

Application of a New Class of Enriched Nonexpansive Mappings to a Class of Differential Equations

Sri Bharathi R¹, Ashis Bera¹, Hasanen A. Hammad^{2,3}, Manuel De la Sen^{4,*}

¹Department of Mathematics, Vellore Institute of Technology Chennai, India

²Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

³Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

⁴Institute of Research and Development of Processes, Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country 48940, Leioa (Bizkaia), Spain

*Corresponding author: manuel.delasen@ehu.eus

Abstract. This paper presents the concept of enriched Chatterjea-Suzuki-(C) type nonexpansive mappings in real Banach spaces. For this class, strong and weak convergence theorems are derived using a modified ECM-iterative scheme. To illustrate the theoretical findings, a numerical example is presented, with convergence comparisons against existing schemes supported by graphical and tabular evidence. Furthermore, the practical relevance of the results is demonstrated by applying them to a delay differential equation and a nonlinear fractional integro-differential equation, where the existence and uniqueness of solutions are also addressed.

1. INTRODUCTION

Fixed point theory is a core mathematical concept with broad applications in various fields of sciences and technologies. Over recent decades, mathematicians have focused on establishing the existence and approximation of fixed points for various classes of nonexpansive mappings, following the foundational works of Browder [9], Göhde [20] and Kirk [29]. Many eminent mathematicians came up with novel notions and enriched this discipline with massive contributions (see, [1, 21, 23, 48, 49] and references therein). Now, we recall the following definitions.

Definition 1.1. Suppose Z is a Banach space with $\emptyset \neq S \subset Z$. For all $c, d, e \in S$, a mapping $\Omega : S \rightarrow S$ is defined as

Received: Aug. 21, 2026.

2020 Mathematics Subject Classification. 47H10; 54A20.

Key words and phrases. fixed point; enriched Chatterjea-Suzuki-(C) type nonexpansive; weak convergence; strong convergence; Banach space; delay differential equation; fractional integro-differential equation.

(a) *nonexpansive if*

$$\|\Omega c - \Omega d\| \leq \|c - d\|.$$

(b) *quasinonexpansive if $F(\Omega)$ is nonempty and the inequality*

$$\|\Omega c - e\| \leq \|c - e\|,$$

holds for $e \in F(\Omega)$.

In 1972, Chatterjea [17] considered mappings $\Omega : S \rightarrow S$ satisfying

$$\|\Omega c - \Omega d\| \leq k[\|c - \Omega d\| + \|d - \Omega c\|] \quad \text{for all } c, d \in S,$$

for some $k \in [0, \frac{1}{2})$, a condition that is independent of the Banach contraction condition and does not require continuity. In 2008, Suzuki [42] introduced a class of mappings wider than the nonexpansive ones, defined by condition (C):

$$\frac{1}{2}\|c - \Omega c\| \leq \|c - d\| \quad \text{implies} \quad \|\Omega c - \Omega d\| \leq \|c - d\|,$$

and obtained existence and convergence results for it. Combining these two ideas, Karapinar and Taş [31] introduced the Chatterjea-Suzuki-(C) condition, in which the right-hand side of condition (C) is replaced by $\frac{1}{2}[\|\Omega c - d\| + \|c - \Omega d\|]$. Fixed point results for these and related classes of nonexpansive type mappings have since been obtained by many authors (see [13, 14, 28, 30, 38, 40, 48]).

A major breakthrough came in 2019 when Berinde [8] put forward the concept of enriched nonexpansive mappings. In 2021, Berinde and Păcurar [12] introduced enriched Chatterjea contractions and approximated their fixed points by the Krasnoselskij iteration. This work was further developed in 2022 by Ullah et al. [51], who formulated enriched Suzuki nonexpansive (ESN) mappings and examined fixed point existence along with convergence behavior in Hilbert spaces. Around the same period, Shukla and Pant [44] generalized this direction by presenting Suzuki enriched nonexpansive and enriched quasinonexpansive mappings within the Banach space framework, along with associated convergence results. In 2023, Ali and Jubair [2] put forward a broader class known as enriched Berinde nonexpansive mappings. Recently, Bera et al. [15] introduced enriched contractions in pair, established common fixed point results for them, and applied these results to a pair of nonlinear matrix equations. Motivated by these developments, we introduced enriched Reich-Suzuki type nonexpansive mappings in [11] in 2024, and in the present work, we put forward a new class of enriched Chatterjea-Suzuki-(C) type nonexpansive mappings in Banach spaces.

Throughout this work, let Z denote a Banach space with a nonempty subset S , and let $c_1 \in S$ be a given starting point. A number of iterative schemes have been developed in the literature to approximate fixed points of a mapping $\Omega : S \rightarrow S$, some of which are presented below.

(1) **Picard iteration (Picard, 1890)** [37]:

$$c_{n+1} = \Omega c_n, \quad \forall n \in \mathbb{N}. \quad (1.1)$$

(2) **Mann iteration (Mann, 1953) [34]:**

$$c_{n+1} = (1 - \lambda_n)c_n + \lambda_n\Omega c_n, \quad \forall n \in \mathbb{N}. \quad (1.2)$$

(3) **Ishikawa iteration (Ishikawa, 1974) [27]:**

$$\begin{cases} d_n = (1 - \mu_n)c_n + \mu_n\Omega c_n, \\ c_{n+1} = (1 - \lambda_n)c_n + \lambda_n\Omega d_n, \end{cases} \quad \forall n \in \mathbb{N}. \quad (1.3)$$

(4) **Noor iteration (Noor, 2000) [35]:**

$$\begin{cases} e_n = (1 - \nu_n)c_n + \nu_n\Omega c_n, \\ d_n = (1 - \mu_n)c_n + \mu_n\Omega e_n, \\ c_{n+1} = (1 - \lambda_n)c_n + \lambda_n\Omega d_n, \end{cases} \quad \forall n \in \mathbb{N}. \quad (1.4)$$

(5) **S-iteration (Agarwal et al., 2007) [4]:**

$$\begin{cases} d_n = (1 - \mu_n)c_n + \mu_n\Omega c_n, \\ c_{n+1} = (1 - \lambda_n)\Omega c_n + \lambda_n\Omega d_n, \end{cases} \quad \forall n \in \mathbb{N}. \quad (1.5)$$

which satisfy the condition $\sum_{n=1}^{\infty} \lambda_n \mu_n (1 - \mu_n) = \infty$.

(6) **Abbas-Nazir iteration (Abbas and Nazir, 2014) [3]:**

$$\begin{cases} e_n = (1 - \nu_n)c_n + \nu_n\Omega c_n, \\ d_n = (1 - \mu_n)\Omega c_n + \mu_n\Omega e_n, \\ c_{n+1} = (1 - \lambda_n)\Omega d_n + \lambda_n\Omega e_n, \end{cases} \quad \forall n \in \mathbb{N}. \quad (1.6)$$

(7) **Picard–S iteration (Gursoy and Karakaya, 2014) [22]:**

$$\begin{cases} e_n = (1 - \mu_n)c_n + \mu_n\Omega c_n, \\ d_n = (1 - \lambda_n)\Omega c_n + \lambda_n\Omega e_n, \\ c_{n+1} = \Omega d_n, \end{cases} \quad \forall n \in \mathbb{N}. \quad (1.7)$$

(8) **Vatan two-step iteration (Karakaya et al., 2015) [32]:**

$$\begin{cases} d_n = \Omega((1 - \mu_n)c_n + \mu_n\Omega c_n), \\ c_{n+1} = \Omega((1 - \lambda_n)d_n + \lambda_n\Omega d_n), \end{cases} \quad \forall n \in \mathbb{N}. \quad (1.8)$$

(9) **Thakur iteration (Thakur et al., 2016) [46]:**

$$\begin{cases} e_n = (1 - \mu_n)c_n + \mu_n\Omega c_n, \\ d_n = \Omega((1 - \lambda_n)c_n + \lambda_n e_n), \\ c_{n+1} = \Omega d_n, \end{cases} \quad \forall n \in \mathbb{N}. \quad (1.9)$$

(10) **M-iteration (Ullah and Arshad, 2018) [47]:**

$$\begin{cases} e_n = \Omega((1 - \mu_n)c_n + \mu_n\Omega c_n), \\ d_n = \Omega((1 - \lambda_n)e_n + \lambda_n\Omega e_n), \\ c_{n+1} = \Omega d_n, \forall n \in \mathbb{N}. \end{cases} \quad (1.10)$$

(11) **Piri iteration (Piri et al., 2019) [39]:**

$$\begin{cases} e_n = \Omega((1 - \mu_n)c_n + \mu_n\Omega c_n), \\ d_n = \Omega e_n, \\ c_{n+1} = (1 - \lambda_n)\Omega e_n + \lambda_n\Omega d_n, \forall n \in \mathbb{N}. \end{cases} \quad (1.11)$$

(12) **Garodia-Uddin iteration (Garodia and Uddin, 2020) [24]:**

$$\begin{cases} e_n = \Omega[(1 - \mu_n)c_n + \mu_n\Omega c_n], \\ d_n = \Omega[(1 - \lambda_n)\Omega c_n + \lambda_n\Omega e_n], \\ c_{n+1} = \Omega d_n, \forall n \in \mathbb{N}. \end{cases} \quad (1.12)$$

(13) **Z-iteration (Temir and Korkut, 2022) [45]:**

$$\begin{cases} e_n = \Omega[(1 - \mu_n)c_n + \mu_n\Omega c_n], \\ d_n = \Omega e_n, \\ c_{n+1} = \Omega[(1 - \lambda_n)\Omega c_n + \lambda_n\Omega d_n], \forall n \in \mathbb{N}. \end{cases} \quad (1.13)$$

(14) **PV algorithm (Gautam and Vineet, 2024) [25]:**

$$\begin{cases} d_n = \Omega((1 - \lambda_n)\Omega^2 c_n + \lambda_n\Omega c_n), \\ c_{n+1} = \Omega d_n. \end{cases} \quad (1.14)$$

(15) **ECM iteration (Nour-Eddine et al., 2024) [26]:**

$$\begin{cases} f_n = (1 - v_n)c_n + v_n\Omega c_n \\ e_n = \Omega(\mu_n\Omega f_n + (1 - \mu_n)\Omega c_n), \\ d_n = \Omega e_n, \\ c_{n+1} = \Omega(\lambda_n\Omega d_n + (1 - \lambda_n)\Omega e_n), \forall n \in \mathbb{N}. \end{cases} \quad (1.15)$$

where $\{\lambda_n\}$, $\{\mu_n\}$ and $\{v_n\} \in [0, 1]$.

Motivated by recent developments, we propose a modified ECM iteration for approximating fixed points of an enriched class of mappings. Set $\Omega_\gamma : S \rightarrow S$ as $\Omega_\gamma c = (1 - \gamma)c + \gamma\Omega c$. The modified ECM-iteration is formulated as follows:

$$\begin{cases} f_n = (1 - v_n)c_n + v_n\Omega_\gamma c_n \\ e_n = \Omega_\gamma(\mu_n\Omega_\gamma f_n + (1 - \mu_n)\Omega_\gamma c_n), \\ d_n = \Omega_\gamma e_n, \\ c_{n+1} = \Omega_\gamma(\lambda_n\Omega_\gamma d_n + (1 - \lambda_n)\Omega_\gamma e_n), \forall n \in \mathbb{N} \end{cases} \quad (1.16)$$

where, $\gamma = \frac{1}{\kappa+1}$, $\kappa \geq 0$ and $\{\lambda_n\}$, $\{\mu_n\}$ and $\{v_n\} \in [0, 1]$.

