

On Transcendental Entire Solutions of System of Non-Linear Complex Differential-Difference Equations

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Abstract. Using Nevanlinna theory of the value distribution of meromorphic functions, we investigate the problem of transcendental entire solutions of one type of system of complex differential-difference equations, and provide the explicit form of all transcendental entire function solutions to the system of equations, as well as several existence conditions for the system. In fact, we consider a broader class of system, which contains the corresponding system for equation studied in the reference *Scientia Sinica Mathematica* 2019, 49 by Liu and Gao. These findings extend the existing results of complex differential (difference) equations to the systems of differential-difference equations.

1. INTRODUCTION AND RESULTS

In this paper, we use the standard notations of the Nevanlinna's value distribution theory (e.g. see refs. [2, 7, 8, 17]), i.e. $N(r, f)$ and $T(r, f)$ denote the counting function and the characteristic function of f , respectively. For the integrated counting function for distinct poles of f we use the notation $\bar{N}(r, f)$. In addition, we will use the notation $\sigma(f)$ to denote the order of growth of meromorphic function $f(z)$. Moreover, we say that a meromorphic function g is small with respect to f if $T(r, g) = S(r, f)$, where $S(r, f) = o(T(r, f))$ outside a possible exceptional set of finite linear measure.

Fermat's last theorem [13] states that there is no nonzero rational numbers x, y and an integer $n \geq 3$ such that $x^n + y^n = 1$. The equation $x^2 + y^2 = 1$ does admit nontrivial rational solutions. Replacing x, y in above equation by entire or meromorphic functions f, g , Fermat type functional

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equations were studied by Gross [5,6] and many others thereafter, such as [12,14]. Yang [14] study the Fermat type equation

$$a(z)f(z)^n + b(z)g(z)^m = 1, \quad (1.1)$$

where $a(z), b(z)$ are small with respect to $f(z)$, and obtained the following result.

Theorem A. ([14]) Suppose m, n are two positive integers and satisfy $\frac{1}{m} + \frac{1}{n} < 1$, then there is no nonconstant entire solution $f(z)$ and $g(z)$ satisfy equation (1.1).

The above theorem implies that there is no nonconstant entire solutions with the assumption of $n > 2, m > 2$ in (1.1). However, when $m = n = 2$ and $g(z)$ has a specific relationship with $f(z)$ in (1.1), the question of whether we can obtain the exact expression of the entire function solution deserves our consideration. For example, in 2004, Yang and Li [15] considered the entire solutions of

$$(f'(z))^2 + f(z)^2 = 1. \quad (1.2)$$

In fact, they considered a generalization of the above equation. The solutions of (1.2) can be described as follows.

Theorem B. ([15]) Transcendental meromorphic solutions of (1.2) satisfy $f(z) = \frac{1}{2}(Pe^{-iz} + \frac{1}{P}e^{iz})$, where P is a nonzero constant.

Let $e^A = P$ in Theorem B. The solutions of (1.2) also can be written as $f(z) = \sin(z + Ai + \frac{\pi}{2}) = \cos(z + Ai)$.

In 2012, Liu et al. ([10]) considered Fermat type differential-difference equation

$$(f'(z))^2 + f(z+c)^2 = 1, \quad (1.3)$$

and obtained the following result.

Theorem C. ([10]) The transcendental entire solutions with finite order of the differential-difference equation (1.3) must satisfy $f(z) = \sin(z \pm iB)$, where B is a constant and $c = 2k\pi$ or $c = 2k\pi + \pi$, k is an integer.

In 2016, Chen et al. [1] obtain the following conclusion.

Theorem D. ([1]) Suppose that $P(z)$ and $Q(z)$ are two nonzero polynomials. If differential-difference equation

$$(f'(z))^2 + P^2(z)f(z+c)^2 = Q(z), \quad (1.4)$$

admits a transcendental entire solution $f(z)$ of finite order. Then $P(z) \equiv A (\neq 0)$ and $Q(z) \equiv pq (\neq 0)$, and

$$f(z) = \frac{pe^{az+b} - qe^{-(az+b)}}{2a}, \quad (1.5)$$

where $a = \pm iA$, $A = \frac{(-1)^k k\pi}{c}$, $b \in \mathbb{C}$, $p, q, c \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{Z}$.

Gao et al. had discussed the existence or growth of some types of systems of complex difference equations, and obtained some results (see [3,4,9]). For example, in 2016, Gao (see [4]) considered the system of differential-difference equations of the form

$$\begin{cases} (w_1'(z))^2 + w_2(z+c)^2 = 1, \\ (w_2'(z))^2 + w_1(z+c)^2 = 1, \end{cases} \quad (1.6)$$

and obtained the following result.

Theorem E. ([4]) *Let $(w_1(z), w_2(z))$ be a pair transcendental entire solution with $\sigma(w_1(z), w_2(z)) < \infty$ of the system of differential-difference equations (1.6). Then $(w_1(z), w_2(z))$ satisfy*

$$(w_1(z), w_2(z)) = (\sin(z - bi), \sin(z - b_1i))$$

or

$$(w_1(z), w_2(z)) = (\sin(z + bi), \sin(z + b_1i))$$

where b, b_1 are constants, $c = k\pi, k$ is an integer.

The order of growth of meromorphic solution (w_1, w_2) of system (1.6) is defined by

$$\begin{aligned} \sigma &= \sigma(w_1, w_2) = \max\{\sigma(w_1), \sigma(w_2)\}, \\ \sigma(w_k) &= \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, w_k)}{\log r}, k = 1, 2. \end{aligned}$$

The coefficients involved in the system (1.6) are all constant 1, which is the simplest case. Later, Gao further studied the system with polynomials. For example, in 2017, Gao (see [3]) considered the system of differential-difference equations of the form

$$\begin{cases} (w_1'(z))^2 + w_2(z+c)^2 = Q_1(z), \\ (w_2'(z))^2 + w_1(z+c)^2 = Q_2(z), \end{cases} \quad (1.7)$$

where Q_i ($i = 1, 2$) are non-zeros polynomials and obtained the following result.

Theorem F. ([3]) *Let (w_1, w_2) be a pair transcendental entire solution with $\sigma(w_1, w_2) < \infty$ of the system of differential-difference equations (1.7). Then $Q_1(z) = c_{11}c_{12}$, $Q_2(z) = c_{21}c_{22}$, and*

$$w_1(z) = \frac{c_{11}e^{az+b_1} - c_{12}e^{-az-b_1}}{2a}, \quad w_2(z) = \frac{c_{21}e^{az+b_2} - c_{22}e^{-az-b_2}}{2a},$$

where $a^4 = 1$, $b_1, b_2, c_{kj} (\neq 0, k, j = 1, 2)$ are constants.

