.

1. INTRODUCTION

Fixed point theory is one of the most active branches of nonlinear analysis, mainly because a large number of problems arising in differential equations, integral equations, optimization and mathematical economics can be reformulated as the problem of locating a fixed point of a suitably chosen operator. The starting point of the metric theory is the classical contraction principle of Banach [1], which asserts that every self-mapping f of a complete metric space satisfying $d(fw_1, fw_2) \leq \eta d(w_1, w_2)$ with $\eta \in [0, 1)$ has exactly one fixed point, reachable by Picard iteration. The elegance of this statement, together with the restrictiveness of the Lipschitz condition it imposes, has motivated an enormous literature devoted to weakening its hypotheses. Kannan [2]

Received: Aug. 25, 2026.

2020 *Mathematics Subject Classification.* 47H10; 54H25; 54E50; 45D05.

Key words and phrases. integral type contraction; ω_t -distance mapping; b -metric space; altering distance function; comparison function; fixed point; Volterra integral equation.

and Chatterjea [3] replaced the distance between the points by the distances between the points and their images, thereby producing contractive conditions that do not force the continuity of f . A different and particularly fruitful device was proposed by Khan, Swaleh and Sessa [4], who introduced the notion of an *altering distance function*, that is, a continuous nondecreasing function θ vanishing only at the origin, and controlled $\theta(d(fw_1, fw_2))$ rather than $d(fw_1, fw_2)$ itself. Geraghty [5] obtained a genuine generalization of Banach's theorem by allowing the contraction constant to be replaced by a function whose values approach 1 only along sequences tending to zero, while Jleli and Samet [6] achieved a similar aim by means of a new class of auxiliary θ -functions.

A second, essentially independent, line of generalization concerns the contractive inequality itself rather than the constants appearing in it. Branciari [9] was the first to observe that the Banach condition may be recast in an integral form, namely $\int_0^{d(fw_1, fw_2)} \phi(w) dw \leq \eta \int_0^{d(w_1, w_2)} \phi(w) dw$, where ϕ is a nonnegative Lebesgue integrable mapping which is summable on every compact subset of $[0, +\infty)$ and whose integral over every nondegenerate interval is strictly positive. Choosing $\phi \equiv 1$ recovers the classical principle, so that the integral formulation is a proper extension; more importantly, it permits contractive behaviour that is violated pointwise but restored after averaging. Liu, Li, Kang and Cho [10] developed this circle of ideas systematically and established, among other things, the continuity property of the mapping $c \mapsto \int_0^c \phi(w) dw$ along convergent sequences, a technical tool which is indispensable in the present paper.

In parallel with the modification of the contractive condition, considerable effort has been devoted to enlarging the ambient space. Bakhtin [11], and independently Czerwik [12], introduced the concept of a b -metric space by relaxing the triangle inequality to $d_b(w_1, w_2) \leq s [d_b(w_1, w_3) + d_b(w_3, w_2)]$ for some constant $s \geq 1$. This apparently modest relaxation has substantial consequences: a b -metric need not be a continuous function of its arguments, and open balls need not be open sets, so that many arguments valid in metric spaces have to be rebuilt from scratch. Fixed point results in this framework and in its numerous descendants have been obtained by Aghajani et al. [13], by Bataihah, Qawasmeh and Shatnawi [14], by Ahmad et al. [15] in the bipolar graphical setting, by Kamran et al. [17], who proposed the extended b -metric spaces, and by Ali et al. [18] for multi-valued contractions in b -gauge spaces; the interplay between b -metric and G -metric structures is analysed in Shatanawi and Bataihah [19], and by Al-Omeri [16], the property (P) in ordered metric spaces in the neutrosophic setting. Recent works on MR -metric spaces have established several fixed point results and applications [7, 8, 52].

A third and more recent strategy consists in leaving the underlying space untouched while replacing the metric *inside* the contractive inequality by an auxiliary, generally nonsymmetric, distance. This idea originates with Kada et al. [20], who introduced the ω -distance in order to derive non-convex minimization theorems, and it has since been transferred to a variety of settings under the names Ω -distance, generalized Ω -distance and modified ω -distance. Abodayeh et al. [21] and Abu-Irwaq, Shatanawi, Bataihah and Nuseir [22] established fixed point theorems for nonlinear

contractions through generalized Ω -distance mappings; common fixed point results in G -metric spaces equipped with an Ω -distance are due to Abu-Irwaq et al. [23]; cyclic and Suzuki type conditions were treated by Shatanawi et al. [24], by Shatanawi et al. [25] and by Shatanawi et al. [26]. In the quasi-metric framework, Qawasmeh, Tallafha and Shatanawi [27] and Abodayeh et al. [28] obtained fixed and common fixed point theorems by means of modified ω -distance mappings.

The notion which is central to the present work, the ω_t -distance, was introduced by Hussain et al. [29] precisely in order to carry the ω -distance methodology into metric type, and in particular b -metric, spaces. An ω_t -distance is a nonnegative function t on $\mathcal{T} \times \mathcal{T}$ which obeys a triangle inequality with the same constant s as the underlying b -metric, which is s -lower semicontinuous in its second variable, and which separates points in the weak sense expressed by axiom $(\omega_t 3)$. Neither $t(w, w) = 0$ nor the symmetry of t is postulated, so that an ω_t -distance is strictly more general than the b -metric that supports it, and consequently every contractive condition written in terms of t is weaker than the corresponding condition written in terms of d_b . Fixed point results formulated in this language have been obtained by Bataihah et al. [30] through H -simulation functions, with applications to matrix equations, by Al-Tawil et al. [31] for Geraghty type contractions, and by Qawasmeh [32,33] for interpolative contractions and for G_b -metric spaces; a related contraction in extended b -metric spaces is due to Qawasmeh and Hatamleh [34].

