

A Characterization of Bessel and Jacobi Orthogonal Polynomials via a Three-Parameter Differential Raising Operator

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Abstract. Let $\mathcal{A}_{\xi_1, \xi_2, \mu} := x(xD + \mu\mathbb{I}) + \xi_1\mathbb{I} + \xi_2D$ be a first-order differential raising operator, where $(\xi_1, \xi_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$, $D = d/dx$, and $\mathbb{I}$ is the identity operator on the polynomial space. We characterize all monic orthogonal polynomial sequences for which the normalized sequence generated by $\mathcal{A}_{\xi_1, \xi_2, \mu}$ is also orthogonal. Under the appropriate regularity conditions, the resulting sequences are scaled Bessel polynomials when $\xi_2 = 0$ and scaled Jacobi polynomials when $\xi_2 \neq 0$, with the Gegenbauer polynomials arising precisely in the subcase $\xi_1 = 0$. Some structural properties and connection relations are also established.

1. INTRODUCTION

An orthogonal polynomial sequence $\{p_n\}_{n \geq 0}$ is said to be *D-classical* when it satisfies the criterion established by Hahn: the derived sequence $\{Dp_n\}_{n \geq 0}$ is itself orthogonal [16, 25]. As shown in [19, 20], the Hermite, Laguerre, Bessel, and Jacobi families constitute the complete list of orthogonal polynomial sequences with this property.

More generally, consider a linear operator $\mathcal{O}$ acting on the space of polynomials such that a polynomial of degree n is mapped to a polynomial of degree $n + n_0$, where n_0 is a fixed integer (with $n \geq 0$ if $n_0 \geq 0$, and $n \geq n_0$ if $n_0 < 0$). A sequence $\{p_n\}_{n \geq 0}$ of orthogonal polynomials will be called *O-classical* whenever the transformed sequence $\{\mathcal{O}p_n\}_{n \geq 0}$ preserves orthogonality.

In the present work, our analysis identifies two well-known classical families: the Bessel and Jacobi polynomials, with the Gegenbauer polynomials arising as the symmetric Jacobi subcase.

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These families play a prominent role in several branches of physics, particularly in problems involving spherical or rotational symmetry, as well as in quantum mechanics, where they arise in the study of angular momentum and wave functions on spherical domains [18, 21, 24]. Then, we consider the raising operator, $\mathcal{A}_{\xi_1, \xi_2, \mu} := x(xD + \mu\mathbb{I}) + \xi_1\mathbb{I} + \xi_2D$, where ξ_1, ξ_2 and μ are nonzero parameters and $\mathbb{I}$ represents the identity operator. We describe all the $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical orthogonal polynomial sequences.

Indeed, if $\xi_2 = 0$, we proved that the scaled Bessel polynomial sequence $\{a^{-n}B_n^{(\frac{\mu}{2}+1)}((ax))\}_{n \geq 0}$, where $a = (\frac{\xi_1}{2})^{-1}$, $\mu/2 \neq -n$ and $\mu \neq -n - 1$ for $n \geq 1$, is the unique $\mathcal{A}_{\xi_1, 0, \mu}$ -classical orthogonal polynomial sequence.

If $\xi_2 \neq 0$, we proved that the scaled Jacobi polynomial sequence $\left\{a^{-n}P_n^{\left(\frac{\mu-a\xi_1}{2}, \frac{\mu+a\xi_1}{2}\right)}(ax)\right\}_{n \geq 0}$ where $a^2 = -\xi_2^{-1}$, $(\frac{\mu-a\xi_1}{2}, \frac{\mu+a\xi_1}{2}) \neq -n, \mu \neq -n - 1, n \geq 1$, is the unique $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical orthogonal polynomial sequence. Here the parameters are assumed to satisfy the appropriate regularity conditions. The Gegenbauer family is recovered precisely when $\xi_1 = 0$, for which $A = B$.

The concept of O -classical orthogonal polynomials has been studied by many authors in the literature; see [1–14, 17, 22, 23].

The paper is organized as follows. Section 2 introduces the notation and preliminary results used throughout the paper. Section 3 defines $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical orthogonal polynomial sequences, establishes their structural relations and classification, and derives the associated second-order differential eigenoperator. Finally, Section 4 summarizes the main results and outlines directions for future research.

2. PRELIMINARIES

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients. The algebraic dual space of $\mathbb{P}$ will be represented by $\mathbb{P}'$. We denote by $\langle u, p \rangle$ the action of $u \in \mathbb{P}'$ on $p \in \mathbb{P}$ and by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the sequence of moments of u with respect to the polynomial sequence $\{x^n\}_{n \geq 0}$.