Differential and difference equations are powerful tools for depicting the changing laws of natural phenomena. Many nonlinear phenomena in engineering technology, medicine, ecology, and mathematical physics can be characterized by establishing corresponding mathematical models with the aid of differential and difference equations. In reality, when there are disturbances, the coefficients of the equations reflecting natural phenomena are generally no longer constants but become variable coefficients. In fact, the study of complex problems is often accompanied by equations with variable coefficients. When the constant 1 on the right-hand side of the system

of equations (1.6) becomes polynomials $Q_1(z), Q_2(z)$, the process of studying the solutions of the system (1.7) becomes more complex and challenging.

In 2019, Liu et al. consider the form of the entire function solutions for a certain class of complex differential-difference equations

$$(w''(z))^2 + w(z+c)^2 = Q(z), \quad (1.8)$$

where $Q(z)$ is a polynomial, $c \neq 0 \in \mathbb{C}$. They obtain the following result.

Theorem G. ([11]) *Let $w_1(z)$ be a pair transcendental entire solution with $\sigma(w_1, w_2) < \infty$ of the differential-difference equation (1.8). Then $Q_1(z) = c_1 c_2$ is a constant, and $w_1(z)$ satisfies*

$$w_1(z) = \frac{c_1 e^{az+b} + c_2 e^{-az-b}}{2a^2},$$

where a, b are constant, $a^4 = 1, c = \frac{\ln(-ia^2) + 2k\pi i}{a}$, k is an integer.

In fact, the increase in the order of the derivative has made our study of the transcendental entire function solutions of the system of equations much more difficult. For instance, when studying the first-order derivative, there is a discussion about $Q'P - QP' + QPh'$ (see $R_1(z)$ in the following Lemma 2.3), which contains 3 terms product of polynomials; when studying the second-order derivative, there is a discussion about $P^2Q'' - PP''Q - PP'Qh' + 2P^2Q'h' + P^2Qh'' - 2PP'Q + 2(P')^2Q - PP'Qh' + P^2Q(h')^2$, which contains 9 terms product of polynomials (see $R_2(z)$ in the following Lemma 2.3). The increase in the number of terms makes the research more complex. This article will attempt to conduct a study of the second-order derivative under the condition of having two pairs of polynomials.

After comparing and analyzing the theorems A-G, we naturally raised the following two questions.

(1) How to determine the specific forms of $w_1(z)$ and $w_2(z)$ in Theorems F? That is to say, what are the values of c_{11}, c_{12}, c_{21} and c_{22} respectively? Can the relationship between these coefficients be expressed in a more concise form?

(2) Consider the corresponding system for equation (1.8), or a broader class of system, that is, the following system (1.9), what conditions need to be met in order for it to have a pair finite-order transcendental entire solution?

The aim of our paper is to investigate transcendental entire solutions with finite order of the systems of differential-difference equations of the form

$$\begin{cases} (w_1''(z))^2 + P_2^2(z)w_2(z+c)^2 = Q_1(z), \\ (w_2''(z))^2 + P_1^2(z)w_1(z+c)^2 = Q_2(z), \end{cases} \quad (1.9)$$

where $P_i(z), Q_i(z)$ ($i = 1, 2$) are non-zeros polynomials.

Theorem 1.1. *Let (w_1, w_2) be a pair transcendental entire solution with $\sigma(w_1, w_2) < \infty$ of the system of differential-difference equations (1.9). Then $P_i(z), Q_i(z)$ ($i = 1, 2$) must reduce to nonzero constants, and denoted by p_i, q_i respectively, and there exists a non-zero constant a that satisfies $\frac{Q_2}{Q_1} = \frac{a^4}{p_2^2} = \frac{p_1^2}{a^4}$ and*

$e^{4ac} = 1$. Further, all transcendental entire solutions with $\sigma(w_1, w_2) < \infty$ of system (1.9) are of the forms and can only be of the forms

$$\begin{aligned} w_1(z) &= \frac{1}{2a^2}(de^{az} + \frac{Q_1}{d}e^{-az}), \\ w_2(z) &= \frac{1}{2ip_2}(de^{-ac}e^{az} - \frac{Q_1}{d}e^{ac}e^{-az}), \end{aligned}$$

where d is an arbitrary non-zero constant.

Remark 1.1. Theorem 1.1 answers the two questions raised above.

(1) Unlike Theorem F, which presents a solution form with undetermined constants c_{11}, c_{12}, c_{21} and c_{22} , Theorem 1.1 explicitly specifies all parameters within the solutions, and no longer contains any unknown parameters. Consequently, no illustrative examples are required to specify these parameters, unlike in [3].

(2) Theorem 1.1 establishes the necessary and sufficient conditions for the existence of finite order transcendental entire solutions, namely: all the polynomial coefficients $Q_i(z), P_i(z)$ ($i = 1, 2$) have been reduced to constants, and denoted by Q_i, p_i respectively, and there exists a non-zero constant a that satisfies $\frac{Q_2}{Q_1} = \frac{a^4}{p_2^2} = \frac{p_1^2}{a^4}$ and $e^{4ac} = 1$. In contrast, Theorem F does not give such conditions.

Remark 1.2. Actually, from the formula (3.39) in the following proof process of Theorem 1.1, we have $\frac{Q_2}{Q_1} = \frac{p_1^2}{a^4} = \frac{a^4}{p_2^2} = \pm \frac{p_1}{p_2}, a^8 = p_1^2 p_2^2$. Hence $\frac{Q_2}{Q_1} = \frac{a^4}{p_2^2} = \frac{p_1^2}{a^4}$ appears in Theorem 1.1 can be replaced by one of the following terms equivalently,

$$\begin{cases} \frac{Q_2}{Q_1} = \frac{a^4}{p_2^2} = \pm \frac{p_1}{p_2}; & \frac{Q_2}{Q_1} = \frac{a^4}{p_2^2}, a^8 = p_1^2 p_2^2; \\ \frac{Q_2}{Q_1} = \frac{p_1^2}{a^4} = \pm \frac{p_1}{p_2}; & \frac{Q_2}{Q_1} = \frac{p_1^2}{a^4}, a^8 = p_1^2 p_2^2; \\ & \frac{Q_2}{Q_1} = \pm \frac{p_1}{p_2}, a^8 = p_1^2 p_2^2. \end{cases}$$

The following corollaries 1.1-1.2 can be easily derived from Theorem 1.1 and Remark 1.2, which establish several sufficient conditions for the non-existence of finite order transcendental entire solutions to the system (1.9).

Corollary 1.1. System (1.9) has no transcendental entire solutions of finite order, if at least one of P_1, P_2, Q_1, Q_2 is a non-constant polynomial.