Beyond the development of generalized contractive mappings and iterative schemes, fixed point theory remains a cornerstone for addressing nonlinear problems in applied mathematics. Specifically, modified Ćirić-Reich-Rus type contractions have been utilized to establish existence and uniqueness results for variational inequality problems [5]. Furthermore, the advancement of implicit iterative techniques, such as Jungck-type schemes, has significantly improved stability, data dependency analysis, and convergence behavior, particularly in the study of nonlinear functional integral equations with delays [33]. From an application perspective, recent studies have demonstrated that fixed point approaches can be effectively combined with Green's function techniques to analyze fractional differential equations [52]. In particular, fractional Volterra-Fredholm integro-differential equations involving Caputo derivatives have been employed to model complex phenomena such as blood flow dynamics, where iterative schemes ensure convergence to the unique solution [6].

This paper focuses on approximating the fixed point of an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping within a Banach space. We support our findings with computational results through a numerical example. In particular, we analyze the convergence patterns of various well established iterative schemes, comparing their performance using a table and a graphical illustration. Our study demonstrates that the modified ECM-iterative scheme achieves faster convergence. Furthermore, the theoretical results are applied to study the convergence of the ECM-iterative scheme for a delay differential equation and to establish the existence and uniqueness of solutions for a nonlinear fractional integro-differential equation.

2. PRELIMINARIES

This section highlights some essential definitions and theorems that provide the necessary foundation for our results. We begin with the concept of fixed points in normed linear spaces, followed by key geometric properties that play a crucial role in our analysis.

In a normed linear space Z , a point $c \in Z$ is a fixed point of a mapping $\Omega : Z \rightarrow Z$ if $\Omega c = c$. The set of all fixed points is denoted by $F(\Omega)$.

Definition 2.1. [16] A Banach space Z is uniformly convex if, for every $0 < \epsilon \leq 2$ there exists a $\delta > 0$ such that

$$\left\| \frac{c + d}{2} \right\| \leq 1 - \delta,$$

whenever $c, d \in Z$ satisfy $\|c\| = \|d\| = 1$ and $\|c - d\| > \epsilon$.

Definition 2.2. [36] A Banach space Z is said to satisfy the Opial property if, for every sequence $\{c_n\}$ in Z that converges weakly to $\varsigma \in Z$,

$$\liminf_{n \rightarrow \infty} \|c_n - \varsigma\| < \liminf_{n \rightarrow \infty} \|c_n - \varsigma^*\| \quad \text{for all } \varsigma^* \in Z \text{ with } \varsigma^* \neq \varsigma.$$

Definition 2.3. [19] Suppose Z is a Banach space with $\emptyset \neq S \subset Z$ and $\{c_n\}$ is a bounded sequence in Z . Define:

(i) Asymptotic radius of $\{c_n\}$ at c :

$$r(c, \{c_n\}) := \limsup_{n \rightarrow \infty} \|c_n - c\|.$$

(ii) Asymptotic radius of $\{c_n\}$ relative to S :

$$r(S, \{c_n\}) := \inf \{r(c, \{c_n\}) : c \in S\}.$$

(iii) Asymptotic centre of $\{c_n\}$ relative to S :

$$A(S, \{c_n\}) := \{c \in S : r(c, \{c_n\}) = r(S, \{c_n\})\}.$$

Definition 2.4. [7] Consider a self-mapping Ω on a Banach space Z . A γ -Krasnoselskii (averaged) mapping is defined as

$$\Omega_\gamma c = (1 - \gamma)c + \gamma\Omega c, \quad \forall c \in Z,$$

where $\gamma \in (0, 1]$.

Remark 2.1. The fixed point set of the γ -Krasnoselskii mapping Ω_γ is identical to that of Ω ; that is,

$$F(\Omega_\gamma) = F(\Omega).$$

Berinde and Păcurar coined the term ‘enriched’ to describe an enlargement of the traditional classes of contraction and nonexpansive mappings.

Definition 2.5. [10] Consider a normed linear space denoted as $(S, \|\cdot\|)$. A mapping $\Omega : S \rightarrow S$ is termed as an enriched contraction provided there exist constants $\kappa \in [0, \infty)$ and $\theta \in [0, \kappa + 1)$ such that

$$\|\kappa(c - d) + \Omega c - \Omega d\| \leq \theta \|c - d\|, \quad \forall c, d \in S. \quad (2.1)$$

Definition 2.6. [8] Consider a normed linear space $(S, \|\cdot\|)$ and a mapping $\Omega : S \rightarrow S$ is known as an enriched nonexpansive mapping provided there exists $\kappa \in [0, \infty)$ such that

$$\|\kappa(c - d) + \Omega c - \Omega d\| \leq (\kappa + 1) \|c - d\|, \quad \forall c, d \in S. \quad (2.2)$$

Substituting $\kappa = 0$ in equation (2.2) yields nonexpansive mappings, which are a subset of enriched nonexpansive mappings (but not vice versa, as shown in [8]). According to [8], enriched nonexpansive and quasinonexpansive mappings are entirely unrelated.

Theorem 2.1. [43] Consider a sequence $\{c_n\} \in \mathbb{R}^+$ that satisfies the inequality:

$$c_{n+1} \leq (1 - v_n)c_n.$$

where $\{v_n\} \in (0, 1)$ and $\sum_{n=0}^{\infty} v_n = \infty$. Then $\lim_{n \rightarrow \infty} c_n = 0$.

Definition 2.7. [31] Let Z be a Banach space with a nonempty subset S . A mapping $\Omega : S \rightarrow S$ satisfies the Chatterjea-Suzuki-(C) condition if, for all $c, d \in S$

$$\frac{1}{2}\|c - \Omega c\| \leq \|c - d\| \text{ implies } \|\Omega c - \Omega d\| \leq \frac{1}{2} [\|\Omega c - d\| + \|c - \Omega d\|]. \quad (2.3)$$

Theorem 2.2. [50] If Ω satisfies Chatterjea-Suzuki-(C) condition with a fixed point, then Ω is a quasi-nonexpansive mapping.

Theorem 2.3. [41] Suppose Z is a Banach space with uniform convexity, and let $\{\mu_n\}$ be a sequence such that for all $n \in \mathbb{N}$,

$$0 < \lambda \leq \mu_n \leq \nu < 1.$$

Suppose that $\{c_n\}$ and $\{d_n\}$ are sequences in Z satisfying

$$\limsup_{n \rightarrow \infty} \|c_n\| \leq \epsilon, \limsup_{n \rightarrow \infty} \|d_n\| \leq \epsilon \text{ and } \lim_{n \rightarrow \infty} \|(1 - \mu_n)d_n + \mu_n c_n\| = \epsilon$$

for some $\epsilon \geq 0$, then $\lim_{n \rightarrow \infty} \|d_n - c_n\| = 0$.

Theorem 2.4. [31] Assume that Z is a Banach space which contains a nonempty subset S . If a mapping $\Omega : S \rightarrow S$ satisfies Chatterjea-Suzuki-(C) condition, then $\|c - \Omega d\| \leq 5\|c - \Omega c\| + \|c - d\|$.

3. CONVERGENCE ANALYSIS OF ENRICHED TYPE NONEXPANSIVE MAPPING

This section presents the class of enriched Chatterjea-Suzuki-(C) type nonexpansive mappings.

Definition 3.1. Consider a normed linear space $(S, \|\cdot\|)$ and if there exists $\kappa \in [0, \infty)$ such that

$$\|\kappa(c - d) + \Omega c - \Omega d\| \leq \frac{1}{2} [\|(\kappa + 1)(d - c) + c - \Omega c\| + \|(\kappa + 1)(c - d) + d - \Omega d\|] \quad (3.1)$$

$\forall c, d \in S$, then a mapping $\Omega : S \rightarrow S$ is called an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping.

Lemma 3.1. Let S be a nonempty convex subset of Z and $\Omega : S \rightarrow S$ be an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping, then the γ -Krasnoselskii mapping $\Omega_\gamma : Z \rightarrow Z$ with $\gamma = \frac{1}{\kappa + 1}$ satisfies

$$\|\Omega_\gamma c - \Omega_\gamma d\| \leq \frac{1}{2} [\|d - \Omega_\gamma c\| + \|c - \Omega_\gamma d\|] \text{ for all } c, d \in Z. \quad (3.2)$$

In particular, Ω_γ satisfies the Chatterjea-Suzuki-(C) condition.

Proof. Put $\gamma = \frac{1}{\kappa + 1} \in (0, 1]$, so that $\kappa = \frac{1}{\gamma} - 1$ and $\kappa + 1 = \frac{1}{\gamma}$. For all $c, d \in Z$ we have

$$\begin{aligned} \kappa(c - d) + \Omega c - \Omega d &= \frac{1}{\gamma} \left[(1 - \gamma)(c - d) + \gamma(\Omega c - \Omega d) \right] = \frac{1}{\gamma} (\Omega_\gamma c - \Omega_\gamma d), \\ (\kappa + 1)(d - c) + c - \Omega c &= \frac{1}{\gamma} \left[(d - c) + \gamma(c - \Omega c) \right] = \frac{1}{\gamma} (d - \Omega_\gamma c), \end{aligned}$$

$$(\kappa + 1)(c - d) + d - \Omega d = \frac{1}{\gamma} \left[(c - d) + \gamma(d - \Omega d) \right] = \frac{1}{\gamma} (c - \Omega_\gamma d).$$

Substituting these three identities in (3.1) and multiplying by γ , we obtain (3.2). Since (3.2) holds for all $c, d \in S$, it holds in particular whenever $\frac{1}{2}\|c - \Omega_\gamma c\| \leq \|c - d\|$. Thus Ω_γ satisfies the Chatterjea-Suzuki-(C) condition. $\square$

Theorem 3.1. Let S be a nonempty bounded convex subset of a Banach space Z , and let $\Omega : S \rightarrow S$ be an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping with $\gamma = \frac{1}{\kappa+1}$ and define a sequence $\{c_n\}$ in S by $c_1 \in S$ and

$$c_{n+1} = \gamma \Omega c_n + (1 - \gamma)c_n, \quad n \in \mathbb{N}.$$

Then $\lim_{n \rightarrow \infty} \|\Omega c_n - c_n\| = 0$.