The two developments just described, integral type contractions on the one hand and auxiliary distance mappings on the other, have only recently begun to be combined. Shatnawi and Bataihah [35] proved fixed point theorems for contractions of integral type through a generalized Ω -distance, and Bataihah and Hazaymeh [36] obtained integral type results in neutrosophic fuzzy metric spaces. Related refinements of the notion of distance itself include the Γ -distance and Γ_k -distance mappings studied by Qawasmeh et al. [37], by Bataihah et al. [38] and by Bataihah et al. [39], the new distance spaces of Bataihah and Qawasmeh [40] and of Bataihah [41], the simulation function approach of Hazaymeh and Bataihah [42], and the mixed-power type contractions of Bataihah and Shatnawi [43]. Coincidence and cyclic type results in ordered and modular spaces, which share the same methodology, may be found in Aydi, Karapinar and Postolache [44], Bejenaru and Postolache [45] and Chandok and Postolache [46].

Finally, the same programme has been carried out in fuzzy and neutrosophic environments, where the distance takes values in a lattice of membership degrees rather than in $[0, +\infty)$. We mention the generalized fuzzy soft expert sets of Hazaymeh et al. [47], the neutrosophic double controlled metric spaces of Uddin, Ishtiaq, Hussain, Javed, Al Sulami and Ahmed [48], the quasi contractions of Bataihah and Hazaymeh [49], the nonlinear contractions of Hazaymeh and Bataihah [50], the neutrosophic MR-metric spaces of Malkawi and Rabaiah [51] and of Qawasmeh and Malkawi [52], the Geraghty type results with equivalent distance of Bataihah [53], and the common fixed point results of Bataihah et al. [54], and NF-L contractions [55].

Against this background, the following question arises naturally and, to the best of our knowledge, has not yet been answered: *can an integral type contractive condition be imposed on an ω_t -distance defined*

over a b -metric space, with the integrand governed by an altering distance function and the resulting integral compressed by a nonlinear comparison function, in such a way that existence and uniqueness of the fixed point are still guaranteed? The present paper answers this question in the affirmative. We begin by introducing the class Γ of continuous non-decreasing comparison mappings with pointwise vanishing iterates, and we record its two elementary but essential properties, namely $\gamma(a) < a$ for every $a > 0$ and $\gamma(0) = 0$ (Remark 3.1). Building on this class we define the notion of an (ω_t, Θ) integral contraction, in which the altering distance function θ acts on the upper limit of integration while the comparison function γ acts on the value of the integral; both the class of Branciari [9] and the Ω -distance integral contractions of [35] are recovered as particular cases. Our main result, Theorem 3.1, asserts that a self-mapping of a complete b -metric space endowed with an ω_t -distance and satisfying this condition has exactly one fixed point, under either the continuity of f or a Kada type infimum condition, the latter alternative being what makes the result applicable to discontinuous operators. Specializing the comparison function then yields two readily verifiable corollaries, and we exhibit an operator of Volterra type which satisfies our hypotheses. Finally, we apply the main theorem to prove the existence and uniqueness of the solution of a Volterra integral equation on $C([0, 1], \mathbb{R})$, thereby demonstrating that the abstract framework is not vacuous.

2. PRELIMINARIES

Bakhtin [11] established the idea of a b -metric space as a refinement of the classical metric space and employed this notion to obtain several significant fixed point theorems. The concept is defined as follows.

Definition 2.1 ([11]). Let $\mathcal{T}$ be a nonempty set, let $s \geq 1$ be a constant and let $d_b : \mathcal{T} \times \mathcal{T} \rightarrow [0, \infty)$ be a function. Then d_b is called a b -metric with base s if it fulfills the following properties for all $w_1, w_2, w_3 \in \mathcal{T}$:

- (d_b1) $d_b(w_1, w_2) = 0 \iff w_1 = w_2$;
- (d_b2) $d_b(w_1, w_2) = d_b(w_2, w_1)$;
- (d_b3) $d_b(w_1, w_2) \leq s[d_b(w_1, w_3) + d_b(w_3, w_2)]$.

In this case the triple $(\mathcal{T}, d_b, s)$ is called a b -metric space.

From this point on, $\mathcal{T}$ denotes a nonempty set, $(\mathcal{T}, d_b, s)$ denotes a b -metric space, $f : \mathcal{T} \rightarrow \mathcal{T}$ denotes a self-mapping and

$$\mathcal{T}_f = \{w \in \mathcal{T} : fw = w\}$$

denotes the set of all fixed points of f .

Definition 2.2. Let $\{w_n\}$ be a sequence in $(\mathcal{T}, d_b, s)$. Then:

- (1) $\{w_n\}$ is a b -Cauchy sequence if $\lim_{n,m \rightarrow \infty} d_b(w_n, w_m) = 0$;
- (2) $\{w_n\}$ is b -convergent to $u \in \mathcal{T}$ if $\lim_{n \rightarrow \infty} d_b(w_n, u) = 0$; in this case we write $w_n \rightarrow u$;
- (3) $(\mathcal{T}, d_b, s)$ is b -complete if every b -Cauchy sequence in $\mathcal{T}$ is b -convergent to some element of $\mathcal{T}$.

It is well known that limits of b -convergent sequences are unique, even though d_b need not be a continuous function of its arguments.