Corollary 1.2. Suppose $Q_i, p_i (i = 1, 2)$ are non-zero constants, then system

$$\begin{cases} (w_1''(z))^2 + p_2^2 w_2(z+c)^2 = Q_1, \\ (w_2''(z))^2 + p_1^2 w_1(z+c)^2 = Q_2 \end{cases} \quad (1.10)$$

has no transcendental entire solutions of finite order, if one of the following conditions is satisfied:

- (I) there does not exist an a that satisfies both $\frac{Q_2}{Q_1} = \frac{a^4}{p_2^2} = \frac{p_1^2}{a^4}$ and $e^{4ac} = 1$;
- (II) there does not exist an a that satisfies both $\frac{Q_2}{Q_1} = \frac{a^4}{p_2^2} = \pm \frac{p_1}{p_2}$ and $e^{4ac} = 1$;

- (III) there does not exist an a that satisfies both $\frac{Q_2}{Q_1} = \frac{a^4}{p_2^2}, a^8 = p_1^2 p_2^2$ and $e^{4ac} = 1$;
 (IV) there does not exist an a that satisfies both $\frac{Q_2}{Q_1} = \frac{p_1^2}{a^4} = \pm \frac{p_1}{p_2}$ and $e^{4ac} = 1$;
 (V) there does not exist an a that satisfies both $\frac{Q_2}{Q_1} = \frac{p_1^2}{a^4}, a^8 = p_1^2 p_2^2$ and $e^{4ac} = 1$;
 (VI) there does not exist an a that satisfies both $\frac{Q_2}{Q_1} = \pm \frac{p_1}{p_2}, a^8 = p_1^2 p_2^2$ and $e^{4ac} = 1$.

Remark 1.3. If $c = 0$, then system (1.9) becomes

$$\begin{cases} (w_1''(z))^2 + P_2^2(z)w_2(z)^2 = Q_1(z), \\ (w_2''(z))^2 + P_1^2(z)w_1(z)^2 = Q_2(z), \end{cases} \quad (1.11)$$

where $Q_i(z), P_i(z)$ ($i = 1, 2$) are non-zeros polynomials. Note that $e^{4ac} = 1$ for any non-zero constant a , and we have the following corollary.

Corollary 1.3. Let (w_1, w_2) be a pair transcendental entire solution with $\sigma(w_1, w_2) < \infty$ of the system of differential-difference equations (1.11). Then $Q_i(z), P_i(z)$ ($i = 1, 2$) must reduce to nonzero constants, and denoted by Q_i, p_i respectively, and there exists a non-zero constant a that satisfies $\frac{Q_2}{Q_1} = \frac{a^4}{p_2^2} = \frac{p_1^2}{a^4}$. Further, all transcendental entire solutions with $\sigma(w_1, w_2) < \infty$ of system (1.11) are of the forms and can only be of the forms

$$\begin{aligned} w_1(z) &= \frac{1}{2a^2}(de^{az} + \frac{Q_1}{d}e^{-az}), \\ w_2(z) &= \frac{1}{2ip_2}(de^{az} - \frac{Q_1}{d}e^{-az}), \end{aligned}$$

where d is an arbitrary non-zero constant.

2. LEMMA FOR THE PROOF OF THEOREM

Lemma 2.1. ([16]) Let $f_j(z)$ be meromorphic functions, $f_k(z)$ ($k = 1, 2, \dots, n-1$) being nonconstant, satisfying $\sum_{j=1}^n f_j = 1$ and $n \geq 3$. If $f_n(z) \not\equiv 0$ and

$$\sum_{j=1}^n N(r, \frac{1}{f_j}) + (n-1) \sum_{j=1}^n \bar{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $\lambda < 1$ and $k = 1, 2, \dots, n-1$, then $f_n \equiv 1$.

Lemma 2.2. ([16]) Suppose that $n \geq 2$ and let $f_1(z), \dots, f_n(z)$ be meromorphic functions and $g_1(z), \dots, g_n(z)$ be entire functions such that

- (i) $\sum_{j=1}^n f_j(z) \exp\{g_j(z)\} = 0$;
- (ii) when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not constant;
- (iii) when $1 \leq j \leq n, 1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\} \quad (r \rightarrow \infty, r \notin E),$$

where $E \subset (1, \infty)$ has finite linear measure or logarithmic measure.

Then $f_j(z) \equiv 0, \quad j = 1, \dots, n$.

Lemma 2.3. Let $H(z) = \frac{Q(z)}{P(z)}e^{h(z)}$, where $Q(z), P(z)$ are non-zero polynomials, $h(z)$ is a non-constant polynomial. Suppose that

$$H''(z) = \frac{R_2(z)}{P(z)^3}e^{h(z)},$$

then $R_2(z) \not\equiv 0$, is a polynomial with degree

$$\deg R_2(z) = 2 \deg P(z) + \deg Q(z) + 2(\deg h(z) - 1).$$

Proof. Obviously, we have

$$H'(z) = \frac{R_1(z)}{P(z)^2}e^{h(z)}, \quad R_1(z) := Q'(z)P(z) - Q(z)P'(z) + Q(z)P(z)h'(z).$$

Note that $Q(z), P(z)$ are non-zero polynomial, $\deg h(z) \geq 1$, if $Q(z)$ and $P(z)$ are constants, then $R_1(z) = Q(z)P(z)h'(z)$ is a polynomial, with degree $\deg h(z) - 1$; if at least one of $Q(z)$ and $P(z)$ is not a constant, we have $\deg(Q'(z)P(z) - Q(z)P'(z)) \leq \deg(Q(z)P(z)) - 1 < \deg(Q(z)P(z)) \leq \deg(Q(z)P(z)h'(z))$. Hence $R_1(z) \not\equiv 0$, is a polynomial with degree $\deg P(z) + \deg Q(z) + \deg h(z) - 1$.

Repeat the operation as the previous step, we have

$$\begin{aligned} H''(z) &= \frac{R_2(z)}{P(z)^3}e^{h(z)}, \\ R_2(z) &:= R_1'(z)P(z) - 2R_1(z)P'(z) + R_1(z)P(z)h'(z) \\ &= P(z)^2Q''(z) - P(z)P''(z)Q(z) - 2P(z)P'(z)Q(z)h'(z) + 2P(z)^2Q'(z)h'(z) \\ &\quad + P(z)^2Q(z)h''(z) - 2P(z)P'(z)Q(z) + 2P'(z)^2Q(z) + P(z)^2Q(z)(h'(z))^2. \end{aligned}$$

By comparing the degrees of each term on the right-hand side of the above equation, it can be observed that only the last term $P^2(z)Q(z)(h'(z))^2$ has the highest degree, and we have

$$\deg R_2(z) = 2 \deg P(z) + \deg Q(z) + 2(\deg h(z) - 1).$$

Further, if $P(z) \equiv p$ is a constant, $h(z) = \pm az + c_1, a \neq 0$, then $R_2(z)$ becomes

$$R_2(z) = p^2[(\pm a)^2Q(z) \pm 2aQ'(z) + Q''(z)].$$

3. PROOF OF THEOREM 1.1

Suppose that (w_1, w_2) is a pair transcendental entire solution with $\sigma(w_1, w_2) < \infty$ to the system of differential-difference equations (1.9).