Proof. From Lemma 3.1,

$$\|\Omega_\gamma c_n - \Omega_\gamma c_{n+1}\| \leq \frac{1}{2} \left[\|c_{n+1} - \Omega_\gamma c_n\| + \|c_n - \Omega_\gamma c_{n+1}\| \right]. \quad (3.3)$$

Since, we have

$$c_{n+1} = \Omega_\gamma c_n = \gamma \Omega c_n + (1 - \gamma)c_n,$$

and hence we get $\|c_{n+1} - \Omega_\gamma c_n\| = 0$. Applying triangular inequality in (3.3),

$$\|\Omega_\gamma c_n - \Omega_\gamma c_{n+1}\| \leq \frac{1}{2} \left[\|c_n - \Omega_\gamma c_n\| + \|\Omega_\gamma c_n - \Omega_\gamma c_{n+1}\| \right].$$

Hence, $\|\Omega_\gamma c_n - \Omega_\gamma c_{n+1}\| \leq \|c_n - c_{n+1}\|$. Thus, writing $s_n = \|c_{n+1} - c_n\|$, we have $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$, so $\{s_n\}$ converges to some $s \geq 0$. For $k \geq 0$ put $D_{n,k} = \|c_{n+k} - c_n\|$. Applying Lemma 3.1 with $c = c_n$ and $d = c_{n+k}$, where $k \geq 1$, we get

$$D_{n+1,k} = \|\Omega_\gamma c_n - \Omega_\gamma c_{n+k}\| \leq \frac{1}{2} \left[\|c_{n+k} - c_{n+1}\| + \|c_n - c_{n+k+1}\| \right] = \frac{1}{2} (D_{n+1,k-1} + D_{n,k+1}),$$

and therefore $D_{n,k+1} \geq 2D_{n+1,k} - D_{n+1,k-1}$. We claim that $\lim_{n \rightarrow \infty} D_{n,k} = ks$ for every $k \geq 0$. This is clear for $k = 0$ and $k = 1$, because $D_{n,0} = 0$ and $D_{n,1} = s_n$. If the claim holds for $k - 1$ and k , where $k \geq 1$, then the last inequality yields $\liminf_{n \rightarrow \infty} D_{n,k+1} \geq 2ks - (k - 1)s = (k + 1)s$, while the triangle inequality

and the monotonicity of $\{s_n\}$ give $D_{n,k+1} \leq \sum_{i=0}^k s_{n+i} \leq (k + 1)s_n$, and hence $\limsup_{n \rightarrow \infty} D_{n,k+1} \leq (k + 1)s$.

Consequently $\lim_{n \rightarrow \infty} D_{n,k+1} = (k + 1)s$, and the claim follows by induction. Since S is bounded, $D_{n,k} \leq \text{diam}(S) < \infty$ for all n and k , so $ks \leq \text{diam}(S)$ for every $k \geq 0$, which is possible only if $s = 0$. Therefore

$$\lim_{n \rightarrow \infty} \|\Omega_\gamma c_n - c_n\| = \lim_{n \rightarrow \infty} s_n = 0.$$

Since $\Omega_\gamma c_n - c_n = \gamma(\Omega c_n - c_n)$ and $\gamma > 0$, it follows that $\lim_{n \rightarrow \infty} \|\Omega c_n - c_n\| = 0$. $\square$

Theorem 3.2. Let Z be a Banach space that contains a compact convex subset S , and let Ω be an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping defined on S . Construct a sequence $\{c_n\}$ in S by choosing an initial point $c_1 \in S$ and setting

$$c_{n+1} = \gamma \Omega c_n + (1 - \gamma)c_n, \quad n \in \mathbb{N},$$

where $\gamma = \frac{1}{\kappa+1}$. Then $\{c_n\}$ converges strongly to a fixed point of Ω .

Proof. Since a compact set is bounded, Theorem 3.1 gives $\lim_{n \rightarrow \infty} \|\Omega c_n - c_n\| = 0$. Multiplying on both sides by γ , we obtain

$$\lim_{n \rightarrow \infty} \|\Omega_\gamma c_n - c_n\| = 0.$$

Since S is compact, any sequence within it has a convergent subsequence. Hence a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ converges to a point $\varsigma \in S$. According to Theorem 2.4 we have,

$$\|c_{n_k} - \Omega_\gamma \varsigma\| \leq 5\|c_{n_k} - \Omega_\gamma c_{n_k}\| + \|c_{n_k} - \varsigma\|, \quad \forall k \in \mathbb{N}.$$

Therefore, $\{c_{n_k}\} \rightarrow \Omega_\gamma \varsigma$, which implies

$$\Omega_\gamma \varsigma = \varsigma. \quad (3.4)$$

Consider,

$$\|\Omega_\gamma c_n - \Omega_\gamma \varsigma\| \leq \frac{1}{2} [\|\varsigma - \Omega_\gamma c_n\| + \|c_n - \Omega_\gamma \varsigma\|].$$

Using triangular inequality and by (3.4), we have

$$\|\Omega_\gamma c_n - \varsigma\| \leq \frac{1}{2} [\|\varsigma - \Omega_\gamma c_n\| + \|c_n - \varsigma\|].$$

Given that, $c_{n+1} = \gamma \Omega c_n + (1 - \gamma)c_n$ and thus we have

$$\|c_{n+1} - \varsigma\| \leq \|c_n - \varsigma\|, \quad \text{for } n \in \mathbb{N}. \quad (3.5)$$

By (3.5), the sequence $\{\|c_n - \varsigma\|\}$ is nonincreasing, so $\lim_{n \rightarrow \infty} \|c_n - \varsigma\|$ exists. Since $\|c_{n_k} - \varsigma\| \rightarrow 0$ as $k \rightarrow \infty$, this limit is 0. Hence $\{c_n\}$ converges strongly to ς , and $\varsigma \in F(\Omega)$ by (3.4) and Remark 2.1. $\square$

Theorem 3.3. Suppose Z is a Banach space and $\Omega : Z \rightarrow Z$ represents an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping. If Ω possesses a fixed point, then γ -krasnoselskii mapping $\Omega_\gamma : Z \rightarrow Z$ is a quasinonexpansive mapping.

Proof. By Lemma 3.1, for all $c, d \in Z$,

$$\|\Omega_\gamma c - \Omega_\gamma d\| \leq \frac{1}{2} [\|d - \Omega_\gamma c\| + \|c - \Omega_\gamma d\|].$$

Let $d \in F(\Omega)$. By Remark 2.1, $\Omega_\gamma d = d$, so the inequality becomes

$$\|\Omega_\gamma c - d\| \leq \frac{1}{2} [\|d - \Omega_\gamma c\| + \|c - d\|],$$

that is, $\|\Omega_\gamma c - d\| \leq \|c - d\|$ for all $c \in Z$. Hence Ω_γ is a quasinonexpansive mapping. $\square$

Theorem 3.4. Let Z be a Banach space satisfying Opial's condition, let S be a nonempty closed convex subset of Z , and let $\Omega : S \rightarrow S$ be an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping. Then the mapping $I - \Omega$ is demiclosed at zero.

Proof. Using Lemma 3.1, Ω_γ satisfies Chatterjea-Suzuki-(C) condition for $\gamma = \frac{1}{\kappa+1}$. Consider a sequence $\{c_n\}$ that weakly converges to $\varsigma \in S$, and the sequence $\{(I - \Omega)c_n\}$ converges strongly to 0, it follows that $\lim_{n \rightarrow \infty} \|c_n - \Omega c_n\| = 0$. We aim to prove that $(I - \Omega)\varsigma = 0$. Consider,

$$\|c_n - \Omega_\gamma c_n\| = \|c_n - (1 - \gamma)c_n - \gamma\Omega c_n\| = \gamma\|c_n - \Omega c_n\|.$$

As $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \|c_n - \Omega_\gamma c_n\| = \gamma \lim_{n \rightarrow \infty} \|c_n - \Omega c_n\| = 0.$$

Then by Theorem 2.4, we have

$$\|c_n - \Omega_\gamma \varsigma\| \leq 5\|c_n - \Omega_\gamma c_n\| + \|c_n - \varsigma\|.$$

Taking $\liminf_{n \rightarrow \infty}$ on the above inequality, we get

$$\liminf_{n \rightarrow \infty} \|c_n - \Omega_\gamma \varsigma\| \leq \liminf_{n \rightarrow \infty} \|c_n - \varsigma\|.$$

If $\Omega_\gamma \varsigma \neq \varsigma$, applying Opial's condition gives

$$\liminf_{n \rightarrow \infty} \|c_n - \varsigma\| < \liminf_{n \rightarrow \infty} \|c_n - \Omega_\gamma \varsigma\|,$$

which contradicts the previous inequality. Thus, $\Omega_\gamma \varsigma = \varsigma$, since $\{c_n\}$ weakly converges to ς . Hence, $\gamma(\varsigma - \Omega\varsigma) = 0$. Since $\gamma \in (0, 1]$, we obtain, $\Omega\varsigma - \varsigma = 0$. Therefore, $I - \Omega$ is demiclosed at zero. $\square$

4. CONVERGENCE RESULTS VIA MODIFIED ECM-ITERATION

This section establishes both strong and weak convergence results through the modified ECM-iteration.

Theorem 4.1. *Let S be a closed, convex subset of a uniformly convex Banach space Z , and $\Omega : S \rightarrow S$ an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping with $F(\Omega) \neq \emptyset$. If $c_1 \in S$ is arbitrarily chosen, $\{c_n\}$ is the sequence constructed by the modified ECM-iterative scheme (1.16), then, for every $\varsigma \in F(\Omega)$, $\lim_{n \rightarrow \infty} \|c_n - \varsigma\|$ exists.*

Proof. Suppose $\varsigma \in F(\Omega)$, we have $\varsigma \in F(\Omega_\gamma)$. By Lemma 3.1, $\Omega_\gamma : S \rightarrow S$ satisfies Chatterjea-Suzuki-(C) condition, where $\gamma = \frac{1}{\kappa+1}$. By (3.1),

$$\|\kappa(c - d) + \Omega c - \Omega d\| \leq \frac{1}{2} [\|(\kappa + 1)(d - c) + c - \Omega c\| + \|(\kappa + 1)(c - d) + d - \Omega d\|],$$

for all $c, d \in S$. Using (1.16) and Theorem 2.2 we have,

$$\begin{aligned} \|f_n - \varsigma\| &= \|(1 - v_n)(c_n - \varsigma) + v_n(\Omega_\gamma c_n - \varsigma)\| \\ &\leq (1 - v_n)\|c_n - \varsigma\| + v_n\|\Omega_\gamma c_n - \varsigma\| \\ &\leq (1 - v_n)\|c_n - \varsigma\| + v_n\|c_n - \varsigma\| \\ &= \|c_n - \varsigma\|, \end{aligned}$$

$$\|e_n - \varsigma\| = \|\Omega_\gamma [\mu_n \Omega_\gamma f_n + (1 - \mu_n) \Omega_\gamma c_n] - \varsigma\|$$

$$\begin{aligned}
&\leq \|\mu_n \Omega_\gamma f_n + (1 - \mu_n) \Omega_\gamma c_n - \varsigma\| \\
&= \|\mu_n (\Omega_\gamma f_n - \varsigma) + (1 - \mu_n) (\Omega_\gamma c_n - \varsigma)\| \\
&\leq \mu_n \|\Omega_\gamma f_n - \varsigma\| + (1 - \mu_n) \|\Omega_\gamma c_n - \varsigma\| \\
&\leq \mu_n \|f_n - \varsigma\| + (1 - \mu_n) \|c_n - \varsigma\| \\
&\leq \mu_n \|c_n - \varsigma\| + (1 - \mu_n) \|c_n - \varsigma\| \\
&= \|c_n - \varsigma\|.
\end{aligned}$$

$$\|d_n - \varsigma\| = \|\Omega_\gamma e_n - \varsigma\| \leq \|e_n - \varsigma\| \leq \|c_n - \varsigma\|,$$

$$\begin{aligned}
\|c_{n+1} - \varsigma\| &= \|\Omega_\gamma [\lambda_n \Omega_\gamma d_n + (1 - \lambda_n) \Omega_\gamma e_n] - \varsigma\| \\
&\leq \|\lambda_n \Omega_\gamma d_n + (1 - \lambda_n) \Omega_\gamma e_n - \varsigma\| \\
&= \|\lambda_n (\Omega_\gamma d_n - \varsigma) + (1 - \lambda_n) (\Omega_\gamma e_n - \varsigma)\| \\
&\leq \lambda_n \|\Omega_\gamma d_n - \varsigma\| + (1 - \lambda_n) \|\Omega_\gamma e_n - \varsigma\| \\
&\leq \lambda_n \|d_n - \varsigma\| + (1 - \lambda_n) \|e_n - \varsigma\| \\
&\leq \lambda_n \|c_n - \varsigma\| + (1 - \lambda_n) \|c_n - \varsigma\| \\
&= \|c_n - \varsigma\|
\end{aligned}$$

