

Wandering Vectors and Frame Representations in Quaternionic Hilbert Spaces

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Abstract. We develop the theory of wandering vectors and frame representations in separable right quaternionic Hilbert spaces. We introduce the local commutant and obtain operator-theoretic characterizations of complete wandering vectors and normalized tight frame vectors. We prove that every complete normalized tight frame vector admits a dilation to a complete wandering vector in a larger quaternionic Hilbert space. Finally, we show that every quaternionic frame representation of a discrete group is unitarily equivalent to a subrepresentation of the left regular representation, yielding a canonical model for such representations.

1. INTRODUCTION

Frame theory provides a flexible and robust framework for representing elements in Hilbert spaces through redundant systems. Since the pioneering work of Duffin and Schaeffer [7], frames have been extensively developed and applied in various areas of mathematics and engineering. A comprehensive account of the theory of frames and Riesz bases can be found in [5], while the broader perspective and structural aspects of frame theory are discussed in [3]. In particular, the interaction between frames and group representations has led to significant developments in understanding the structure of Hilbert spaces and operator systems [10].

A key concept in this direction is that of *wandering vectors*, which arise naturally in the study of unitary systems and group representations. Wandering vectors generate orthonormal systems under the action of a unitary family and play a fundamental role in the study of frame representations. The theory of wandering vectors and their connection with unitary systems was systematically

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investigated by Dai and Larson [9], where fundamental structural results were established in the setting of complex Hilbert spaces.

There has been growing interest in extending classical Hilbert space theory to the quaternionic setting. Quaternionic Hilbert spaces provide a natural framework for non-commutative analysis and have found applications in mathematical physics and operator theory. Foundational aspects of quaternionic Hilbert spaces and their operator theory is developed in [1,6,8,11], while the broader context of quaternionic analysis is discussed in [2]. The non-commutativity of the scalar field give rise to additional subtleties, particularly in the development of operator-theoretic concepts and frame structures. In recent years, the study of frame theory in quaternionic Hilbert spaces has received considerable attention. In particular, quaternionic frames and their properties have been studied in [4,12], where several fundamental results analogous to the complex case were obtained. However, the interaction between quaternionic frame theory and unitary systems, especially in the context of wandering vectors and operator-theoretic characterizations, remains relatively unexplored.

Motivated by these developments, this paper seeks to generalize the theory of wandering vectors and frame representations from complex Hilbert spaces to separable right quaternionic Hilbert spaces. We introduce and study wandering vectors associated with unitary systems of right-linear operators and develop their structural properties in the quaternionic framework. In particular, we define the local commutant at a vector and use it to characterize wandering vectors and frame vectors in terms of operator-theoretic conditions.

One of the main contributions of this work is the characterization of complete wandering vectors and normalized tight frame vectors via operators in the local commutant. We show that the set of complete wandering vectors can be completely described through the action of unitary operators in the commutant, thereby extending classical results to the quaternionic setting. Moreover, we prove a dilation theorem, demonstrating that every complete normalized tight frame vector admits a dilation to a complete wandering vector in a larger quaternionic Hilbert space. This provides a quaternionic analogue of the frame dilation theory known in the complex setting. We also investigate quaternionic frame representations of discrete groups and establish their unitary equivalence to a subrepresentation of the left regular representation. This result provides a canonical realization of quaternionic frame representations and highlights the fundamental role of the regular representation in this theory.

The paper is structured as follows. Section 2 introduces the foundational concepts of quaternionic Hilbert spaces, operators, and frames. Section 3 focuses on wandering vectors and their operator-theoretic characterization through local commutants. In Section 4, we develop the dilation theory and establish the extension of frame vectors to wandering vectors and discuss quaternionic frame representations and their equivalence, culminating in the realization theorem via the left regular representation. Finally, examples are provided to illustrate the developed theory.

2. PRELIMINARIES

The quaternionic field $\mathfrak{H} = \{u = u_0 + u_1i + u_2j + u_3k, \quad u_0, u_1, u_2, u_3 \in \mathbb{R}\}$ is a non-commutative division algebra over $\mathbb{R}$, where the imaginary units i, j, k satisfy

$$i^2 = j^2 = k^2 = -1, \quad ijk = -1.$$

The conjugate of u is defined by

$$\bar{u} = u_0 - u_1i - u_2j - u_3k,$$

and its modulus by

$$|u| = \sqrt{u_0^2 + u_1^2 + u_2^2 + u_3^2}.$$

For a detailed account of quaternionic analysis, we refer to [2, 6, 11].

