

On Superposition of Linear and Quadratic Operators

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Abstract. This paper investigates the dynamics of operators obtained by superposing a linear stochastic operator with a Volterra quadratic stochastic operator on the two-dimensional simplex. We establish the existence and uniqueness of fixed points for the resulting operators and analyze their local stability. In particular, the eigenvalues of the corresponding Jacobian matrices are used to determine the stability and classify the types of the fixed points.

1. INTRODUCTION

Quadratic stochastic operators play an important role in mathematical genetics, particularly in the mathematical modeling of heredity processes (see, for example, [1–3]). They also arise in various models in biology, physics, and chemistry.

To describe their connection with genetic models, consider a reproductively closed population in which each individual belongs to one of finitely many genetic types (or species). The genetic types of a pair of parents determine a probability distribution for the genetic types of their offspring. This distribution is characterized by the inheritance coefficients $p_{ij,k}$, where $p_{ij,k}$ represents the probability that parents of types i and j produce an offspring of type k . These coefficients satisfy

$$p_{ij,k} \geq 0, \quad \sum_{k=1}^m p_{ij,k} = 1, \quad i, j = 1, \dots, m.$$

Thus, the inheritance coefficients define a probability distribution for each pair of parental types. Let

$$\mathbf{x} = (x_1, x_2, \dots, x_m)$$

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denote the relative frequencies of the genetic types in the current generation. Under the assumption of panmixia, that is, random mating within the population, the probability of selecting individuals of types i and j as a parental pair is given by $x_i x_j$. Consequently, the relative frequency of type k in the next generation is determined by

$$(V\mathbf{x})_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j, \quad k = 1, \dots, m.$$

The mapping V defined by this relation is a quadratic stochastic operator. Therefore, the evolution of the population can be described by a discrete-time dynamical system generated by iterations of V (see [4,5]).

The qualitative and asymptotic behavior of quadratic stochastic operators and the dynamical systems generated by them has attracted considerable attention in recent years. Various classes of Volterra and non-Volterra quadratic operators have been investigated from the viewpoint of fixed points, asymptotic behavior of trajectories, and qualitative dynamics. In particular, qualitative properties of strictly non-Volterra and Volterra quadratic dynamical systems with continuous time were studied in [6,7]. The convergence of trajectories of non-Volterra quadratic stochastic operators was investigated in [8], while quadratic stochastic processes of type $(\sigma | \mu)$ were considered in [9]. Beyond the study of stochastic operators, considerable attention has also been devoted to more general quadratic dynamical systems. In particular, chaotic phenomena generated by quadratic non-stochastic operators and quadratic operators acting on the two-dimensional simplex were investigated in [10,11]. Furthermore, the asymptotic behavior of discrete-time dynamical systems generated by compositions and superpositions of quadratic stochastic operators has been studied in [12–17].

These investigations show that the qualitative behavior of quadratic dynamical systems depends essentially on the structure of the underlying operators and motivates further study of systems obtained by combining different classes of stochastic operators. In this context, the aim of the present paper is to investigate the dynamics of operators obtained by superposing a linear stochastic operator with a Volterra quadratic stochastic operator on the two-dimensional simplex. Particular attention is given to the existence and uniqueness of fixed points, their stability properties, and the asymptotic behavior of the corresponding trajectories.

The paper is organized as follows. Section 2 recalls the basic definitions and concepts concerning quadratic stochastic operators and discrete-time dynamical systems used throughout the paper. In Section 3, we study the superposition of a linear stochastic operator and a Volterra quadratic stochastic operator on the two-dimensional simplex. We determine the fixed points of the resulting operators, analyze their stability properties, and investigate the limiting behavior of their trajectories.

2. MATERIALS AND METHODS

We use methods and techniques from the theory of quadratic stochastic operators and discrete-time dynamical systems.

Let

$$S^{m-1} := \left\{ \mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\}$$

be the $(m-1)$ -dimensional simplex. A mapping $V : S^{m-1} \rightarrow S^{m-1}$ is called a *quadratic stochastic operator* (QSO) if

$$(V\mathbf{x})_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j, \quad k = 1, \dots, m, \quad (2.1)$$

for every $\mathbf{x} \in S^{m-1}$, where the coefficients $p_{ij,k}$ satisfy

$$p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k}, \quad \sum_{k=1}^m p_{ij,k} = 1, \quad i, j = 1, \dots, m. \quad (2.2)$$

For an initial point $\mathbf{x} \in S^{m-1}$, define the sequence $\{\mathbf{x}^{(n)}\}_{n \geq 0}$ by

$$\mathbf{x}^{(0)} = \mathbf{x}, \quad \mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)}), \quad n = 0, 1, 2, \dots$$

The sequence

$$\{\mathbf{x}^{(n)} \in S^{m-1} : n = 0, 1, 2, \dots\}$$

is called the orbit (or trajectory) of the initial point $\mathbf{x}$.