Let us define the following operations in $\mathbb{P}'$ [19]. For a linear functional u , any polynomial g , and any $(a, b) \in \mathbb{C} \setminus \{0\} \times \mathbb{C}$, let $Du = u'$, gu , $\tau_{-b}u$ and $h_a u$ be the linear functionals defined by duality

$$\begin{aligned} \langle u', f \rangle &:= -\langle u, f' \rangle, & \langle gu, f \rangle &:= \langle u, gf \rangle, \\ \langle h_a u, f \rangle &:= \langle u, h_a f \rangle = \langle u, f(ax) \rangle, & \langle \tau_{-b} u, f \rangle &:= \langle u, \tau_b f \rangle = \langle u, f(x-b) \rangle. \end{aligned}$$

A linear functional u is said to be *normalized* when its zeroth moment satisfies $(u)_0 = 1$.

For convenience, we recall several fundamental notions that will be used throughout this work:

- A **monic polynomial sequence (MPS)** is a sequence of polynomials $\{P_n\}_{n \geq 0}$ in which each polynomial P_n has leading coefficient equal to 1; equivalently, the coefficient of x^n in $P_n(x)$ is 1.
- The **dual sequence** $\{u_n\}_{n \geq 0}$ associated with a given MPS $\{P_n\}_{n \geq 0}$ is the unique family of linear functionals on the polynomial space $\mathbb{P}$ satisfying the biorthogonality conditions

$$\langle u_n, P_m \rangle = \delta_{n,m}, \quad n, m \geq 0,$$

where $\delta_{n,m}$ denotes the Kronecker delta.

- The first element u_0 of the dual sequence is referred to as the **canonical form** attached to the monic sequence $\{P_n\}_{n \geq 0}$. This functional determines the entire dual sequence and encodes the orthogonality (or quasi-orthogonality) properties of the polynomials.

Lemma 2.1. [19,20] Let $\{P_n\}_{n \geq 0}$ be a MPS, for any $u \in \mathbb{P}'$ and any integer $m \geq 1$, the following statements are equivalent:

- (i) $\langle u, P_{m-1} \rangle \neq 0, \langle u, P_n \rangle = 0, n \geq m$.
- (ii) $\exists \lambda_k \in \mathbb{C}, 0 \leq k \leq m-1, \lambda_{m-1} \neq 0$ such that $u = \sum_{k=0}^{m-1} \lambda_k u_k$.

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n \geq 0}$ of $\{P_n^{[1]}\}_{n \geq 0}$ where $P_n^{[1]}(x) := (n+1)^{-1}P'_{n+1}(x), n \geq 0$, satisfies [19]

$$Du_n^{[1]} = -(n+1)u_{n+1}, \quad n \geq 0.$$

Similarly, the dual sequence $\{\tilde{u}_n\}_{n \geq 0}$ of $\{\tilde{P}_n\}_{n \geq 0}$, where $\tilde{P}_n(x) := a^{-n}P_n(ax+b)$ with $(a,b) \in \mathbb{C} \setminus \{0\} \times \mathbb{C}$, is given by [19]

$$\tilde{u}_n = a^n(h_{a^{-1}} \circ \tau_{-b})u_n, \quad n \geq 0.$$

A linear functional u is said to be *regular* [19] if there exists a polynomial sequence $\{P_n\}_{n \geq 0}$, with $\deg P_n = n$, such that

$$\langle u, P_n P_m \rangle = r_n \delta_{n,m}, \quad n, m \geq 0,$$

where $r_n \neq 0$ for every $n \geq 0$. The polynomials may be normalized to be monic, in which case $\{P_n\}_{n \geq 0}$ is called the monic orthogonal polynomial sequence (MOPS) associated with u . Since $P_0 = 1$, regularity implies

$$(u)_0 := \langle u, 1 \rangle = r_0 \neq 0.$$

Hence u can be written as

$$u = (u)_0 u_0,$$

where $u_0 := (u)_0^{-1}u$ is the corresponding normalized regular functional, satisfying $\langle u_0, 1 \rangle = 1$.

Moreover, a regular functional has no nonzero polynomial annihilator: if $F \in \mathbb{P}$ satisfies $Fu = 0$, then necessarily $F = 0$ [19].

Proposition 2.1. [19] Let $\{P_n\}_{n \geq 0}$ be a MOPS with $\deg P_n = n, n \geq 0$, and let $\{u_n\}_{n \geq 0}$ be its dual sequence. The following statements are equivalent.