From this,

$$\|f_n - \varsigma\| \leq \|c_n - \varsigma\| \quad (4.1)$$

$$\|e_n - \varsigma\| \leq \|c_n - \varsigma\| \quad (4.2)$$

$$\|d_n - \varsigma\| \leq \|c_n - \varsigma\| \quad (4.3)$$

$$\|c_{n+1} - \varsigma\| \leq \|c_n - \varsigma\| \quad (4.4)$$

we get, $\{\|c_n - \varsigma\|\}$ is bounded and nonincreasing. Thus, $\lim_{n \rightarrow \infty} \|c_n - \varsigma\|$ exists for each $\varsigma \in F(\Omega)$. $\square$

Theorem 4.2. Let S be a nonempty closed convex subset of a uniformly convex Banach space Z and let $\Omega : S \rightarrow S$ be an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping. Let $\{c_n\}$ be generated by (1.16) with $\gamma = \frac{1}{\kappa+1}$, where $\{\lambda_n\} \subset [0, 1]$ and there exist constants a, b such that

$$0 < a \leq \mu_n \leq 1 \quad \text{and} \quad 0 < a \leq \nu_n \leq b < 1 \quad \text{for all } n \in \mathbb{N}.$$

Then $F(\Omega) \neq \emptyset$ if and only if $\{c_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|c_n - \Omega_\gamma c_n\| = 0$.

Proof. Suppose that $F(\Omega) \neq \emptyset$ and let $\varsigma \in F(\Omega)$. By Theorem 4.1, $\{c_n\}$ is bounded and

$$\epsilon := \lim_{n \rightarrow \infty} \|c_n - \varsigma\|$$

exists. Since Ω_γ is quasicontractive by Theorem 3.3, $\|\Omega_\gamma c_n - \varsigma\| \leq \|c_n - \varsigma\|$, and therefore

$$\limsup_{n \rightarrow \infty} \|\Omega_\gamma c_n - \varsigma\| \leq \epsilon.$$

Since $d_n = \Omega_\gamma e_n$, the last line of (1.16) gives $c_{n+1} = \Omega_\gamma(\lambda_n \Omega_\gamma d_n + (1 - \lambda_n)d_n)$. Hence, using the quasinonexpansiveness of Ω_γ ,

$$\begin{aligned} \|c_{n+1} - \varsigma\| &\leq \|\lambda_n(\Omega_\gamma d_n - \varsigma) + (1 - \lambda_n)(d_n - \varsigma)\| \\ &\leq \lambda_n \|\Omega_\gamma d_n - \varsigma\| + (1 - \lambda_n) \|d_n - \varsigma\| \\ &\leq \|d_n - \varsigma\| = \|\Omega_\gamma e_n - \varsigma\| \leq \|e_n - \varsigma\|. \end{aligned}$$

From the proof of Theorem 4.1, $\|e_n - \varsigma\| \leq \mu_n \|f_n - \varsigma\| + (1 - \mu_n) \|c_n - \varsigma\|$. Combining the last two estimates,

$$\|c_{n+1} - \varsigma\| \leq \mu_n \|f_n - \varsigma\| + (1 - \mu_n) \|c_n - \varsigma\|,$$

that is, $\mu_n (\|c_n - \varsigma\| - \|f_n - \varsigma\|) \leq \|c_n - \varsigma\| - \|c_{n+1} - \varsigma\|$. Since $\|f_n - \varsigma\| \leq \|c_n - \varsigma\|$ by (4.1) and $\mu_n \geq a > 0$,

$$0 \leq \|c_n - \varsigma\| - \|f_n - \varsigma\| \leq \frac{1}{a} (\|c_n - \varsigma\| - \|c_{n+1} - \varsigma\|) \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence $\lim_{n \rightarrow \infty} \|f_n - \varsigma\| = \epsilon$, that is,

$$\lim_{n \rightarrow \infty} \|(1 - \nu_n)(c_n - \varsigma) + \nu_n(\Omega_\gamma c_n - \varsigma)\| = \epsilon.$$

Applying Theorem 2.3 to the sequences $\{\Omega_\gamma c_n - \varsigma\}$, $\{c_n - \varsigma\}$ and $\{\nu_n\}$, we conclude that

$$\lim_{n \rightarrow \infty} \|c_n - \Omega_\gamma c_n\| = 0.$$

On the other hand, assume $\{c_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|c_n - \Omega_\gamma c_n\| = 0$. Since Z is uniformly convex and S is nonempty, closed and convex, the asymptotic centre $A(S, \{c_n\})$ is a singleton (see [19]), say $A(S, \{c_n\}) = \{\varsigma\}$. By Lemma 3.1, Ω_γ satisfies the Chatterjea-Suzuki-(C) condition, so Theorem 2.4 gives

$$\begin{aligned} r(\Omega_\gamma \varsigma, \{c_n\}) &= \limsup_{n \rightarrow \infty} \|c_n - \Omega_\gamma \varsigma\| \\ &\leq 5 \limsup_{n \rightarrow \infty} \|c_n - \Omega_\gamma c_n\| + \limsup_{n \rightarrow \infty} \|c_n - \varsigma\| \\ &= \limsup_{n \rightarrow \infty} \|c_n - \varsigma\| = r(\varsigma, \{c_n\}). \end{aligned}$$

Since $\Omega_\gamma \varsigma \in S$, it follows that $\Omega_\gamma \varsigma \in A(S, \{c_n\})$, and so $\Omega_\gamma \varsigma = \varsigma$. By Remark 2.1, $\varsigma \in F(\Omega_\gamma) = F(\Omega)$. Hence $F(\Omega) \neq \emptyset$. $\square$

Theorem 4.3. Suppose that Ω is an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping on a nonempty compact convex subset S of a uniformly convex Banach space Z , with a nonempty fixed point set $F(\Omega)$. Then the sequence $\{c_n\}$ defined by (1.16), where $\{\lambda_n\} \subset [0, 1]$, $0 < a \leq \mu_n \leq 1$ and $0 < a \leq \nu_n \leq b < 1$ for all $n \in \mathbb{N}$, converges strongly to a point in $F(\Omega)$.

Proof. Since $F(\Omega) \neq \emptyset$, Theorem 4.2 gives $\lim_{n \rightarrow \infty} \|\Omega_\gamma c_n - c_n\| = 0$. Since S is compact, there exist a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ and a point $\varsigma \in S$ such that $c_{n_k} \rightarrow \varsigma$ as $k \rightarrow \infty$. By Theorem 2.4,

$$\|c_{n_k} - \Omega_\gamma \varsigma\| \leq 5 \|\Omega_\gamma c_{n_k} - c_{n_k}\| + \|c_{n_k} - \varsigma\| \longrightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Thus $c_{n_k} \rightarrow \Omega_\gamma \varsigma$, and by uniqueness of the limit $\Omega_\gamma \varsigma = \varsigma$. By Remark 2.1, $\varsigma \in F(\Omega)$. By Theorem 4.1, $\lim_{n \rightarrow \infty} \|c_n - \varsigma\|$ exists, and since $\|c_{n_k} - \varsigma\| \rightarrow 0$. Hence $\{c_n\}$ converges strongly to the fixed point ς of Ω . $\square$

Theorem 4.4. Assume Z is a real uniformly convex Banach space with Opial's property, and let S be a nonempty, closed, convex subset of Z . If Ω is an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping on S with $F(\Omega) \neq \emptyset$ and $\{c_n\}$ is the sequence defined by (1.16), where $\{\lambda_n\} \subset [0, 1]$, $0 < a \leq \mu_n \leq 1$ and $0 < a \leq \nu_n \leq b < 1$ for all $n \in \mathbb{N}$, then $\{c_n\}$ converges weakly to a fixed point of Ω .

Proof. By Theorem 4.1, $\{c_n\}$ is bounded, and by Theorem 4.2,

$$\lim_{n \rightarrow \infty} \|c_n - \Omega_\gamma c_n\| = 0, \quad \text{so} \quad \lim_{n \rightarrow \infty} \|c_n - \Omega c_n\| = \frac{1}{\gamma} \lim_{n \rightarrow \infty} \|c_n - \Omega_\gamma c_n\| = 0.$$

Since Z is uniformly convex, it is reflexive, so the bounded sequence $\{c_n\}$ has weakly convergent subsequences, and their limits lie in S because S is closed and convex. Let ς_1 and ς_2 be weak limits of subsequences $\{c_{n_k}\}$ and $\{c_{n_l}\}$ of $\{c_n\}$, respectively. By Theorem 3.4, $I - \Omega$ is demiclosed at zero, so $(I - \Omega)\varsigma_1 = 0$, showing $\varsigma_1 \in F(\Omega)$. A similar argument shows that $\varsigma_2 \in F(\Omega)$. Our next step is to prove that $\varsigma_1 = \varsigma_2$. Let if possible, $\varsigma_1 \neq \varsigma_2$. Then we obtain,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|c_n - \varsigma_1\| &= \liminf_{k \rightarrow \infty} \|c_{n_k} - \varsigma_1\| \\ &< \liminf_{k \rightarrow \infty} \|c_{n_k} - \varsigma_2\| \\ &= \lim_{n \rightarrow \infty} \|c_n - \varsigma_2\| \\ &= \liminf_{l \rightarrow \infty} \|c_{n_l} - \varsigma_2\| \\ &< \liminf_{l \rightarrow \infty} \|c_{n_l} - \varsigma_1\| \\ &= \lim_{n \rightarrow \infty} \|c_n - \varsigma_1\|, \end{aligned}$$

which leads to a contradiction. Thus $\varsigma_1 = \varsigma_2$. Hence all weakly convergent subsequences of $\{c_n\}$ have the same limit, which lies in $F(\Omega)$, and so $\{c_n\}$ converges weakly to an element of $F(\Omega)$. $\square$

5. NUMERICAL RESULTS

In this section, we show by an example that the modified ECM-iterative scheme (1.16) converges faster than several existing iterations for an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping.

