

A Frailty-Augmented Weibull–Poisson Competing Risk Model for k -out-of- n System Reliability: Properties, Bayesian Estimation, and Simulation

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Abstract. This paper introduces the Weibull–Poisson frailty (WPF) distribution, a four-parameter lifetime model in which a component fails at the earliest of the failure times of a zero-truncated Poisson number of latent causes, each cause having a Weibull failure time modulated by its own gamma frailty. The system lifetime is the k -th order statistic of n independent component lifetimes, so the model applies directly to k -out-of- n systems. The survival, density and hazard functions are available in closed form, and the model nests six distributions, including the Weibull–Poisson and gamma-frailty Weibull models. We characterize the possible hazard shapes (decreasing, increasing and unimodal), establish a stochastic ordering in the scale, competing-risk and frailty parameters, derive the condition under which the moments are finite, and show that ignoring the frailty while keeping the other parameters fixed always underestimates the mean time to failure (MTTF), most strongly for components with a decreasing hazard. Bayesian estimation is carried out with an adaptive Metropolis-within-Gibbs sampler. A Monte Carlo study shows that the Bayesian estimator recovers the shape parameter and the MTTF accurately and is far more stable than maximum likelihood, whereas the scale and competing-risk parameters are only weakly identified. Two applications illustrate the model: the frailty component is not supported by Aarset’s device-lifetime data, but it significantly improves the fit to Badar and Priest’s carbon fiber strengths.

1. INTRODUCTION

A k -out-of- n system consists of n components and, in this paper, fails as soon as k of them have failed (the k -out-of- n :F convention); $k = 1$ gives a series system and $k = n$ a parallel system. Such redundancy structures are among the most widely studied in reliability engineering [1, 2] and arise in power generation networks, aerospace actuator assemblies, parallel processing stations and redundant communication links. When the component lifetimes $T_1, \dots, T_n$ are independent and identically distributed (i.i.d.), the system lifetime is the k -th order statistic $T_{(k)}$, so its reliability,

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hazard rate and mean time to failure (MTTF) are inherited from the component lifetime distribution. An accurate component model is therefore essential for maintenance scheduling, warranty reserve calculation and spare-parts planning.

Component lifetimes are usually modeled by the exponential or Weibull distribution. These models ignore two features that are common in field data. The first is the presence of several latent failure causes—mechanical wear, corrosion, thermal degradation, electrical overstress—any one of which ends the life of the component. If the number of causes is random and each cause has its own failure time, the component lifetime is the minimum of a random number of latent times, and its distribution is obtained by compounding a continuous lifetime law with a discrete distribution for the number of causes. This approach was introduced by Adamidis and Loukas [3] with the exponential–geometric distribution and was extended to the exponential–Poisson [4], Weibull–geometric [5], Weibull–Poisson (WP) [6, 7] and Weibull–power-series families [8, 9]. Recent work extends such compound models to k -th order statistics [10, 11] and combines them with gamma frailty [12–15].

The second feature is unobserved heterogeneity. Nominally identical components differ in material microstructure, installation conditions or operating environment in ways that are not recorded. Frailty models [16, 17] represent this heterogeneity by a positive random effect Z that multiplies the hazard rate. Gamma frailty is the most common choice, because the marginal survival function remains in closed form and the frailty variance φ measures the degree of heterogeneity directly. Ignoring heterogeneity distorts the population hazard: components whose individual hazards increase can produce a population hazard that rises and then falls [16].

The present paper combines these two features within the order-statistic framework of k -out-of- n systems. We introduce the Weibull–Poisson frailty (WPF) distribution, in which the number of latent causes follows a zero-truncated Poisson law and each cause has a Weibull failure time modulated by its own gamma frailty. The main contributions are as follows: (i) the component and system survival, density and hazard functions are obtained in closed form, and the model is shown to nest six distributions, including the WP and gamma-frailty Weibull (GFW) models; (ii) the possible hazard shapes are characterized and a stochastic ordering in the scale, competing-risk and frailty parameters is established; (iii) the sensitivity of the system MTTF to the frailty is quantified; (iv) a Bayesian estimation procedure is developed and compared with maximum likelihood in a Monte Carlo study; and (v) the model is applied to two benchmark datasets.

The paper is organized as follows. Section 2 defines the model and derives its distributional properties. Section 3 presents the theoretical results and examines identifiability. Section 4 describes Bayesian estimation. Section 5 reports the simulation study and Section 6 the applications. Section 7 discusses the results and their limitations, and Section 8 concludes.

2. THE WEIBULL-POISSON FRAILITY DISTRIBUTION

2.1. Construction. For component i in a k -out-of- n system, let the number of latent competing failure causes follow a zero-truncated Poisson distribution:

$$\Pr(N_i = m) = \frac{\lambda^m e^{-\lambda}}{m! (1 - e^{-\lambda})}, \quad m = 1, 2, 3, \dots, \quad \lambda > 0, \quad (2.1)$$

with mean $\mathbb{E}[N_i] = \lambda / (1 - e^{-\lambda})$ and probability generating function $G(s) = (e^{\lambda s} - 1) / (e^{\lambda} - 1)$. Cause j of component i has a latent failure time X_{ij} and a gamma frailty Z_{ij} with shape and rate $1/\varphi$, so that $\mathbb{E}[Z_{ij}] = 1$ and $\text{Var}[Z_{ij}] = \varphi \geq 0$ ($Z_{ij} \equiv 1$ when $\varphi = 0$). Given Z_{ij} , the hazard of X_{ij} is the Weibull hazard multiplied by Z_{ij} :

$$\Pr(X_{ij} > t \mid Z_{ij}) = \exp\{-Z_{ij}(\beta t)^\alpha\}.$$

The frailties Z_{ij} are i.i.d. across causes and components and independent of N_i , so they represent unobserved heterogeneity in the susceptibility of each latent failure mechanism. The component lifetime is $T_i = \min_{j \leq N_i} X_{ij}$.

2.2. Survival Function. Marginalizing over Z_{ij} with the gamma Laplace transform gives the Lomax-type baseline survival $S_0(t) = \Pr(X_{ij} > t) = (1 + \varphi(\beta t)^\alpha)^{-1/\varphi}$. Since the X_{ij} are i.i.d., $\Pr(T_i > t \mid N_i = m) = S_0(t)^m$, and averaging over the zero-truncated Poisson weights, $S_T(t) = \mathbb{E}[S_0(t)^{N_i}] = G(S_0(t))$, i.e.

$$S_T(t; \boldsymbol{\theta}) = \frac{e^{-\lambda[1 - (1 + \varphi(\beta t)^\alpha)^{-1/\varphi}] - e^{-\lambda}}}{1 - e^{-\lambda}} \quad (2.2)$$

for $t > 0$, where $\boldsymbol{\theta} = (\alpha, \beta, \lambda, \varphi)$ with $\alpha, \beta, \lambda > 0$ and $\varphi \geq 0$ ($\varphi = 0$ being understood as the limit $S_0(t) = e^{-(\beta t)^\alpha}$). The system lifetime $Y = T_{(k)}$ (the k -th order statistic of n i.i.d. copies of T) has:

$$S_Y(t) = \sum_{j=0}^{k-1} \binom{n}{j} [1 - S_T(t)]^j [S_T(t)]^{n-j}, \quad (2.3)$$

$$f_Y(t) = n \binom{n-1}{k-1} f_T(t) [1 - S_T(t)]^{k-1} [S_T(t)]^{n-k}, \quad (2.4)$$

where $f_T(t) = -dS_T/dt$ is given in Appendix A. Both expressions are in closed form.

2.3. Tail Behavior and Moment Existence. For $\varphi > 0$, $S_0(t) \sim (\varphi \beta^\alpha)^{-1/\varphi} t^{-\alpha/\varphi}$ as $t \rightarrow \infty$, and since $S_T = G(S_0) \sim \lambda S_0 / (e^\lambda - 1)$, the component survival has a power-law tail with index α/φ . In the upper tail $S_Y \sim \binom{n}{k-1} S_T^{n-k+1}$, so $\mathbb{E}[Y^r] = \int_0^\infty r t^{r-1} S_Y(t) dt$ is finite if and only if

$$r < \frac{\alpha}{\varphi} (n - k + 1). \quad (2.5)$$

In particular, the system MTTF exists if and only if $(\alpha/\varphi)(n - k + 1) > 1$; for a single component ($k = n = 1$) this reduces to $\alpha > \varphi$. When $\varphi = 0$ the tail is of Weibull type and all moments are finite. All moments reported in this paper satisfy condition (2.5).