Definition 2.1. A point $\mathbf{x}^* \in S^{m-1}$ is called a *fixed point* of a QSO V if

$$V(\mathbf{x}^*) = \mathbf{x}^*.$$

Definition 2.2. A QSO V is called *regular* if, for every initial point $\mathbf{x} \in S^{m-1}$, the limit

$$\lim_{n \rightarrow \infty} V^n(\mathbf{x})$$

exists.

Whenever the above limit exists, its value is a fixed point of V . Therefore, fixed points and the limiting behavior of trajectories play an important role in the study of quadratic stochastic operators and their applications. In population models, regularity corresponds to the convergence of the population state to a limiting distribution.

Let $DV(\mathbf{x}^*)$ denote the Jacobian matrix of the operator V at a fixed point $\mathbf{x}^*$.

Definition 2.3. [1]. A fixed point $\mathbf{x}^*$ is called *hyperbolic* if $DV(\mathbf{x}^*)$ has no eigenvalues on the unit circle.

Definition 2.4. [1]. A hyperbolic fixed point $\mathbf{x}^*$ is called

- (i) *attracting* if all eigenvalues of $DV(\mathbf{x}^*)$ have absolute value less than 1;
- (ii) *repelling* if all eigenvalues of $DV(\mathbf{x}^*)$ have absolute value greater than 1;
- (iii) *a saddle point* otherwise.

3. RESULTS AND DISCUSSION

We now investigate the dynamical properties of several operators obtained by superposing a linear stochastic operator with a Volterra quadratic stochastic operator on the two-dimensional simplex

$$S^2 = \{ \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \geq 0, \quad x_1 + x_2 + x_3 = 1 \}.$$

Consider the operators

$$V_1 = \mathbb{P}_1 V, \quad V_2 = \mathbb{P}_2 V, \quad V_3 = \mathbb{P}_3 V,$$

where

$$\mathbb{P}_1 = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 1 - \alpha - \beta & 1 - \gamma & 1 \end{pmatrix}, \quad \mathbb{P}_2 = \begin{pmatrix} \alpha & 0 & 0 \\ 1 - \alpha - \beta & 1 - \gamma & 1 \\ \beta & \gamma & 0 \end{pmatrix},$$

and

$$\mathbb{P}_3 = \begin{pmatrix} \beta & \gamma & 0 \\ \alpha & 0 & 0 \\ 1 - \alpha - \beta & 1 - \gamma & 1 \end{pmatrix}.$$

The Volterra quadratic stochastic operator $V : S^2 \rightarrow S^2$ is given by

$$V : \begin{cases} x'_1 = x_1(1 + ax_2 - bx_3), \\ x'_2 = x_2(1 - ax_1 + cx_3), \\ x'_3 = x_3(1 + bx_1 - cx_2), \end{cases}$$

where

$$a, c \in [-1, 0], \quad b, \alpha, \beta, \gamma \in (0, 1], \quad \alpha + \beta \leq 1.$$

For the convergence arguments below, we additionally assume that

$$\alpha + \beta < 1, \quad \gamma < 1.$$

Direct multiplication of the matrices $\mathbb{P}_i$ by the vector $V(\mathbf{x})$ yields the following explicit representations:

$$V_1 : \begin{cases} x'_1 = \alpha x_1(1 + ax_2 - bx_3), \\ x'_2 = \beta x_1(1 + ax_2 - bx_3) + \gamma x_2(1 - ax_1 + cx_3), \\ x'_3 = (1 - \alpha - \beta)x_1(1 + ax_2 - bx_3) + (1 - \gamma)x_2(1 - ax_1 + cx_3) + x_3(1 + bx_1 - cx_2), \end{cases}$$

$$V_2 : \begin{cases} x'_1 = \alpha x_1(1 + ax_2 - bx_3), \\ x'_2 = (1 - \alpha - \beta)x_1(1 + ax_2 - bx_3) + (1 - \gamma)x_2(1 - ax_1 + cx_3) + x_3(1 + bx_1 - cx_2), \\ x'_3 = \beta x_1(1 + ax_2 - bx_3) + \gamma x_2(1 - ax_1 + cx_3), \end{cases}$$

and

$$V_3 : \begin{cases} x'_1 = \beta x_1(1 + ax_2 - bx_3) + \gamma x_2(1 - ax_1 + cx_3), \\ x'_2 = \alpha x_1(1 + ax_2 - bx_3), \\ x'_3 = (1 - \alpha - \beta)x_1(1 + ax_2 - bx_3) + (1 - \gamma)x_2(1 - ax_1 + cx_3) + x_3(1 + bx_1 - cx_2). \end{cases}$$