The concept of an ω_t -distance mapping was introduced by Hussain et al. [29], who used it to obtain several significant fixed point theorems. Recall that a function $g : \mathcal{T} \rightarrow [0, \infty)$ is s -lower semicontinuous at $u \in \mathcal{T}$ if $g(u) \leq s \liminf_{n \rightarrow \infty} g(w_n)$ whenever $w_n \rightarrow u$.

Definition 2.3 ([29]). A function $t : \mathcal{T} \times \mathcal{T} \rightarrow [0, \infty)$ on $(\mathcal{T}, d_b, s)$ is called an ω_t -distance mapping with base $s \geq 1$ if t fulfills the following properties:

- (ω_t1) $t(w_1, w_2) \leq s[t(w_1, w_3) + t(w_3, w_2)]$ for all $w_1, w_2, w_3 \in \mathcal{T}$;
- (ω_t2) for every $w \in \mathcal{T}$ the function $t(w, \cdot) : \mathcal{T} \rightarrow [0, \infty)$ is s -lower semicontinuous;
- (ω_t3) for every $\epsilon > 0$ there exists $\alpha > 0$ such that, for all $w_1, w_2, w_3 \in \mathcal{T}$,

$$t(w_1, w_2) < \alpha \text{ and } t(w_1, w_3) < \alpha \implies d_b(w_2, w_3) < \epsilon.$$

The following lemma is the bridge between convergence with respect to t and convergence with respect to d_b . It is exactly the point at which axiom (ω_t3) is used, and it will be applied in the proof of Theorem 3.1.

Lemma 2.1 ([29]). Let t be an ω_t -distance on $(\mathcal{T}, d_b, s)$ and let $\{w_n\}$ be a sequence in $\mathcal{T}$ with $\lim_{n, m \rightarrow +\infty} t(w_n, w_m) = 0$. Then $\{w_n\}$ is a b -Cauchy sequence.

Proof. Let $\epsilon > 0$ and let $\alpha > 0$ be the constant provided by (ω_t3) for this ϵ . By hypothesis there is $N \in \mathbb{N}$ with $t(w_n, w_m) < \alpha$ for all $n, m \geq N$. Fix $p, q \geq N$. Applying (ω_t3) to $t(w_N, w_p) < \alpha$ and $t(w_N, w_q) < \alpha$ yields $d_b(w_p, w_q) < \epsilon$. Since $\epsilon > 0$ was arbitrary, $\{w_n\}$ is b -Cauchy. $\square$

Definition 2.4 ([4]). A function $\theta : [0, \infty) \rightarrow [0, \infty)$ is said to be an altering distance function if

- (1) θ is non-decreasing and continuous;
- (2) $\theta(a) = 0 \iff a = 0$.

From now on, Θ denotes the set of all altering distance functions. We shall repeatedly use the following elementary consequence of Definition 2.4, which we record once and for all.

Remark 2.1. Let $\theta \in \Theta$ and let $a, b \geq 0$.

- (1) If $\theta(a) < \theta(b)$ then $a < b$. Indeed, $a \geq b$ would give $\theta(a) \geq \theta(b)$.
- (2) If $\{a_n\} \subset [0, \infty)$ and $\theta(a_n) \rightarrow 0$ then $a_n \rightarrow 0$. Indeed, if $a_n \geq c > 0$ for infinitely many n , then $\theta(a_n) \geq \theta(c) > 0$ for those n .

3. FIXED POINT RESULTS FOR (ω_t, Θ) INTEGRAL CONTRACTIONS

Before we introduce our main results, it is essential to introduce the following two classes of functions.

Definition 3.1. (see [10]) Let

$$\Phi = \left\{ \phi : [0, +\infty) \rightarrow [0, +\infty) : \phi \text{ is Lebesgue integrable and summable on every} \right.$$

compact subset of $[0, +\infty)$, and

$$0 < \int_0^\alpha \phi(w) dw < +\infty \text{ for every } \alpha > 0\},$$

$$\Gamma = \left\{ \gamma : [0, +\infty) \rightarrow [0, +\infty) : \gamma \text{ is continuous and non-decreasing, and} \right. \\ \left. \lim_{n \rightarrow +\infty} \gamma^n(a) = 0 \text{ for every } a > 0 \right\},$$

where γ^n denotes the n -th iterate of γ .

Remark 3.1. If $\gamma \in \Gamma$, then:

- (1) $\gamma(0) = 0$;
- (2) $\gamma(a) < a$ for every $a > 0$.

Proof. (1) Fix any $a > 0$. Since γ is non-decreasing and $\gamma^n(a) \geq 0$, we have $0 \leq \gamma(0) \leq \gamma(\gamma^n(a)) = \gamma^{n+1}(a)$ for every $n \in \mathbb{N}$. Letting $n \rightarrow +\infty$ gives $\gamma(0) = 0$.
 (2) Assume to the contrary that, $\gamma(a_0) \geq a_0$ for some $a_0 > 0$. Since γ is non-decreasing, an easy induction gives $\gamma^{n+1}(a_0) = \gamma(\gamma^n(a_0)) \geq \gamma^n(a_0) \geq a_0$ for every n , so that $\gamma^n(a_0) \geq a_0 > 0$ for all n . This contradicts $\gamma^n(a_0) \rightarrow 0$. □

The following two lemmas are of particular importance and will be instrumental in the sequel.