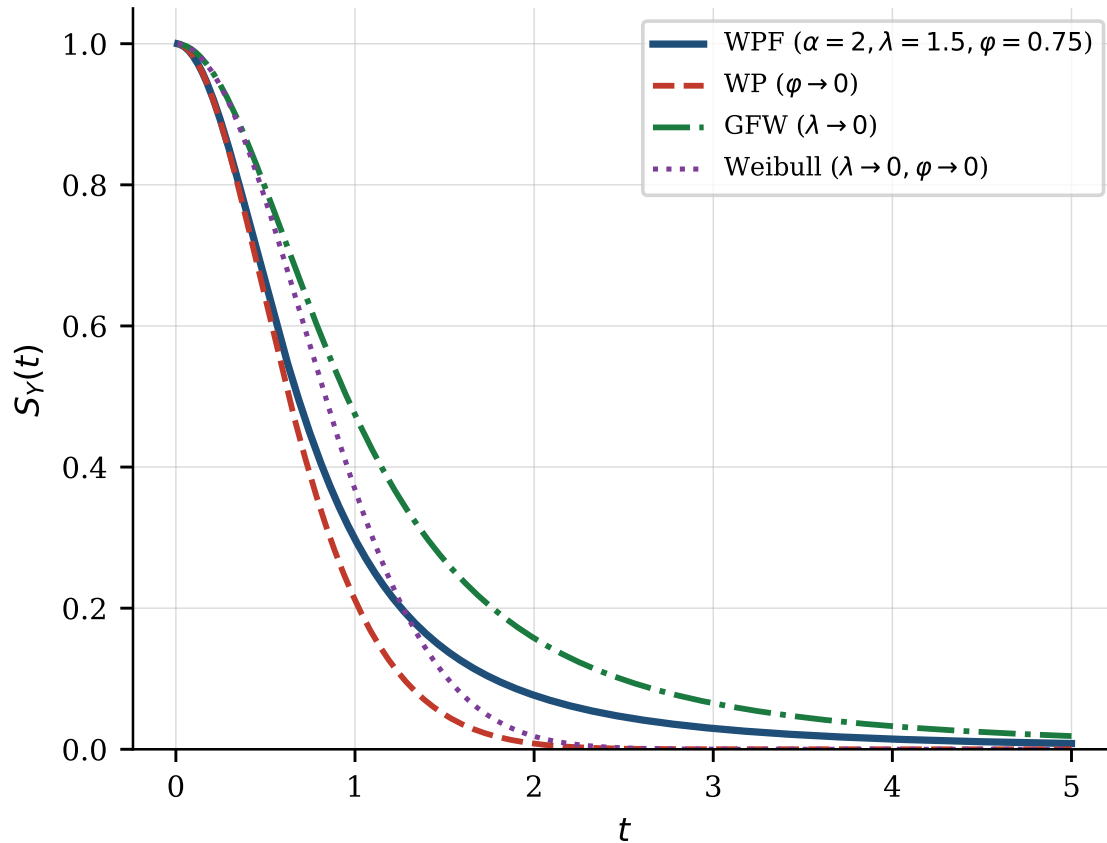


FIGURE 1. WPF survival function $S_Y(t)$ and three of its nested special cases ($k = 1$, $n = 1$, $\beta = 1$, $\alpha = 2$, $\lambda = 1.5$, $\varphi = 0.75$). Adding frailty ($\varphi > 0$) lengthens the tail relative to WP; adding competing risks ($\lambda > 0$) shortens the tail relative to the gamma-frailty Weibull model.

2.4. Special Cases. Table 1 lists the six distributions nested by WPF and the corresponding limiting conditions; EP denotes the exponential–Poisson distribution [4]. Figure 1 compares the survival function of WPF with three of its sub-models. For example, at $(\alpha, \beta, \lambda, \varphi) = (2, 1, 1.5, 0.75)$ and $t = 1$, the survival probability is $S_T = 0.2977$ under WPF, 0.2115 in the WP limit ($\varphi \rightarrow 0$), 0.4742 in the GFW limit ($\lambda \rightarrow 0$) and $0.3679 = e^{-1}$ for the standard Weibull ($\varphi \rightarrow 0$, $\lambda \rightarrow 0$).

2.5. Distributional Properties. Table 2 reports distributional properties computed at $\alpha = 2$, $\beta = 1$, $\lambda = 1.5$, $\varphi = 0.75$, $k = 2$, $n = 5$. Table 3 reports point values of S_Y , f_Y , the hazard rate $h_Y = f_Y/S_Y$ and the mean residual life (MRL) $m(t) = \mathbb{E}[Y - t \mid Y > t] = S_Y(t)^{-1} \int_t^\infty S_Y(u) du$. The r -th raw moment is:

$$\mathbb{E}[Y^r] = \sum_{j=0}^{k-1} \binom{n}{j} \sum_{l=0}^j \binom{j}{l} (-1)^l I_{n-j+l}(r, \boldsymbol{\theta}), \quad (2.6)$$

where $I_m(r, \boldsymbol{\theta}) = \int_0^\infty r t^{r-1} S_T(t)^m dt$. For $\varphi = 0$ these integrals have the series form given in Appendix A; for $\varphi > 0$ they, as well as the entropies and the MRL, are evaluated by adaptive numerical quadrature on $(0, \infty)$. All values in Tables 2–3 were confirmed in high-precision arithmetic.

TABLE 1. Special Cases of the WPF Distribution

Special case	Condition	Reduces to	Ref.
WP	$\varphi \rightarrow 0,$ $k = n = 1$	WP distribution	[6, 7]
WP k -th ord.	$\varphi \rightarrow 0$	WP order statistic	New
GFW	$\lambda \rightarrow 0,$ $k = n = 1$	Burr XII (gamma- frailty Weibull)	[17]
EP frailty	$\alpha = 1,$ $k = n = 1$	EP-gamma frailty	New
Standard Weibull	$\varphi, \lambda \rightarrow 0,$ $k = n = 1$	$W(\alpha, \beta)$	Classical
k -out-of- n W.	$\varphi, \lambda \rightarrow 0$	Weibull ord. stat.	[2]

TABLE 2. Distributional Properties. $(\alpha, \beta, \lambda, \varphi, k, n) = (2, 1, 1.5, 0.75, 2, 5)$.

Property	Value	Expression
$\mathbb{E}[Y]$ (MTTF)	0.5047	$\int_0^\infty S_Y(t) dt$
$\mathbb{E}[Y^2]$	0.3045	Eq. (2.6), $r = 2$
$\text{Var}[Y]$	0.0497	$\mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$
Std. deviation	0.2230	$\sqrt{\text{Var}[Y]}$
Skewness	1.0892	Standardized 3rd central moment
Kurtosis	5.5557	Standardized 4th central moment
Shannon entropy	-0.1616	$-\mathbb{E}[\log f_Y(Y)]$
Rényi ($\rho = 0.5$)	0.0938	$(1 - \rho)^{-1} \log \int f_Y^\rho$
MRL at $t = 0.5$	0.1967	$S_Y(0.5)^{-1} \int_{0.5}^\infty S_Y$
MRL at $t = 1.0$	0.1755	$S_Y(1.0)^{-1} \int_{1.0}^\infty S_Y$

Figure 3 illustrates how each parameter shapes $S_Y(t)$ and $h_Y(t)$. Increasing the shape α steepens the survival curve, the curves crossing near $t = 1/\beta$; increasing λ (competing risks) reduces

TABLE 3. Point Values of $S_Y(t)$, $f_Y(t)$, $h_Y(t)$, MRL. Same parameters as Table 2.

t	$S_Y(t)$	$f_Y(t)$	$h_Y(t)$	$m(t)$
0.50	0.4470	1.8580	4.1569	0.1967
1.00	0.0299	0.1763	5.8890	0.1755
2.00	0.0002	0.0007	4.4178	0.2463
3.00	<0.0001	<0.0001	3.2310	0.3392

lifetime; increasing φ (frailty) creates a heavier tail. The k -out-of- n structure is illustrated in Fig. 4: the series system ($k = 1$) has the shortest lifetime and highest hazard rate, while the parallel system ($k = n$) has the longest lifetime.

3. MAIN THEORETICAL RESULTS

Proposition 3.1 (Hazard shapes). *Let $h_T = f_T/S_T$ be the component hazard and h_Y the system hazard.*

- (i) *If $\alpha \leq 1$, h_T is decreasing, and so is h_Y for a series system ($k = 1$).*
- (ii) *If $\alpha > 1$, then $h_T(0^+) = 0$. If moreover $\varphi = 0$, then $h_T(t) \rightarrow \infty$ as $t \rightarrow \infty$.*
- (iii) *If $\alpha > 1$ and $\varphi > 0$, then $h_T(t) \sim \alpha/(\varphi t) \rightarrow 0$ as $t \rightarrow \infty$, so h_T is not monotone: it increases from zero and eventually decreases.*
- (iv) *If $k \geq 2$ and $\varphi > 0$, then $h_Y(t) \rightarrow 0$ as $t \rightarrow \infty$; if moreover $k\alpha > 1$, then $h_Y(0^+) = 0$, so the system hazard is not monotone.*