We rewrite (1.9) as follows

$$\begin{cases} (w_1''(z) + iP_2(z)w_2(z+c))(w_1''(z) - iP_2(z)w_2(z+c)) = Q_1(z), \\ (w_2''(z) + iP_1(z)w_1(z+c))(w_2''(z) - iP_1(z)w_1(z+c)) = Q_2(z). \end{cases}$$

Then

$$\begin{aligned} w_1''(z) + iP_2(z)w_2(z+c), & \quad w_1''(z) - iP_2(z)w_2(z+c), \\ w_2''(z) + iP_1(z)w_1(z+c), & \quad w_2''(z) - iP_1(z)w_1(z+c) \end{aligned}$$

both have finite number of zeros since $Q_1(z)$, $Q_2(z)$ have finite zeros. By Hadamard's factorization theorem [16], we can suppose that

$$\begin{cases} w_1''(z) + iP_2(z)w_2(z+c) = Q_{11}(z)e^{h_1(z)}, \\ w_1''(z) - iP_2(z)w_2(z+c) = Q_{12}(z)e^{-h_1(z)}, \\ w_2''(z) + iP_1(z)w_1(z+c) = Q_{21}(z)e^{h_2(z)}, \\ w_2''(z) - iP_1(z)w_1(z+c) = Q_{22}(z)e^{-h_2(z)}, \end{cases} \quad (3.1)$$

where $h_1(z)$, $h_2(z)$ are polynomials, $Q_1(z) = Q_{11}(z)Q_{12}(z)$, $Q_2(z) = Q_{21}(z)Q_{22}(z)$, $Q_{kj}(z)$ ($k, j = 1, 2$) are nonzero polynomials and $\deg Q_{kj} \geq 0$.

From (3.1), we obtain

$$w_1''(z) = \frac{Q_{11}(z)e^{h_1(z)} + Q_{12}(z)e^{-h_1(z)}}{2}, \quad (3.2)$$

$$w_1(z+c) = \frac{Q_{21}(z)e^{h_2(z)} - Q_{22}(z)e^{-h_2(z)}}{2iP_1(z)}, \quad (3.3)$$

$$w_2''(z) = \frac{Q_{21}(z)e^{h_2(z)} + Q_{22}(z)e^{-h_2(z)}}{2}, \quad (3.4)$$

$$w_2(z+c) = \frac{Q_{11}(z)e^{h_1(z)} - Q_{12}(z)e^{-h_1(z)}}{2iP_2(z)}. \quad (3.5)$$

By (3.3) and (3.5), we have

$$w_1''(z+c) = \frac{R_{21}(z)}{2iP_1(z)^3}e^{h_2(z)} - \frac{R_{22}(z)}{2iP_1(z)^3}e^{-h_2(z)}, \quad (3.6)$$

$$w_2''(z+c) = \frac{S_{21}(z)}{2iP_2(z)^3}e^{h_1(z)} - \frac{S_{22}(z)}{2iP_2(z)^3}e^{-h_1(z)}. \quad (3.7)$$

where R_{vj}, S_{vj} ($v = 1, 2; j = 1, 2$) are polynomials and defined as follows

$$\begin{cases} R_{21}(z) := R'_{11}(z)P_1(z) - 2R_{11}(z)P'_1(z) + R_{11}(z)P_1(z)h'_2(z), \\ R_{11}(z) := Q'_{21}(z)P_1(z) - Q_{21}(z)P'_1(z) + Q_{21}(z)P_1(z)h'_2(z), \\ R_{22}(z) := R'_{12}(z)P_1(z) - 2R_{12}(z)P'_1(z) - R_{12}(z)P_1(z)h'_2(z), \\ R_{12}(z) := Q'_{22}(z)P_1(z) - Q_{22}(z)P'_1(z) - Q_{22}(z)P_1(z)h'_2(z), \end{cases} \quad (3.8)$$

and

$$\begin{cases} S_{21}(z) := S'_{11}(z)P_2(z) - 2S_{11}(z)P'_2(z) + S_{11}(z)P_2(z)h'_1(z), \\ S_{11}(z) := Q'_{11}(z)P_2(z) - Q_{11}(z)P'_2(z) + Q_{11}(z)P_2(z)h'_1(z), \\ S_{22}(z) := S'_{12}(z)P_2(z) - 2S_{12}(z)P'_2(z) - S_{12}(z)P_2(z)h'_1(z), \\ S_{12}(z) := Q'_{12}(z)P_2(z) - Q_{12}(z)P'_2(z) - Q_{12}(z)P_2(z)h'_1(z). \end{cases} \quad (3.9)$$

By Lemma 2.3, for $j = 1, 2$, we have

$$R_{2j}(z) \neq 0, \quad S_{2j}(z) \neq 0.$$

and

$$\begin{aligned}\deg R_{2j} &= 2 \deg P_1 + \deg Q_{2j} + 2(\deg h_2 - 1), \\ \deg S_{2j} &= 2 \deg P_2 + \deg Q_{1j} + 2(\deg h_1 - 1).\end{aligned}\quad (3.10)$$

Combing (3.2) and (3.6), (3.4) and (3.7), we get

$$B_{11}(z)e^{h_1(z+c)-h_2(z)} + B_{12}(z)e^{h_1(z+c)+h_2(z)} + B_{13}(z)e^{2h_1(z+c)} + B_{14}e^{h_{10}(z)} \equiv 0, \quad (3.11)$$

$$B_{21}(z)e^{h_2(z+c)-h_1(z)} + B_{22}(z)e^{h_2(z+c)+h_1(z)} + B_{23}(z)e^{2h_2(z+c)} + B_{24}e^{h_{20}(z)} \equiv 0, \quad (3.12)$$

where $h_{10}(z) \equiv 0$, $h_{20}(z) \equiv 0$,

$$\begin{cases} B_{11}(z) = R_{22}(z), \\ B_{12}(z) = -R_{21}(z), \\ B_{13}(z) = iP_1(z)^3 Q_{11}(z+c), \\ B_{14}(z) = iP_1(z)^3 Q_{12}(z+c), \\ B_{21}(z) = S_{22}(z), \\ B_{22}(z) = -S_{21}(z), \\ B_{23}(z) = iP_2(z)^3 Q_{21}(z+c), \\ B_{24}(z) = iP_2(z)^3 Q_{22}(z+c). \end{cases}$$

