

A Study on the Apostol Fibonacci-Sigmoid Polynomials of Higher Order α **So Young Park¹, Hui Young Lee^{1,*}, Seongju Do²**¹*Department of Mathematics, Hannam University Daejeon 34430, Korea*²*Department of Computational Science Korea Atomic Energy Research Institute Daejeon 34057, Korea***Corresponding author: hylee@hnu.kr*

Abstract. In this paper, we introduce the Apostol Fibonacci-Sigmoid polynomials of higher order α by means of a generating function involving the golden exponential function. Using the generalized binomial and golden binomial theorem, we derive explicit representations and several fundamental identities for these polynomials. We also establish recurrence relations, addition formulas and differential properties within the framework of Fibonacci calculus. Furthermore, symmetric identity is obtained through suitably constructed generating functions. These results broaden the theoretical framework of Fibonacci sigmoid polynomials and pave the way for future investigations into higher-order Fibonacci-type special polynomials.

1. INTRODUCTION

The construction of higher-order polynomial families has become an active area of research in recent years. Such generalizations not only extend known results but also reveal deeper relationships among special polynomials. In particular, the generalized binomial theorem provides an effective method for defining and analyzing higher-order generating functions. This approach has been successfully applied to numerous classical polynomial families. Inspired by these studies, we investigate the Fibonacci sigmoid polynomials of higher order α .

The study of Fibonacci-type analogues of classical functions and polynomials has recently attracted growing interest. By introducing the Fibonacci factorial and Fibonacci exponential function in place of the classical factorial and exponential function, one can develop various families of special polynomials possessing distinctive structural and analytical features. For completeness, we recall below the definitions of the Fibonacci factorial and the golden exponential function.

The Fibonacci factorial is defined as [3, 4]

$$F_n! = F_n F_{n-1} F_{n-2} \cdots F_1, \quad F_0! = 1$$

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where F_n denotes Fibonacci number.

From the definition, the Fibonacci binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{F_n!}{F_k! F_{n-k}!} \quad (0 \leq k \leq n).$$

The golden exponential functions of the first and second types are given, respectively [5]

$$e_F^x = \sum_{n=0}^{\infty} \frac{x^n}{F_n!},$$

$$E_F^x = \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{x^n}{F_n!}.$$

The following property holds:

$$e_F^x E_F^y = e_F^{(x+y)_F} = \sum_{n=0}^{\infty} \frac{(x+y)_F^n}{F_n!}.$$

From golden binomial expansion, we have

$$(x+y)_F^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\binom{k}{2}} x^{n-k} y^k.$$

The golden derivative is defined by

$$D_F^x(f(x)) = \frac{f(\varphi x) - f(\varphi' x)}{(\varphi - \varphi')x}. \quad [3,4]$$

where φ and φ' are called the golden and silver ratio, respectively.

From the definition, the following relations hold:

$$D_F^x(e_F^{kx}) = k e_F^{kx}, \quad D_F^x(x^n) = F_n x^{n-1}.$$

The Fibonacci sigmoid polynomials $S_{n,F}(x)$ are defined by

$$\sum_{n=0}^{\infty} S_{n,F}(x) \frac{t^n}{F_n!} = \frac{1}{1 + e_F^{-t}} e_F^{xt}. \quad [1,2]$$

By setting $x = 0$ in the above definition, we obtain the Fibonacci sigmoid numbers $S_{n,F}$.

2. GENERAL PROPERTIES OF THE APOSTOL-TYPE FIBONACCI SIGMOID POLYNOMIALS OF HIGHER ORDER

Definition 2.1. For $\alpha, x, t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ the Apostol-type Fibonacci sigmoid polynomials of higher order α , $S_{n,F}^{(\alpha)}(x, \lambda)$ defined by

$$\sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!} = \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{xt},$$

$$\sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(\lambda) \frac{t^n}{F_n!} = \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha}.$$

When $\alpha = 1$ and $\lambda = 1$ in Definition 2.1, we recover the Fibonacci sigmoid polynomials $S_{n,F}(x)$.