Proof. (i) Given Z_{ij} , the Weibull law with $\alpha \leq 1$ has a decreasing failure rate (DFR); gamma mixtures of DFR laws are DFR, so S_0 is DFR. The minimum of m i.i.d. DFR lifetimes has hazard mh_0 and is DFR, and a mixture over N_i of DFR laws is DFR [2], so h_T is decreasing; a series system is again a minimum of i.i.d. DFR lifetimes. (ii) From (A.1), $f_T(t) = O(t^{\alpha-1}) \rightarrow 0$ while $S_T(0) = 1$. (ii)–(iii) As $t \rightarrow \infty$, $S_0 \rightarrow 0$ and $S_T = (e^{\lambda S_0} - 1)/(e^\lambda - 1) \sim \lambda S_0/(e^\lambda - 1)$, so $h_T(t) \sim h_0(t) = \alpha\beta^\alpha t^{\alpha-1}/(1 + \varphi(\beta t)^\alpha)$, which tends to ∞ when $\varphi = 0$ and $\alpha > 1$, and behaves as $\alpha/(\varphi t)$ when $\varphi > 0$. (iv) As $t \rightarrow \infty$, $S_Y \sim \binom{n}{k-1} S_T^{n-k+1}$, so $h_Y \sim (n-k+1)h_T \rightarrow 0$, because $h_T \sim h_0 \rightarrow 0$ whenever $\varphi > 0$. As $t \rightarrow 0$, $1 - S_T(t) \sim ct^\alpha$ and $f_T(t) \sim c\alpha t^{\alpha-1}$ with $c = \lambda\beta^\alpha/(1 - e^{-\lambda})$, so $h_Y(t) \sim f_Y(t) = O(t^{k\alpha-1}) \rightarrow 0$ when $k\alpha > 1$. $\square$

Proposition 3.1 establishes the behavior of the hazard at the endpoints and its monotonicity under the stated conditions; it does not establish that a non-monotone hazard has a single interior maximum. In every configuration we examined, the non-monotone hazards in (iii) and (iv) are unimodal, which is a numerical observation rather than a general result; when $k\alpha < 1$ the system hazard instead starts from infinity. As $\varphi \rightarrow 0$ the mode moves to the right, so for small φ the hazard is increasing over the practically relevant range. All shapes are illustrated in Figs. 2–4.

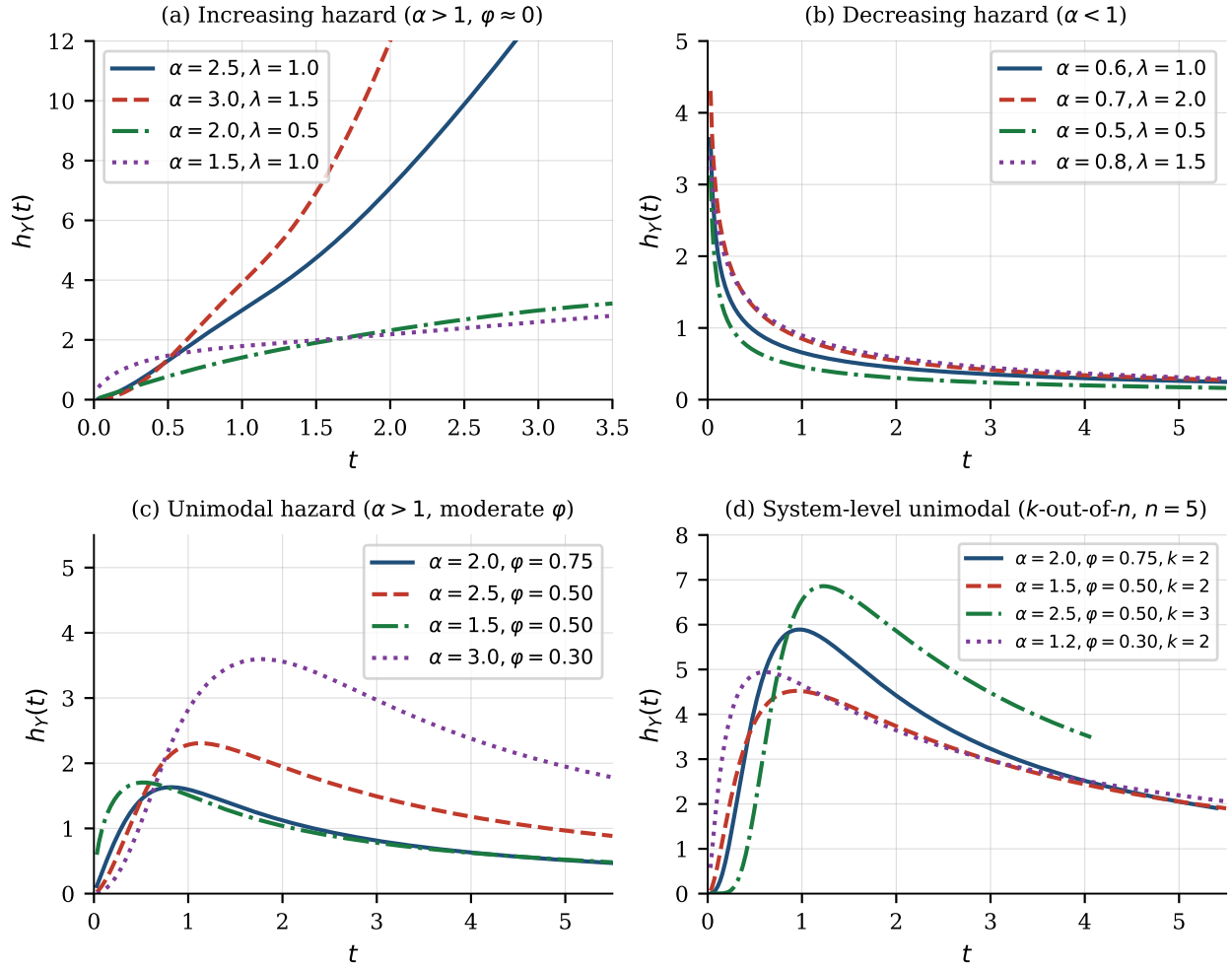


FIGURE 2. WPF hazard rate $h_Y(t)$: (a) increasing ($\alpha > 1, \varphi \approx 0$); (b) decreasing ($\alpha < 1$); (c) unimodal at the component level ($\alpha > 1$, moderate φ); (d) non-monotone hazards at the system level (k -out-of- n , $n = 5$).

Proposition 3.2 (Stochastic ordering). Let $Y_1 \sim \text{WPF}(\alpha, \beta_1, \lambda_1, \varphi_1, k, n)$ and $Y_2 \sim \text{WPF}(\alpha, \beta_2, \lambda_2, \varphi_2, k, n)$ share the same α, k and n . If $\beta_1 \geq \beta_2$, $\lambda_1 \geq \lambda_2$ and $\varphi_1 \leq \varphi_2$, then $Y_1 \leq_{\text{st}} Y_2$, i.e. $S_{Y_1}(t) \leq S_{Y_2}(t)$ for all $t > 0$.

Proof. $S_0(t) = (1 + \varphi(\beta t)^\alpha)^{-1/\varphi}$ is decreasing in β and increasing in φ , because $x \mapsto (1 + \varphi x)^{1/\varphi}$ is decreasing in φ for $x > 0$. $G(s) = (e^{\lambda s} - 1)/(e^\lambda - 1)$ is increasing in s and, for $s \in (0, 1)$, decreasing in λ . Hence $S_T = G(S_0)$ is decreasing in β and λ and increasing in φ . Finally, $S_Y = \Pr\{\text{Bin}(n, 1 - S_T) \leq k - 1\}$ is increasing in S_T [2]. $\square$

Corollary 3.1. More latent failure causes per component (larger λ) stochastically reduce system lifetime. A larger frailty variance φ , with $\mathbb{E}[Z_{ij}] = 1$ held fixed, stochastically increases it: heterogeneity thickens the right tail of every cause-specific lifetime (Fig. 3). Reducing production heterogeneity is therefore not,