- (i) $\{P_n\}_{n \geq 0}$ is orthogonal with respect to u_0 .
- (ii) $u_n = \langle u_0, P_n^2 \rangle^{-1} P_n u_0$, $n \geq 0$.
- (iii) $\{P_n\}_{n \geq 0}$ satisfies the third-order linear recurrence relation

$$\begin{cases} P_0(x) = 1, & P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), & n \geq 0, \end{cases} \quad (2.1)$$

where $\beta_n = \langle u_0, xP_n^2 \rangle \langle u_0, P_n^2 \rangle^{-1}$, $n \geq 0$ and $\gamma_{n+1} = \langle u_0, P_{n+1}^2 \rangle \langle u_0, P_n^2 \rangle^{-1} \neq 0$, $n \geq 0$.

If $\{P_n\}_{n \geq 0}$ is a MOPS with respect to the regular form u_0 , then $\{\tilde{P}_n\}_{n \geq 0}$ is a MOPS with respect to the regular form $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$, and satisfies [19,20]

$$\begin{cases} \tilde{P}_0(x) = 1, & \tilde{P}_1(x) = x - \tilde{\beta}_0, \\ \tilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{P}_n(x), & n \geq 0, \end{cases}$$

where $\tilde{\beta}_n = a^{-1}(\beta_n - b)$ and $\tilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$.

An orthogonal polynomial sequence $\{P_n\}_{n \geq 0}$ is called D -classical, if $\{P_n^{[1]}\}_{n \geq 0}$ is also orthogonal (Hermite, Laguerre, Bessel or Jacobi), [16,19]. A second characterization of these polynomials, which will play the leading role in the sequel, is that they are the only polynomial solutions of the second-order differential equation (Bochner [15])

$$\phi(x)P''_{n+1}(x) - \psi(x)P'_{n+1}(x) = \lambda_n P_{n+1}(x), \quad n \geq 0, \quad (2.2)$$

where ϕ, ψ are polynomials, ϕ monic, $\deg \phi = t \leq 2$, $\deg \psi = 1$ and $\lambda_n = (n+1)\left(\frac{1}{2}\phi''(0)n - \psi'(0)\right) \neq 0$, $n \geq 0$ (see also [19]).

If $\{P_n\}_{n \geq 0}$ is a D -classical sequence satisfying (2.2), then $\{\tilde{P}_n\}_{n \geq 0}$ is also a D -classical sequence and satisfies (see [19])

$$\tilde{\phi}(x)\tilde{P}''_{n+1}(x) - \tilde{\psi}(x)\tilde{P}'_{n+1}(x) = \tilde{\lambda}_n \tilde{P}_{n+1}(x), \quad n \geq 0, \quad (2.3)$$

where $\tilde{\phi}(x) = a^{-t}\phi(ax+b)$, $\tilde{\psi}(x) = a^{1-t}\psi(ax+b)$ and $\tilde{\lambda}_n = a^{2-t}\lambda_n$.

We now recall some basic properties of D -classical orthogonal polynomials, focusing on two families: the Bessel polynomials (C1) and the Jacobi polynomials (C2).

Bessel Orthogonal Polynomials (C1): For $n \geq 0$ and $\alpha \neq -\frac{n}{2}$, the Bessel orthogonal polynomials are denoted by $P_n(x) = B_n^{(\alpha)}(x)$, with $u_0 = \mathcal{B}^{(\alpha)}$. The coefficients are given by:

$$\begin{aligned} \beta_0 &= -\frac{1}{\alpha}, & \beta_n &= \frac{1-\alpha}{(n+\alpha-1)(n+\alpha)}, & n \geq 0, \\ \gamma_n &= -\frac{n(n+2\alpha-2)}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)}, & n \geq 1. \end{aligned}$$

The polynomials ϕ and ψ are x^2 and $-2(\alpha x + 1)$, respectively, and λ_n are $(n+1)(n+2\alpha)$ for $n \geq 0$. [19,20].

Jacobi Orthogonal Polynomials (C2): For $n \geq 0$ and $\alpha, \beta \neq -n$, $\alpha + \beta \neq -n-1$, $n \geq 1$, the Jacobi orthogonal polynomials are denoted by $P_n(x) = J_n^{(\alpha, \beta)}(x)$, with $u_0 = \mathcal{J}^{(\alpha, \beta)}$. The coefficients are

given by [19,20]

$$\beta_0 = \frac{\alpha - \beta}{\alpha + \beta + 2}, \quad \beta_n = \frac{\alpha^2 - \beta^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)},$$

$$\gamma_n = \frac{4n(n + \alpha + \beta)(n + \alpha)(n + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)}, \quad n \geq 1.$$

$\phi(x) = x^2 - 1$, $\psi(x) = -(\alpha + \beta + 2)x + \alpha - \beta$, $\lambda_n = (n + 1)(n + \alpha + \beta + 2)$ for $n \geq 0$ [19,20].