Let $\mathcal{H}^{rt}(\mathfrak{H})$ denote a right quaternionic vector space. The quaternionic inner product on $\mathcal{H}^{rt}(\mathfrak{H})$ is a mapping

$$\langle \cdot | \cdot \rangle : \mathcal{H}^{rt}(\mathfrak{H}) \times \mathcal{H}^{rt}(\mathfrak{H}) \rightarrow \mathfrak{H},$$

such that, for all $x_1, x_2, x_3 \in \mathcal{H}^{rt}(\mathfrak{H})$ and $q \in \mathfrak{H}$ the following properties are satisfied:

- (1) $\langle x_1 | x_2 + x_3 \rangle = \langle x_1 | x_2 \rangle + \langle x_1 | x_3 \rangle,$
- (2) $\langle x_1 | x_2q \rangle = \langle x_1 | x_2 \rangle q,$
- (3) $\langle x_1 | x_2 \rangle = \overline{\langle x_2 | x_1 \rangle},$
- (4) $\langle x_1 | x_1 \rangle \geq 0$ and $\langle x_1 | x_1 \rangle = 0$ if and only if $x_1 = 0$.

The space $\mathcal{H}^{rt}(\mathfrak{H})$ is referred to as *right quaternionic Hilbert space* (right QHS) provided it is complete under the norm $\|x_1\| = \sqrt{\langle x_1 | x_1 \rangle}$. For further details, see [1, 8].

We define an operator $L : \mathcal{H}^{rt}(\mathfrak{H}) \rightarrow \mathcal{H}^{rt}(\mathfrak{H})$ to be *right-linear* if

$$L(x_1q + x_2) = L(x_1)q + L(x_2), \quad x_1, x_2 \in \mathcal{H}^{rt}(\mathfrak{H}), \quad q \in \mathfrak{H}.$$

We say a right-linear operator L on $\mathcal{H}^{rt}(\mathfrak{H})$ is *bounded* if for some $0 < M \in \mathbb{R}$, $\|Lx\| \leq M\|x\|$ for all $x \in \mathcal{H}^{rt}(\mathfrak{H})$. We denote by $\mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$, the set of all bounded right-linear operators on $\mathcal{H}^{rt}(\mathfrak{H})$. For $L \in \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$, the adjoint L^* is defined by

$$\langle Lx_1 | x_2 \rangle = \langle x_1 | L^*x_2 \rangle, \quad x_1, x_2 \in \mathcal{H}^{rt}(\mathfrak{H}).$$

An operator $U \in \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$ is called *unitary* if $U^*U = UU^* = I$.

Hilbert (orthonormal) basis for $\mathcal{H}^{rt}(\mathfrak{H})$ is a sequence $\{e_n\}_{n \in \mathbb{N}} \subset \mathcal{H}^{rt}(\mathfrak{H})$ such that

$$\langle e_m | e_n \rangle = \delta_{m,n}, \quad \text{and} \quad \overline{\text{span}}\{e_n\} = \mathcal{H}^{rt}(\mathfrak{H}).$$

A *quaternionic Riesz basis* is a sequence $\{z_n\}_{n \in \mathbb{N}} \subset \mathcal{H}^{rt}(\mathfrak{H})$ such that $z_n = Te_n$, for some orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ where $T \in \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$ and is invertible.

Frames provide a redundant yet stable representation of elements in $\mathcal{H}^{rt}(\mathfrak{H})$. Frame theory in Hilbert spaces originates from [7] and was further advanced in [3, 5]. The quaternionic frame theory has been studied in [4, 12].