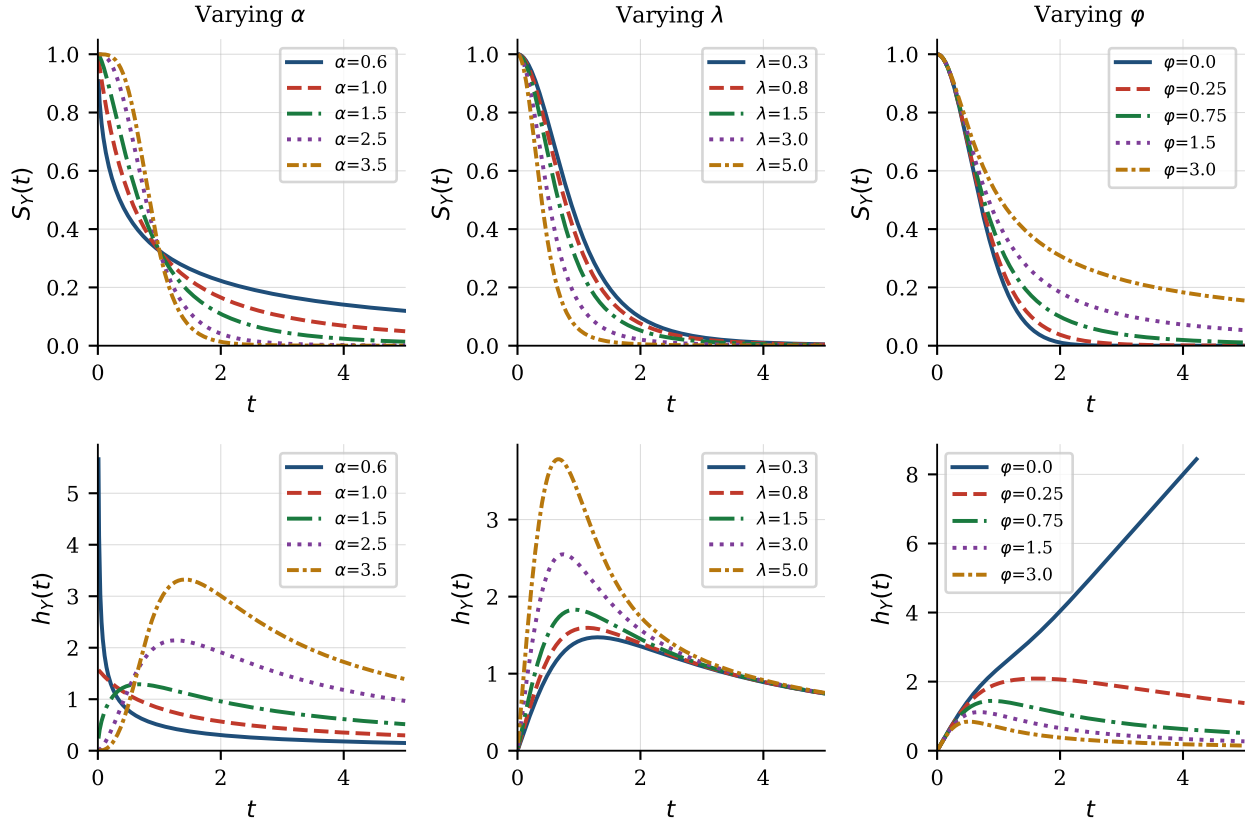


FIGURE 3. Effect of α (left), λ (center), and ϕ (right) on $S_Y(t)$ (top row) and $h_Y(t)$ (bottom row). Baseline: $\alpha = 2$, $\beta = 1$, $\lambda = 1$, $\phi = 0.5$, $k = 1$, $n = 1$. The hazard is decreasing for $\alpha \leq 1$ and unimodal for $\alpha > 1$ when $\phi > 0$; higher λ shortens lifetime; higher ϕ creates a heavier tail and turns the increasing hazard ($\phi = 0$) into a unimodal one.

by itself, a reliability improvement in the sense of $\leq_{st}$: it shortens the right tail of the lifetime distribution rather than lengthening lifetimes.

3.1. Fisher information and identifiability. The Fisher information matrix $\mathbf{I}(\theta)$, with entries

$$I_{rs}(\theta) = \mathbb{E} \left[\partial_{\theta_r} \log f_Y(Y; \theta) \partial_{\theta_s} \log f_Y(Y; \theta) \right],$$

has no closed form. We evaluated it numerically (Appendix B) on a grid of 54 interior points ($\alpha \in \{0.8, 1.5, 2.5\}$, $\beta = 1$, $\lambda \in \{0.5, 1.5, 3\}$, $\phi \in \{0.25, 0.75, 1.5\}$, $(k, n) \in \{(1, 1), (2, 5)\}$). It is positive definite at every grid point, supporting local identifiability of all four parameters at these points (global identifiability is not established), but it is ill-conditioned: the smallest eigenvalue of the correlation-scaled matrix lies between 2.5×10^{-6} and 7.4×10^{-3} . Some combinations of (β, λ) are therefore nearly interchangeable, which explains the slow mixing and the persistent estimation bias for these two parameters in Sections 4–5. At the boundaries $\lambda \rightarrow 0$ and $\phi \rightarrow 0$ the model reduces to a three-parameter sub-model and the matrix becomes singular.

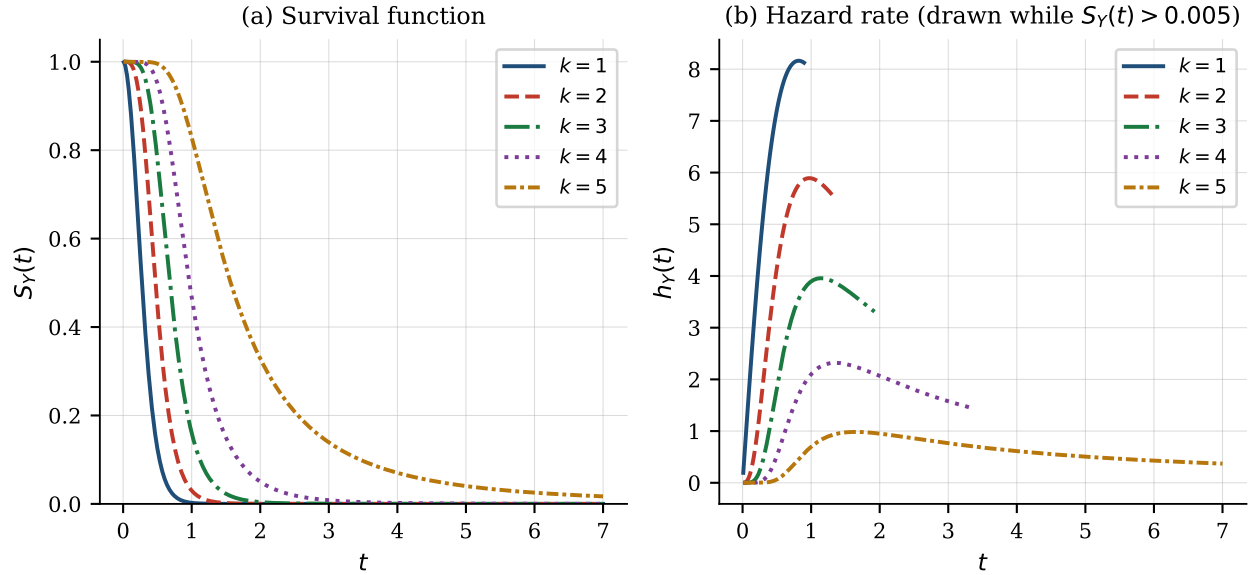


FIGURE 4. WPF system survival (a) and hazard (b) for $k = 1, \dots, 5$ out of $n = 5$. $\alpha = 2.0, \beta = 1.0, \lambda = 1.5, \varphi = 0.75$. In (b), each curve is plotted only while $S_Y(t) > 0.005$ to avoid numerical artifacts when the system lifetime is nearly exhausted. The series system ($k = 1$) has the highest hazard, peaking at $h_Y \approx 8.1$; the parallel system ($k = 5$) has the lowest.

Proposition 3.3 (Sensitivity of MTTF to the frailty). Fix $(\alpha, \beta, \lambda, k, n)$ and let $M(\varphi) = \int_0^\infty S_Y(t; \varphi) dt$ be the system MTTF, and let $\varphi_0 > 0$ be the true frailty variance, with $M(\varphi_0)$ finite, i.e. $(\alpha/\varphi_0)(n - k + 1) > 1$ by (2.5). The fixed-parameter bias, i.e. the error in the MTTF incurred by setting the frailty variance to zero while α, β and λ keep their true values, is

$$\text{Bias}(\varphi_0) = M(0) - M(\varphi_0) = B \varphi_0 + O(\varphi_0^2), \quad B = -M'(0). \quad (3.1)$$

It is strictly negative for every $\varphi_0 > 0$, and the first-order coefficient $B = B(\alpha, \beta, \lambda, k, n)$ satisfies $B \leq 0$.

Proof. By Proposition 3.2, $S_Y(t; \varphi)$ is increasing in φ , and strictly so for $t > 0$ because S_0 is strictly increasing in φ and G and the binomial map are strictly increasing. Hence $M(\varphi_0) > M(0)$ and $M'(0) \geq 0$. The expansion follows by differentiating under the integral sign. $\square$

Remark 3.1. Table 5 reports the relative fixed-parameter bias $100 \text{Bias}(\varphi_0) / M(\varphi_0)$ computed exactly. At $\varphi = 0.75$, the MTTF is underestimated by 21.0% for $\alpha = 0.8$ and 4.9% for $\alpha = 2.5$; at $\varphi = 1.5, \alpha = 0.8$, it reaches 41.0%. In all cases computed, the magnitude of the bias decreases as α increases (Fig. 5(b)). The fixed-parameter bias describes the situation in which (α, β, λ) are taken as known—for example from engineering specifications or accelerated tests—and heterogeneity is ignored. It differs from the bias obtained when a WP model is fitted by maximum likelihood to data that contain frailty, because the fitted (α, β, λ) then adjust to the heavier tail. For example, at $\alpha = 1.5$ and $\varphi = 0.75$, setting $\varphi = 0$ with the true (α, β, λ) underestimates the MTTF by 9.1% (Table 5), whereas fitting the WP model by maximum likelihood underestimates it by only 1.0% on average (Section 5).