Remark 2.1. *There are some special cases of Jacobi polynomials. When $\alpha = \beta$, the Jacobi polynomials reduce to the Gegenbauer (or ultraspherical) polynomials.*

3. THE $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -CLASSICAL ORTHOGONAL POLYNOMIALS

In this section, we introduce the notion of $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical orthogonal polynomial sequences and derive the main relations connecting the original and transformed sequences. These relations are then used to classify all such sequences and to establish the associated differential eigenvalue equation.

Recall the operator

$$\mathcal{A}_{\xi_1, \xi_2, \mu} : \mathbb{P} \longrightarrow \mathbb{P},$$

$$f \longmapsto \mathcal{A}_{\xi_1, \xi_2, \mu}(f) = (x^2 + \xi_2)f' + (\mu x + \xi_1)f.$$

Since $\mathcal{A}_{\xi_1, \xi_2, \mu}$ raises the degree of a polynomial by one, it naturally generates a new polynomial sequence from a given one. This observation motivates the following definition.

Definition 3.1. *We call a sequence $\{P_n\}_{n \geq 0}$ of orthogonal polynomials $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical if there exists a sequence $\{P_n^{[1]}\}_{n \geq 0}$ of orthogonal polynomials such that $\mathcal{A}_{\xi_1, \xi_2, \mu}P_n = P_{n+1}^{[1]}$, $n \geq 0$.*

For any MPS $\{P_n\}_{n \geq 0}$ we define the MPS $\{P_n^{[1]}\}_{n \geq 0}$, given by

$$P_{n+1}^{[1]}(x) := \frac{\mathcal{A}_{\xi_1, \xi_2, \mu}P_n(x)}{n + \mu}, \quad n \geq 0,$$

or equivalently

$$(n + \mu)P_{n+1}^{[1]}(x) := (x^2 + \xi_2)P'_n(x) + (\mu x + \xi_1)P_n(x), \quad n \geq 0, \quad (3.1)$$

with initial value $P_0^{[1]}(x) = 1$.

Our next goal is to describe all the $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical polynomial sequences. Note that we need $\xi = (\xi_1, \xi_2) \neq (0, 0)$ to ensure that $\{P_n^{[1]}\}_{n \geq 0}$ is an orthogonal sequence. Indeed, if we suppose that $\xi = (\xi_1, \xi_2) = (0, 0)$, the relation (3.1) becomes, for $x = 0$, $P_{n+1}^{[1]}(0) = 0$, $n \geq 0$, which contradicts the orthogonality of $\{P_n^{[1]}\}_{n \geq 0}$.

The operator $\mathcal{A}_{\xi_1, \xi_2, \mu}$ raises the degree of any polynomial. Such an operator is called a *raising operator*. By transposition of the operator $\mathcal{A}_{\xi}$, we get

$${}^t\mathcal{A}_{\xi_1, \xi_2, \mu} = -\mathcal{A}_{-\xi_1, \xi_2, 2-\mu}. \quad (3.2)$$

To prove this, let $f \in \mathbb{P}$ and $u \in \mathbb{P}'$. Then

$$\langle u, \mathcal{A}_{\xi_1, \xi_2, \mu} f \rangle = \langle {}^t \mathcal{A}_{\xi_1, \xi_2, \mu} u, f \rangle.$$

It holds that

$$\begin{aligned} \langle u, \mathcal{A}_{\xi_1, \xi_2, \mu} f \rangle &= \langle u, (x^2 + \xi_2)f' + (\mu x + \xi_1)f \rangle \\ &= \langle u, (x^2 + \xi_2)f' \rangle + \langle u, (\mu x + \xi_1)f \rangle \\ &= \langle (x^2 + \xi_2)u, f' \rangle + \langle (\mu x + \xi_1)u, f \rangle \\ &= -\langle [(x^2 + \xi_2)u]', f \rangle + \langle (\mu x + \xi_1)u, f \rangle \\ &= -\langle 2xu + (x^2 + \xi_2)u', f \rangle + \langle (\mu x + \xi_1)u, f \rangle \\ &= \langle -(x^2 + \xi_2)u' - ((2 - \mu)x - \xi_1)u, f \rangle = \langle -\mathcal{A}_{-\xi_1, \xi_2, 2-\mu} u, f \rangle. \end{aligned}$$

Denote by $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ the dual sequences corresponding to $\{P_n\}_{n \geq 0}$ and $\{P_n^{[1]}\}_{n \geq 0}$, respectively. Then

$$\langle v_{n+1}, P_{m+1}^{[1]} \rangle = \delta_{n,m}, \quad n, m \geq 0.$$

Hence,

$$\langle v_{n+1}, \mathcal{A}_{\xi_1, \xi_2, \mu} P_m \rangle = (m + \mu)\delta_{n,m}, \quad n, m \geq 0.$$