Example 5.1. Let $S = [-3, 3]$ and suppose, $\Omega : S \rightarrow S$ defined as follows:

$$\Omega(c) = \begin{cases} \frac{c-5}{3} & \text{when } c = 2 \\ \frac{4-c}{7} & \text{when } c \neq 2 \end{cases}$$

Then Ω is an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping, but not a nonexpansive mapping.

Verification: First let us show that Ω is not nonexpansive mapping. Now let us consider, $c = 2$ and $d \neq 2$.

$$\|\Omega c - \Omega d\| = \left| \frac{c-5}{3} - \frac{4-d}{7} \right| = \left| \frac{3d-33}{21} \right|, \quad |c-d| = |2-d|$$

Let us take $d = \frac{3}{2}$, then

$$\|\Omega c - \Omega d\| = \left| \frac{\frac{3}{2}-11}{7} \right| = \left| \frac{19}{14} \right| > |c-d| = |2-d| = \left| \frac{1}{2} \right|$$

From this, we can say that Ω is not a nonexpansive mapping.

We set $\kappa = 1$ and proceed to verify that Ω satisfies the enriched Chatterjea-Suzuki-(C) type nonexpansive condition. For this purpose, we analyze the following three cases:

Case 1: Consider, $c = d = 2$,

$$\|\kappa(c-d) + \Omega c - \Omega d\| = \left| c-d + \frac{c-5}{3} - \frac{d-5}{3} \right| = 0 \quad (5.1)$$

Now, consider

$$\begin{aligned} \frac{1}{2} [\|(\kappa+1)(d-c) + c - \Omega c\| + \|(\kappa+1)(c-d) + d - \Omega d\|] &= \frac{1}{2} [\|c - \Omega c\| + \|d - \Omega d\|] \\ &= \frac{1}{2} \left[\left| c - \frac{c-5}{3} \right| + \left| d - \frac{d-5}{3} \right| \right] = 3 \end{aligned} \quad (5.2)$$

From (5.1) and (5.2),

$$\|\kappa(c-d) + \Omega c - \Omega d\| < \frac{1}{2} [\|(\kappa+1)(d-c) + c - \Omega c\| + \|(\kappa+1)(c-d) + d - \Omega d\|]$$

holds, for $c = d = 2$.

Case 2: Consider $c, d \neq 2$

$$\|\kappa(c-d) + \Omega c - \Omega d\| = \left| c-d + \frac{4-c}{7} - \frac{4-d}{7} \right| = \frac{6}{7} |c-d| \quad (5.3)$$

Now, consider

$$\begin{aligned} \frac{1}{2} [\|(\kappa+1)(d-c) + c - \Omega c\| + \|(\kappa+1)(c-d) + d - \Omega d\|] \\ &= \frac{1}{2} \left[\left| 2(d-c) + c - \frac{4-c}{7} \right| + \left| 2(c-d) + d - \frac{4-d}{7} \right| \right] \\ &= \frac{1}{2} \left[\left| \frac{14c-6d-4}{7} \right| + \left| \frac{14d-6c-4}{7} \right| \right] \end{aligned} \quad (5.4)$$

Subcase 1: If $c = d$, then we have

$$\|\kappa(c-d) + \Omega c - \Omega d\| = \frac{6}{7} |c-d| = 0.$$

Now, consider

$$\frac{1}{2} [\|(\kappa+1)(d-c) + c - \Omega c\| + \|(\kappa+1)(c-d) + d - \Omega d\|] = \frac{1}{2} \left[\left| \frac{14c-6d-4}{7} \right| + \left| \frac{14d-6c-4}{7} \right| \right] \geq 0.$$

Hence, we get

$$\|\kappa(c-d) + \Omega c - \Omega d\| \leq \frac{1}{2} \left[\|(\kappa+1)(d-c) + c - \Omega c\| + \|(\kappa+1)(c-d) + d - \Omega d\| \right]$$

for all $c, d \in S \setminus \{2\}$ with $c = d$.

Subcase 3: If $c \neq d$, then we have

$$\|\kappa(c-d) + \Omega c - \Omega d\| = \left| c-d + \frac{4-c}{7} - \frac{4-d}{7} \right| = \frac{6}{7} |c-d|. \quad (5.5)$$

Now, consider

$$\begin{aligned} & \frac{1}{2} \left[\|(\kappa+1)(d-c) + c - \Omega c\| + \|(\kappa+1)(c-d) + d - \Omega d\| \right] \\ &= \frac{1}{2} \left[\left| 2(d-c) + c - \frac{4-c}{7} \right| + \left| 2(c-d) + d - \frac{4-d}{7} \right| \right] \\ &= \frac{1}{2} \left[\left| \frac{14c-6d-4}{7} \right| + \left| \frac{14d-6c-4}{7} \right| \right]. \end{aligned} \quad (5.6)$$

By the triangle inequality,

$$\left| \frac{14c-6d-4}{7} \right| + \left| \frac{14d-6c-4}{7} \right| \geq \left| \frac{(14c-6d-4) - (14d-6c-4)}{7} \right| = \frac{20}{7} |c-d|,$$

and thus, from (5.5) and (5.6),

$$\begin{aligned} \|\kappa(c-d) + \Omega c - \Omega d\| &= \frac{6}{7} |c-d| \\ &< \frac{10}{7} |c-d| \\ &< \frac{1}{2} \left[\left| \frac{14c-6d-4}{7} \right| + \left| \frac{14d-6c-4}{7} \right| \right] \\ &= \frac{1}{2} \left[\|(\kappa+1)(d-c) + c - \Omega c\| + \|(\kappa+1)(c-d) + d - \Omega d\| \right] \end{aligned}$$

for all $c, d \in S \setminus \{2\}$ with $c \neq d$.

Case 2: Consider, $c = 2$; $d \neq 2$

$$\|\kappa(c-d) + \Omega c - \Omega d\| = \left| c-d + \frac{c-5}{3} - \frac{4-d}{7} \right| = \left| \frac{28c-18d-47}{21} \right| = \frac{|9-18d|}{21} = \frac{3}{7} |2d-1| \quad (5.7)$$

Now, consider

$$\begin{aligned} & \frac{1}{2} \left[\|(\kappa+1)(d-c) + c - \Omega c\| + \|(\kappa+1)(c-d) + d - \Omega d\| \right] \\ &= \frac{1}{2} \left[\left| 2(d-c) + c - \frac{c-5}{3} \right| + \left| 2(c-d) + d - \frac{4-d}{7} \right| \right] \\ &= \frac{1}{2} \left[\left| \frac{6d-4c+5}{3} \right| + \left| \frac{14c-6d-4}{7} \right| \right] = \frac{1}{2} |2d-1| + \frac{12-3d}{7} \end{aligned} \quad (5.8)$$

From (5.7) and (5.8),

$$\|\kappa(c-d) + \Omega c - \Omega d\| = \frac{3}{7} |2d-1|$$

$$\begin{aligned}
&< \frac{1}{2}|2d-1| \\
&< \frac{1}{2}|2d-1| + \frac{12-3d}{7} \\
&= \frac{1}{2} \left[\|(\kappa+1)(d-c) + c - \Omega c\| + \|(\kappa+1)(c-d) + d - \Omega d\| \right]
\end{aligned}$$

for $c = 2$ and all $d \in S \setminus \{2\}$.

Thus, Ω is an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping.

Now, we will explore the comparison of convergence among several recent iterations. Suppose that the sequences $\{\lambda_n\}, \{\mu_n\}, \{v_n\}$ given by

$$\lambda_n = \frac{1}{2}; \mu_n = \frac{2}{5}; v_n = \frac{9}{10}$$

where $\{\lambda_n\}, \{\mu_n\}, \{v_n\} \in (0, 1)$ and initial value is 1.

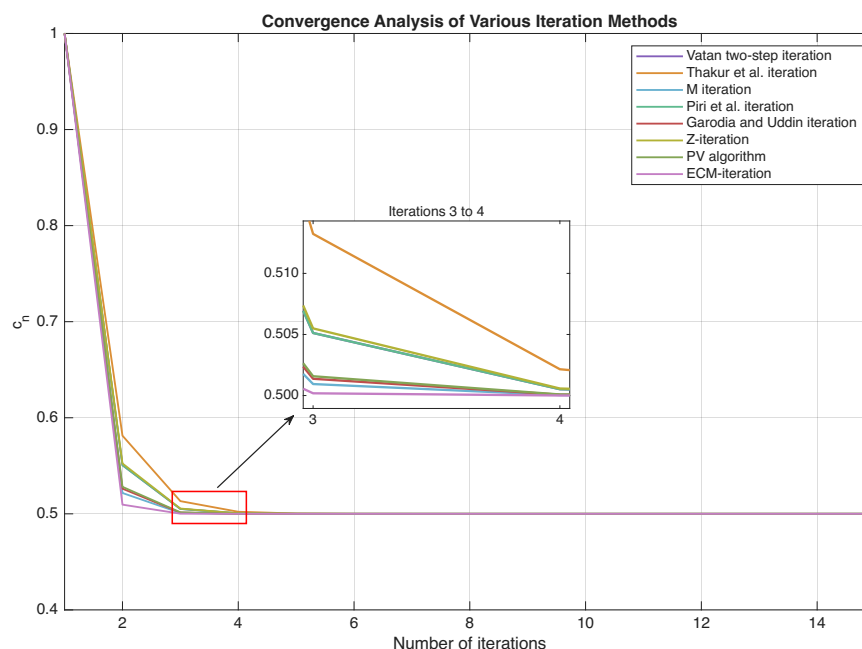


FIGURE 1. A comparative illustration of selective iterative schemes with initial value $c_1 = 1$.

Table 1 and Figure 1 compare the modified ECM-iteration with the Vatan, Thakur, M, Piri, Garodia–Uddin, Z and PV iterations, where Ω is replaced by Ω_γ in each of these schemes. We take $\kappa = 1$, so that $\gamma = \frac{1}{2}$, with $\lambda_n = \frac{1}{2}$, $\mu_n = \frac{2}{5}$, $v_n = \frac{9}{10}$ and $c_1 = 1$. All the sequences converge to the fixed point 0.5. The modified ECM-iteration is the first to agree with 0.5 to eight decimal places, at $n = 6$; the M-iteration needs $n = 7$, and the remaining schemes need more than seven iterations. The enlarged view of iterations 3 to 4 in Figure 1 shows that the curve of the modified ECM-iteration lies below all the others and is closest to 0.5. Hence, for this example, the modified ECM-iteration converges faster than the other schemes.

n	Vatan	Thakur	M	Piri	Garodia & Uddin	Z	PV	ECM
1	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
2	0.55060392	0.58134111	0.52168739	0.55060392	0.52618552	0.55242458	0.52811329	0.50956999
3	0.50512151	0.51323275	0.50094069	0.50512151	0.50137136	0.50549667	0.50158071	0.50018317
4	0.50051834	0.50215273	0.50004080	0.50051834	0.50007182	0.50057632	0.50008888	0.50000351
5	0.50005246	0.50035021	0.50000177	0.50005246	0.50000376	0.50006043	0.50000500	0.50000007
6	0.50000531	0.50005697	0.50000008	0.50000531	0.50000020	0.50000634	0.50000028	0.50000000
7	0.50000054	0.50000927	0.50000000	0.50000054	0.50000001	0.50000066	0.50000002	0.50000000

TABLE 1. Iterative comparison of successive approximations toward the fixed point 0.5.