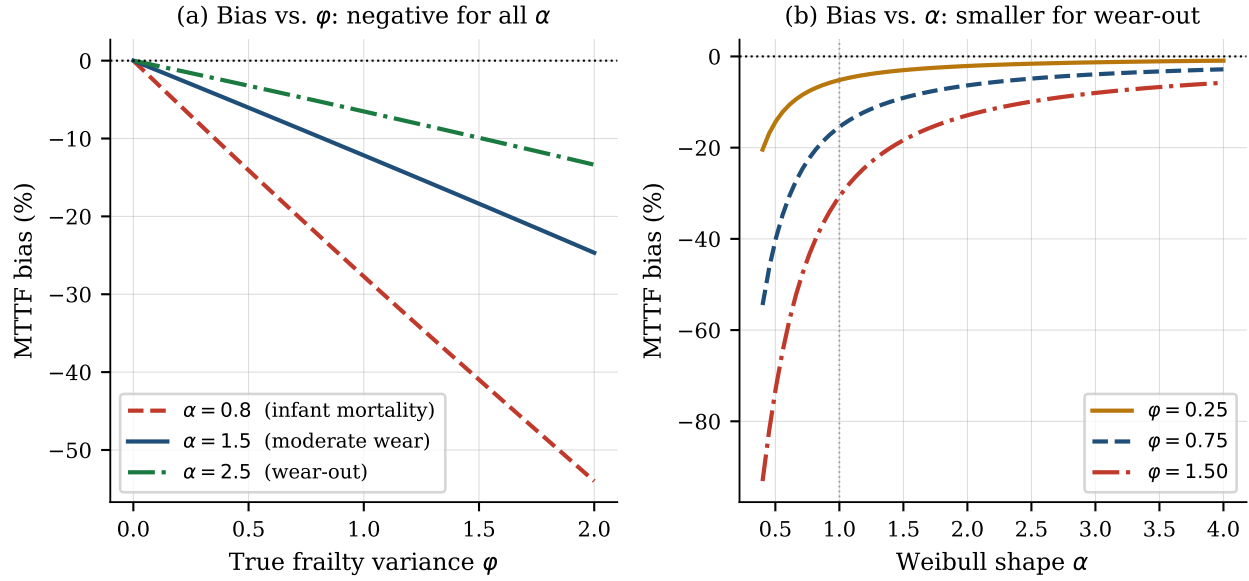


FIGURE 5. Fixed-parameter MTTF bias (%) from setting $\varphi = 0$ (Proposition 3.3). (a) Bias vs. φ for three values of α : always negative, larger in magnitude for infant-mortality components. (b) Bias vs. α : approaches zero as α increases. $\lambda = 1.5$, $\beta = 1.0$, $k = 2$, $n = 5$.

4. BAYESIAN ESTIMATION

4.1. Prior Distributions and Motivation. Throughout Sections 4–6, N denotes the number of observed lifetimes (n remains the number of components). Bayesian inference is adopted for three reasons specific to this model. First, positivity constraints on all four parameters are natural for log-normal and gamma priors but create boundary difficulties for frequentist confidence intervals in small samples—common in accelerated life testing. Second, the near non-identifiability of (β, λ) (Section 3.1) makes the maximum likelihood estimates (MLEs) unstable in small samples (Section 5); weakly informative priors regularize them. Third, the posterior predictive survival curve (4.2) integrates naturally over parameter uncertainty, yielding credible bands directly interpretable for engineering decisions.

We use log-normal (LogN) and gamma (Ga, shape–rate parameterization) priors:

$$\begin{aligned} \alpha &\sim \text{LogN}(\log 2, 0.5^2), & \beta &\sim \text{Ga}(1, 0.01), \\ \lambda &\sim \text{Ga}(1, 1), & \varphi &\sim \text{Ga}(1, 2). \end{aligned} \quad (4.1)$$

The log-normal prior on α is weakly informative (prior median 2, 90% mass on (0.88, 4.55)). The Gamma(1,2) prior on φ is exponential with mean 0.5 and mode at zero, expressing a mild preference for the simpler WP model.

4.2. Sampler. None of the full conditional distributions has a standard form—the gamma priors are not conjugate to the WPF likelihood—so we use an adaptive Metropolis-within-Gibbs sampler.

Each iteration updates α , β , λ and φ in turn by a Gaussian random walk on the log scale; the acceptance ratio includes the Jacobian term $\log \theta' - \log \theta$ of this transformation. During burn-in the proposal scales are adapted every 100 iterations toward an acceptance rate of 0.234 [18]. For the real-data analyses we run 4 chains of 10,000 burn-in and 100,000 retained iterations; the long chains are needed because λ mixes slowly (Section 3.1). Convergence is assessed by requiring a potential scale reduction factor $\hat{R} < 1.01$ [19,20] and an effective sample size (ESS) above 500 for each parameter.

4.3. Posterior Predictive Reliability. The posterior predictive survival function at mission time t_m is:

$$\hat{S}_Y(t_m) = \frac{1}{S} \sum_{s=1}^S S_Y(t_m; \boldsymbol{\theta}^{(s)}), \quad (4.2)$$

where $\{\boldsymbol{\theta}^{(s)}\}_{s=1}^S$ are retained posterior draws. The 2.5th and 97.5th empirical percentiles of $\{S_Y(t_m; \boldsymbol{\theta}^{(s)})\}$ form a 95% pointwise credible band; credible intervals (CrIs) for the parameters are obtained in the same way from their marginal draws. This propagates full parameter uncertainty into the reliability estimate.

5. MONTE CARLO SIMULATION STUDY

5.1. Experimental Design. We study one representative configuration, $\alpha = 1.5$, $\beta = 1$, $\lambda = 1.5$, $\varphi = 0.75$, $k = 2$, $n = 5$, at sample sizes $N \in \{30, 75, 150, 300\}$, with $R = 200$ replications per sample size. Data are generated by inverting S_Y numerically. Type I right censoring is applied at the 0.8 quantile of the true distribution, giving 20% censoring. In each replication we compute (a) the posterior mean and 95% equal-tailed credible interval from one chain of 2,000 burn-in and 5,000 retained iterations, and (b) the MLE, obtained by quasi-Newton maximization from several starting values, with 95% Wald intervals built on the log scale from the observed information matrix. Starting values are drawn within $\pm 20\%$ (α, β) and $\pm 50\%$ (λ, φ) of the true values, which favors both estimators. When the observed information matrix is singular (16.5% of replications at $N = 75$), the Wald interval is counted as not covering; in a further 28% of replications the interval for λ is unbounded above and covers trivially, which inflates its empirical coverage. In this configuration the MTTF is finite whenever $\varphi < 4\alpha$ by (2.5); this condition held for every posterior draw we inspected. We report the relative bias $\text{RB}(\hat{\theta}) = (\bar{\hat{\theta}} - \theta_0)/\theta_0 \times 100\%$, the root mean squared error (RMSE), and the coverage probability (CP).

5.2. Parameter Recovery. Table 4 reports results at $N = 75$. The Bayesian estimator recovers α ($\text{RB} = +1.8\%$) and the system MTTF ($\text{RB} = +1.3\%$) well, and its credible intervals attain at least the nominal coverage for all four parameters. The scale β is overestimated by about 25% and λ underestimated by about 48%; Fig. 6(a) shows that these biases decrease only slowly with N (to 23% and 45% at $N = 300$), reflecting the near non-identifiability of (β, λ) noted in Section 3.1, with the λ posterior pulled toward its prior mean of one. The frailty variance is estimated with moderate bias (-15%). Maximum likelihood is much less stable: the estimates occasionally diverge, which yields an RB of $+107\%$ for α and inflates the RMSE of λ and φ by two orders of magnitude, and

TABLE 4. Parameter Recovery: $N = 75$, $\alpha = 1.5$, $\beta = 1.0$, $\lambda = 1.5$, $\varphi = 0.75$, $k = 2$, $n = 5$, Censoring 20%, $R = 200$.

Par.	Bayesian			MLE		
	RB%	RMSE	CP%	RB%	RMSE	CP%
α	+1.8	0.153	99.0	+106.8	19.44	79.5
β	+25.5	0.298	96.0	+39.0	0.958	63.0
λ	-48.3	0.765	95.0	+1873	104.6	83.5
φ	-15.2	0.229	100.0	+653	47.77	76.0
MTTF	+1.3	—	—	+5.6	—	—

RB: relative bias; CP: coverage of 95% credible (Bayes) or Wald (MLE) intervals.

MLE: a few divergent estimates of λ, φ dominate RB and RMSE.

TABLE 5. Fixed-Parameter MTTF Bias (%) from Setting $\varphi = 0$. $\lambda = 1.5$, $\beta = 1.0$, $k = 2$, $n = 5$.

φ	Weibull shape α		
	0.8 (infant)	1.5 (moderate)	2.5 (wear-out)
0.00	0.0	0.0	0.0
0.25	-7.1	-3.0	-1.6
0.75	-21.0	-9.1	-4.9
1.50	-41.0	-18.4	-9.9

$100 [M(0) - M(\varphi)] / M(\varphi)$, other parameters fixed.

Magnitude decreasing in α , increasing in φ .

the Wald coverage is only 63–84%. When the misspecified WP model ($\varphi = 0$) is fitted to the same data by maximum likelihood, the MTTF is underestimated by only 1.0% on average. By contrast, setting $\varphi = 0$ while keeping α, β and λ at their true values underestimates it by 9.1% (Table 5); the fitted WP parameters compensate for most of the omitted frailty (see Remark 3.1).