Definition 2.1 ([12]). Let Λ be a countable index set and $\{z_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{H}^{rt}(\mathfrak{H})$. Then $\{z_\lambda\}_{\lambda \in \Lambda}$ is called a quaternionic frame for $\mathcal{H}^{rt}(\mathfrak{H})$ if there exist constants $A_1, B_1 \in \mathbb{R}, 0 < A_1 \leq B_1 < \infty$ satisfying the inequality,

$$A_1 \|x\|^2 \leq \sum_{\lambda \in \Lambda} |\langle z_\lambda | x \rangle|^2 \leq B_1 \|x\|^2, \quad x \in \mathcal{H}^{rt}(\mathfrak{H}).$$

For a countable group $\mathcal{G}$, define

$$\ell^2(\mathcal{G}, \mathfrak{H}) = \left\{ h : \mathcal{G} \rightarrow \mathfrak{H} \left| \sum_{a \in \mathcal{G}} |h(a)|^2 < \infty \right. \right\}$$

The space $\ell^2(\mathcal{G}, \mathfrak{H})$, is a right quaternionic Hilbert space with inner product

$$\langle h_1 | h_2 \rangle = \sum_{a \in \mathcal{G}} \overline{h_1(a)} h_2(a).$$

A mapping $\Pi : \mathcal{G} \rightarrow \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$ is called a *unitary representation* of $\mathcal{G}$ if

$$\Pi(a_1 a_2) = \Pi(a_1) \Pi(a_2), \quad \Pi(e) = I, \quad a_1, a_2, e \in \mathcal{G}.$$

and each $\Pi(a)$ is unitary. The left regular representation λ^L of $\mathcal{G}$ on $\ell^2(\mathcal{G}, \mathfrak{H})$ is defined by

$$(\lambda^L(a)f)(b) = f(a^{-1}b), \quad a, b \in \mathcal{G}.$$

For further details on frame representations and their connections with group representations, we refer to [9, 10].

3. WANDERING VECTORS AND LOCAL COMMUTANT

The theory of wandering vectors for unitary systems is well established in complex Hilbert spaces. We extend this concept to right quaternionic Hilbert spaces, taking into consideration that the underlying skew field $\mathfrak{H}$ is non-commutative.

Definition 3.1. Let $\mathcal{H}^{rt}(\mathfrak{H})$ be a right QHS. A collection $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})) \subseteq \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$ is called a *unitary system* if:

- (1) every $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary operator on $\mathcal{H}^{rt}(\mathfrak{H})$, and
- (2) the identity operator $I \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$.

Definition 3.2. Let $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ be a unitary system on the right QHS $\mathcal{H}^{rt}(\mathfrak{H})$. A *wandering vector* for the unitary system $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, is a vector $\zeta \in \mathcal{H}^{rt}(\mathfrak{H})$, for which the set

$$\{U\zeta\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))}$$

is orthonormal in $\mathcal{H}^{rt}(\mathfrak{H})$. We call $\zeta \in \mathcal{H}^{rt}(\mathfrak{H})$ a *complete wandering vector* if this set forms a Hilbert(orthonormal) basis for $\mathcal{H}^{rt}(\mathfrak{H})$. We define $W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ as the collection of complete wandering vectors in right QHS $\mathcal{H}^{rt}(\mathfrak{H})$ i.e., $W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))) = \{\zeta \in \mathcal{H}^{rt}(\mathfrak{H}) : \{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\} \text{ is a Hilbert basis for } \mathcal{H}^{rt}(\mathfrak{H})\}$. We call vector $\zeta \in \mathcal{H}^{rt}(\mathfrak{H})$ a *normalized tight frame vector* (respectively, *tight frame vector*, *frame vector*) for $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ if the set

$$\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$$

forms a normalized tight frame (respectively, tight frame, frame) for the linear space

$$\overline{\text{span}}\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}.$$

If this set forms a normalized tight frame (respectively, tight frame, frame) for $\mathcal{H}^{rt}(\mathfrak{H})$, then ζ is called a complete normalized frame vector (respectively, complete tight frame vector, complete frame vector).

The notion of local commutants is fundamental to analyzing the structure of frame vectors associated with unitary systems. In the quaternionic setting, this concept provides a natural framework to analyze the interaction between operators and orbit structures generated by unitary systems. We give the definition of local commutant in the right QHS.

Definition 3.3. Let $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ denote a unitary system acting on the right QHS $\mathcal{H}^{rt}(\mathfrak{H})$ and let $\zeta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. The local commutant at ζ denoted by $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$, is the set

$$C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))) = \{R \in \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H})) : R(U\zeta) = U(R\zeta), U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}.$$

It is immediate that the commutant $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$ is contained in $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$.