Differentiating both sides of the generating function of order α in Definition 2.1 with respect to t yields several polynomials and numbers. We introduce some of these below.

$$\begin{aligned} S_{0,F}^{(\alpha)}(x, \lambda) &= (1 + \lambda)^{-\alpha}, \\ S_{1,F}^{(\alpha)}(x, \lambda) &= (1 + \lambda)^{-\alpha-1} \left((1 + \lambda)x + \alpha\lambda \right), \\ S_{2,F}^{(\alpha)}(x, \lambda) &= (1 + \lambda)^{-\alpha-2} \left((1 + \lambda)^2 x^2 + \alpha\lambda(1 + \lambda)x - \alpha\lambda(1 + \lambda) + \frac{\alpha(\alpha + 1)}{2} \lambda^2 \right), \\ S_{0,F}^{(\alpha)}(\lambda) &= (1 + \lambda)^{-\alpha}, \\ S_{1,F}^{(\alpha)}(\lambda) &= \alpha\lambda(1 + \lambda)^{-\alpha-1}, \\ S_{2,F}^{(\alpha)}(\lambda) &= -\alpha\lambda(1 + \lambda)^{-\alpha-1} + \frac{\alpha(\alpha + 1)}{2} \lambda^2 (1 + \lambda)^{-\alpha-2}. \end{aligned}$$

Theorem 2.1. (Addition formula) For $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} (1) \quad S_{n,F}^{(\alpha)}(x + y, \lambda) &= \sum_{k=0}^n (-1)^{\binom{k}{2}} \left[\begin{matrix} n \\ k \end{matrix} \right]_F y^k S_{n-k,F}^{(\alpha)}(x, \lambda), \\ (2) \quad S_{n,F}^{(\alpha)}(x + y, \lambda) &= \sum_{l=0}^n \sum_{k=0}^l \left[\begin{matrix} n \\ l \end{matrix} \right]_F \left[\begin{matrix} l \\ k \end{matrix} \right]_F S_{n-l,F}^{(\alpha)}(\lambda) (-1)^{\binom{k}{2}} x^{l-k} y^k. \end{aligned}$$

Proof. (1) From the Definition 2.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x + y, \lambda) \frac{t^n}{F_n!} &= \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{(x+y)Ft} \\ &= \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{xt} E_F^{yt} \\ &= \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!} \times \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{(yt)^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_F (-1)^{\binom{k}{2}} y^k S_{n-k,F}^{(\alpha)}(x, \lambda) \right) \frac{t^n}{F_n!}. \end{aligned}$$

The desired result follows by comparing the coefficients of $\frac{t^n}{F_n!}$ on both sides.

(2) From the golden binomial identity, it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x+y, \lambda) \frac{t^n}{F_n!} &= \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{(x+y)_F t} \\ &= \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(\lambda) \frac{t^n}{F_n!} \times \sum_{n=0}^{\infty} (x+y)_F^n \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(\lambda) \frac{t^n}{F_n!} \times \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\binom{k}{2}} x^{n-k} y^k \right) \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \sum_{k=0}^l \begin{bmatrix} n \\ l \end{bmatrix}_F \begin{bmatrix} l \\ k \end{bmatrix}_F S_{n-l,F}^{(\alpha)}(\lambda) (-1)^{\binom{k}{2}} x^{l-k} y^k \right) \frac{t^n}{F_n!}. \end{aligned}$$

□

Theorem 2.2. For $\alpha, \beta \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we obtain

$$\begin{aligned} (1) \quad S_{n,F}^{(\alpha+\beta)}(x, \lambda) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_F S_{n-k,F}^{(\alpha)}(x, \lambda) S_{k,F}^{(\beta)}(\lambda), \\ (2) \quad S_{n,F}^{(\alpha+\beta)}(x, \lambda) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_F S_{n-k,F}^{(\beta)}(x, \lambda) S_{k,F}^{(\alpha)}(\lambda). \end{aligned}$$

Proof. (1) Using Definition 2.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} S_{n,F}^{(\alpha+\beta)}(x, \lambda) \frac{t^n}{F_n!} &= \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\beta} e_F^{xt} \\ &= \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!} \times \sum_{n=0}^{\infty} S_{n,F}^{(\beta)}(\lambda) \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_F S_{n-k,F}^{(\alpha)}(x, \lambda) S_{k,F}^{(\beta)}(\lambda) \right) \frac{t^n}{F_n!}. \end{aligned}$$

For Equation (2), the expansion proceeds in the same way as for Equation (1).