Using (3.2), we obtain

$$\langle {}^t \mathcal{A}_{\xi_1, \xi_2, \mu} v_{n+1}, P_m \rangle = (m + \mu)\delta_{n,m}, \quad n, m \geq 0.$$

According to Lemma 2.1 and (3.2), we obtain that

$$(x^2 + \xi_2)v'_{n+1} + ((2 - \mu)x - \xi_1)v_{n+1} = -(n + \mu)u_n, \quad n \geq 0. \quad (3.3)$$

Assume that $\{P_n\}_{n \geq 0}$ and $\{P_n^{[1]}\}_{n \geq 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1, P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \gamma_{n+1} \neq 0, n \geq 0, \end{cases} \quad (3.4)$$

$$\begin{cases} P_0^{[1]}(x) = 1, P_1^{[1]}(x) = x - \rho_0, \\ P_{n+2}^{[1]}(x) = (x - \rho_{n+1})P_{n+1}^{[1]}(x) - \varrho_{n+1}P_n^{[1]}(x), \varrho_{n+1} \neq 0, n \geq 0. \end{cases} \quad (3.5)$$

Next, a first result will be deduced as a consequence of the relations (3.1), (3.4) and (3.5).

Proposition 3.1. *The sequences $\{P_n\}_{n \geq 0}$ and $\{P_n^{[1]}\}_{n \geq 0}$ satisfy the following finite type relation*

$$(x^2 + \xi_2)P_n(x) = P_{n+2}^{[1]}(x) + \theta_n P_{n+1}^{[1]}(x) + \omega_n P_n^{[1]}(x), \quad n \geq 0, \quad (3.6)$$

where

$$\begin{aligned} \theta_n &= (n + \mu)(\beta_n - \rho_{n+1}), \quad n \geq 0, \\ \omega_n &= (n + \mu - 1)\gamma_n - (n + \mu)\varrho_{n+1}, \quad n \geq 0, \end{aligned}$$

with the convention $\gamma_0 = 0$.

Proof. By differentiating (3.4), we obtain

$$P'_{n+2}(x) = (x - \beta_{n+1})P'_{n+1}(x) - \gamma_{n+1}P'_n(x) + P_{n+1}(x), \quad n \geq 0.$$

Multiply the last equation by $x^2 + \xi_2$ and the relation (3.4) by $\mu x + \xi_1$, take the sum of the two resulting equations, and substitute (3.1). Then, we obtain

$$\begin{aligned} (n+2+\mu)P_{n+3}^{[1]}(x) &= (n+1+\mu)(x-\beta_{n+1})P_{n+2}^{[1]}(x) \\ &\quad - (n+\mu)\gamma_{n+1}P_{n+1}^{[1]}(x) + (x^2 + \xi_2)P_{n+1}(x), \quad n \geq 0. \end{aligned}$$

Using the three-term recurrence relation (3.5), we get

$$\begin{aligned} (x^2 + \xi_2)P_{n+1}(x) &= P_{n+3}^{[1]}(x) + (n+1+\mu)(\beta_{n+1} - \rho_{n+2})P_{n+2}^{[1]}(x) \\ &\quad + ((n+\mu)\gamma_{n+1} - (n+1+\mu)\varrho_{n+2})P_{n+1}^{[1]}(x), \quad n \geq 0. \end{aligned}$$

In fact, this result is valid for $n+1$ replaced by n . More precisely, we have for $n \geq 0$

$$(x^2 + \xi_2)P_n(x) = P_{n+2}^{[1]}(x) + (n+\mu)(\beta_n - \rho_{n+1})P_{n+1}^{[1]}(x) + ((n-1+\mu)\gamma_n - (n+\mu)\varrho_{n+1})P_n^{[1]}(x),$$

with the convention $\gamma_0 = 0$. Hence the desired result. $\square$

Now we establish, in the next lemma, an algebraic relation between the forms u_0 and v_0 .

Lemma 3.1. *The forms u_0 and v_0 satisfy the following relation*

$$(x^2 + \xi_2)v_0 = -\mu\varrho_1u_0.$$

Proof. According to Proposition 3.1, we obtain

$$\langle (x^2 + \xi_2)v_0, P_n \rangle = 0, \quad n \geq 1. \quad (3.7)$$

On the other hand, by (3.6) with $n = 0$, we have

$$x^2 + \xi_2 = P_2^{[1]} + \mu(\beta_0 - \rho_1)P_1^{[1]} - \mu\varrho_1,$$

and then

$$\begin{aligned} \langle (x^2 + \xi_2)v_0, P_0 \rangle &= \langle v_0, P_2^{[1]} + \mu(\beta_0 - \rho_1)P_1^{[1]} \rangle - \mu\varrho_1(v_0)_0 \\ &= -\mu\varrho_1, \end{aligned} \quad (3.8)$$

since $\{P_n^{[1]}\}_{n \geq 0}$ is orthogonal with respect to the form v_0 , where v_0 is assumed to be normalized. According to Lemma 2.1 and using (3.7) and (3.8), we obtain the desired result. $\square$

It is clear that the formula (3.1) is a first-order differential equation satisfied by $\{P_n\}_{n \geq 0}$. Based on the last lemma, we obtain a first-order differential equation satisfied by $\{P_n^{[1]}\}_{n \geq 0}$.