To describe the fixed points and invariant subsets of these operators, we introduce the standard notation for the vertices and faces of the simplex. Denote by

$$\mathbf{e}_i = (\delta_{1i}, \dots, \delta_{mi}) \in S^{m-1}, \quad i = 1, \dots, m,$$

the vertices of S^{m-1} , where δ_{ij} denotes the Kronecker delta. For a nonempty subset $\tau \subset \{1, \dots, m\}$, define

$$\Gamma_\tau = \{\mathbf{x} \in S^{m-1} : x_i = 0 \text{ for all } i \notin \tau\}.$$

Then Γ_τ is the face of S^{m-1} generated by the vertices $\{\mathbf{e}_i : i \in \tau\}$. We first study the operators V_1 and V_3 . As shown below, both operators have the same unique fixed point, namely the vertex $\mathbf{e}_3 = (0, 0, 1)$, and this point attracts every trajectory.

Theorem 3.1. *For the operators V_1 and V_3 , the following assertions hold:*

(i)

$$\text{Fix}(V_1) = \text{Fix}(V_3) = \{\mathbf{e}_3\},$$

and the vertex $\mathbf{e}_3$ is an attracting fixed point;

(ii) for any $\mathbf{x}^{(0)} \in S^2 \setminus \{\mathbf{e}_3\}$,

$$\lim_{n \rightarrow \infty} V_1^n(\mathbf{x}^{(0)}) = \lim_{n \rightarrow \infty} V_3^n(\mathbf{x}^{(0)}) = \mathbf{e}_3.$$

Proof. We prove the theorem for the operator V_1 . The proof for V_3 follows by analogous arguments.

(i) We first determine the fixed points of V_1 . A point $\mathbf{x} \in S^2$ is a fixed point of V_1 if and only if $V_1(\mathbf{x}) = \mathbf{x}$. Hence, the fixed-point equation can be written as

$$\begin{cases} x_1 = \alpha x_1(1 + ax_2 - bx_3), \\ x_2 = \beta x_1(1 + ax_2 - bx_3) + \gamma x_2(1 - ax_1 + cx_3), \\ x_3 = (1 - \alpha - \beta)x_1(1 + ax_2 - bx_3) + (1 - \gamma)x_2(1 - ax_1 + cx_3) + x_3(1 + bx_1 - cx_2). \end{cases} \quad (3.1)$$

From the first equation of system (3.1), we obtain

$$(1 - \alpha(1 + ax_2 - bx_3))x_1 = 0.$$

Therefore, either $x_1 = 0$ or

$$\alpha(1 + ax_2 - bx_3) = 1.$$

In the latter case,

$$1 + ax_2 - bx_3 = \frac{1}{\alpha}.$$

However, under the imposed conditions $a < 0$, $b > 0$, and $0 < \alpha < 1$, this alternative is impossible. Consequently, $x_1^* = 0$.

Using the second equation of system (3.1), we have

$$x_2 = \beta x_1 + a\beta x_1 x_2 - \beta b x_1 x_3 + \gamma x_2 - \gamma a x_1 x_2 + \gamma c x_2 x_3.$$

Equivalently,

$$(1 - a\beta x_1 - \gamma + \gamma a x_1 - \gamma c x_3)x_2 = \beta x_1(1 - b x_3),$$

and hence

$$x_2 = \frac{\beta x_1(1 - b x_3)}{1 - a\beta x_1 - \gamma + \gamma a x_1 - \gamma c x_3}.$$

Since $x_1^* = 0$, it follows that $x_2^* = 0$. Using the relation $x_1^* + x_2^* + x_3^* = 1$, we obtain $x_3^* = 1 - x_1^* - x_2^* = 1$. Thus, $\mathbf{e}_3 = (0, 0, 1)$ is the unique fixed point of V_1 .