Lemma 3.1 ([10]). Let $\phi \in \Phi$ and let $\{a_n\}$ be a sequence in $[0, +\infty)$ with $a_n \rightarrow a$. Then

$$\lim_{n \rightarrow +\infty} \int_0^{a_n} \phi(w) dw = \int_0^a \phi(w) dw. \quad (3.1)$$

Lemma 3.2 ([35]). Let $\phi \in \Phi$ and put $\Lambda(c) = \int_0^c \phi(w) dw$ for $c \geq 0$. Then Λ is strictly increasing on $[0, +\infty)$, $\Lambda(0) = 0$ and $\Lambda(c) > 0$ for every $c > 0$. Consequently, for $c_1, c_2 \geq 0$,

$$\Lambda(c_1) < \Lambda(c_2) \iff c_1 < c_2, \quad \Lambda(c_1) = 0 \iff c_1 = 0.$$

Proof. For $0 \leq c_1 < c_2$ we have $\Lambda(c_2) - \Lambda(c_1) = \int_{c_1}^{c_2} \phi(w) dw > 0$, because $\phi \in \Phi$ has a strictly positive integral over every nondegenerate interval. Hence Λ is strictly increasing, and the stated equivalences follow at once. □

Throughout the remainder of the paper we keep the notation $\Lambda(c) = \int_0^c \phi(w) dw$. We now present the concept of an (ω_t, Θ) integral contraction.

Definition 3.2. Let t be an ω_t -distance with base s on $(\mathcal{T}, d_b, s)$. A self-mapping f of $\mathcal{T}$ is called an (ω_t, Θ) integral contraction if there exist $\phi \in \Phi$, $\gamma \in \Gamma$ and $\theta \in \Theta$ such that

$$\int_0^{\theta(st(fw_1, fw_2))} \phi(w) dw \leq \gamma \left(\int_0^{\theta(t(w_1, w_2))} \phi(w) dw \right) \quad \text{for all } w_1, w_2 \in \mathcal{T}. \quad (3.2)$$

Remark 3.2. The composition in (3.2) is $\theta(s t(fw_1, fw_2))$, that is, the altering distance function is evaluated at the number $s t(fw_1, fw_2)$. Taking $\theta = \text{id}$, $s = 1$ and $\gamma(a) = \eta a$ reduces (3.2) to the integral contraction of Branciari [9].

Lemma 3.3. Let $f : \mathcal{T} \rightarrow \mathcal{T}$ be an (ω_t, Θ) integral contraction.

Then for all $w_1, w_2 \in \mathcal{T}$,

$$t(fw_1, fw_2) \leq \frac{1}{s} t(w_1, w_2), \quad (3.3)$$

and the inequality is strict whenever $t(w_1, w_2) > 0$.

Proof. Fix $w_1, w_2 \in \mathcal{T}$ and abbreviate $A = \theta(s t(fw_1, fw_2))$ and $B = \theta(t(w_1, w_2))$, so that (3.2) reads $\Lambda(A) \leq \gamma(\Lambda(B))$.

Case 1: $t(w_1, w_2) = 0$. Then $B = \theta(0) = 0$ and $\Lambda(B) = 0$, so Remark 3.1(1) gives $\Lambda(A) \leq \gamma(0) = 0$. By Lemma 3.2, $A = 0$, hence $\theta(s t(fw_1, fw_2)) = 0$ and therefore $t(fw_1, fw_2) = 0 = \frac{1}{s} t(w_1, w_2)$, so (3.3) holds with equality.

Case 2: $t(w_1, w_2) > 0$. Then $B > 0$ and, by Lemma 3.2, $\Lambda(B) > 0$. Remark 3.1(2) therefore gives

$$\Lambda(A) \leq \gamma(\Lambda(B)) < \Lambda(B).$$

By Lemma 3.2 this forces $A < B$, i.e. $\theta(s t(fw_1, fw_2)) < \theta(t(w_1, w_2))$, and Remark 2.1(1) yields $s t(fw_1, fw_2) < t(w_1, w_2)$, which is the strict form of (3.3). $\square$

Lemma 3.4. If $f : \mathcal{T} \rightarrow \mathcal{T}$ is an (ω_t, Θ) integral contraction, then $\mathcal{T}_f$ contains at most one element. Moreover $t(w, w) = 0$ for every $w \in \mathcal{T}_f$.

Proof. We first show that $t(w, w) = 0$ whenever $w \in \mathcal{T}_f$. Suppose, on the contrary, that $t(w, w) > 0$ for some $w \in \mathcal{T}_f$. Since $fw = w$, Lemma 3.3 applied to the pair (w, w) gives the strict inequality

$$t(w, w) = t(fw, fw) < \frac{1}{s} t(w, w) \leq t(w, w),$$

where the last step uses $s \geq 1$. This is a contradiction, and therefore $t(w, w) = 0$ for all $w \in \mathcal{T}_f$.

Now let $w_1, w_2 \in \mathcal{T}_f$. If $t(w_1, w_2) > 0$, then Lemma 3.3 applied to the pair (w_1, w_2) gives

$$t(w_1, w_2) = t(fw_1, fw_2) < \frac{1}{s} t(w_1, w_2) \leq t(w_1, w_2),$$

again a contradiction. Hence $t(w_1, w_2) = 0$. Since also $t(w_1, w_1) = 0$, condition $(\omega_t 3)$ applied with first argument w_1 shows that $d_b(w_1, w_2) < \epsilon$ for every $\epsilon > 0$, so that $d_b(w_1, w_2) = 0$ and therefore $w_1 = w_2$. $\square$

Theorem 3.1. Let t be an ω_t -distance with base s on a complete b -metric space $(\mathcal{T}, d_b, s)$ and let $f : \mathcal{T} \rightarrow \mathcal{T}$ be an (ω_t, Θ) integral contraction. Assume that at least one of the following conditions is satisfied:

- (1) f is continuous;
- (2) for every $u \in \mathcal{T}$ with $fu \neq u$,

$$0 < \inf \{ t(w, u) + t(fw, u) : w \in \mathcal{T} \}.$$

Then $\mathcal{T}_f$ is a singleton.