6. APPLICATIONS

In this section, we utilize our principal results to analyze the convergence behavior of the ECM-iterative scheme for a nonlinear delay differential equation and to establish the existence and uniqueness of solutions to a nonlinear fractional integro-differential equation.

6.1. Application to a Delay Differential Equation. We examine a nonlinear delay differential equation and apply our main result to study the convergence of the ECM-iterative scheme.

Consider a delay differential equation

$$c'(t) = f(t, c(t), c(t - \xi)), \quad \forall t \in [t_0, b], \quad (6.1)$$

with initial condition

$$c(t) = Y(t), \quad t \in [t_0 - \xi, t_0]. \quad (6.2)$$

Assume the following conditions hold:

- (D₁) $t_0, b \in \mathbb{R}$ with $t_0 < b$, and $\xi > 0$;
- (D₂) $f \in C([t_0, b] \times \mathbb{R}^2, \mathbb{R})$;
- (D₃) $Y \in C([t_0 - \xi, t_0], \mathbb{R})$;
- (D₄) there exists $L_f > 0$ such that

$$|f(t, p_1, p_2) - f(t, q_1, q_2)| \leq L_f(|p_1 - q_1| + |p_2 - q_2|),$$

for all $p_\psi, q_\psi \in \mathbb{R}$, $\psi = 1, 2$; $t \in [t_0, b]$;

- (D₅) $2L_f(b - t_0) < 1$.

An equivalent formulation of the problem (6.1)-(6.2) is given by the integral equation below:

$$c(t) = \begin{cases} Y(t), & t \in [t_0 - \xi, t_0], \\ Y(t_0) + \int_{t_0}^t f(p, c(p), c(p - \xi)) dp, & t \in [t_0, b]. \end{cases}$$

The existence and uniqueness of a solution to problem (6.1)-(6.2) were established by Coman et al. [18] via the Picard iterative scheme. We extend this analysis by establishing the convergence of the ECM-iterative scheme to the same solution.

Theorem 6.1. [18] *Under the given conditions (D_1) – (D_5) , the problem (6.1)–(6.2) possesses a unique solution say $\varsigma \in C([t_0 - \xi, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$. Moreover, the Picard iterative scheme converges to ς for any $c_0 \in C([t_0 - \xi, b], \mathbb{R})$.*

Building upon the above existence result, we now establish that the ECM-iteration (1.15) converges to the unique solution ς for an arbitrary initial function, thus confirming the practical effectiveness of ECM-iterative framework.

Theorem 6.2. *When conditions (D_1) – (D_5) are satisfied, the sequence $\{c_n\}$ generated by iterative scheme (1.15) converges to the solution of problem (6.1)–(6.2), say $\varsigma \in C([t_0 - \xi, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$ for any $c_0 \in C([t_0 - \xi, b], \mathbb{R})$.*

Proof. Consider the sequence $\{c_n\}$ that is generated using the iterative scheme (1.15) for the operator,

$$\Omega c(t) = \begin{cases} Y(t), & t \in [t_0 - \xi, t_0], \\ Y(t_0) + \int_{t_0}^t f(p, c(p), c(p - \xi)) dp, & t \in [t_0, b]. \end{cases}$$

where $\{\lambda_n\}, \{\mu_n\} \in (0, 1)$ for all $n \in \mathbb{N}$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$. Let ς denote the fixed point of Ω , so that $\Omega \varsigma = \varsigma$. We now show that $c_n \rightarrow \varsigma$ as $n \rightarrow \infty$.

For $t \in [t_0 - \xi, t_0]$, convergence follows immediately from the definition of Ω . We therefore focus on $t \in [t_0, b]$. Using (1.15) and condition (D_4) ,

$$\begin{aligned} \|f_n - \varsigma\|_{\infty} &= \|(1 - \nu_n)c_n + \nu_n \Omega c_n - \varsigma\|_{\infty} \\ &\leq (1 - \nu_n)\|c_n - \varsigma\|_{\infty} + \nu_n \|\Omega c_n - \Omega \varsigma\|_{\infty} \\ &= (1 - \nu_n)\|c_n - \varsigma\|_{\infty} + \nu_n \max_{t \in [t_0 - \xi, b]} |\Omega c_n(t) - \Omega \varsigma(t)| \\ &= (1 - \nu_n)\|c_n - \varsigma\|_{\infty} + \nu_n \max_{t \in [t_0 - \xi, b]} \left| \int_{t_0}^t f(p, c_n(p), c_n(p - \xi)) - f(p, \varsigma(p), \varsigma(p - \xi)) dp \right| \\ &\leq (1 - \nu_n)\|c_n - \varsigma\|_{\infty} + \nu_n \max_{t \in [t_0 - \xi, b]} \int_{t_0}^t |f(p, c_n(p), c_n(p - \xi)) - f(p, \varsigma(p), \varsigma(p - \xi))| dp \\ &\leq (1 - \nu_n)\|c_n - \varsigma\|_{\infty} + \nu_n \max_{t \in [t_0 - \xi, b]} \int_{t_0}^t L_f (|c_n(p) - \varsigma(p)| + |c_n(p - \xi) - \varsigma(p - \xi)|) dp \\ &\leq (1 - \nu_n)\|c_n - \varsigma\|_{\infty} + \nu_n \int_{t_0}^t L_f (\|c_n - \varsigma\|_{\infty} + \|c_n - \varsigma\|_{\infty}) dp \\ &\leq (1 - \nu_n)\|c_n - \varsigma\|_{\infty} + 2\nu_n L_f (b - t_0) \|c_n - \varsigma\|_{\infty} \\ &= (1 - (1 - 2L_f(b - t_0))\nu_n) \|c_n - \varsigma\|_{\infty} \end{aligned}$$

Therefore,

$$\|f_n - \varsigma\|_\infty \leq \left(1 - (1 - 2L_f(b - t_0))\nu_n\right) \|c_n - \varsigma\|_\infty \quad (6.3)$$

Let $g_n = \mu_n \Omega f_n + (1 - \mu_n) \Omega c_n$, so that $e_n = \Omega g_n$. Then

$$\begin{aligned} \|g_n - \varsigma\|_\infty &\leq \|\mu_n \Omega f_n + (1 - \mu_n) \Omega c_n - \varsigma\|_\infty \\ &\leq \mu_n \|\Omega f_n - \varsigma\|_\infty + (1 - \mu_n) \|\Omega c_n - \varsigma\|_\infty \end{aligned} \quad (6.4)$$

Using condition (D_4) and by (6.3),

$$\begin{aligned} \|\Omega f_n - \varsigma\|_\infty &= \max_{t \in [t_0 - \xi, b]} |\Omega f_n(t) - \Omega \varsigma(t)| \\ &= \max_{t \in [t_0 - \xi, b]} \left| \int_{t_0}^t f(p, f_n(p), f_n(p - \xi)) dp - \int_{t_0}^t f(p, \varsigma(p), \varsigma(p - \xi)) dp \right| \\ &\leq \max_{t \in [t_0 - \xi, b]} \int_{t_0}^t |f(p, f_n(p), f_n(p - \xi)) - f(p, \varsigma(p), \varsigma(p - \xi))| dp \\ &\leq \max_{t \in [t_0 - \xi, b]} \int_{t_0}^t L_f (|f_n(p) - \varsigma(p)| + |f_n(p - \xi) - \varsigma(p - \xi)|) dp \\ &\leq \int_{t_0}^t L_f \left(\max_{t \in [t_0 - \xi, b]} |f_n(p) - \varsigma(p)| + \max_{t \in [t_0 - \xi, b]} |f_n(p - \xi) - \varsigma(p - \xi)| \right) dp \\ &= \int_{t_0}^t L_f (\|f_n - \varsigma\|_\infty + \|f_n - \varsigma\|_\infty) dp \\ &\leq 2L_f(b - t_0) \|f_n - \varsigma\|_\infty \\ &\leq \left(1 - (1 - 2L_f(b - t_0))\nu_n\right) \|c_n - \varsigma\|_\infty \end{aligned}$$

Using condition (D_5) ,

$$\|\Omega f_n - \varsigma\|_\infty \leq \left(1 - (1 - 2L_f(b - t_0))\nu_n\right) \|c_n - \varsigma\|_\infty \quad (6.5)$$

Again by condition (D_4) ,

$$\begin{aligned} \|\Omega c_n - \varsigma\|_\infty &= \max_{t \in [t_0 - \xi, b]} |\Omega c_n(t) - \Omega \varsigma(t)| \\ &= \max_{t \in [t_0 - \xi, b]} \left| \int_{t_0}^t f(p, c_n(p), c_n(p - \xi)) dp - \int_{t_0}^t f(p, \varsigma(p), \varsigma(p - \xi)) dp \right| \\ &\leq \max_{t \in [t_0 - \xi, b]} \int_{t_0}^t |f(p, c_n(p), c_n(p - \xi)) - f(p, \varsigma(p), \varsigma(p - \xi))| dp \\ &\leq \max_{t \in [t_0 - \xi, b]} \int_{t_0}^t L_f (|c_n(p) - \varsigma(p)| + |c_n(p - \xi) - \varsigma(p - \xi)|) dp \\ &\leq \int_{t_0}^t L_f \max_{t \in [t_0 - \xi, b]} (|c_n(p) - \varsigma(p)| + |c_n(p - \xi) - \varsigma(p - \xi)|) dp \\ &= \int_{t_0}^t L_f (\|c_n - \varsigma\|_\infty + \|c_n - \varsigma\|_\infty) dp \end{aligned}$$

$$\leq 2L_f(b - t_0)\|c_n - \varsigma\|_\infty.$$

So, by condition (D_5) , we get

$$\|\Omega c_n - \varsigma\|_\infty \leq \|c_n - \varsigma\|_\infty. \quad (6.6)$$

Substituting (6.5) and (6.6) in (6.4), we get

$$\begin{aligned} \|g_n - \varsigma\|_\infty &\leq \mu_n \left(1 - (1 - 2L_f(b - t_0))\nu_n\right) \|c_n - \varsigma\|_\infty + (1 - \mu_n) \|c_n - \varsigma\|_\infty \\ &\leq \left(1 - (1 - 2L_f(b - t_0))\mu_n\nu_n\right) \|c_n - \varsigma\|_\infty. \end{aligned}$$

Similar to the above inequality (6.5), we obtain, $\|\Omega g_n - \varsigma\|_\infty \leq \|g_n - \varsigma\|_\infty$ implies,

$$\begin{aligned} \|e_n - \varsigma\|_\infty &= \|\Omega g_n - \varsigma\|_\infty \\ &\leq \left(1 - (1 - 2L_f(b - t_0))\mu_n\nu_n\right) \|c_n - \varsigma\|_\infty. \end{aligned} \quad (6.7)$$

Similar to the above inequality (6.5), we obtain, $\|\Omega e_n - \varsigma\|_\infty \leq \|e_n - \varsigma\|_\infty$ implies,

$$\|d_n - \varsigma\|_\infty = \|\Omega e_n - \varsigma\|_\infty \leq \|e_n - \varsigma\|_\infty \leq \left(1 - (1 - 2L_f(b - t_0))\mu_n\nu_n\right) \|c_n - \varsigma\|_\infty. \quad (6.8)$$