□

Theorem 2.3. For $\alpha, x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we get

$$D_F^x \left(S_{n,F}^{(\alpha)}(x, \lambda) \right) = F_n S_{n-1,F}^{(\alpha)}(x, \lambda).$$

Proof. Using the golden derivative, we obtain

$$\begin{aligned} D_F^x \left(\sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!} \right) &= D_F^x \left[\left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{(xt)} \right] \\ &= t \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{(xt)} \\ &= \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^{n+1}}{F_n!} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} S_{n-1,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_{n-1}!} \\
&= \sum_{n=0}^{\infty} F_n S_{n-1,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!}.
\end{aligned}$$

□

Corollary 2.1. For $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we get

$$D_F^{x,(k)} \left(S_{n,F}^{(\alpha)}(x, \lambda) \right) = \frac{F_n!}{F_{n-k}!} S_{n-k,F}^{(\alpha)}(x, \lambda).$$

Theorem 2.4. (Recurrence relation) For $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we get

$$\begin{aligned}
(1) \quad S_{n,F}^{(\alpha-1)}(x, \lambda) &= \lambda \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^k S_{n-k,F}^{(\alpha)}(x, \lambda) + S_{n,F}^{(\alpha)}(x, \lambda), \\
(2) \quad S_{n,F}^{(\alpha-1)}(x, \lambda) &= \lambda \sum_{l=0}^n \sum_{k=0}^l \begin{bmatrix} n \\ l \end{bmatrix}_F \begin{bmatrix} l \\ k \end{bmatrix}_F S_{n-l,F}^{(\alpha)}(\lambda) (-1)^k x^{l-k} + S_{n,F}^{(\alpha)}(x, \lambda).
\end{aligned}$$

Proof. (1) By using the generating function in Definition 2.1 and the identity

$$\begin{aligned}
&\left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha-1} e_F^{xt} \\
&= (1 + \lambda e_F^{-t}) \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{xt} \\
&= \lambda e_F^{-t} \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{xt} + \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{xt}.
\end{aligned}$$

Hence, we get

$$\sum_{n=0}^{\infty} S_{n,F}^{(\alpha-1)}(x, \lambda) \frac{t^n}{F_n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_F \lambda (-1)^k S_{n-k,F}^{(\alpha)}(x, \lambda) \right) \frac{t^n}{F_n!} + \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!}.$$

The desired result follows by comparing the coefficients of $\frac{t^n}{F_n!}$ on both sides.

(2) By using the generating function in Definition 2.1 and the identity

$$\begin{aligned}
&\left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha-1} e_F^{xt} \\
&= (1 + \lambda e_F^{-t}) \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{xt} \\
&= \lambda \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} \times e_F^{xt} e_F^{-t} + \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{\alpha} e_F^{xt} \\
&= \lambda \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(\lambda) \frac{t^n}{F_n!} \times \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^k x^{n-k} \right) \frac{t^n}{F_n!} + \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!}.
\end{aligned}$$

□

Hence, we get

$$\sum_{n=0}^{\infty} S_{n,F}^{(\alpha-1)}(x, \lambda) \frac{t^n}{F_n!} = \sum_{n=0}^{\infty} \left(\lambda \sum_{l=0}^n \sum_{k=0}^l \begin{bmatrix} n \\ l \end{bmatrix}_F \begin{bmatrix} l \\ k \end{bmatrix}_F S_{n-l,F}^{(\alpha)}(\lambda) (-1)^k x^{l-k} + S_{n,F}^{(\alpha)}(x, \lambda) \right) \frac{t^n}{F_n!}.$$

By the generalized binomial theorem, we have

$$(1+x)^\alpha = \sum_{m=0}^{\infty} \binom{\alpha}{m} x^m, \quad \alpha \in \mathbb{R}$$

Note that,

$$\binom{\alpha}{m} = \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!}.$$

By setting $x = \lambda e_F^{-t}$, we obtain the following expression.

$$(\lambda e_F^{-t} + 1)^\alpha = \sum_{m=0}^{\infty} \binom{\alpha}{m} \lambda^m (e_F^{-t})^m.$$

The power of the golden exponential has different properties compared to the usual exponentiation. Since it cannot be simply expressed as $(e^x)^m = e^{mx}$, we aim to define the power of the golden exponential through the following process and symbolize it for use.