Proof. Let $w_0 \in \mathcal{T}$ be arbitrary and define the Picard sequence $\{w_n\}$ recursively by $w_{n+1} = fw_n$, $n \in \mathbb{N} \cup \{0\}$.

Step 1 (a degenerate case). Assume first that $t(w_r, w_{r+1}) = 0$ for some r . Putting $w_1 = w_r$ and $w_2 = w_{r+1}$ in (3.2) and using

$\theta(0) = 0$, $\Lambda(0) = 0$ and $\gamma(0) = 0$ (Remark 3.1),

$$\begin{aligned} \int_0^{\theta(st(w_{r+1}, w_{r+2}))} \phi(w) dw &= \int_0^{\theta(st(fw_r, fw_{r+1}))} \phi(w) dw \\ &\leq \gamma \left(\int_0^{\theta(t(w_r, w_{r+1}))} \phi(w) dw \right). \end{aligned}$$

By Lemma 3.2, $\theta(st(w_{r+1}, w_{r+2})) = 0$, hence

$$t(w_{r+1}, w_{r+2}) = 0. \quad (3.4)$$

By $(\omega_t 1)$ and $t(w_r, w_{r+1}) = 0$,

$$t(w_r, w_{r+2}) \leq s[t(w_r, w_{r+1}) + t(w_{r+1}, w_{r+2})] = 0. \quad (3.5)$$

Now $t(w_r, w_{r+1}) = 0$ and $t(w_r, w_{r+2}) = 0$ have the same first argument w_r , so $(\omega_t 3)$ gives $d_b(w_{r+1}, w_{r+2}) < \epsilon$ for every $\epsilon > 0$, whence $w_{r+1} = w_{r+2} = fw_{r+1}$. Thus $w_{r+1} \in \mathcal{T}_f$ and, by Lemma 3.4, $\mathcal{T}_f = \{w_{r+1}\}$ and we are done.

Step 2 (the sequence $\{w_n\}$ is t -asymptotically regular). From now on we may therefore assume that

$$t(w_n, w_{n+1}) > 0 \quad \text{for every } n \in \mathbb{N} \cup \{0\}. \quad (3.6)$$

Since $s \geq 1$ and θ is non-decreasing, for every n

$$\Lambda(\theta(t(w_n, w_{n+1}))) \leq \Lambda(\theta(st(w_n, w_{n+1}))). \quad (3.7)$$

Using (3.2), then (3.7) together with the fact that γ is non-decreasing, and iterating,

$$\begin{aligned} \Lambda(\theta(st(w_n, w_{n+1}))) &\leq \gamma \left(\Lambda(\theta(t(w_{n-1}, w_n))) \right) \\ &\leq \gamma^2 \left(\Lambda(\theta(t(w_{n-2}, w_{n-1}))) \right) \\ &\vdots \\ &\leq \gamma^n \left(\Lambda(\theta(t(w_0, w_1))) \right). \end{aligned}$$

By (3.6) and Lemma 3.2, $\Lambda(\theta(t(w_0, w_1))) > 0$, so the right-hand side tends to 0 as $n \rightarrow +\infty$ and hence

$$\lim_{n \rightarrow +\infty} \Lambda(\theta(st(w_n, w_{n+1}))) = 0. \quad (3.8)$$

We claim that $t(w_n, w_{n+1}) \rightarrow 0$. If not, there are $c > 0$ and a subsequence with $t(w_{n_j}, w_{n_j+1}) \geq c$; since Λ and θ are non-decreasing, $\Lambda(\theta(s t(w_{n_j}, w_{n_j+1}))) \geq \Lambda(\theta(c)) > 0$, contradicting (3.8). Therefore

$$\lim_{n \rightarrow +\infty} t(w_n, w_{n+1}) = 0. \quad (3.9)$$

Applying (3.2) to the reversed pairs (w_n, w_{n-1}) and repeating the same argument starting from $\Lambda(\theta(t(w_1, w_0)))$, we obtain in the same way

$$\lim_{n \rightarrow +\infty} t(w_{n+1}, w_n) = 0. \quad (3.10)$$

Step 3 ($\{w_n\}$ is b -Cauchy). We prove that

$$\lim_{n, m \rightarrow \infty} t(w_n, w_m) = 0. \quad (3.11)$$

Suppose (3.11) fails. Then there exist $\epsilon > 0$ and integers n_k, m_k with $k < n_k < m_k$ and

$$t(w_{n_k}, w_{m_k}) \geq \epsilon. \quad (3.12)$$

For each k we choose m_k to be the *smallest* index greater than n_k for which (3.12) holds. By (3.9) we have $t(w_{n_k}, w_{n_k+1}) < \epsilon$ for all large k , so $m_k > n_k + 1$ for all large k , and the minimality of m_k gives

$$t(w_{n_k}, w_{m_k-1}) < \epsilon. \quad (3.13)$$

Discarding finitely many terms we may assume $n_k \geq 2$ and $m_k > n_k + 1$ for every k . Put

$$\alpha_k = t(w_{n_k-1}, w_{m_k}).$$

Using (3.12), Lemma 3.3 (applied to the pair (w_{n_k-1}, w_{m_k-1})) and $(\omega_t 1)$ with the intermediate point w_{m_k} ,

$$\epsilon \leq t(w_{n_k}, w_{m_k}) = t(fw_{n_k-1}, fw_{m_k-1}) \leq \frac{1}{s} t(w_{n_k-1}, w_{m_k-1}) \leq \alpha_k + t(w_{m_k}, w_{m_k-1}).$$