The concept of the local commutant serves as an effective tool in studying the operator-theoretic structure underlying unitary systems. It enables a precise description of how different frame vectors are related through operators in the unitary system. The following result establishes such a characterization in the quaternionic setting.

Proposition 3.1. Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary system of right linear operators acting on right QHS $\mathcal{H}^{rt}(\mathfrak{H})$. Suppose $\zeta_1, \zeta_2 \in \mathcal{H}^{rt}(\mathfrak{H})$ such that the sets $\{\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\zeta_1\}$ and $\{\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\zeta_2\}$ form complete normalized tight frames for $\mathcal{H}^{rt}(\mathfrak{H})$. Then the frames $\{U\zeta_1 : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ and $\{U\zeta_2 : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ are unitarily equivalent if and only if $W\zeta_1 = \zeta_2$, for some unitary operator $W \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$.

Proof. Suppose the frame $\{\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\zeta_1\}$ is unitarily equivalent to $\{\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\zeta_2\}$. Then $W(U\zeta_1) = U\zeta_2$, for some unitary operator $W \in \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$ and for all $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$. Since $I \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, it follows that $W\zeta_1 = \zeta_2$. Hence, for all $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, $W(U\zeta_1) = U(W\zeta_1)$, which implies that $WU = UW$ on the cyclic set generated by ζ_1 . Since ζ_1 is a complete frame vector, this extends to $\mathcal{H}^{rt}(\mathfrak{H})$, and therefore $W \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$.

Conversely, if $W\zeta_1 = \zeta_2$ for a unitary operator $W \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$, then

$$W(U\zeta_1) = U(W\zeta_1) = U\zeta_2,$$

for all $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, showing that the frames are unitarily equivalent. $\square$

The local commutant associated with a complete wandering vector exhibits strong structural properties. In particular, the following result shows that such vectors act as separating vectors and establishes a relationship between the local commutant and the commutant of the unitary system.

Lemma 3.1. Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})) \subseteq \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary system of right linear operators acting on right QHS $\mathcal{H}^{rt}(\mathfrak{H})$. If $0 \neq \zeta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$, then ζ separates $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Moreover, if $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a semigroup, then $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))) = \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$.

Proof. Let $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ be such that $L\zeta = 0$. Then for any $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, $L(U\zeta) = U(L\zeta) = 0$. Since ζ is a complete wandering vector, the set $\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ forms a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$. Therefore, L vanishes on a Hilbert basis of $\mathcal{H}^{rt}(\mathfrak{H})$, and therefore $L = 0$. This shows that $\zeta \in \mathcal{H}^{rt}(\mathfrak{H})$ separates $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$.

Next, assume that $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a semigroup. It is clear that

$$\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))' \subseteq C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))).$$

To prove the reverse inclusion, let $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Then for all $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, $L(U\zeta) = U(L\zeta)$. Let $V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$. Since $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a semigroup, $VU \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, and therefore $L(VU\zeta) = VU(L\zeta)$. Also,

$$LV(U\zeta) = VL(U\zeta), \quad \text{for all } U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})).$$

Since $\{U\zeta\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))}$ is a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$, it follows that $LV = VL$ on $\mathcal{H}^{rt}(\mathfrak{H})$. Therefore $L \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$, and hence

$$C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))) = \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))'. \quad \square$$

We now describe the structure of complete wandering vectors associated with a unitary system using the local commutant. The following result shows that all such vectors can be generated via unitary operators belonging to the local commutant.

Theorem 3.1. Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary system of right linear operators acting on the right QHS $\mathcal{H}^{rt}(\mathfrak{H})$, and let $\zeta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Then

$$W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))) = \{T\zeta : T \in \mathcal{U}(C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))))\},$$

where $\mathcal{U}(C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))))$ is the collection of all unitary operators contained in the set $(C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))))$.

Proof. Let $T \in \mathcal{U}(C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))))$ and set $\eta = T\zeta$. Then for any $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, $U\eta = UT\zeta = TU\zeta$, since $T \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. As ζ is a complete wandering vector, the set $\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ forms a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$. Since T is unitary, it preserves orthonormality, and therefore

$$\{U\eta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\} = \{TU\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$$

is also a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$. Thus $\eta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$.