Further, (3.11) and (3.12) can be rewritten as

$$A_{11}(z)e^{h_1(z+c)-h_2(z)} + A_{12}(z)e^{h_1(z+c)+h_2(z)} + A_{13}(z)e^{2h_1(z+c)} \equiv -1, \quad (3.13)$$

$$A_{21}(z)e^{h_2(z+c)-h_1(z)} + A_{22}(z)e^{h_2(z+c)+h_1(z)} + A_{23}(z)e^{2h_2(z+c)} \equiv -1, \quad (3.14)$$

where

$$\begin{cases} A_{11}(z) = \frac{R_{22}(z)}{iP_1(z)^3 Q_{12}(z+c)}, \\ A_{12}(z) = \frac{-R_{21}(z)}{iP_1(z)^3 Q_{12}(z+c)}, \\ A_{13}(z) = \frac{Q_{11}(z+c)}{Q_{12}(z+c)}, \end{cases}$$

and

$$\begin{cases} A_{21}(z) = \frac{S_{22}(z)}{iP_2(z)^3 Q_{22}(z+c)}, \\ A_{22}(z) = \frac{-S_{21}(z)}{iP_2(z)^3 Q_{22}(z+c)}, \\ A_{23}(z) = \frac{Q_{21}(z+c)}{Q_{22}(z+c)}. \end{cases}$$

By (3.2) and (3.3), we have $\sigma(w_1(z)) = \sigma(w_1'(z)) = \deg h_1$ and $\sigma(w_1(z)) = \sigma(w_1(z+c)) = \deg h_2$, hence $\deg h_1(z) = \deg h_2(z)$.

Now, we **assert that** $\deg h_1(z) = \deg h_2(z) = 1$. Otherwise, we set $\deg h_1(z) = \deg h_2(z) := r \geq 2$, then, in order to ensure that the condition (ii) in Lemma 2.2 is satisfied and to apply Lemma 2.2, we can split our proof into the three cases: (1) $\deg(h_1(z+c) \pm h_2(z)) \geq 1$; (2) $h_1(z+c) - h_2(z) = d_1$; (3) $h_1(z+c) + h_2(z) = d_2$, where d_1, d_2 are two constants.

Case 1. If $\deg(h_1(z+c) \pm h_2(z)) \geq 1$, then $\deg \pm(h_1(z+c) \pm h_2(z)) \geq 1$ and $\deg(-2h_2(z)) = \deg(2h_1(z+c)) = r \geq 2$. Noting that $e^{\pm(h_1(z+c) \pm h_2(z))}$, $e^{-2h_2(z)}$, $e^{2h_1(z+c)}$ are all have regular order, and $\sigma(B_{1i}(z)) = 0$ ($i = 1, 2, 3, 4$). Hence, for $i = 1, 2, 3, 4$, we have

$$\begin{cases} T(r, B_{1i}(z)) = o(T(r, e^{\pm(h_1(z+c) \pm h_2(z))})), \\ T(r, B_{1i}(z)) = o(T(r, e^{-2h_2(z)})), \\ T(r, B_{1i}(z)) = o(T(r, e^{2h_1(z+c)})). \end{cases} \quad (3.15)$$

By Lemma 2.2, (3.11) and (3.15), we obtain

$$B_{1i}(z) \equiv 0, \quad i = 1, 2, 3, 4,$$

that is, $B_{14}(z) = iP_1(z)^3 Q_{12}(z+c) \equiv 0$, a contradiction.

Case 2. If $h_1(z+c) - h_2(z) \equiv d_1$, then (3.12) can be rewritten as

$$B_{21}(z)e^{h_1(z+2c)-h_1(z)-d_1} + B_{22}(z)e^{h_1(z+2c)+h_1(z)-d_1} + B_{23}(z)e^{2h_1(z+2c)-2d_1} + B_{24}(z) \equiv 0, \quad (3.16)$$

by the same reasoning as using in Case 1, applying Lemma 2.2 to (3.16), we can obtain $B_{24}(z) = iP_1(z)^3 Q_{22}(z+c) \equiv 0$, a contradiction.

Case 3. If $h_1(z+c) + h_2(z) \equiv d_2$, then (3.12) can be rewritten as

$$B_{21}(z)e^{-h_1(z)-h_1(z+2c)+d_2} + B_{22}(z)e^{-h_1(z+2c)+h_1(z)+d_2} + B_{23}(z)e^{-2h_1(z+2c)+2d_2} + B_{24}(z) \equiv 0, \quad (3.17)$$

by the same reasoning as using in Case 1, applying Lemma 2.2 to (3.17), we can obtain $B_{24}(z) = iP_2(z)^3 Q_{22}(z+c) \equiv 0$, a contradiction.

Thus, we can obtain $\deg h_1(z) = \deg h_2(z) = 1$. Suppose that

$$h_1(z) = az + c_1 \quad (a \neq 0, c_1 \in \mathbb{C}).$$

Then, we can split our proof into the following three cases: Case I: $h_1(z) - h_2(z)$ is a constant; Case II: $h_1(z) + h_2(z)$ is a constant; Case III: $\deg(h_1(z) \pm h_2(z)) \geq 1$.

Case I. If $h_1(z) - h_2(z)$ is a constant, then suppose that $h_2(z) = az + c_2$ ($c_2 \in \mathbb{C}$). According to Lemma 2.1 and (3.13), we obtain

$$\frac{-iP_1(z)^3 Q_{12}(z+c)}{R_{22}(z)} = \frac{R_{21}(z)}{iP_1(z)^3 Q_{11}(z+c)} = e^{ac+c_1-c_2}. \quad (3.18)$$

Similarly, according to Lemma 2.1 and (3.14), we obtain

$$\frac{-iP_2(z)^3 Q_{22}(z+c)}{S_{22}(z)} = \frac{S_{21}(z)}{iP_2(z)^3 Q_{21}(z+c)} = e^{ac+c_2-c_1}. \quad (3.19)$$