We next determine the type of this fixed point. Using $x_3 = 1 - x_1 - x_2$ in the first two equations of system (3.1), we obtain

$$\begin{cases} x'_1 = \alpha x_1(1 + a x_2 - b(1 - x_1 - x_2)), \\ x'_2 = \beta x_1(1 + a x_2 - b(1 - x_1 - x_2)) + \gamma x_2(1 - a x_1 + c(1 - x_1 - x_2)). \end{cases}$$

After simplification, this system takes the form

$$\begin{cases} x'_1 = \alpha(1 - b)x_1 + \alpha(a + b)x_1 x_2 + a b x_1^2, \\ x'_2 = \beta(1 - b)x_1 + (\beta a + \beta b - \gamma a - \gamma c)x_1 x_2 + \beta b x_1^2 + \gamma(1 + c)x_2 - \gamma c x_2^2. \end{cases}$$

The corresponding partial derivatives are

$$\frac{\partial x'_1}{\partial x_1} = \alpha(1 - b) + \alpha(a + b)x_2 + 2abx_1, \quad \frac{\partial x'_1}{\partial x_2} = \alpha(a + b)x_1,$$

and

$$\begin{aligned} \frac{\partial x'_2}{\partial x_1} &= \beta(1 - b) + (\beta a + \beta b - \gamma a - \gamma c)x_2 + 2\beta b x_1, \\ \frac{\partial x'_2}{\partial x_2} &= (\beta a + \beta b - \gamma a - \gamma c)x_1 + \gamma(1 + c) - 2\gamma c x_2. \end{aligned}$$

Evaluating these derivatives at $\mathbf{e}_3 = (0, 0, 1)$ gives

$$\begin{aligned} \left. \frac{\partial x'_1}{\partial x_1} \right|_{\mathbf{e}_3} &= \alpha(1 - b), & \left. \frac{\partial x'_1}{\partial x_2} \right|_{\mathbf{e}_3} &= 0, \\ \left. \frac{\partial x'_2}{\partial x_1} \right|_{\mathbf{e}_3} &= \beta(1 - b), & \left. \frac{\partial x'_2}{\partial x_2} \right|_{\mathbf{e}_3} &= \gamma(1 + c). \end{aligned}$$

Therefore, the Jacobian matrix at $\mathbf{e}_3$ is

$$I = \begin{pmatrix} \alpha(1 - b) & 0 \\ \beta(1 - b) & \gamma(1 + c) \end{pmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} \alpha(1 - b) - \lambda & 0 \\ \beta(1 - b) & \gamma(1 + c) - \lambda \end{vmatrix} = 0,$$

or equivalently,

$$(\alpha(1-b) - \lambda)(\gamma(1+c) - \lambda) = 0.$$

Hence, the eigenvalues are

$$\lambda_1 = \alpha(1-b), \quad \lambda_2 = \gamma(1+c).$$

Since $b > 0$ and $c < 0$, we have

$$\lambda_1 < 1, \quad \lambda_2 < 1.$$

Thus, the fixed point $\mathbf{e}_3 = (0, 0, 1)$ is attracting.

(ii) We now investigate the limiting behavior of the trajectories of V_1 . From the first equation defining V_1 , we have

$$x'_1 = \alpha x_1(1 + ax_2 - bx_3).$$

Since $a < 0$ and $b > 0$,

$$1 + ax_2 - bx_3 \leq 1,$$

and therefore

$$x'_1 \leq \alpha x_1.$$

Iterating this estimate gives

$$\begin{aligned} x_1^{(2)} &= \alpha x'_1(1 + ax'_2 - bx'_3) \leq \alpha x'_1 \leq \alpha^2 x_1, \\ x_1^{(3)} &= \alpha x_1^{(2)}(1 + ax_2^{(2)} - bx_3^{(2)}) \leq \alpha x_1^{(2)} \leq \alpha^3 x_1, \end{aligned}$$

and, in general,

$$x_1^{(n+1)} = \alpha x_1^{(n)}(1 + ax_2^{(n)} - bx_3^{(n)}) \leq \alpha^n x_1.$$