5.3. MTTF Sensitivity to the Frailty. Table 5 reports the fixed-parameter MTTF bias of Proposition 3.3, computed exactly by numerical quadrature ($\lambda = 1.5$, $\beta = 1$, $k = 2$, $n = 5$). The bias is always negative and grows in magnitude with φ ; infant-mortality components ($\alpha = 0.8$) are the most sensitive.

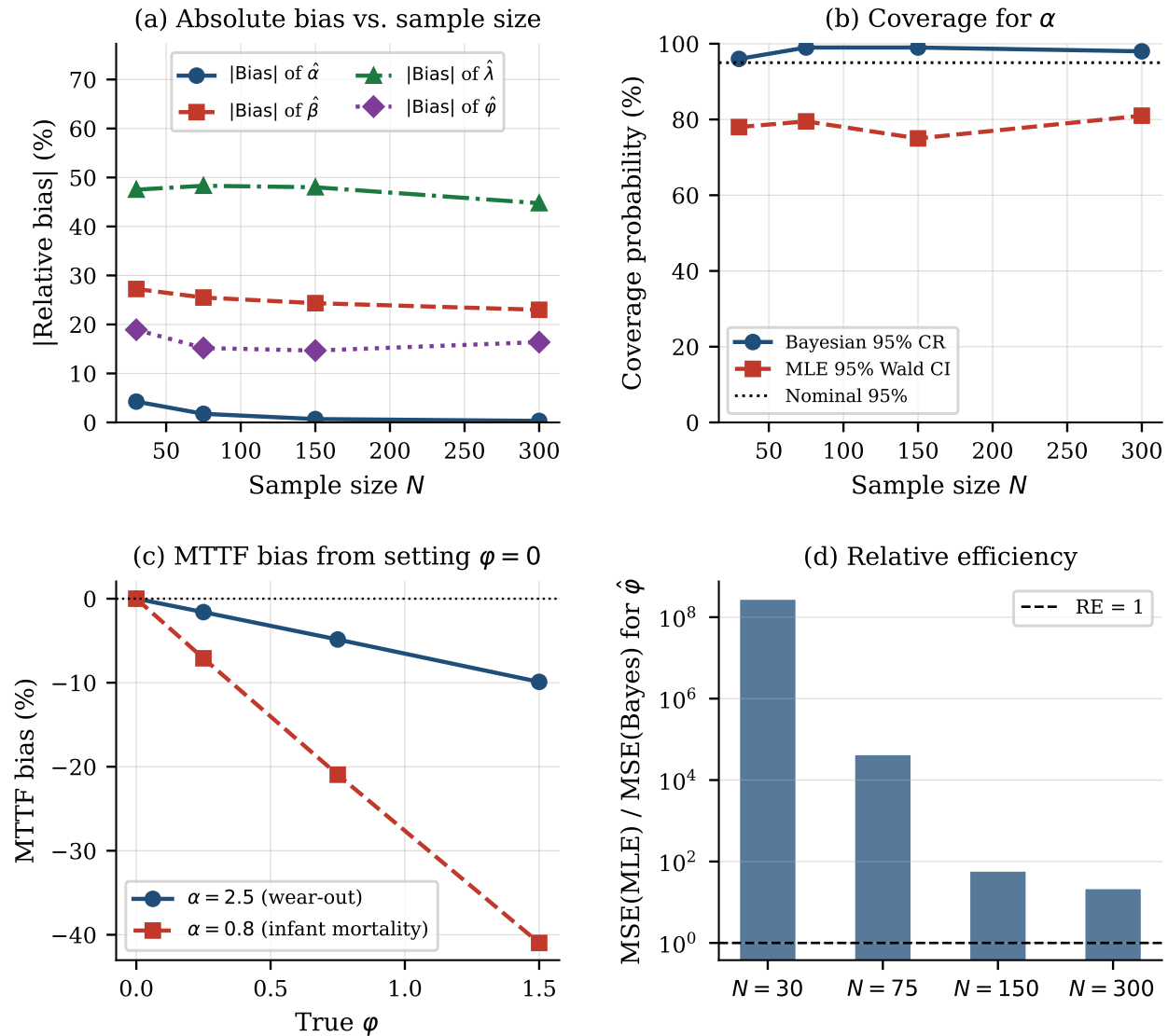


FIGURE 6. Monte Carlo simulation results ($R = 200$ replications per sample size, $\alpha = 1.5$, $\beta = 1.0$, $\lambda = 1.5$, $\phi = 0.75$, $k = 2$, $n = 5$, 20% censoring). (a) Absolute relative bias of the posterior means. (b) Coverage of 95% intervals for α . (c) Fixed-parameter MTTF bias (Proposition 3.3). (d) Relative efficiency $\text{MSE}(\text{MLE})/\text{MSE}(\text{Bayes})$ for $\hat{\phi}$ (log scale).

Figure 6 summarizes the simulation: panel (a) shows the absolute relative bias of the posterior means against N ; panel (b) compares the coverage of the Bayesian and Wald intervals for α ; panel (c) displays the fixed-parameter MTTF bias of Table 5; and panel (d) reports the relative efficiency of the two estimators of ϕ , measured by the ratio of their mean squared errors (MSEs), $\text{MSE}(\text{MLE})/\text{MSE}(\text{Bayes})$, which falls from about 3×10^8 at $N = 30$ to 22 at $N = 300$.

TABLE 6. Model Comparison: Aarset (1987) 50-Device Dataset.

Model	d	$\hat{\ell}$	AIC	BIC	K-S	p -value
WPF (proposed)	4	-241.00	490.00	497.65	0.193	0.042
WP ($\varphi = 0$)	3	-241.00	488.00	493.74	0.193	0.042
GFW ($\lambda \rightarrow 0$)	3	-241.00	488.00	493.74	0.193	0.042
W	2	-241.00	486.00	489.83	0.193	0.042

All models reach the Weibull maximum ($\hat{\alpha} = 0.949$, $\hat{\beta} = 0.0223$; $\hat{\lambda}, \hat{\phi} \rightarrow 0$).

6. APPLICATIONS

Two publicly available real datasets are analyzed at the component level ($k = n = 1$). For each, we fit WPF and its three sub-models—WP ($\varphi = 0$), GFW ($\lambda \rightarrow 0$) and the standard Weibull (W)—by maximum likelihood, and WPF by the Bayesian method of Section 4. Models are compared by the maximized log-likelihood $\hat{\ell}$, the Akaike and Bayesian information criteria (AIC and BIC) and the Kolmogorov–Smirnov (K–S) statistic; d denotes the number of parameters. Because the parameters are estimated from the same data, the ordinary K–S p -values are optimistic and are reported only as descriptive measures of fit, not as formal tests.

6.1. Dataset 1: Aarset (1987) 50-Device Lifetimes. The Aarset [21] dataset contains $N = 50$ electronic device lifetimes (hours); all observations are complete.

The maximum likelihood estimates of λ and φ converge to the boundary, so all four models attain the Weibull maximum $\hat{\ell} = -241.00$ ($\hat{\alpha} = 0.949$, $\hat{\beta} = 0.0223$), and the standard Weibull is preferred by AIC and BIC (Table 6). The fit is poor for every model (K–S statistic 0.193, with a small descriptive p -value of 0.042), because the Aarset data have a bathtub-shaped hazard [21]. At the component level, WPF cannot produce this shape: by Proposition 3.1(i)–(ii), its hazard is either decreasing ($\alpha \leq 1$) or starts from zero ($\alpha > 1$). A bathtub hazard requires models such as the exponentiated or additive Weibull [22, 23]. The fitted curves are shown in Fig. 7. The Bayesian estimates (Table 7) are consistent with this: α is centered at 1.00 (95% CrI [0.78, 1.25]), the frailty variance is small (0.087, CrI [0.002, 0.33]), and the posterior of λ (mean 0.38) lies below its prior mean of one but remains wide. Because the MTTF is infinite when $\alpha \leq \varphi$, which occurs in 27 of the 400,000 posterior draws, its posterior mean is not defined; the posterior median MTTF is 47.7 h (95% CrI [35.2, 70.5]), against a sample mean of 45.7 h.

6.2. Dataset 2: Badar and Priest (1982) Carbon Fiber Strengths. The Badar and Priest [24] dataset contains $N = 63$ single carbon fiber strength measurements (GPa) at a gauge length of 10 mm, with no censoring. Carbon fiber bundles are a natural k -out-of- n structure—a bundle of n fibers fails when the k -th breaks—and we fit WPF ($k = 1$, $n = 1$) at the individual fiber level. Results are in Tables 8–9 and Figs. 8–9.

TABLE 7. Bayesian Estimates: WPF, Aarset (1987) Dataset (4 chains $\times$ 100,000 draws after 10,000 burn-in; SD: posterior standard deviation; *posterior mean not defined, median 47.74 h).