Similar to the above inequality (6.5), we obtain, $\|\Omega d_n - \varsigma\|_\infty \leq \|d_n - \varsigma\|_\infty$ implies,

$$\|\Omega d_n - \varsigma\|_\infty \leq \|d_n - \varsigma\|_\infty \leq \left(1 - (1 - 2L_f(b - t_0))\mu_n\nu_n\right) \|c_n - \varsigma\|_\infty. \quad (6.9)$$

Let $h_n = \lambda_n \Omega d_n + (1 - \lambda_n) \Omega e_n$, so that $c_{n+1} = \Omega h_n$. Then

$$\|h_n - \varsigma\|_\infty = \|\lambda_n \Omega d_n + (1 - \lambda_n) \Omega e_n - \varsigma\|_\infty \leq \lambda_n \|\Omega d_n - \varsigma\|_\infty + (1 - \lambda_n) \|\Omega e_n - \varsigma\|_\infty. \quad (6.10)$$

Applying (6.8) and (6.9) in (6.10) we get

$$\|h_n - \varsigma\|_\infty \leq \lambda_n \|\Omega d_n - \varsigma\|_\infty + (1 - \lambda_n) \|\Omega e_n - \varsigma\|_\infty \leq \left(1 - (1 - 2L_f(b - t_0))\mu_n\nu_n\right) \|c_n - \varsigma\|_\infty. \quad (6.11)$$

Consider,

$$\begin{aligned} \|c_{n+1} - \varsigma\|_\infty &= \|\Omega h_n - \varsigma\|_\infty \\ &= \max_{t \in [t_0 - \xi, b]} |\Omega h_n(t) - \Omega \varsigma(t)| \\ &= \max_{t \in [t_0 - \xi, b]} \left| \int_{t_0}^t f(p, h_n(p), h_n(p - \xi)) dp - \int_{t_0}^t f(p, \varsigma(p), \varsigma(p - \xi)) dp \right| \\ &\leq \max_{t \in [t_0 - \xi, b]} \int_{t_0}^t |f(p, h_n(p), h_n(p - \xi)) - f(p, \varsigma(p), \varsigma(p - \xi))| dp \\ &\leq \max_{t \in [t_0 - \xi, b]} \int_{t_0}^t L_f \left(|h_n(p) - \varsigma(p)| + |h_n(p - \xi) - \varsigma(p - \xi)| \right) dp \\ &\leq \int_{t_0}^t L_f \max_{u \in [t_0 - \xi, b]} \left(|h_n(p) - \varsigma(p)| + |h_n(p - \xi) - \varsigma(p - \xi)| \right) dp \\ &= \int_{t_0}^t L_f (\|h_n - \varsigma\|_\infty + \|h_n - \varsigma\|_\infty) dp \\ &\leq 2L_f(b - t_0) \|h_n - \varsigma\|_\infty \end{aligned}$$

$$\leq \|h_n - \varsigma\|_\infty.$$

Since $0 \leq (1 - 2L_f(b - t_0))\mu_n\nu_n \leq 1$, inequality (6.11) gives $\|h_n - \varsigma\|_\infty \leq \|c_n - \varsigma\|_\infty$. Hence, with $\theta := 2L_f(b - t_0)$,

$$\|c_{n+1} - \varsigma\|_\infty \leq \theta \|c_n - \varsigma\|_\infty \quad \text{for all } n \in \mathbb{N},$$

and by induction $\|c_{n+1} - \varsigma\|_\infty \leq \theta^n \|c_1 - \varsigma\|_\infty$. Since $\theta < 1$ by (D_5) , we obtain $\lim_{n \rightarrow \infty} \|c_n - \varsigma\|_\infty = 0$. $\square$

6.2. Application to a Fractional Integro-Differential Equation. Now, we apply our main results to study a class of nonlinear fractional integro-differential equations.

Let $S = C([0, 1], \mathbb{R})$ be the Banach space endowed with the norm

$$\|c\| = \sup_{t \in [0, 1]} |c(t)|.$$

Consider the nonlinear fractional integro-differential equation

$$D_t^\alpha c(t) = f(t, c(t)) + \int_0^t K_1(t, s, c(s)) ds + \int_0^1 K_2(t, s, c(s)) ds, \quad t \in [0, 1], \quad (6.12)$$

subject to the initial condition

$$c(0) = c_0,$$

where $0 < \alpha < 1$ and D_t^α denotes the Caputo fractional derivative.

Assume that the functions

$$f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}, \quad K_i : [0, 1] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}, \quad i = 1, 2,$$

are continuous.

Using the properties of the Caputo fractional derivative, equation (6.12) is equivalent to the integral equation

$$c(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s, c(s)) + \int_0^s K_1(s, r, c(r)) dr + \int_0^1 K_2(s, r, c(r)) dr \right] ds. \quad (6.13)$$

Define the operator $\Omega : S \rightarrow S$ by

$$(\Omega c)(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s, c(s)) + \int_0^s K_1(s, r, c(r)) dr + \int_0^1 K_2(s, r, c(r)) dr \right] ds. \quad (6.14)$$

Theorem 6.3. Let $\Omega : S \rightarrow S$ be the operator defined by (6.14). Suppose the following conditions hold:

(F₁) There exists $L > 0$ such that

$$|f(t, c) - f(t, d)| \leq L|c - d|$$

for all $t \in [0, 1]$ and $c, d \in \mathbb{R}$.

(F₂) There exist $M_1, M_2 > 0$ such that

$$|K_i(t, s, c) - K_i(t, s, d)| \leq M_i|c - d|, \quad i = 1, 2.$$

(F₃) The constants satisfy

$$\frac{L + M_1 + M_2}{\Gamma(\alpha + 1)} < 1.$$

Then the operator $\Omega : S \rightarrow S$ is an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping.

Proof. Let $c, d \in S$. Using (F₂) in (6.14),

$$|\Omega c(t) - \Omega d(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [L|c(s) - d(s)| + M_1|c(s) - d(s)| + M_2|c(s) - d(s)|] ds.$$

Thus,

$$|\Omega c(t) - \Omega d(t)| \leq \frac{L + M_1 + M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |c(s) - d(s)| ds.$$

Since $|c(s) - d(s)| \leq \|c - d\|$, we obtain

$$|\Omega c(t) - \Omega d(t)| \leq \frac{L + M_1 + M_2}{\Gamma(\alpha)} \|c - d\| \int_0^t (t-s)^{\alpha-1} ds.$$

Using

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha},$$

we obtain

$$|\Omega c(t) - \Omega d(t)| \leq \frac{L + M_1 + M_2}{\Gamma(\alpha + 1)} \|c - d\|.$$

Taking supremum,

$$\|\Omega c - \Omega d\| \leq \lambda \|c - d\|, \text{ where } \lambda = \frac{L + M_1 + M_2}{\Gamma(\alpha + 1)}.$$

Using triangle inequality,

$$\|\kappa(c - d) + \Omega c - \Omega d\| \leq \kappa \|c - d\| + \|\Omega c - \Omega d\|.$$

Hence

$$\|\kappa(c - d) + \Omega c - \Omega d\| \leq (\kappa + \lambda) \|c - d\|. \quad (6.15)$$

Using triangle inequality,

$$\|(\kappa + 1)(d - c) + c - \Omega c\| \geq |(\kappa + 1)\|c - d\| - \|c - \Omega c\||. \quad (6.16)$$

Similarly,

$$\|(\kappa + 1)(c - d) + d - \Omega d\| \geq |(\kappa + 1)\|c - d\| - \|d - \Omega d\||. \quad (6.17)$$

Adding (6.16) and (6.17), we obtain

$$\begin{aligned} & \frac{1}{2} \left[\|(\kappa + 1)(d - c) + c - \Omega c\| + \|(\kappa + 1)(c - d) + d - \Omega d\| \right] \\ & \geq (\kappa + 1) \|c - d\| - \frac{1}{2} (\|c - \Omega c\| + \|d - \Omega d\|). \end{aligned}$$

Using (F_3) , we have

$$\begin{aligned}(\kappa + \lambda)\|c - d\| &\leq (\kappa + 1)\|c - d\| - \frac{1}{2}(\|c - \Omega c\| + \|d - \Omega d\|) \\ &\leq \frac{1}{2}\left[\|(\kappa + 1)(d - c) + c - \Omega c\| + \|(\kappa + 1)(c - d) + d - \Omega d\|\right].\end{aligned}$$

Thus, by using (6.15)

$$\|\kappa(c - d) + \Omega c - \Omega d\| \leq \frac{1}{2}\left[\|(\kappa + 1)(d - c) + c - \Omega c\| + \|(\kappa + 1)(c - d) + d - \Omega d\|\right].$$

Hence Ω satisfies the enriched Chatterjea-Suzuki-(C) type nonexpansive. $\square$

Theorem 6.4. *The fractional integro-differential equation (6.12) admits a unique solution in $C([0, 1], \mathbb{R})$.*

Proof. From the previous theorem, the operator Ω is an enriched Chatterjea-Suzuki-(C) type nonexpansive mapping on the Banach space S .

Since S is complete and Ω is a self-map satisfying the enriched Chatterjea-Suzuki-(C) condition, all the assumptions of our main fixed point theorem are satisfied.

Hence Ω has a unique fixed point $\varsigma \in S$, that is

$$\Omega\varsigma = \varsigma.$$

Therefore, the fractional integro-differential equation (6.12) possesses a unique solution. $\square$

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] J. Ali, F. Ali, Approximation of Common Fixed Points and the Solution of Image Recovery Problem, *Results Math.* 74 (2019), 130. <https://doi.org/10.1007/s00025-019-1053-4>.
- [2] J. Ali, M. Jubair, Existence and Estimation of the Fixed Points of Enriched Berinde Nonexpansive Mappings, *Miskolc Math. Notes* 24 (2023), 541–552. <https://doi.org/10.18514/mmn.2023.3973>.
- [3] M. Abbas, T. Nazir, A New Faster Iteration Process Applied to Constrained Minimization and Feasibility Problems, *Mat. Vesn.* 66 (2014), 223–234.
- [4] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative Construction of Fixed Points of Nearly Asymptotically Nonexpansive Mappings, *J. Nonlinear Convex Anal.* 8 (2007), 61–79.
- [5] R. Anjum, N. Hussain, H. Alamri, B. Ali, Fixed Point Existence and Applications in Variational Inequalities via Modified Ćirić-Reich-Rus Contractions, *Res. Math.* 12 (2025), 2535083. <https://doi.org/10.1080/27684830.2025.2535083>.
- [6] S. Ajmal, H. Iqbal, R. Anjum, A. Turab, N. Hussain, On the Convergence of Some Iterative Schemes for Weak Contractions with Applications to Fractional-Order Blood Flow Models, *Numer. Algorithms* 103 (2026), 1435–1467. <https://doi.org/10.1007/s11075-025-02271-x>.
- [7] J.B. Baillon, R.E. Bruck, S. Reich, On the Asymptotic Behavior of Nonexpansive Mappings and Semigroups in Banach Spaces, *Houst. J. Math.* 4 (1978), 1–9.