Passing to the limit inferior as $k \rightarrow +\infty$ and using (3.10) we conclude

$$\liminf_{k \rightarrow +\infty} \alpha_k \geq \epsilon. \quad (3.14)$$

On the other hand, applying Lemma 3.3 to the pair (w_{n_k-2}, w_{m_k-1}) and then $(\omega_t 1)$ twice,

$$\begin{aligned} \alpha_k &= t(fw_{n_k-2}, fw_{m_k-1}) \leq \frac{1}{s} t(w_{n_k-2}, w_{m_k-1}) \leq t(w_{n_k-2}, w_{n_k}) + t(w_{n_k}, w_{m_k-1}) \\ &\leq s[t(w_{n_k-2}, w_{n_k-1}) + t(w_{n_k-1}, w_{n_k})] + t(w_{n_k}, w_{m_k-1}). \end{aligned}$$

Passing to the limit superior as $k \rightarrow +\infty$, using (3.9) for the bracketed terms and (3.13) for the last term, we obtain

$$\limsup_{k \rightarrow +\infty} \alpha_k \leq \epsilon. \quad (3.15)$$

Combining (3.14) and (3.15),

$$\lim_{k \rightarrow +\infty} \alpha_k = \epsilon. \quad (3.16)$$

Next put $\beta_k = t(w_{n_k}, w_{m_k+1})$. Lemma 3.3 applied to the pair (w_{n_k-1}, w_{m_k}) gives $\beta_k = t(fw_{n_k-1}, fw_{m_k}) \leq \frac{1}{s}\alpha_k$, hence by (3.16)

$$\limsup_{k \rightarrow +\infty} \beta_k \leq \frac{\epsilon}{s}. \quad (3.17)$$

Furthermore, by (3.12) and $(\omega_t 1)$,

$$\epsilon \leq t(w_{n_k}, w_{m_k}) \leq s[\beta_k + t(w_{m_k+1}, w_{m_k})], \quad (3.18)$$

and taking the limit inferior and using (3.10) yields

$$\liminf_{k \rightarrow +\infty} \beta_k \geq \frac{\epsilon}{s}. \quad (3.19)$$

By (3.17) and (3.19),

$$\lim_{k \rightarrow +\infty} \beta_k = \frac{\epsilon}{s}. \quad (3.20)$$

Finally, setting $w_1 = w_{n_k-1}$ and $w_2 = w_{m_k}$ in (3.2) gives

$$\int_0^{\theta(s\beta_k)} \phi(w) dw = \int_0^{\theta(st(fw_{n_k-1}, fw_{m_k}))} \phi(w) dw \leq \gamma \left(\int_0^{\theta(\alpha_k)} \phi(w) dw \right).$$

Letting $k \rightarrow +\infty$ and using (3.16), (3.20), the continuity of θ and of γ , and Lemma 3.1, we obtain, since $s \cdot \frac{\epsilon}{s} = \epsilon$,

$$\Lambda(\theta(\epsilon)) \leq \gamma(\Lambda(\theta(\epsilon))) < \Lambda(\theta(\epsilon)), \quad (3.21)$$

where the strict inequality follows from Remark 3.1(2) because $\theta(\epsilon) > 0$ and hence $\Lambda(\theta(\epsilon)) > 0$ by Lemma 3.2. This is a contradiction, and (3.11) is proved.

By Lemma 2.1, $\{w_n\}$ is a b -Cauchy sequence. Since $(\mathcal{T}, d_b, s)$ is complete, there exists $u \in \mathcal{T}$ with $w_n \rightarrow u$.

Step 4 (u is a fixed point). Assume first that condition (1) holds. Then f is continuous, so $w_{n+1} = fw_n \rightarrow fu$; since also $w_{n+1} \rightarrow u$ and limits are unique in a b -metric space, $fu = u$.

Assume now that condition (2) holds. Let $\epsilon > 0$. By (3.11) there is $N \in \mathbb{N}$ such that $t(w_n, w_m) \leq \epsilon$ for all $n, m \geq N$. Since $w_m \rightarrow u$ and $t(w_n, \cdot)$ is s -lower semicontinuous by $(\omega_t 2)$,

$$t(w_n, u) \leq s \liminf_{m \rightarrow \infty} t(w_n, w_m) \leq s\epsilon \quad \text{for all } n \geq N.$$

If $fu \neq u$, then condition (2) together with $fw_n = w_{n+1}$ gives

$$0 < \inf \{t(w, u) + t(fw, u) : w \in \mathcal{T}\} \leq \inf_{n \geq N} \{t(w_n, u) + t(w_{n+1}, u)\} \leq 2s\epsilon.$$

As $\epsilon > 0$ was arbitrary, the infimum on the left is 0, a contradiction. Therefore $fu = u$.

In either case $\mathcal{T}_f \neq \emptyset$, and Lemma 3.4 shows that $\mathcal{T}_f$ is a singleton. $\square$

Taking $\gamma : [0, +\infty) \rightarrow [0, +\infty)$ given by $\gamma(a) = \eta a$ with $\eta \in [0, 1)$ — which clearly belongs to Γ — we obtain the following corollary.