Par.	Mean	SD	2.5%	97.5%	$\hat{R}$	ESS
α	1.000	0.121	0.779	1.250	1.000	60252
β	0.021	0.004	0.013	0.030	1.000	15945
λ	0.381	0.441	0.009	1.464	1.000	4144
φ	0.087	0.092	0.002	0.332	1.000	39789
MTTF (h)*	—	—	35.18	70.49	—	—

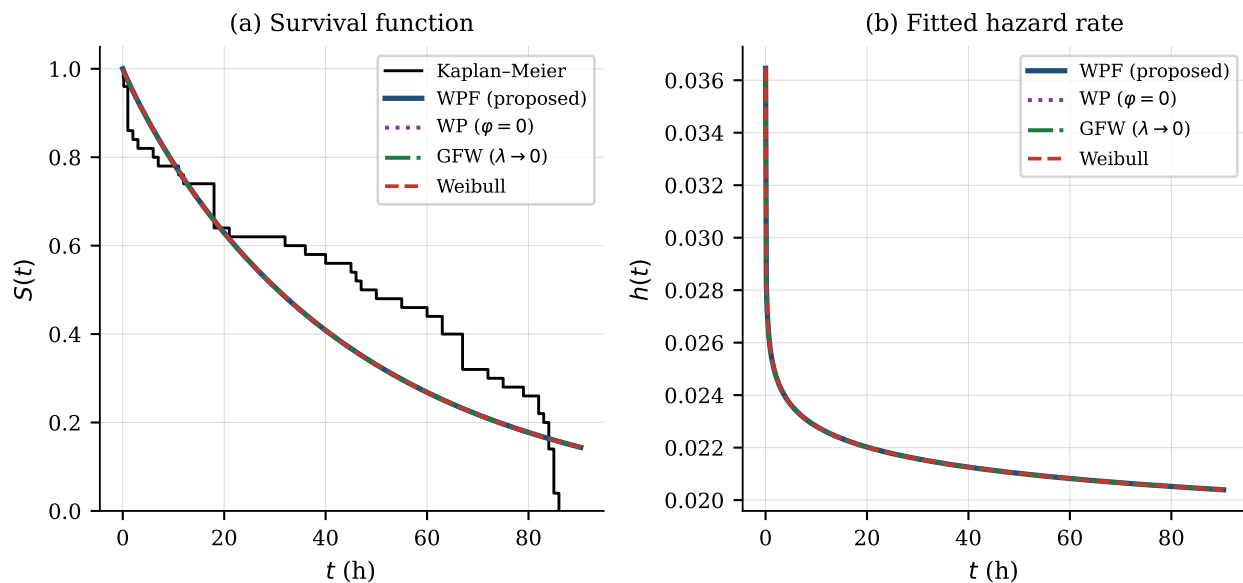


FIGURE 7. Aarset (1987) dataset: (a) fitted survival functions and the Kaplan–Meier estimate; (b) fitted hazard rates. All four models coincide at the Weibull fit ($\hat{\alpha} = 0.949 < 1$, decreasing hazard), which misses the bathtub shape of the data.

The WPF maximum likelihood estimates are $\hat{\alpha} = 8.43$, $\hat{\beta} = 0.330$, $\hat{\lambda} = 0.14$ and $\hat{\varphi} = 0.93$. The competing-risk component adds almost nothing ($\hat{\ell}$ for WPF and GFW agree to two decimals), whereas the frailty component is clearly supported: the likelihood-ratio statistic for GFW against the Weibull is 8.23, with $p \approx 0.002$ from the $\frac{1}{2}\chi_0^2 + \frac{1}{2}\chi_1^2$ boundary distribution. GFW therefore has the lowest AIC and BIC, and the frailty turns the Weibull's monotone hazard into a unimodal one (Fig. 8(b)). The K–S statistics are similar for all four models (0.088–0.091). The Bayesian posterior (Table 9) agrees: φ has mean 0.48 with 95% CrI [0.04, 1.26], bounded away from zero, while λ remains weakly identified (CrI [0.03, 3.64]). The posterior predictive survival curve closely follows the Kaplan–Meier estimate (Fig. 9(a)). Every posterior draw satisfies $\alpha > \varphi$ (the smallest ratio

TABLE 8. Model Comparison: Badar and Priest (1982) Carbon Fiber Strength Dataset ($N = 63$, all complete).

Model	d	$\hat{\ell}$	AIC	BIC	K-S	p -value
WPF (proposed)	4	-57.84	123.68	132.25	0.091	0.635
WP ($\varphi = 0$)	3	-59.17	124.34	130.77	0.090	0.660
GFW ($\lambda \rightarrow 0$)	3	-57.84	121.68	128.11	0.091	0.638
W	2	-61.96	127.91	132.20	0.088	0.686

WPF MLE: $\hat{\alpha} = 8.43$, $\hat{\beta} = 0.330$, $\hat{\lambda} = 0.14$, $\hat{\varphi} = 0.93$; GFW: $\hat{\alpha} = 8.49$, $\hat{\beta} = 0.333$, $\hat{\varphi} = 0.95$.

TABLE 9. Bayesian Estimates: WPF, Carbon Fiber Dataset (4 chains $\times$ 100,000 draws after 10,000 burn-in; SD: posterior standard deviation).

Par.	Mean	SD	2.5%	97.5%	$\hat{R}$	ESS
α	6.670	1.026	4.958	8.973	1.000	12065
β	0.302	0.021	0.255	0.337	1.000	3903
λ	1.035	1.001	0.027	3.641	1.000	3581
φ	0.478	0.321	0.040	1.262	1.000	11306
Mean strength (GPa)	3.043	0.088	2.874	3.224	—	—

α/φ is 2.35), so the mean strength exists; its posterior mean is 3.04 GPa (CrI [2.87, 3.22]), against a sample mean of 3.06 GPa.

7. DISCUSSION

The four parameters of WPF carry separable engineering interpretations. The Weibull shape α governs the failure rate trend—decreasing for $\alpha \leq 1$ (infant mortality), and rising from zero for $\alpha > 1$ —and determines the sensitivity of the MTTF to the frailty. The scale β sets the characteristic life. The Poisson parameter λ governs the number of latent failure causes, whose mean is $\lambda/(1 - e^{-\lambda})$: about 1.3 for $\lambda = 0.5$ and 3.2 for $\lambda = 3$. The frailty variance φ measures unobserved heterogeneity in the susceptibility of the failure mechanisms; large φ implies a population of apparently identical components that is highly dispersed, which lengthens the right tail and turns an increasing hazard into a unimodal one.

Proposition 3.3 and Table 5 show that the system MTTF is sensitive to the frailty when the remaining parameters are fixed: at $\varphi = 0.75$ and $\alpha = 0.8$, an analyst who ignores heterogeneity predicts an MTTF 21.0% shorter than the true one, which would lead to conservative but costly maintenance schedules and spare-parts stocks. When the frailty-free model is re-estimated from

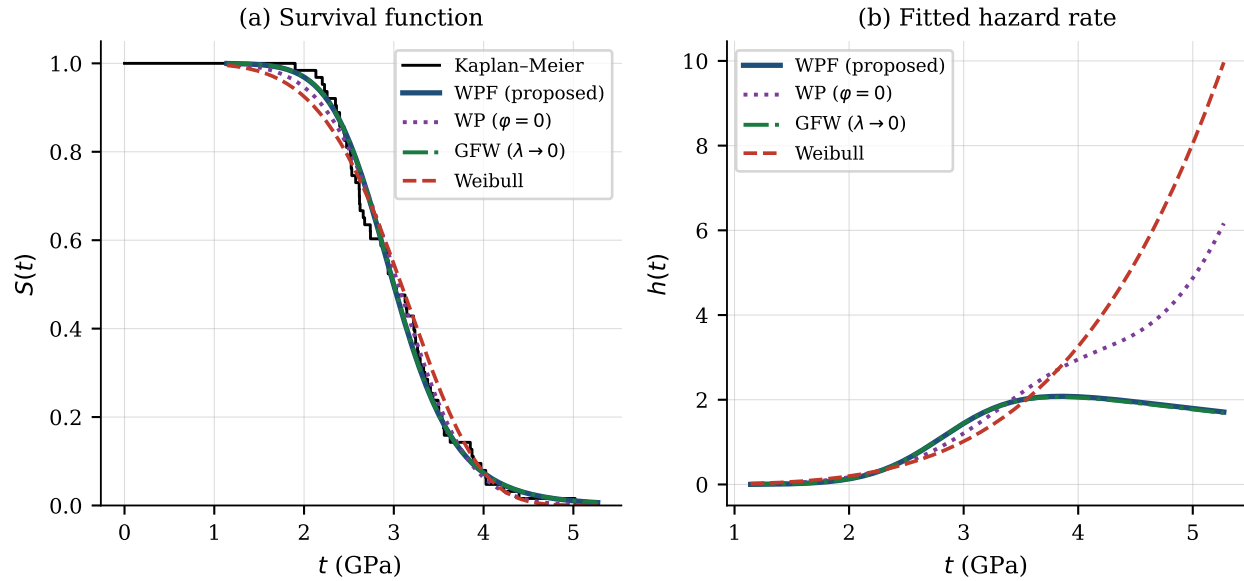


FIGURE 8. Carbon fiber dataset: (a) fitted survival functions and the Kaplan–Meier estimate; (b) fitted hazard rates. WPF and GFW coincide and give a unimodal hazard; WP and the Weibull give increasing hazards.