- [8] V. Berinde, Approximating Fixed Points of Enriched Nonexpansive Mappings by Krasnoselskij Iteration in Hilbert Spaces, *Carpath. J. Math.* 35 (2019), 293–304. <https://doi.org/10.37193/cjm.2019.03.04>.
- [9] F.E. Browder, Nonexpansive Nonlinear Operators in a Banach Space, *Proc. Natl. Acad. Sci. USA* 54 (1965), 1041–1044. <https://doi.org/10.1073/pnas.54.4.1041>.
- [10] V. Berinde, M. Păcurar, Approximating Fixed Points of Enriched Contractions in Banach Spaces, *J. Fixed Point Theory Appl.* 22 (2020), 38. <https://doi.org/10.1007/s11784-020-0769-9>.
- [11] R. Sri Bharathi, A. Bera, Existence of Fixed Points Results via New Enriched Type of Nonexpansive Maps and Application to Delay Differential Equations, *J. Appl. Math. Comput.* 70 (2024), 5829–5855. <https://doi.org/10.1007/s12190-024-02162-9>.
- [12] V. Berinde, M. Păcurar, Approximating Fixed Points of Enriched Chatterjea Contractions by Krasnoselskij Iterative Algorithm in Banach Spaces, *J. Fixed Point Theory Appl.* 23 (2021), 66. <https://doi.org/10.1007/s11784-021-00904-x>.
- [13] A. Bera, A. Chanda, L.K. Dey, J. Ali, Iterative Approximation of Fixed Points of a General Class of Non-Expansive Mappings in Hyperbolic Metric Spaces, *J. Appl. Math. Comput.* 68 (2022), 1817–1839. <https://doi.org/10.1007/s12190-021-01592-z>.
- [14] A. Bera, L.K. Dey, H. Garai, S. Raha, Common Fixed Points via Asymptotic Contraction and Application to Matrix Equations, *Comput. Appl. Math.* 39 (2020), 301. <https://doi.org/10.1007/s40314-020-01358-6>.
- [15] A. Bera, İ.M. Erhan, R. Sri Bharathi, S. Sarkar, Fixed Points Results via Enriched Contraction and Application to Matrix Equations, *Fixed Point Theory* 27 (2026), 165–178.
- [16] J.A. Clarkson, Uniformly Convex Spaces, *Trans. Am. Math. Soc.* 40 (1936), 396–414. <https://doi.org/10.1090/s0002-9947-1936-1501880-4>.
- [17] S.K. Chatterjea, Fixed-Point Theorems, *C. R. Acad. Bulg. Sci.* 25 (1972), 727–730.
- [18] G.H. Coman, G. Pavel, I. Rus, I.A. Rus, *Introduction in the Theory of Operational Equation*, Ed. Dacia, Cluj-Napoca, 1976.
- [19] M. Edelstein, The Construction of an Asymptotic Center with a Fixed-Point Property, *Bull. Am. Math. Soc.* 78 (1972), 206–208. <https://doi.org/10.1090/s0002-9904-1972-12918-5>.
- [20] D. Göhde, Zum Prinzip der kontraktiven Abbildung, *Math. Nachr.* 30 (1965), 251–258. <https://doi.org/10.1002/mana.19650300312>.
- [21] M. Greguš, A Fixed Point Theorem in Banach Space, *Boll. Unione Mat. Ital.* 17 (1980), 193–198.
- [22] F. Gürsoy, V. Karakaya, A Picard-S Hybrid Type Iteration Method for Solving a Differential Equation with Retarded Argument, *arXiv:1403.2546*, (2014). <https://arxiv.org/abs/1403.2546>.
- [23] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, 1984.
- [24] C. Garodia, I. Uddin, A New Iterative Method for Solving Split Feasibility Problem, *J. Appl. Anal. Comput.* 10 (2020), 986–1004. <https://doi.org/10.11948/20190179>.
- [25] P. Gautam, Vineet, Convergence and Stability of a Novel Iterative Algorithm for Weak Contraction in Banach Spaces, *Rend. Circ. Mat. Palermo* 73 (2024), 1537–1554. <https://doi.org/10.1007/s12215-023-00991-4>.
- [26] N.-E. El Harmouchi, R. Outass, K. Chaira, J. Mouline, Fixed Point Approximation via a New Faster Iteration Process in Banach Spaces with an Application, *Adv. Fixed Point Theory* 14 (2024), 2. <https://doi.org/10.28919/afpt/8306>.
- [27] S. Ishikawa, Fixed Points by a New Iteration Method, *Proc. Am. Math. Soc.* 44 (1974), 147–150. <https://doi.org/10.1090/s0002-9939-1974-0336469-5>.
- [28] R. Kannan, Fixed Point Theorems in Reflexive Banach Spaces, *Proc. Am. Math. Soc.* 38 (1973), 111–118. <https://doi.org/10.1090/s0002-9939-1973-0313896-2>.
- [29] W.A. Kirk, A Fixed Point Theorem for Mappings Which Do Not Increase Distances, *Am. Math. Mon.* 72 (1965), 1004–1006. <https://doi.org/10.2307/2313345>.
- [30] S.H. Khan, H. Fukhar-ud-din, Weak and Strong Convergence of a Scheme with Errors for Two Nonexpansive Mappings, *Nonlinear Anal. Theory Methods Appl.* 61 (2005), 1295–1301. <https://doi.org/10.1016/j.na.2005.01.081>.

- [31] E. Karapınar, K. Taş, Generalized (C)-Conditions and Related Fixed Point Theorems, *Comput. Math. Appl.* 61 (2011), 3370–3380. <https://doi.org/10.1016/j.camwa.2011.04.035>.
- [32] V. Karakaya, Y. Atalan, K. Dogan, N.E.H. Bouzara, Convergence Analysis for a New Faster Iteration Method, *arXiv:1507.00200*, (2015). <https://arxiv.org/abs/1507.00200>.
- [33] K. Kumar, H. Alamri, V. Kumar, F. Gürsoy, I.K. Argyros, et al., A Robust Implicit Jungck Iterative Scheme: Stability, Data Dependency, and Applications to Nonlinear Integral Equations, *Bound. Value Probl.* 2025 (2025), 84. <https://doi.org/10.1186/s13661-025-02080-0>.
- [34] W.R. Mann, Mean Value Methods in Iteration, *Proc. Am. Math. Soc.* 4 (1953), 506–510. <https://doi.org/10.1090/s0002-9939-1953-0054846-3>.
- [35] M.A. Noor, New Approximation Schemes for General Variational Inequalities, *J. Math. Anal. Appl.* 251 (2000), 217–229. <https://doi.org/10.1006/jmaa.2000.7042>.
- [36] Z. Opial, Weak Convergence of the Sequence of Successive Approximations for Nonexpansive Mappings, *Bull. Am. Math. Soc.* 73 (1967), 591–597. <https://doi.org/10.1090/s0002-9904-1967-11761-0>.
- [37] E. Picard, Mémoire sur la Théorie des Équations aux Dérivées Partielles et la Méthode des Approximations Successives, *J. Math. Pures Appl.* 6 (1890), 145–210.
- [38] R. Pandey, R. Pant, A. Al-Rawashdeh, Fixed Point Results for a Class of Monotone Nonexpansive Type Mappings in Hyperbolic Spaces, *J. Funct. Spaces* 2018 (2018), 5850181. <https://doi.org/10.1155/2018/5850181>.
- [39] H. Piri, B. Daraby, S. Rahrovi, M. Ghasemi, Approximating Fixed Points of Generalized α -Nonexpansive Mappings in Banach Spaces by New Faster Iteration Process, *Numer. Algorithms* 81 (2019), 1129–1148. <https://doi.org/10.1007/s11075-018-0588-x>.
- [40] S. Reich, Kannan's Fixed Point Theorem, *Boll. Unione Mat. Ital.* 4 (1971), 1–11.
- [41] J. Schu, Weak and Strong Convergence to Fixed Points of Asymptotically Nonexpansive Mappings, *Bull. Aust. Math. Soc.* 43 (1991), 153–159. <https://doi.org/10.1017/s0004972700028884>.
- [42] T. Suzuki, Fixed Point Theorems and Convergence Theorems for Some Generalized Nonexpansive Mappings, *J. Math. Anal. Appl.* 340 (2008), 1088–1095. <https://doi.org/10.1016/j.jmaa.2007.09.023>.
- [43] Ş.M. Şoltuz, D. Otrocol, Classical Results via Mann-Ishikawa Iteration, *J. Numer. Anal. Approx. Theory* 36 (2007), 193–197. <https://doi.org/10.33993/jnaat362-868>.
- [44] R. Shukla, R. Pant, Some Fixed Point Results for Enriched Nonexpansive Type Mappings in Banach Spaces, *Appl. Gen. Topol.* 23 (2022), 31–43. <https://doi.org/10.4995/agt.2022.16165>.
- [45] S. Temir, Ö. Korkut, Approximating Fixed Points of Generalized α -Nonexpansive Mappings by the New Iteration Process, *J. Math. Sci. Model.* 5 (2022), 35–39. <https://doi.org/10.33187/jmsm.993823>.
- [46] B.S. Thakur, D. Thakur, M. Postolache, A New Iterative Scheme for Numerical Reckoning Fixed Points of Suzuki's Generalized Nonexpansive Mappings, *Appl. Math. Comput.* 275 (2016), 147–155. <https://doi.org/10.1016/j.amc.2015.11.065>.
- [47] K. Ullah, M. Arshad, Numerical Reckoning Fixed Points for Suzuki's Generalized Nonexpansive Mappings via New Iteration Process, *Filomat* 32 (2018), 187–196. <https://doi.org/10.2298/fil1801187u>.
- [48] I. Uddin, J. Ali, J.J. Nieto, An Iteration Scheme for a Family of Multivalued Mappings in CAT(0) Spaces with an Application to Image Recovery, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* 112 (2018), 373–384. <https://doi.org/10.1007/s13398-017-0382-y>.
- [49] I. Uddin, C. Garodia, J.J. Nieto, Mann Iteration for Monotone Nonexpansive Mappings in Ordered CAT(0) Space with an Application to Integral Equations, *J. Inequal. Appl.* 2018 (2018), 339. <https://doi.org/10.1186/s13660-018-1925-2>.
- [50] K. Ullah, J. Ahmad, M. Arshad, Z. Ma, Approximating Fixed Points Using a Faster Iterative Method and Application to Split Feasibility Problems, *Computation* 9 (2021), 90. <https://doi.org/10.3390/computation9080090>.

- [51] K. Ullah, J. Ahmad, M. Arshad, Z. Ma, Approximation of Fixed Points for Enriched Suzuki Nonexpansive Operators with an Application in Hilbert Spaces, *Axioms* 11 (2022), 14. <https://doi.org/10.3390/axioms11010014>.
- [52] H. Ullah, K. Ullah, A. Asghar, M. Tariq, H. Ahmad, et al., New Approach of Generalized α -Nonexpansive Mappings via HK-Iteration Process with Applications to Fractional Differential Equation and Boundary Value Problems, *Contemp. Math.* 7 (2026), 1662–1682. <https://doi.org/10.37256/cm.7220268645>.