$$\begin{aligned} (e_F^{-t})^2 &= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{F_n!} \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k_1=0}^n \begin{bmatrix} n \\ k_1 \end{bmatrix}_F (-1)^{k_1} \right) \frac{t^n}{F_n!} \\ (e_F^{-t})^3 &= \sum_{n=0}^{\infty} \left(\sum_{k_1=0}^n \begin{bmatrix} n \\ k_1 \end{bmatrix}_F (-1)^{k_1} \right) \frac{t^n}{F_n!} \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k_2=0}^n \sum_{k_1=0}^{n-k_2} \begin{bmatrix} n \\ k_2 \end{bmatrix}_F \begin{bmatrix} n-k_2 \\ k_1 \end{bmatrix}_F (-1)^{k_1+k_2} \right) \frac{t^n}{F_n!} \end{aligned}$$

Repeating this process m times, we obtain

$$\begin{aligned} (e_F^{-t})^m &= \sum_{n=0}^{\infty} \left(\sum_{k_{m-1}=0}^n \sum_{k_{m-2}=0}^{n-k_{m-1}} \sum_{k_{m-3}=0}^{n-k_{m-1}-k_{m-2}} \cdots \sum_{k_1=0}^{n-k_{m-1}-\cdots-k_{m-2}} \right. \\ &\quad \left. \begin{bmatrix} n \\ k_{m-1} \end{bmatrix}_F \begin{bmatrix} n-k_{m-1} \\ k_{m-2} \end{bmatrix}_F \begin{bmatrix} n-k_{m-1}-k_{m-2} \\ k_{m-3} \end{bmatrix}_F \cdots \begin{bmatrix} n-k_{m-1}-\cdots-k_2 \\ k_1 \end{bmatrix}_F (-1)^{k_1+\cdots+k_{m-1}} \right) \frac{t^n}{F_n!}. \end{aligned}$$

Definition 2.2. For $m \in \mathbb{N}$, we define $B_m(n, k_1, k_2, \dots, k_{m-1})$ as follows:

$$B_m(n, k_1, k_2, \dots, k_{m-1})$$

$$= \left(\sum_{k_{m-1}=0}^n \sum_{k_{m-2}=0}^{n-k_{m-1}} \sum_{k_{m-3}=0}^{n-k_{m-1}-k_{m-2}} \cdots \sum_{k_1=0}^{n-k_2-\cdots-k_{m-1}} \begin{bmatrix} n \\ k_{m-1} \end{bmatrix}_F \begin{bmatrix} n-k_{m-1} \\ k_{m-2} \end{bmatrix}_F \right. \\ \left. \begin{bmatrix} n-k_{m-1}-k_{m-2} \\ k_{m-3} \end{bmatrix}_F \cdots \begin{bmatrix} n-k_{m-1}-\cdots-k_2 \\ k_1 \end{bmatrix}_F \right).$$

Therefore, for $m \in \mathbb{N}$, $(e_F^{-t})^m$ is expressed as follows:

$$(e_F^{-t})^m = \sum_{n=0}^{\infty} (-1)^n B_m(n, k_1, k_2, \dots, k_{m-1}) \frac{t^n}{F_n!}.$$

Corollary 2.2. For $\alpha \in \mathbb{R}$, we get

$$(\lambda e_F^{-t} + 1)^\alpha = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \binom{\alpha}{m} \lambda^m (-1)^n B_m(n, k_1, k_2, \dots, k_{m-1}) \right) \frac{t^n}{F_n!}.$$

Theorem 2.5. For $n \geq 1$, $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we get

$$0 = \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_F S_{n-l,F}^{(\alpha)}(\lambda) \left(\sum_{m=0}^{\infty} \binom{\alpha}{m} \lambda^m (-1)^l B_m(l, k_1, k_2, \dots, k_{m-1}) \right).$$

Proof. Using the Definition 2.7, we have

$$\begin{aligned} 1 &= \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^\alpha \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^{-\alpha} \\ &= \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^\alpha (1 + \lambda e_F^{-t})^\alpha \\ &= \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(\lambda) \frac{t^n}{F_n!} \times \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \binom{\alpha}{m} \lambda^m (-1)^n B_m(n, k_1, k_2, \dots, k_{m-1}) \right) \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_F S_{n-l,F}^{(\alpha)}(\lambda) \sum_{m=0}^{\infty} \binom{\alpha}{m} \lambda^m (-1)^l B_m(l, k_1, k_2, \dots, k_{m-1}) \right) \frac{t^n}{F_n!}. \end{aligned}$$