Conversely, let $\eta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Then

$$\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\} \quad \text{and} \quad \{U\eta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$$

are Hilbert bases for $\mathcal{H}^{rt}(\mathfrak{H})$. Let $T : \mathcal{H}^{rt}(\mathfrak{H}) \rightarrow \mathcal{H}^{rt}(\mathfrak{H})$ be defined by

$$T(U\zeta) = U\eta, \quad U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})).$$

Then T is a unitary operator on $\mathcal{H}^{rt}(\mathfrak{H})$. Clearly, $T\zeta = \eta$. Moreover, for any $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$,

$$T(U\zeta) = U\eta = U(T\zeta),$$

therefore $T \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Thus $T \in \mathcal{U}(C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))))$. Hence,

$$W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))) = \{T\zeta : T \in \mathcal{U}(C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))))\}.$$

□

We now show that the parametrization of complete wandering vectors via unitary operators in the local commutant is faithful. In particular, the following result establishes the injectivity of the associated mapping.

Theorem 3.2. *Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary system of right linear operators acting on the right QHS $\mathcal{H}^{rt}(\mathfrak{H})$, and $0 \neq \zeta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Define the mapping*

$$\Phi_\zeta : \mathcal{U}(C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))) \rightarrow W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))), \quad \Phi_\zeta(T) = T\zeta.$$

Then Φ_ζ is injective.

Proof. Let $T_1, T_2 \in \mathcal{U}(C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))))$ be such that $\Phi_\zeta(T_1) = \Phi_\zeta(T_2)$, that is, $T_1\zeta = T_2\zeta$. Then for any $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$,

$$T_1(U\zeta) = U(T_1\zeta) = U(T_2\zeta) = T_2(U\zeta),$$

since $T_1, T_2 \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Thus,

$$T_1(U\zeta) = T_2(U\zeta), \quad \text{for all } U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})).$$

Since ζ is a complete wandering vector, the set $\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ forms a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$. Therefore, $T_1 = T_2$ on $\mathcal{H}^{rt}(\mathfrak{H})$, and hence Φ_ζ is injective. □

We now present a canonical example that illustrates the structure of complete wandering vectors associated with a unitary system. In particular, this example highlights the role of the local commutant in parametrizing all such vectors in a concrete setting.

Example 3.1. *Suppose $\mathcal{H}^{rt}(\mathfrak{H})$ is a right QHS and $\{e_j\}_{j \in \mathbb{Z}}$ is a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$. Let $U : \mathcal{H}^{rt}(\mathfrak{H}) \rightarrow \mathcal{H}^{rt}(\mathfrak{H})$ be a right-linear operator defined by*

$$Ue_j = e_{j+1}, \quad \text{for all } j \in \mathbb{Z},$$

and extend U linearly to $\mathcal{H}^{rt}(\mathfrak{H})$. Then the operator U is unitary on $\mathcal{H}^{rt}(\mathfrak{H})$. Consider the unitary system

$$\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})) = \{U^j : j \in \mathbb{Z}\}$$

generated by U . Note that $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ forms a unitary group. For each $j \in \mathbb{Z}$, the vector $e_j \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$, hence

$$W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))) = \{Te_0 : T \in \mathcal{U}(C_{e_0}(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))))\}.$$

Example 3.1 illustrates the structure of complete wandering vectors generated by a unitary system. In particular, it shows that once a single complete wandering vector is fixed, all others can be obtained through unitary operators belonging to the corresponding local commutant.

It is known (Theorem 3.1) that whenever $W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))) \neq \emptyset$ and $\zeta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$, the vector ζ together with the unitary operators in $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ completely characterizes the set $W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. This viewpoint naturally extends to the study of quaternionic Riesz bases.

We now extend the preceding results from Hilbert bases to the more general setting of quaternionic Riesz bases. This provides a broader framework for understanding orbit structures generated by unitary systems under bounded invertible perturbations.

Theorem 3.3. Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary system of right linear operators acting on a right QHS $\mathcal{H}^{rt}(\mathfrak{H})$ and $0 \neq \zeta \in \mathcal{H}^{rt}(\mathfrak{H})$ is such that the set $\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ forms a quaternionic Riesz basis for $\mathcal{H}^{rt}(\mathfrak{H})$. Then, for $\gamma \in \mathcal{H}^{rt}(\mathfrak{H})$, the set $\{U\gamma : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ forms a quaternionic Riesz basis for $\mathcal{H}^{rt}(\mathfrak{H})$ if and only if $\gamma = T\zeta$ for some bounded invertible right-linear operator, $T \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$.