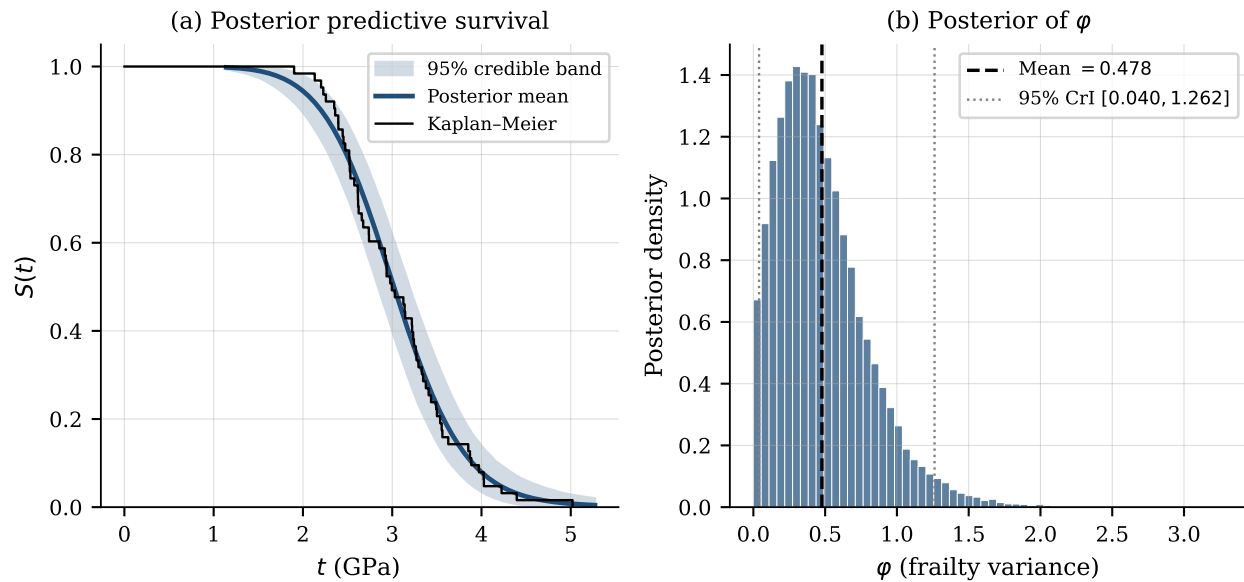


FIGURE 9. Carbon fiber dataset, WPF model: (a) posterior predictive survival with 95% pointwise credible band and the Kaplan–Meier estimate; (b) posterior density of the frailty variance ϕ .

data, however, the other parameters largely compensate, and the MTTF bias is small (Section 5). The practical value of the frailty term therefore lies less in the mean lifetime than in the shape of the hazard and the right tail, which drive tail-based quantities such as the mean residual life and upper quantiles.

The two applications show when the frailty term matters. The Aarset data do not support either extension and lie outside the model's range of hazard shapes. The carbon fiber data clearly support the frailty component, which improves the fit over the Weibull by 4.1 log-likelihood units and changes the hazard from increasing to unimodal; the competing-risk component is not identified by these data.

Seven limitations apply. First, each latent cause has its own frailty; a frailty shared by all causes of a component would induce dependence between the causes but no longer yields the closed form (2.2). Second, frailty is time-invariant; degradation processes require a state-space extension. Third, zero-truncated Poisson mixing imposes underdispersion (variance below the mean) on the competing risk count. Negative-binomial mixing [12, 13] would allow overdispersion. Fourth, β and λ are only weakly identified (Section 3.1), so λ should be interpreted with care unless prior information about the number of failure mechanisms is available; identifiability was checked locally on a grid, not globally. Fifth, the simulation covers a single parameter configuration over four sample sizes, so its conclusions about the two estimators may not extend to other designs. Sixth, both applications are at the component level, and their K-S p -values are not calibrated for parameter estimation; a dataset of k -out-of- n system lifetimes would be needed to illustrate the system model directly. Seventh, because the frailty produces a power-law tail, the MTTF and higher moments exist only under condition (2.5), which must be checked before reporting moment-based summaries.

8. CONCLUSION

This paper introduces the Weibull–Poisson frailty (WPF) distribution as a four-parameter model for k -out-of- n system reliability under competing risks and unobserved heterogeneity. Its survival, density and hazard functions are in closed form, and it nests six distributions, four of them previously published (Fig. 1, Table 1). Three propositions characterize the hazard shape (decreasing, increasing, or unimodal), establish the stochastic ordering in $(\beta, \lambda, \varphi)$, and quantify the fixed-parameter MTTF bias from ignoring the frailty, which is always negative, with a magnitude that decreases with α and increases with φ (Table 5, Fig. 5). A numerical study of the Fisher information shows local identifiability but near-collinearity of β and λ . The moments of the system lifetime exist only below a threshold set by α/φ and $n - k + 1$. In the simulated configuration, Bayesian estimation with an adaptive Metropolis-within-Gibbs sampler recovers the shape parameter and the MTTF accurately and is far more stable than maximum likelihood. On real data, the frailty component substantially improves the fit to carbon fiber strengths, whereas Aarset's bathtub-shaped data are beyond the model's range.

APPENDIX A. COMPONENT DENSITY AND MOMENTS

Differentiating $S_T(t)$ in (2.2):

$$f_T(t) = \frac{\lambda \alpha \beta^\alpha t^{\alpha-1} (1 + \varphi(\beta t)^\alpha)^{-1/\varphi-1} e^{-\lambda[1-(1+\varphi(\beta t)^\alpha)^{-1/\varphi}]}}{1 - e^{-\lambda}}. \quad (\text{A.1})$$

As $\varphi \rightarrow 0$: $(1 + \varphi(\beta t)^\alpha)^{-1/\varphi-1} \rightarrow e^{-(\beta t)^\alpha}$, and (A.1) reduces to the Weibull–Poisson density [6, 7]. For $\varphi = 0$, write $S_T^m = (e^\lambda - 1)^{-m} \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} e^{j\lambda S_0}$ with $S_0 = e^{-(\beta t)^\alpha}$, expand $e^{j\lambda S_0}$ in powers of S_0 , and integrate term by term using $\int_0^\infty r t^{r-1} e^{-l(\beta t)^\alpha} dt = \Gamma(1 + r/\alpha) \beta^{-r} l^{-r/\alpha}$. The constant terms cancel, and

$$I_m(r, \theta) = \frac{\Gamma(1 + r/\alpha)}{\beta^r (e^\lambda - 1)^m} \sum_{j=1}^m \binom{m}{j} (-1)^{m-j} \sum_{l=1}^\infty \frac{(j\lambda)^l}{l! l^{r/\alpha}}.$$

For $\varphi > 0$, I_m is evaluated by adaptive numerical quadrature on $(0, \infty)$. For small λ , S_T and f_T should be evaluated in the equivalent form $S_T = (e^{\lambda S_0} - 1)/(e^\lambda - 1)$ with a cancellation-free evaluation of $e^x - 1$; form (2.2) loses precision as $\lambda \rightarrow 0$ and can produce spurious likelihood maxima.

APPENDIX B. NUMERICAL EVALUATION OF THE FISHER INFORMATION

We use the outer-product form $\mathbf{I}(\theta) = \mathbb{E}[\mathbf{s}(Y)\mathbf{s}(Y)^\top]$ with score $\mathbf{s} = \partial \log f_Y / \partial \theta$ obtained by central differences (relative step 10^{-5}). The expectation is computed on the probability scale, $\mathbb{E}[g(Y)] = \int_0^1 g(Q_Y(u)) du$, with 400-node Gauss–Legendre quadrature, where Q_Y is the quantile function. Positive definiteness is assessed through the smallest eigenvalue of the correlation-scaled matrix $D^{-1/2} \mathbf{I} D^{-1/2}$, $D = \text{diag}(\mathbf{I})$.

APPENDIX C. DATASETS USED IN THE APPLICATIONS

For complete reproducibility, the two datasets used in Section 6 are reproduced in full below. All values are exactly as published in the original sources.

D1. Aarset (1987) – Lifetimes of 50 Electronic Devices (hours).

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

$N = 50$; all observations are complete (no censoring). Summary: min = 0.1 h, max = 86 h, mean = 45.69 h, median = 48.5 h.

D2. Badar and Priest (1982) – Carbon Fiber Strengths (GPa), 10 mm Gauge Length. Single carbon fibers tested under tension at a gauge length of 10 mm [24]. All observations are complete.

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397
2.445	2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614
2.616	2.618	2.624	2.659	2.675	2.738	2.740	2.856	2.917
2.928	2.937	2.937	2.977	2.996	3.030	3.125	3.139	3.145
3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.332	3.346
3.377	3.408	3.435	3.493	3.501	3.537	3.554	3.562	3.628
3.852	3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020

$N = 63$; all observations are complete. Summary: min = 1.901 GPa, max = 5.020 GPa, mean = 3.059 GPa, SD = 0.621 GPa.

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Data Availability: Both datasets analyzed in this study are reproduced in full in Appendix C.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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