Proposition 3.2. *The following fundamental relation holds*

$$(P_{n+1}^{[1]})'(x) = (n+1)P_n(x), \quad n \geq 0. \quad (3.9)$$

Proof. According to Proposition 2.1 (ii), the relation (3.3) can be written as follows

$$(x^2 + \xi_2)[P_{n+1}^{[1]}(x)v_0]' + ((2 - \mu)x - \xi_1)P_{n+1}^{[1]}v_0 = \lambda_n P_n(x)u_0, \quad n \geq 0,$$

where $\lambda_n := -(n + \mu)\langle v_0, (P_{n+1}^{[1]})^2 \rangle \langle u_0, P_n^2 \rangle^{-1}$, $n \geq 0$.

Using the fact that $(fu)' = f'u + fu'$ for all $f \in \mathbb{P}$, $u \in \mathbb{P}'$, the last equation becomes

$$(x^2 + \xi_2)[(P_{n+1}^{[1]})'(x)v_0 + P_{n+1}^{[1]}(x)v_0'] + ((2 - \mu)x - \xi_1)P_{n+1}^{[1]}v_0 = \lambda_n P_n(x)u_0, \quad n \geq 0. \quad (3.10)$$

Taking $n = 0$ in (3.10), we get $(x^2 + \xi_2)v_0' = ((\mu - 2)x + \xi_1)v_0$, ($\lambda_0 = -\mu\varrho_1$). Substituting this relation in (3.10), we obtain

$$(\lambda_n P_n + \mu\varrho_1(P_{n+1}^{[1]}))'u_0 = 0.$$

Using Lemma 3.1 and the fact that $\lambda_0 = -\mu\varrho_1$ and taking into account the regularity of u_0 , we finally obtain $\lambda_0(P_{n+1}^{[1]})'(x) = \lambda_n P_n(x)$, $n \geq 0$. By comparing the degrees in the last equation, we get $\lambda_n = (n + 1)\lambda_0$, $n \geq 0$, and then, $(P_{n+1}^{[1]})'(x) = (n + 1)P_n(x)$, $n \geq 0$. $\square$

According to Proposition 3.2, and based on the Böchner characterization, we get the $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical orthogonal sequence. Now, we will describe all of the $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical polynomial sequences.

Theorem 3.1. *Each of the following assertions is valid:*

Case 1. *If $\xi_2 = 0$.*

The orthogonal polynomial sets which are also $\mathcal{A}_{\xi_1, 0, \mu}$ -classical, up to affine transformations, are the set of the Bessel polynomials $\left\{B_n^{(\frac{\mu}{2})}\right\}_{n \geq 0}$

($\frac{\mu}{2} \neq -n$, $\mu \neq -n - 1$, $n \geq 1$). More precisely, $P_n(x) = a^{-n}B_n^{(\frac{\mu}{2}+1)}(ax)$ and $P_n^{[1]}(x) = a^{-n}B_n^{(\frac{\mu}{2})}(ax)$ where $a = (\frac{\xi_1}{2})^{-1}$.

Case 2. *If $\xi_2 \neq 0$.*

The orthogonal polynomial sequences which are also $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical, up to affine transformations, are

the sequence of Jacobi polynomials $\left\{a^{-n}\mathcal{A}_n^{(\frac{\mu-a\xi_1}{2}, \frac{\mu+a\xi_1}{2})}(ax)\right\}_{n \geq 0}$

($\frac{\mu-a\xi_1}{2}, \frac{\mu+a\xi_1}{2} \neq -n$, $\mu \neq -n - 1$, $n \geq 1$). More precisely, $P_n(x) = a^{-n}\mathcal{A}_n^{(\frac{\mu-a\xi_1}{2}, \frac{\mu+a\xi_1}{2})}(ax)$ and $P_n^{[1]}(x) = a^{-n}\mathcal{A}_n^{(\frac{\mu-a\xi_1}{2}-1, \frac{\mu+a\xi_1}{2}-1)}(ax)$ where $a^2 = -\xi_2^{-1}$.