Proof. If $\gamma = T\zeta$ for some bounded invertible right-linear operator $T \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$, then

$$U\gamma = U(T\zeta) = T(U\zeta), \quad U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})).$$

Thus $\{U\gamma : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ is the image of the Riesz basis $\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ under T , and therefore is a Riesz basis.

Conversely, if both $\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ and $\{U\gamma : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ are Riesz bases for $\mathcal{H}^{rt}(\mathfrak{H})$, then $T(U\zeta) = U\gamma$, $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, for a unique invertible operator $T \in \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$. In particular, $T\zeta = \gamma$, and $T(U\zeta) = U(T\zeta)$, which shows that $T \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. $\square$

We now provide a characterization of normalized tight frame (NTF) vectors associated with a unitary system via operators in the local commutant. This result provides a structural link between frame vectors and partial isometries in the quaternionic framework.

Theorem 3.4. Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary system of right linear operators acting on the right QHS $\mathcal{H}^{rt}(\mathfrak{H})$, and $0 \neq \zeta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Then a vector $\psi \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a NTF for $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ if and only if $L\zeta = \psi$ for a unique operator $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$, where L is a partial isometry.

Proof. Let $\psi \in \mathcal{H}^{rt}(\mathfrak{H})$ be such that the set $\{U\psi\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))}$ forms a NTF for $\overline{\text{span}}(\{U\psi\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))})$. Define a right linear operator $S : \mathcal{H}^{rt}(\mathfrak{H}) \rightarrow \mathcal{H}^{rt}(\mathfrak{H})$ as

$$\begin{aligned} Sz &= \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} U\zeta \langle U\psi | z \rangle, z \in \overline{\text{span}}(\{U\psi\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))}) \quad \text{and} \\ Sz &= 0, z \perp \overline{\text{span}}(\{U\psi\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))}) \end{aligned}$$

Clearly S is an isometry on $\overline{\text{span}}(\{U\psi\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))})$, as $\psi \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a NTF for the subspace $\overline{\text{span}}(\{U\psi\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))})$. Thus S defines a partial isometry on $\mathcal{H}^{rt}(\mathfrak{H})$ and the range $S\mathcal{H}^{rt}(\mathfrak{H})$ is closed. Let $\mathcal{P}$ denote the orthogonal projection from $\mathcal{H}^{rt}(\mathfrak{H})$ onto $S\mathcal{H}^{rt}(\mathfrak{H})$. Define $L = S^* = S^*\mathcal{P}$, then the right linear operator L is the required partial isometry on $\mathcal{H}^{rt}(\mathfrak{H})$ such that $L\zeta = \psi$. Consider $V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, then

$$\begin{aligned} \langle \mathcal{P}U\zeta | SV\psi \rangle &= \langle U\zeta | \mathcal{P}SV\psi \rangle \\ &= \langle U\zeta | SV\psi \rangle \\ &= \langle U\zeta | \sum_{R \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} R\zeta \langle R\psi | V\psi \rangle \rangle \end{aligned}$$

$$= \langle U\psi | V\psi \rangle = \langle SU\psi | SV\psi \rangle.$$

Hence $\mathcal{P}U\zeta = SU\psi$. Next we show $L = S^*\mathcal{P} \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ and $L\zeta = \psi$.

For any $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, we have $S^*SU\psi = S^*\mathcal{P}U\zeta$ and the partial isometry L has $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\psi$ as its initial subspace. This gives $S^*SU\psi = U\psi = LU\zeta$, for each $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$.

Since $I \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, we have $UL\zeta = U\psi = LU\zeta$, therefore $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$.

Further, the final subspace of the right linear operator S is $\mathcal{P}\mathcal{H}^{rt}(\mathfrak{H})(= S\mathcal{H}^{rt}(\mathfrak{H}))$ and S^* defines an isometry on $\mathcal{P}\mathcal{H}^{rt}(\mathfrak{H})$ and vanishes on $\mathcal{P}^\perp\mathcal{H}^{rt}(\mathfrak{H})$. Thus the right linear operator L is a partial isometry on $\mathcal{H}^{rt}(\mathfrak{H})$. Clearly L is unique, since ζ separates $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ as the mapping from $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ to $\mathcal{H}^{rt}(\mathfrak{H})$ is injective.