Corollary 3.1. Let t be an ω_t -distance with base s on a complete b -metric space $(\mathcal{T}, d_b, s)$ and let $f : \mathcal{T} \rightarrow \mathcal{T}$ be a self-mapping. Suppose that there exist $\phi \in \Phi$, $\theta \in \Theta$ and $\eta \in [0, 1)$ such that

$$\int_0^{\theta(st(fw_1, fw_2))} \phi(w) dw \leq \eta \int_0^{\theta(t(w_1, w_2))} \phi(w) dw \quad \text{for all } w_1, w_2 \in \mathcal{T}. \quad (3.22)$$

Assume moreover that at least one of the following holds:

- (1) f is continuous;
- (2) for every $u \in \mathcal{T}$ with $fu \neq u$,

$$0 < \inf \{ t(w, u) + t(fw, u) : w \in \mathcal{T} \}.$$

Then $\mathcal{T}_f$ is a singleton.

Taking $\gamma : [0, +\infty) \rightarrow [0, +\infty)$ given by $\gamma(a) = (1+a)^r - 1$ with $r \in (0, 1)$ we obtain a second corollary. Note that this γ does belong to Γ : it is continuous and non-decreasing, $\gamma(0) = 0$, and $\gamma(a) < a$ for $a > 0$ because $(1+a)^r < 1+a$; hence $\{\gamma^n(a)\}$ is decreasing and its limit $\ell \geq 0$ satisfies $\gamma(\ell) = \ell$, which forces $\ell = 0$.

Corollary 3.2. Let t be an ω_t -distance with base s on a complete b -metric space $(\mathcal{T}, d_b, s)$ and let $f : \mathcal{T} \rightarrow \mathcal{T}$ be a self-mapping. Suppose that there exist $\phi \in \Phi$, $\theta \in \Theta$ and $r \in (0, 1)$ such that

$$\int_0^{\theta(st(fw_1, fw_2))} \phi(w) dw \leq \left(1 + \int_0^{\theta(t(w_1, w_2))} \phi(w) dw \right)^r - 1 \quad \text{for all } w_1, w_2 \in \mathcal{T}. \quad (3.23)$$

Assume moreover that at least one of the following holds:

- (1) f is continuous;
- (2) for every $u \in \mathcal{T}$ with $fu \neq u$,

$$0 < \inf \{ t(w, u) + t(fw, u) : w \in \mathcal{T} \}.$$

Then $\mathcal{T}_f$ is a singleton.

Example 3.1. Let $I = [0, 1]$, let $C(I, \mathbb{R})$ carry the supremum norm $\|\cdot\|_\infty$ and put

$$\mathcal{T} = \{u \in C(I, \mathbb{R}) : \|u\|_\infty \leq 1\}.$$

Fix $m, M \in \mathbb{N}$ with $2m^2 < M^2$ (so that $M \geq 2$) and define $f : \mathcal{T} \rightarrow \mathcal{T}$ by

$$(fu)(x) = \frac{1}{M} \int_0^x (u(w))^m dw, \quad x \in I. \quad (3.24)$$

Then $\mathcal{T}_f$ is a singleton.

Proof. $\mathcal{T}$ is a closed subset of the Banach space $C(I, \mathbb{R})$, hence complete. Moreover f maps $\mathcal{T}$ into $\mathcal{T}$: for $u \in \mathcal{T}$,

$$\|fu\|_\infty \leq \frac{1}{M} \sup_{x \in I} \int_0^x |u(w)|^m dw \leq \frac{1}{M} \leq 1.$$

Define

$$d_b(u, v) = \|u - v\|_\infty^2, \quad u, v \in \mathcal{T}.$$

Since $(a+b)^2 \leq 2(a^2 + b^2)$, d_b is a b -metric with base $s = 2$, and $(\mathcal{T}, d_b, 2)$ is complete. Put $t = d_b$; being a b -metric, t satisfies $(\omega_t 1)$ with $s = 2$, it is s -lower semi-continuous in its second variable, and $(\omega_t 3)$ holds with $\alpha = \epsilon/4$ by $(\omega_t 1)$. Hence t is an ω_t -distance with base 2.

Choose $\theta(a) = a$, $\phi \equiv 1$ (so $\Lambda(c) = c$) and

$$\gamma(a) = \frac{2m^2}{M^2} a,$$

which belongs to Γ because $0 < \frac{2m^2}{M^2} < 1$.

Let $u, v \in \mathcal{T}$. Since $|u|, |v| \leq 1$ on I , the mean value theorem gives for any fixed w $|u^m(w) - v^m(w)| \leq m|u(w) - v(w)|$ point-wise. Hence for every $x \in I$,

$$\begin{aligned} |(fu)(x) - (fv)(x)| &= \frac{1}{M} \left| \int_0^x (u^m(w) - v^m(w)) dw \right| \leq \frac{1}{M} \int_0^x |u^m(w) - v^m(w)| dw \\ &\leq \frac{m}{M} \int_0^x |u(w) - v(w)| dw \leq \frac{m}{M} x \|u - v\|_\infty \leq \frac{m}{M} \|u - v\|_\infty, \end{aligned}$$

because $x \leq 1$. Consequently

$$\|fu - fv\|_\infty^2 \leq \frac{m^2}{M^2} \|u - v\|_\infty^2, \text{ that is } 2t(fu, fv) \leq \frac{2m^2}{M^2} t(u, v).$$

Therefore

$$\int_0^{\theta(2t(fu, fv))} \phi(w) dw = 2t(fu, fv) \leq \frac{2m^2}{M^2} t(u, v) = \gamma \left(\int_0^{\theta(t(u, v))} \phi(w) dw \right),$$

so f is an (ω_t, Θ) integral contraction with base $s = 2$. Finally, f is continuous, since the estimate above shows that it is Lipschitz. Hence condition (1) of Theorem 3.1 holds and Theorem 3.1 establishes that $\mathcal{T}_f$ is a singleton. $\square$