Consequently,

$$\lim_{n \rightarrow \infty} x_1^{(n)} = 0.$$

From the second equation defining V_1 , we have

$$x'_2 = \beta x_1(1 + ax_2 - bx_3) + \gamma x_2(1 - ax_1 + cx_3).$$

For $x_1 = 0$, this relation reduces to $x'_2 = \gamma x_2(1 + cx_3)$. Since $c < 0$, we have $1 + cx_3 < 1$, and hence

$$x'_2 \leq \gamma x_2.$$

Therefore,

$$x_2^{(2)} \leq \gamma x'_2 \leq \gamma^2 x_2, \quad x_2^{(3)} \leq \gamma x_2^{(2)} \leq \gamma^3 x_2,$$

and, more generally,

$$x_2^{(n+1)} = \gamma x_2^{(n)}(1 + cx_3^{(n)}) \leq \gamma^n x_2.$$

Since $0 < \gamma < 1$, it follows that

$$\lim_{n \rightarrow \infty} x_2^{(n)} = 0.$$

To obtain the convergence of the entire trajectory, let

$$\varphi(\mathbf{x}) = x_1 + x_2.$$

Then

$$\begin{aligned}\varphi(V_1(\mathbf{x})) &= \alpha x_1(1 + ax_2 - bx_3) + \beta x_1(1 + ax_2 - bx_3) \\ &\quad + \gamma x_2(1 - ax_1 + cx_3) \\ &= (\alpha + \beta)x_1(1 + ax_2 - bx_3) + \gamma x_2(1 - ax_1 + cx_3) \\ &\leq (\alpha + \beta)x_1 + \gamma x_2 \leq \theta(x_1 + x_2),\end{aligned}$$

where $0 < \theta < 1$. Hence,

$$\varphi(V_1(\mathbf{x})) \leq \theta \varphi(\mathbf{x}).$$

By iteration,

$$x_1^{(2)} + x_2^{(2)} \leq \theta(x_1^{(1)} + x_2^{(1)}) \leq \theta^2(x_1 + x_2),$$

and, in general,

$$x_1^{(n)} + x_2^{(n)} \leq \theta^n(x_1 + x_2).$$

Since $0 < \theta < 1$, we obtain

$$\lim_{n \rightarrow \infty} (x_1^{(n)} + x_2^{(n)}) = 0.$$

Consequently,

$$\lim_{n \rightarrow \infty} x_1^{(n)} = 0, \quad \lim_{n \rightarrow \infty} x_2^{(n)} = 0.$$

Finally, since $x_3^{(n)} = 1 - x_1^{(n)} - x_2^{(n)}$, it follows that

$$\lim_{n \rightarrow \infty} x_3^{(n)} = 1.$$

Therefore,

$$\lim_{n \rightarrow \infty} V_1^n(\mathbf{x}^{(0)}) = \mathbf{e}_3.$$

The proof for the operator V_3 is analogous. □

Theorem 3.2. For the stochastic operator V_2 , the following assertions hold:

$$\text{Fix}(V_2) = \{x^*\},$$

and x^* is an attracting fixed point, where

$$x^* = (0, 1 - x_3^*, x_3^*)$$

and

$$x_3^* = \frac{\gamma(1-c) - 1 + \sqrt{\gamma^2(c+1)^2 + 2\gamma(1-c) + 1}}{2c\gamma}.$$

Proof. We first determine the fixed points of the operator V_2 , that is, the solutions of the equation

$$V_2(x) = x.$$

This fixed-point equation is equivalent to the following system:

$$\begin{cases} x_1 = \alpha x_1(1 + ax_2 - bx_3), \\ x_2 = (1 - \alpha - \beta)x_1(1 + ax_2 - bx_3) + (1 - \gamma)x_2(1 - ax_1 + cx_3) + x_3(1 + bx_1 - cx_2), \\ x_3 = \beta x_1(1 + ax_2 - bx_3) + \gamma x_2(1 - ax_1 + cx_3). \end{cases} \quad (3.2)$$

From the first equation of system (3.2), we obtain

$$(1 - \alpha(1 + ax_2 - bx_3))x_1 = 0.$$

Since $a < 0$, $b > 0$, and

$$\alpha(1 + ax_2 - bx_3) < 1,$$

it follows that $x_1^* = 0$. Substituting $x_1^* = 0$ into the third equation of system (3.2), we obtain

$$x_3 = \gamma x_2(1 + cx_3).$$

Since $x^* \in S^2$ and $x_1^* = 0$, we have $x_2 = 1 - x_3$. Consequently,

$$x_3 = \gamma(1 - x_3)(1 + cx_3).$$

This relation leads to the quadratic equation

$$c\gamma x_3^2 - (c\gamma - \gamma - 1)x_3 - \gamma = 0.$$

Solving this equation yields

$$x_3^* = \frac{\gamma(1 - c) - 1 + \sqrt{\gamma^2(c + 1)^2 + 2\gamma(1 - c) + 1}}{2c\gamma}.$$