Conversely, let the right linear operator L be a partial isometry in $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ and $L\zeta = \psi$, then L^* is an isometry on $L\mathcal{H}^{rt}(\mathfrak{H})$. Now for any $z \in L\mathcal{H}^{rt}(\mathfrak{H})$, we have

$$\begin{aligned} \|z\|^2 &= \|L^*z\|^2 \\ &= \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle L^*z | U\zeta \rangle|^2 \\ &= \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle z | LU\zeta \rangle|^2 \\ &= \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle z | UL\zeta \rangle|^2 = \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle z | U\psi \rangle|^2 \end{aligned}$$

$\implies \psi \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a tight frame for $\overline{\text{span}}\{U\psi\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))}(= L\mathcal{H}^{rt}(\mathfrak{H}))$ with respect to $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ equivalently $\psi \in \mathcal{H}^{rt}(\mathfrak{H})$ is a NTF vector for $\overline{\text{span}}\{U\psi\}_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))}(= L\mathcal{H}^{rt}(\mathfrak{H}))$. $\square$

It follows from the preceding result that complete NTF vectors can be characterized by co-isometries in the local commutant.

Corollary 3.1. Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary system of right linear operators acting on the separable right QHS $\mathcal{H}^{rt}(\mathfrak{H})$, and $0 \neq \zeta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. A vector $\psi \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a NTF for $\mathcal{H}^{rt}(\mathfrak{H})$ associated with the unitary system $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ if and only if $L\zeta = \psi$. for a unique operator $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$, where L is a co-isometry.

Proof. As per Theorem 3.4, $\psi \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a NTF for $\overline{\text{span}}\{U\psi\}$ associated with $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ if and only if $L\zeta = \psi$, where $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ is a unique partial isometry. Moreover, ψ generates a NTF for $\mathcal{H}^{rt}(\mathfrak{H})$ if and only if $\overline{\text{span}}\{U\psi : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\} = \mathcal{H}^{rt}(\mathfrak{H})$. Since $U\psi = L(U\zeta)$, it follows that $\overline{\text{span}}\{U\psi\} = L\mathcal{H}^{rt}(\mathfrak{H})$. Thus completeness holds if and only if $L\mathcal{H}^{rt}(\mathfrak{H}) = \mathcal{H}^{rt}(\mathfrak{H})$, i.e., the range of L is $\mathcal{H}^{rt}(\mathfrak{H})$. This is equivalent to L^* being an isometry, and hence L is a co-isometry. The uniqueness follows from the separating property of ζ for $C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. $\square$

We now establish a structural property of unitary systems admitting NTF vectors. The following theorem shows that the existence of such a frame vector, forces the semigroup to be a group.

Theorem 3.5. Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a semigroup of unitary operators acting on the right QHS $\mathcal{H}^{rt}(\mathfrak{H})$ containing the identity operator I . Suppose a vector $0 \neq \psi \in \mathcal{H}^{rt}(\mathfrak{H})$ for which the set

$$\{U\psi : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$$

forms a NTF for $\mathcal{H}^{rt}(\mathfrak{H})$. Then $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ must be a group.

Proof. Suppose, to the contrary, that $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is not a group. Then for some $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, $U^{-1} \notin \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$. For any $z \in \mathcal{H}^{rt}(\mathfrak{H})$, we have

$$\|z\|^2 = \sum_{V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle z|V\psi \rangle|^2,$$

as $\{V\psi : V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ is a NTF for $\mathcal{H}^{rt}(\mathfrak{H})$.