Hence

$$R_{22}(z) = -ie^{-ac+c_2-c_1} P_1(z)^3 Q_{12}(z+c), \quad (3.20)$$

$$R_{21}(z) = ie^{ac+c_1-c_2}P_1(z)^3Q_{11}(z+c), \quad (3.21)$$

$$S_{22}(z) = -ie^{-ac+c_1-c_2}P_2(z)^3Q_{22}(z+c), \quad (3.22)$$

$$S_{21}(z) = ie^{ac+c_2-c_1}P_2(z)^3Q_{21}(z+c). \quad (3.23)$$

By (3.10), (3.20) and (3.22), we have

$$\deg R_{22} = 2 \deg P_1 + \deg Q_{22} = 3 \deg P_1 + \deg Q_{12},$$

$$\deg S_{22} = 2 \deg P_2 + \deg Q_{12} = 3 \deg P_2 + \deg Q_{22},$$

and thus $\deg P_1 + \deg P_2 = 0$, that is $\deg P_1 = \deg P_2 = 0$, and $P_1(z), P_2(z)$ are nonzero constants. Denote them as $P_1(z) \equiv p_1, P_2(z) \equiv p_2$, By Lemma 2.3, (3.8)-(3.9) indicate

$$R_{21}(z) = p_1^2[a^2Q_{21}(z) + 2aQ'_{21}(z) + Q''_{21}(z)], \quad (3.24)$$

$$R_{22}(z) = p_1^2[(-a)^2Q_{22}(z) - 2aQ'_{22}(z) + Q''_{22}(z)], \quad (3.25)$$

$$S_{21}(z) = p_2^2[a^2Q_{11}(z) + 2aQ'_{11}(z) + Q''_{11}(z)], \quad (3.26)$$

$$S_{22}(z) = p_2^2[(-a)^2Q_{12}(z) - 2aQ'_{12}(z) + Q''_{12}(z)]. \quad (3.27)$$

Thus, (3.20)-(3.23) become

$$(-a)^2Q_{22}(z) - 2aQ'_{22}(z) + Q''_{22}(z) = -ip_1e^{-ac+c_2-c_1}Q_{12}(z+c), \quad (3.28)$$

$$a^2Q_{21}(z) + 2aQ'_{21}(z) + Q''_{21}(z) = ip_1e^{ac+c_1-c_2}Q_{11}(z+c), \quad (3.29)$$

$$(-a)^2Q_{12}(z) - 2aQ'_{12}(z) + Q''_{12}(z) = -ip_2e^{-ac+c_1-c_2}Q_{22}(z+c), \quad (3.30)$$

$$a^2Q_{11}(z) + 2aQ'_{11}(z) + Q''_{11}(z) = ip_2e^{ac+c_2-c_1}Q_{21}(z+c). \quad (3.31)$$

By (3.28) and (3.30), (3.29) and (3.31), we have

$$a^4 = -e^{-2ac}p_1p_2, \quad a^4 = -e^{2ac}p_1p_2,$$

and so

$$a^8 = p_1^2p_2^2, \quad e^{4ac} = 1, \quad ac = \frac{k\pi i}{2}, \quad k \in \mathbb{Z}.$$

By (3.28) and (3.29), or (3.30) and (3.31), we have

$$\deg Q_{22} = \deg Q_{12}, \quad \deg Q_{21} = \deg Q_{11}, \quad \deg Q_1 = \deg Q_2.$$

Let $t := \deg Q_{21} = \deg Q_{11}$ ($t \geq 0$). We assert that $t = 0$ and thus Q_{21}, Q_{11} are reduce to constants. Otherwise, we suppose that $t \geq 1$ and set

$$\begin{cases} Q_{21} = a_t z^t + a_{t-1} z^{t-1} + \cdots + a_0, & a_t \neq 0, \\ Q_{11} = b_t z^t + b_{t-1} z^{t-1} + \cdots + b_0, & b_t \neq 0. \end{cases} \quad (3.32)$$

Then, combining (3.29) and (3.32), we have

$$\begin{aligned} & a^2 a_t z^t + (a^2 a_{t-1} + 2taa_t)z^{t-1} + \dots \\ &= ip_1 e^{ac+c_1-c_2} [b_t z^t + (ctb_t + b_{t-1})z^{t-1} + \dots]. \end{aligned} \quad (3.33)$$

Combining (3.31) and (3.32), we have

$$\begin{aligned} & a^2 b_t z^t + (a^2 b_{t-1} + 2tab_t)z^{t-1} + \dots \\ &= ip_2 e^{ac+c_2-c_1} [a_t z^t + (cta_t + a_{t-1})z^{t-1} + \dots]. \end{aligned} \quad (3.34)$$

Multiply (3.33) with (3.34), and note the coefficients of z^{2t-1} on the both sides, we have

$$\begin{aligned} & a^2 a_t (a^2 b_{t-1} + 2tab_t) + a^2 b_t (a^2 a_{t-1} + 2taa_t) \\ &= -e^{2ac} p_1 p_2 (2cta_t b_t + b_t a_{t-1} + a_t b_{t-1}). \end{aligned} \quad (3.35)$$

Considering $a^4 = -e^{2ac} p_1 p_2$, (3.35) becomes

$$2ta^3(ac-2)a_t b_t = 0.$$

Thus, we have $ac = 2$, is a contradiction for $e^{4ac} = 1$. Hence, $t = 0$ and we have

$$Q_{21} \equiv c_{21} \text{ (consant)}, \quad Q_{11} \equiv c_{11} \text{ (consant)}.$$

By (3.28) and (3.30), and employing a reasoning analogous to that presented above, we have

$$a^4 = -e^{-2ac} p_1 p_2, \quad Q_{22} \equiv c_{22}, \quad Q_{12} \equiv c_{12}.$$

From (3.2) and (3.4), we obtain

$$\begin{cases} w_1(z) = \frac{c_{11}e^{c_1}}{2a^2}e^{az} + \frac{c_{12}e^{-c_1}}{2a^2}e^{-az} + C_1(z), \\ w_2(z) = \frac{c_{21}e^{c_2}}{2a^2}e^{az} + \frac{c_{22}e^{-c_2}}{2a^2}e^{-az} + C_2(z), \end{cases} \quad (3.36)$$

where $c_1, c_2, c_{kj} (\neq 0), k, j = 1, 2$ are constants, $C_1(z), C_2(z)$ are polynomials with degree at most 1.

From (3.3), (3.5) and (3.36), we obtain

$$\begin{cases} w_1(z) = \frac{c_{11}e^{c_1}}{2a^2}e^{az} + \frac{c_{12}e^{-c_1}}{2a^2}e^{-az} + C_1(z) = \frac{c_{21}e^{c_2-ac}}{2ip_1}e^{az} - \frac{c_{22}e^{-c_2+ac}}{2ip_1}e^{-az}, \\ w_2(z) = \frac{c_{21}e^{c_2}}{2a^2}e^{az} + \frac{c_{22}e^{-c_2}}{2a^2}e^{-az} + C_2(z) = \frac{c_{11}e^{c_1-ac}}{2ip_2}e^{az} - \frac{c_{12}e^{-c_1+ac}}{2ip_2}e^{-az}. \end{cases} \quad (3.37)$$