Proof. We will analyze each of the two cases separately.

Case 1. *If $\xi_2 = 0$.*

Differentiating (3.1) and using Proposition 3.2, we obtain the following second-order differential equation satisfied by $\{P_n\}_{n \geq 0}$,

$$x^2 P_n''(x) - (-(2 + \mu)x - \xi_1)P_n'(x) = n(n + \mu + 1)P_n(x), \quad n \geq 0. \quad (3.11)$$

Applying the operator $h_{a^{-1}}$, with $a = (\frac{\xi_1}{2})^{-1}$, to the last equation and using the fact that $h_{a^{-1}}(fg) = (h_{a^{-1}}f)(h_{a^{-1}}g)$ for all $f, g \in \mathbb{P}$, we obtain

$$\begin{aligned} \left(\frac{\xi_1}{2}\right)^2 \left(\frac{\xi_1}{2}\right)^{-2} x^2 P''_{n+1}(x) + \frac{\xi_1}{2} 2 \left(\left(\frac{\xi_1}{2}\right)^{-1} \frac{\mu+2}{2} x + 1\right) P'_{n+1}(x) \\ = (n+1) \left(n + 2 \frac{\mu+2}{2}\right) P_{n+1}(x), \quad n \geq 0. \end{aligned} \quad (3.12)$$

Taking into account (2.3), we observe that (3.12) essentially corresponds to the scaled Bessel polynomial sequence $\left\{a^{-n} B_n^{(\frac{\mu+2}{2})}(ax)\right\}_{n \geq 0}$, where $a = (\frac{\xi_1}{2})^{-1}$. In a similar manner, inserting (3.9)

into (3.1), and taking into account (2.3), we conclude that the sequence $\{P_n^{[1]}\}_{n \geq 0}$ coincides with the scaled Bessel polynomial sequence $\left\{a^{-n} B_n^{(\frac{\mu}{2})}(ax)\right\}_{n \geq 0}$, where $a = (\frac{\xi_1}{2})^{-1}$.

Case 2. If $\xi_2 \neq 0$.

Differentiating (3.1) and using Proposition 3.2, we obtain the following second-order differential equation satisfied by $\{P_n\}_{n \geq 0}$

$$(x^2 + \xi_2)P''_n(x) + ((2 + \mu)x + \xi_1)P'_n(x) = n(n + \mu + 1)P_n(x), \quad n \geq 0. \quad (3.13)$$

Applying the operator $h_{a^{-1}}$, with $a^2 = -\xi_2^{-1}$, to the last equation and using the fact that $h_{a^{-1}}(fg) = (h_{a^{-1}}f)(h_{a^{-1}}g)$ for all $f, g \in \mathbb{P}$, we obtain

$$\begin{aligned} -\xi_2(x^2 - 1)(h_{a^{-1}}P''_n)(x) - a^{-1}(-(2 + \mu)x - \xi_1 a)(h_{a^{-1}}P'_n)(x) \\ = n(n + \mu + 1)(h_{a^{-1}}P_n)(x), \quad n \geq 0. \end{aligned} \quad (3.14)$$

Taking into account (2.3), we observe that (3.14) essentially corresponds to the scaled Jacobi polynomial sequence $\left\{a^{-n} \mathcal{A}_n^{(\frac{\mu-a\xi_1}{2}, \frac{\mu+a\xi_1}{2})}(ax)\right\}_{n \geq 0}$, with $a^2 = -\xi_2^{-1}$.

In a similar manner, inserting (3.9) into (3.1), we obtain

$$(x^2 + \xi_2)(P_n^{[1]})''(x) + (\mu x + \xi_1)(P_n^{[1]})'(x) = (n+1)(n + \mu)P_n^{[1]}(x), \quad n \geq 0.$$

Applying the operator $h_{a^{-1}}$, with $a^2 = -\xi_2^{-1}$, to this equation, we get

$$-\xi_2(x^2 - 1)(h_{a^{-1}}(P_n^{[1]})'')(x) + a^{-1}(\mu x + a\xi_1)(h_{a^{-1}}(P_n^{[1]})')(x) = (n+1)(n + \mu)(h_{a^{-1}}P_n^{[1]})(x), \quad n \geq 0. \quad (3.15)$$

Taking into account (2.3), we conclude that the sequence $\{P_n^{[1]}\}_{n \geq 0}$ coincides with the scaled Jacobi polynomial sequence $\left\{a^{-n} \mathcal{A}_n^{(\frac{\mu-a\xi_1}{2}-1, \frac{\mu+a\xi_1}{2}-1)}(ax)\right\}_{n \geq 0}$, with $a^2 = -\xi_2^{-1}$.