4. APPLICATION

In this section Theorem 3.1 is employed to establish the existence and uniqueness of the solution of a class of Volterra integral equations. Let $I = [0, 1]$, let $\mathcal{T} = C(I, \mathbb{R})$ and consider the Volterra integral equation

$$u(x) = T(x) + \lambda \int_0^x K(x, w, u(w)) dw, \quad x \in I, \quad (4.1)$$

together with the associated operator $f : \mathcal{T} \rightarrow \mathcal{T}$,

$$(fu)(x) = T(x) + \lambda \int_0^x K(x, w, u(w)) dw.$$

Suppose that:

- (1) $T \in C(I, \mathbb{R})$ and $K : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous;
- (2) there is $L > 0$ with $|K(x, w, a) - K(x, w, b)| \leq L|a - b|$ for all $x, w \in I$ and $a, b \in \mathbb{R}$;
- (3) $0 < \lambda < 1$ and $2\lambda L^2 \leq 1$.

Theorem 4.1. *Under assumptions (1)–(3), the Volterra integral equation (4.1) has exactly one solution in $C(I, \mathbb{R})$.*

Proof. By (1) the operator f maps $\mathcal{T}$ into $\mathcal{T}$. Equip $\mathcal{T}$ with

$$d_b(u, v) = t(u, v) = \|u - v\|_\infty^2,$$

which, as in Example 3.1, is a complete b -metric with base $s = 2$ and an ω_t -distance with base 2.

For all $u, v \in \mathcal{T}$ and $x \in I$, assumption (2) gives

$$\begin{aligned} |(fu)(x) - (fv)(x)| &\leq \lambda \int_0^x |K(x, w, u(w)) - K(x, w, v(w))| dw \leq \lambda L \int_0^x |u(w) - v(w)| dw \\ &\leq \lambda x L \|u - v\|_\infty \leq \lambda L \|u - v\|_\infty, \end{aligned}$$

because $x \leq 1$. Hence

$$\|fu - fv\|_\infty^2 \leq \lambda^2 L^2 \|u - v\|_\infty^2,$$

and therefore, using $2\lambda L^2 \leq 1$,

$$2t(fu, fv) \leq 2\lambda^2 L^2 t(u, v) = \lambda(2\lambda L^2)t(u, v) \leq \lambda t(u, v).$$

Choosing $\phi \equiv 1$, $\theta(a) = a$ and $\gamma(a) = \lambda a \in \Gamma$ we obtain

$$\int_0^{\theta(2t(fu, fv))} \phi(w) dw \leq \gamma \left(\int_0^{\theta(t(u, v))} \phi(w) dw \right),$$

so f is an (ω_t, Θ) integral contraction with base 2. The estimate above also shows that f is Lipschitz, hence continuous, so condition (1) of Theorem 3.1 is satisfied. Theorem 3.1 guarantees that $\mathcal{T}_f$ is a singleton, that is, (4.1) has exactly one solution in $C(I, \mathbb{R})$. $\square$

5. CONCLUSION

In this paper we have introduced the notion of an (ω_t, Θ) integral contraction, a contractive condition in which an ω_t -distance t defined over a b -metric space $(\mathcal{T}, d_b, s)$ is processed successively by an altering distance function $\theta \in \Theta$, by an integral with respect to a density $\phi \in \Phi$, and finally by a nonlinear comparison function $\gamma \in \Gamma$. We proved that every self-mapping of a complete b -metric space obeying this condition admits one and only one fixed point, provided that either the mapping is continuous or the Kada type infimum condition of Theorem 3.1(2) is fulfilled. Two corollaries, obtained by choosing γ linear and γ of power type, were derived, an example was given, and the theory was applied to a class of Volterra integral equations, for which existence and uniqueness of a continuous solution were established.

Several features of the result deserve emphasis. First, the ω_t -distance t is neither assumed symmetric nor assumed to vanish on the diagonal, so that the contractive inequality is genuinely weaker than its d_b -counterpart; in particular, the fixed point is obtained without any continuity assumption on the b -metric itself. Second, the presence of the altering distance function θ inside the limit of integration decouples the growth of the integrand from the growth of the distance, which is what allows operators failing the classical integral condition of [9] to be covered. Third, the alternative hypothesis (2) of Theorem 3.1 makes the theorem applicable to discontinuous mappings, a situation in which the Picard–Banach argument is unavailable.

The results of this paper suggest several directions for further research, which we state as open problems.

- (P1) Extend the notion of an (ω_t, Θ) integral contraction to pairs of mappings and obtain common fixed point and coincidence point theorems, in the spirit of [22, 25, 54].
- (P2) Replace the single-valued mapping f by a multivalued operator with closed and bounded values and establish the corresponding integral type theorem, as was done in the b -gauge setting in [18].
- (P3) Study the analogous condition in the frame of G_b -metric and extended b -metric spaces [17, 33, 34], and in the Γ -distance and Γ_k -distance settings [37–39], and determine whether the constant $1/s$ appearing in Lemma 3.3 is optimal.
- (P4) Determine whether the monotonicity assumption on $\gamma \in \Gamma$ (Definition 3.1) can be dispensed with, or replaced by an upper semicontinuity requirement.
- (P5) Develop a neutrosophic and fuzzy counterpart of the present theory, along the lines of [36, 49–52], where the integral has to be interpreted with respect to a membership-valued measure, and exploit the theorem to treat fractional boundary value problems as in [39, 41].

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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