□

Theorem 2.6. For $n \geq 0$, $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we get

$$x^n = \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_F S_{n-l,F}^{(\alpha)}(x, \lambda) \left(\sum_{m=0}^{\infty} \binom{\alpha}{m} \lambda^m (-1)^l B_m(l, k_1, k_2, \dots, k_{m-1}) \right).$$

Proof.

$$\begin{aligned} \left(\frac{1}{1 + \lambda e_F^{-t}} \right)^\alpha e_F^{xt} &= \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!} \\ &\longrightarrow e_F^{xt} = \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!} \times (1 + \lambda e_F^{-t})^\alpha. \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} x^n \frac{t^n}{F_n!} &= \sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(x, \lambda) \frac{t^n}{F_n!} \times \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \binom{\alpha}{m} \lambda^m (-1)^n B_m(n, k_1, k_2, \dots, k_{m-1}) \right) \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \left[\begin{matrix} n \\ l \end{matrix} \right]_F S_{n-l,F}^{(\alpha)}(x, \lambda) \sum_{m=0}^{\infty} \binom{\alpha}{m} \lambda^m (-1)^l B_m(l, k_1, k_2, \dots, k_{m-1}) \right) \frac{t^n}{F_n!}. \end{aligned}$$

The desired result follows by comparing the coefficients of $\frac{t^n}{F_n!}$ on both sides. $\square$

To investigate symmetry properties of the Apostol Fibonacci sigmoid polynomials of order α , we introduce the following special function:

$$G^{(\alpha)}(x, t) = \frac{1}{(\lambda_1 e_F^{-m_1 t} + 1)^\alpha (\lambda_2 e_F^{-m_2 t} + 1)^\alpha} e_F^{m_1 m_2 x t},$$

where α is real number and $\lambda_1, \lambda_2, m_1$ and m_2 are complex numbers.

First type:

$$\begin{aligned} G^{(\alpha)}(x, t) &= \left(\frac{1}{\lambda_1 e_F^{-m_1 t} + 1} \right)^\alpha e_F^{(m_2 x)(m_1 t)} \left(\frac{1}{\lambda_2 e_F^{-m_2 t} + 1} \right)^\alpha \\ &= \left(\sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(\lambda_1, m_2 x) \frac{(m_1 t)^n}{F_n!} \right) \left(\sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(\lambda_2) \frac{(m_2 t)^n}{F_n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_F S_{k,F}^{(\alpha)}(\lambda_1, m_2 x) S_{n-k,F}^{(\alpha)}(\lambda_2) m_1^k m_2^{n-k} \right) \frac{t^n}{F_n!}. \end{aligned}$$

Second type:

$$\begin{aligned} G^{(\alpha)}(x, t) &= \left(\frac{1}{\lambda_2 e_F^{-m_2 t} + 1} \right)^\alpha e_F^{(m_1 x)(m_2 t)} \left(\frac{1}{\lambda_1 e_F^{-m_1 t} + 1} \right)^\alpha \\ &= \left(\sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(\lambda_2, m_1 x) \frac{(m_2 t)^n}{F_n!} \right) \left(\sum_{n=0}^{\infty} S_{n,F}^{(\alpha)}(\lambda_1) \frac{(m_1 t)^n}{F_n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_F S_{k,F}^{(\alpha)}(\lambda_2, m_1 x) S_{n-k,F}^{(\alpha)}(\lambda_1) m_2^k m_1^{n-k} \right) \frac{t^n}{F_n!}. \end{aligned}$$

Theorem 2.7. (Symmetric property) $\alpha \in \mathbb{R}$ and $\lambda_1, \lambda_2, m_1, m_2 \in \mathbb{C}$, we get

$$\begin{aligned} &\sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_F S_{k,F}^{(\alpha)}(m_2 x, \lambda_1) S_{n-k,F}^{(\alpha)}(\lambda_2) m_1^k m_2^{n-k} \\ &= \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_F S_{k,F}^{(\alpha)}(m_1 x, \lambda_2) S_{n-k,F}^{(\alpha)}(\lambda_1) m_2^k m_1^{n-k}. \end{aligned}$$

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