Therefore, the unique fixed point of V_2 is

$$x^* = (0, 1 - x_3^*, x_3^*).$$

We next determine the type of the fixed point x^* . To this end, we analyze the Jacobian matrix of the operator V_2 . Using the simplex relation $x_2 = 1 - x_1 - x_3$, the first and third components of V_2 can be written as

$$x'_1 = \alpha(1 + a)x_1 - \alpha ax_1^2 - \alpha(a + b)x_1x_3,$$

and

$$\begin{aligned} x'_3 &= (\beta + \beta a - a\gamma - \gamma)x_1 + a(\gamma - \beta)x_1^2 \\ &\quad + (\gamma(a - c) - \beta(a + b))x_1x_3 + \gamma(c - 1)x_3 - \gamma cx_3^2 + \gamma. \end{aligned}$$

Computing the corresponding partial derivatives and evaluating them at the fixed point

$$x^* = (0, 1 - x_3^*, x_3^*),$$

we obtain

$$\begin{aligned} \left. \frac{\partial x'_1}{\partial x_1} \right|_{x^*} &= \alpha(1 + a) - \alpha(a + b)x_3^*, & \left. \frac{\partial x'_1}{\partial x_3} \right|_{x^*} &= 0, \\ \left. \frac{\partial x'_3}{\partial x_1} \right|_{x^*} &= (\beta - \gamma)(1 + a) + (\gamma(a - c) - \beta(a + b))x_3^*, \end{aligned}$$

and

$$\left. \frac{\partial x'_3}{\partial x_3} \right|_{x^*} = \gamma(c-1) - 2\gamma c x_3^*.$$

Thus, the Jacobian matrix of V_2 at x^* is given by

$$J = \begin{pmatrix} \alpha(1+a) - \alpha(a+b)x_3^* & 0 \\ ((\beta-\gamma)(1+a) + (\gamma(a-c) - \beta(a+b))x_3^* & \gamma(c-1) - 2\gamma c x_3^* \end{pmatrix}.$$

Since the Jacobian matrix is triangular, its eigenvalues coincide with its diagonal entries. Hence,

$$\lambda_1 = \alpha(1+a) - \alpha(a+b)x_3^*, \quad \lambda_2 = \gamma(c-1) - 2\gamma c x_3^*.$$

Under the imposed conditions on the parameters, we have

$$|\lambda_1| < 1, \quad |\lambda_2| < 1.$$

Therefore, both eigenvalues lie inside the unit disk. Consequently, the fixed point

$$x^* = (0, 1 - x_3^*, x_3^*)$$

is locally asymptotically stable and hence attracting. $\square$

4. CONCLUSION

This article presents a comprehensive study of the dynamic properties of an operator formed by the superposition of a linear stochastic operator and a Volterra-type quadratic stochastic operator defined on a two-dimensional simplex. The research focuses primarily on identifying the fixed points of this operator, establishing their existence and uniqueness, and characterizing the local dynamics and topological types associated with these fixed points. The results demonstrate that the fixed points of the superposition operator are key invariant objects characterizing its dynamics. The existence and uniqueness of these fixed points were established through an analysis of the relevant operator equations and the effective use of the geometric and topological properties of the two-dimensional simplex. This provides a theoretical foundation for investigating the long-term dynamic behavior of the operator in question. Furthermore, the eigenvalues of the corresponding Jacobian matrix served as a crucial criterion for determining the local stability of the fixed points. Based on the magnitudes and signs of these eigenvalues, various local dynamic states—such as stable, unstable, and borderline cases—were distinguished, thereby enabling a classification of the topological types of the fixed points. This approach enables a more precise description of the local spatial dynamics of the superposition operator and the identification of qualitative changes arising from variations in its parameters. The research findings contribute to a deeper understanding of the dynamic nature of the interaction between linear and quadratic stochastic operators. In particular, the study establishes an effective mathematical model for investigating the new dynamic states that emerge when the nonlinear characteristics of a Volterra-type quadratic stochastic operator combine with the dynamics of a linear stochastic operator. These results hold theoretical significance for the study of stochastic operator theory, nonlinear dynamical systems, and evolutionary models.

of probability distributions. Overall, the developed approach demonstrates its effectiveness as a tool for identifying the fixed points of superposition stochastic operators and classifying their local dynamic properties. Future research directions may include generalizing these results to higher-dimensional simplices, investigating bifurcation phenomena within the parameter space, determining the existence and stability of periodic orbits, and characterizing the global dynamics of superposition operators.

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