Applying Lemma 2.2 to (3.37), we have $C_1(z) \equiv 0, C_2(z) \equiv 0$, and

$$\begin{cases} \frac{c_{11}e^{c_1}}{2a^2} = \frac{c_{21}e^{c_2-ac}}{2ip_1}, & \frac{c_{12}e^{-c_1}}{2a^2} = -\frac{c_{22}e^{-c_2+ac}}{2ip_1}, \\ \frac{c_{21}e^{c_2}}{2a^2} = \frac{c_{11}e^{c_1-ac}}{2ip_2}, & \frac{c_{22}e^{-c_2}}{2a^2} = -\frac{c_{12}e^{-c_1+ac}}{2ip_2}. \end{cases}$$

Thus, we have

$$e^{c_2-c_1} = \frac{ip_1}{a^2 e^{-ac}} \frac{c_{11}}{c_{21}} = \frac{a^2 e^{ac}}{-ip_1} \frac{c_{22}}{c_{12}} = \frac{a^2 e^{-ac}}{ip_2} \frac{c_{11}}{c_{21}} = \frac{-ip_2}{a^2 e^{ac}} \frac{c_{22}}{c_{12}}. \quad (3.38)$$

By (3.38), we have

$$\frac{Q_2}{Q_1} = \frac{p_1^2}{a^4} = \frac{a^4}{p_2^2} = \pm \frac{p_1}{p_2}, \quad a^8 = p_1^2 p_2^2, \quad e^{4ac} = 1. \quad (3.39)$$

Setting $c_{11}e^{c_1} := d$, then $c_{12}e^{-c_1} = \frac{Q_1}{d}$. Thus, by (3.37), we have

$$\begin{cases} w_1(z) = \frac{1}{2a^2}(de^{az} + \frac{Q_1}{d}e^{-az}), \\ w_2(z) = \frac{1}{2ip_2}(de^{-ac}e^{az} - \frac{Q_1}{d}e^{ac}e^{-az}). \end{cases} \quad (3.40)$$

where d is a non-zero constant, a is a non-zero constant satisfying (3.39).

Case II. If $h_1(z) + h_2(z)$ is a constant, then suppose that $h_2(z) = -az + c_3$ ($c_3 \in \mathbb{C}$). According to Lemma 2.1 and (3.13), we obtain

$$\frac{-iP_1(z)^3Q_{12}(z+c)}{R_{21}(z)} = \frac{R_{22}(z)}{iP_1(z)^3Q_{11}(z+c)} = e^{ac+c_1+c_3}. \quad (3.41)$$

Similarly, according to Lemma 2.1 and (3.14), we obtain

$$\frac{S_{21}(z)}{-iP_2(z)^3Q_{22}(z+c)} = \frac{iP_2^3Q_{21}(z+c)}{S_{22}(z)} = e^{ac-c_1-c_3}. \quad (3.42)$$

Hence

$$R_{21}(z) = -ie^{-ac-c_1-c_3}P_1(z)^3Q_{12}(z+c), \quad (3.43)$$

$$R_{22}(z) = ie^{ac+c_1+c_3}P_1(z)^3Q_{11}(z+c), \quad (3.44)$$

$$S_{21}(z) = -ie^{ac-c_1-c_3}P_2(z)^3Q_{22}(z+c), \quad (3.45)$$

$$S_{22}(z) = ie^{-ac+c_1+c_3}P_2(z)^3Q_{21}(z+c). \quad (3.46)$$

By (3.10), (3.43) and (3.44), we have

$$\deg R_{21} = 2 \deg P_1 + \deg Q_{21} = 3 \deg P_1 + \deg Q_{12},$$

$$\deg S_{22} = 2 \deg P_2 + \deg Q_{12} = 3 \deg P_2 + \deg Q_{21},$$

and thus $\deg P_1 + \deg P_2 = 0$, that is $\deg P_1 = \deg P_2 = 0$, and $P_1(z), P_2(z)$ are nonzero constants.

Denote them as $P_1(z) \equiv p_1, P_2(z) \equiv p_2$, and we have

$$\deg Q_{21} = \deg Q_{12}, \quad \deg Q_{22} = \deg Q_{11}, \quad \deg Q_1 = \deg Q_2.$$

Considering P_1, P_2 are constants, (3.8)-(3.9) indicate

$$R_{21}(z) = p_1^2[(-a)^2Q_{21}(z) - 2aQ'_{21}(z) + Q''_{21}(z)], \quad (3.47)$$

$$R_{22}(z) = p_1^2[a^2Q_{22}(z) + 2aQ'_{22}(z) + Q''_{22}(z)], \quad (3.48)$$

$$S_{21}(z) = p_2^2[a^2Q_{11}(z) + 2aQ'_{11}(z) + Q''_{11}(z)], \quad (3.49)$$

$$S_{22}(z) = p_2^2[(-a)^2 Q_{12}(z) - 2aQ'_{12}(z) + Q''_{12}(z)]. \quad (3.50)$$

Thus, (3.43)-(3.46) become

$$(-a)^2 Q_{21}(z) - 2aQ'_{21}(z) + Q''_{21}(z) = -ip_1 e^{-ac-c_1-c_3} Q_{12}(z+c), \quad (3.51)$$

$$a^2 Q_{22}(z) + 2aQ'_{22}(z) + Q''_{22}(z) = ip_1 e^{ac+c_1+c_3} Q_{11}(z+c), \quad (3.52)$$

$$a^2 Q_{11}(z) + 2aQ'_{11}(z) + Q''_{11}(z) = -ip_2 e^{ac-c_1-c_3} Q_{22}(z+c), \quad (3.53)$$

$$(-a)^2 Q_{12}(z) - 2aQ'_{12}(z) + Q''_{12}(z) = ip_2 e^{-ac+c_1+c_3} Q_{21}(z+c). \quad (3.54)$$

By (3.51) and (3.54), (3.52) and (3.53), we have

$$a^4 = e^{-2ac} p_1 p_2, \quad a^4 = e^{2ac} p_1 p_2,$$

and so

$$a^8 = p_1^2 p_2^2, \quad e^{4ac} = 1, \quad ac = \frac{k\pi i}{2}, \quad k \in \mathbb{Z}.$$

By (3.51) and (3.52), or (3.53) and (3.54), we have

$$\deg Q_{21} = \deg Q_{12}, \quad \deg Q_{22} = \deg Q_{11}, \quad \deg Q_1 = \deg Q_2.$$

Let $n := \deg Q_{21} = \deg Q_{12}$ ($n \geq 0$). Suppose that $n \geq 1$ and set

$$\begin{cases} Q_{21} = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, & a_n \neq 0, \\ Q_{12} = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0, & b_n \neq 0. \end{cases} \quad (3.55)$$

Then, combining (3.55) and (3.51), we have

$$\begin{aligned} & (-a)^2 a_n z^n + [(-a)^2 a_{n-1} - 2a n a_n] z^{n-1} + \cdots \\ &= -ip_1 e^{-ac-c_1-c_3} [b_n z^n + (c n b_n + b_{n-1}) z^{n-1} + \cdots]. \end{aligned} \quad (3.56)$$