□

The raising operator $\mathcal{A}_{\xi_1, \xi_2, \mu}$, combined with the differentiation operator D , gives rise to a second-order differential operator. The following theorem shows that every polynomial in a $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical sequence is an eigenfunction of this operator.

Theorem 3.2. *There exists a differential linear operator $\mathcal{L}$ of order two, for which the $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical polynomial $P_n(x)$, $n \geq 0$, is an eigenfunction. More precisely, we have:*

$$\mathcal{L}(P_n) = \omega_n P_n, \quad n \geq 0, \quad (3.16)$$

where $\omega_n = (n+1)(n+\mu)$ are the corresponding eigenvalues, and where

$$\mathcal{L} := a_1(x)D^2 + a_2(x)D + a_3(x)\mathbb{I}$$

with

$$a_1(x) = x^2 + \xi_2, \quad a_2(x) = (2+\mu)x + \xi_1, \quad a_3(x) = \mu.$$

Proof. By applying the $\mathcal{A}_{\xi_1, \xi_2, \mu}$ operator, and according to (3.1), we get

$$D \circ \mathcal{A}_{\xi_1, \xi_2, \mu}(P_n)(x) = (n+1)(n+\mu)P_n(x), \quad n \geq 0.$$

This gives, after a simple calculation, the desired result with $\mathcal{L} = D \circ \mathcal{A}_{\xi_1, \xi_2, \mu}$. $\square$

Note that by applying the $\mathcal{L}$ operator to X^n , $n \geq 0$, we obtain

$$\mathcal{L}(X^n) = \omega_n X^n + n\xi_1 X^{n-1} + n(n-1)\xi_2 X^{n-2}, \quad n \geq 0.$$

So, the matrix of the endomorphism $\mathcal{L}$ in the canonical basis $\{X^n\}_{n \geq 0}$ of $\mathbb{P}$ is given by

$$\mathbf{M}_{\mathcal{L}} = \begin{pmatrix} \omega_0 & \xi_1 & 2\xi_2 & 0 & 0 & \cdots & 0 \\ 0 & \omega_1 & 2\xi_1 & & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & & \omega_n & n\xi_1 & n(n-1)\xi_2 & 0 \\ & & & & \ddots & & 0 & \ddots \\ 0 & & & & & & & \ddots \end{pmatrix}$$

Using the relation (3.16), we can write the matrix $\mathbf{M}_{\mathcal{L}}$ in the basis $\{P_n\}_{n \geq 0}$ as follows:

$$\mathbf{L} = \begin{pmatrix} \omega_0 & 0 & \cdots & \cdots & 0 \\ 0 & \omega_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \omega_n & 0 \\ 0 & \cdots & \cdots & 0 & \ddots \end{pmatrix}$$

4. CONCLUSION

We have characterized the $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical orthogonal polynomial sequences through the second-order differential equations satisfied by the polynomials and the Pearson equations satisfied by their corresponding canonical functionals.

If $\xi_2 = 0$, then $\xi_1 \neq 0$, and we have proved that the unique $\mathcal{A}_{\xi_1, 0, \mu}$ -classical MOPS is

$$P_n(x) = a^{-n} B_n^{(\mu/2+1)}(ax), \quad n \geq 0,$$

and that its companion MOPS is

$$P_n^{[1]}(x) = a^{-n} B_n^{(\mu/2)}(ax), \quad n \geq 0,$$

where $\mu \notin \mathbb{Z}_{\leq 0}$.

If $\xi_2 \neq 0$, we have proved that the unique $\mathcal{A}_{\xi_1, \xi_2, \mu}$ -classical MOPS is the scaled Jacobi sequence

$$P_n(x) = a^{-n} J_n^{(A, B)}(ax), \quad n \geq 0,$$

and that its companion MOPS is

$$P_n^{[1]}(x) = a^{-n} J_n^{(A-1, B-1)}(ax), \quad n \geq 0,$$

under the regularity conditions $\mu, A, B \notin \mathbb{Z}_{\leq 0}$. In particular, the Gegenbauer case is recovered precisely when $\xi_1 = 0$.

Moreover, we have derived several structural and spectral properties of these polynomial sequences.

As a direction for future work, we propose studying the analogous classification problem for the following two q -difference operators [13, 22]:

$$\mathcal{H}_{\xi, \mu} := x(xH_q + \mu\mathbb{I}) + \xi\mathbb{I},$$

$$\mathcal{H}_{\xi_1, \xi_2, \mu} := x(xH_q + \mu\mathbb{I}) + \xi_1\mathbb{I} + \xi_2H_q,$$

where H_q denotes the Hahn q -difference operator. The objective is to characterize the orthogonal polynomial sequences for which the normalized sequences generated by these operators remain orthogonal.

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