Combining (3.55) and (3.54), we have

$$\begin{aligned} & (-a)^2 b_n z^n + [a^2 b_{n-1} - 2a n b_n] z^{n-1} + \cdots \\ &= ip_2 e^{-ac+c_1+c_3} [a_n z^n + (c n a_n + a_{n-1}) z^{n-1} + \cdots]. \end{aligned} \quad (3.57)$$

Multiply (3.56) with (3.57), and note the coefficients of z^{2n-1} on the both sides, we have

$$\begin{aligned} & a^k a_n (a^k b_{n-1} + k n a^{k-1} b_n) + a^k b_n (a^k a_{n-1} + k n a^{k-1} a_n) \\ &= b_n a_{n-1} + a_n b_{n-1}. \end{aligned} \quad (3.58)$$

By (3.58) and note that $a^4 = e^{-2ac} p_1 p_2$, we have

$$a^3 n a_n b_n (2 + ac) = 0.$$

Thus, we have $ac = -2$, contradicts $e^{4ac} = 1$. Hence, $n = 0$ and we have

$$Q_{21} \equiv c_{21} (\text{constant}), \quad Q_{12} \equiv c_{12} (\text{constant}).$$

By (3.44) and (3.45), (3.48) and (3.49), and employing a reasoning analogous to that presented above, we have

$$a^4 = e^{2ac} p_1 p_2, \quad Q_{22} \equiv c_{22} (\text{constant}), \quad Q_{11} \equiv c_{11} (\text{constant}).$$

In order to avoid confusion with the solution (w_1, w_2) obtained in Case I, we denote the solution in Case II as $(\hat{w}_1, \hat{w}_2)$, and replace w_1 with $\hat{w}_1$, w_2 with $\hat{w}_2$ in (3.2)-(3.5), respectively. The resulting equations are still coded in their original numbers (3.2)-(3.5). Thus, from (3.2) and (3.4), we obtain

$$\begin{cases} \hat{w}_1(z) = \frac{c_{11}e^{c_1}}{2a^2}e^{az} + \frac{c_{12}e^{-c_1}}{2a^2}e^{-az} + D_1(z), \\ \hat{w}_2(z) = \frac{c_{21}e^{c_2}}{2a^2}e^{az} + \frac{c_{22}e^{-c_2}}{2a^2}e^{-az} + D_2(z), \end{cases} \quad (3.59)$$

where $c_1, c_2, c_{kj} (\neq 0), k, j = 1, 2$ are constants, $D_1(z), D_2(z)$ are polynomials with degree at most 1.

From (3.3), (3.5) and (3.59), we obtain

$$\begin{cases} \hat{w}_1(z) = \frac{c_{11}e^{c_1}}{2a^2}e^{az} + \frac{c_{12}e^{-c_1}}{2a^2}e^{-az} + D_1(z) = -\frac{c_{22}e^{-c_3-ac}}{2ip_1}e^{az} + \frac{c_{21}e^{c_3+ac}}{2ip_1}e^{-az}, \\ \hat{w}_2(z) = \frac{c_{22}e^{-c_3}}{2a^2}e^{az} + \frac{c_{21}e^{c_3}}{2a^2}e^{-az} + D_2(z) = \frac{c_{11}e^{c_1-ac}}{2ip_2}e^{az} - \frac{c_{12}e^{-c_1+ac}}{2ip_2}e^{-az}. \end{cases} \quad (3.60)$$

Applying Lemma 2.2 to (3.60), we have $D_1(z) \equiv 0, D_2(z) \equiv 0$, and

$$\begin{cases} \frac{c_{11}e^{c_1}}{a^2} = \frac{c_{22}e^{-c_3-ac}}{-ip_1}, & \frac{c_{12}e^{-c_1}}{a^2} = \frac{c_{21}e^{c_3+ac}}{ip_1}, \\ \frac{c_{22}e^{-c_3}}{a^2} = \frac{c_{11}e^{c_1-ac}}{ip_2}, & \frac{c_{21}e^{c_3}}{a^2} = -\frac{c_{12}e^{-c_1+ac}}{ip_2}. \end{cases}$$

Thus, we have

$$e^{c_1+c_3} = \frac{a^2e^{-ac}}{-ip_1} \frac{c_{22}}{c_{11}} = \frac{ip_1}{a^2e^{ac}} \frac{c_{12}}{c_{21}} = \frac{ip_2}{a^2e^{-ac}} \frac{c_{22}}{c_{11}} = \frac{a^2e^{ac}}{-ip_2} \frac{c_{12}}{c_{21}}. \quad (3.61)$$

By (3.61), we have

$$\frac{Q_2}{Q_1} = \frac{p_1^2}{a^4} = \frac{a^4}{p_2^2} = \pm \frac{p_1}{p_2}, \quad a^8 = p_1^2 p_2^2, \quad e^{4ac} = 1. \quad (3.62)$$

Setting $c_{11}e^{c_1} := d$, then $c_{12}e^{-c_1} = \frac{Q_1}{d}$. Thus, by (3.60), we have

$$\begin{cases} \hat{w}_1(z) = \frac{1}{2a^2}(de^{az} + \frac{Q_1}{d}e^{-az}), \\ \hat{w}_2(z) = \frac{1}{2ip_2}(de^{-ac}e^{az} - \frac{Q_1}{d}e^{ac}e^{-az}). \end{cases} \quad (3.63)$$

where d is a non-zero constant, a is a non-zero constant satisfying (3.62).

Case III. If $\deg(h_1(z) \pm h_2(z)) \geq 1$. Then, using Lemma 2.2 and by employing a similar reasoning in Case 1, we get $B_{14}(z) \equiv 0$, a contradiction.

Therefore, from the three cases above, and together with (3.40) and (3.63), all transcendental entire solutions with finite order of the system (1.9) are of the forms

$$\begin{cases} w_1(z) = \frac{1}{2a^2}(de^{az} + \frac{Q_1}{d}e^{-az}), \\ w_2(z) = \frac{1}{2ip_2}(de^{-ac}e^{az} - \frac{Q_1}{d}e^{ac}e^{-az}), \end{cases} \quad (3.64)$$

where d is an arbitrary non-zero constant, a is a non-zero constant that satisfies

$$\frac{Q_2}{Q_1} = \frac{a^4}{p_2^2} = \frac{p_1^2}{a^4} \quad \text{and} \quad e^{4ac} = 1. \quad (3.65)$$

By calculation, we can found that, if $w_1(z)$ and $w_2(z)$, which appear in (3.64), satisfying (3.65), then, they must be a pair transcendental entire solution of the system of equations (1.9).

This proves Theorem 1.1.

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