Applying this identity to $U^{-1}z$, and using that U is unitary, we obtain

$$\|z\|^2 = \|U^{-1}z\|^2 = \sum_{V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle U^{-1}z|V\psi \rangle|^2 = \sum_{V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle z|UV\psi \rangle|^2.$$

Thus,

$$\sum_{V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle z|V\psi \rangle|^2 = \sum_{V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle z|UV\psi \rangle|^2.$$

Since $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a semigroup, we have $UV \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ for all $V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, and hence

$$\{UV : V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\} \subseteq \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})).$$

If $I \in \{UV : V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$, then there exists $V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ such that $UV = I$, which implies $V = U^{-1}$. This contradicts the assumption that $U^{-1} \notin \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$. Therefore

$$I \notin \{UV : V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}.$$

Thus, the term corresponding to $\psi = I\psi$ does not appear in the second sum and therefore

$$\langle z|\psi \rangle = 0, \quad \text{for all } z \in \mathcal{H}^{rt}(\mathfrak{H}).$$

Taking $z = \psi$, we obtain $\|\psi\|^2 = 0$, a contradiction. Hence $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ must be a group. $\square$

Corollary 3.2. Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a semigroup of unitary operators acting on the right QHS $\mathcal{H}^{rt}(\mathfrak{H})$, containing the identity operator I . Suppose $0 \neq \zeta \in \mathcal{H}^{rt}(\mathfrak{H})$ for which the set

$$\{U\zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$$

forms a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$. Then $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a group.

Proof. Since every Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$ is a NTF, the result follows directly from Theorem 3.5. $\square$

Let $A \subseteq \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$, we write $w^*(A)$ for the von Neumann algebra generated by A , and by A' the commutant of A .

Consider a von Neumann algebra $S \subseteq \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$, a vector $z \in \mathcal{H}^{rt}(\mathfrak{H})$ is called a *trace vector* for a $S \subseteq \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$ if

$$\langle TVz | z \rangle = \langle VTz | z \rangle, \quad \text{for all } T, V \in S'.$$

It is called *faithful* if the mapping $T \mapsto Tz$ is injective. A finite von Neumann algebra admits a faithful trace vector.

We now relate the existence of complete wandering vectors to the finiteness of von Neumann algebras generated by unitary systems.

Lemma 3.2. *Suppose $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary group of right linear unitary operators acting on the right QHS $\mathcal{H}^{rt}(\mathfrak{H})$ and $0 \neq \zeta \in W(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Then $w^*(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ and $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$ are finite von Neumann algebras.*

Proof. Since $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary group, its commutant $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$ forms a von Neumann algebra, and $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'' = w^*(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. The set $\{U\zeta : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ forms a Hilbert basis as $\zeta \in W(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. We first show that ζ is a trace vector for $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$. Let $A, B \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$. Then

$$\langle AB\zeta | \zeta \rangle = \langle B\zeta | A^*\zeta \rangle.$$

Using the Hilbert basis $\{U\zeta : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ and the standard expansion valid in quaternionic Hilbert spaces, we obtain

$$\langle B\zeta | A^*\zeta \rangle = \sum_{U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))} \langle B\zeta | U\zeta \rangle \langle U\zeta | A^*\zeta \rangle.$$

Since $A, B \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$, they commute with each $U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, and therefore

$$\langle B\zeta | U\zeta \rangle = \langle \zeta | B^*U\zeta \rangle = \langle \zeta | UB^*\zeta \rangle,$$

and similarly for A^* . This yields

$$\langle AB\zeta | \zeta \rangle = \langle BA\zeta | \zeta \rangle,$$

showing that ζ is a trace vector for $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$. Also, since ζ is a complete wandering vector, it separates $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$, i.e., the map $T \mapsto T\zeta$ is injective. Thus ζ is a faithful trace vector for $\mathcal{U}'$, and therefore von Neumann algebra $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$ is finite. Also, since $w^*(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))) = \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))''$, the von Neumann algebra obtained as the bicommutant of a finite von Neumann algebra is finite. Hence von Neumann algebra $w^*(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ is also finite. $\square$

Corollary 3.3. *Suppose $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary group of right linear unitary operators acting on a right QHS $\mathcal{H}^{rt}(\mathfrak{H})$ and $0 \neq \zeta \in W(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. If $\eta \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a NTF for $\mathcal{H}^{rt}(\mathfrak{H})$ then η is necessarily a complete wandering vector.*

Proof. Since $\eta \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a normalized tight frame for $\mathcal{H}^{rt}(\mathfrak{H})$ and $\zeta \in W(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})))$, then $T\zeta = \eta$, for a unique operator $T \in C_\zeta(\mathcal{U}) = \mathcal{U}'$, where T is a co-isometry. Since every

co-isometry in a finite von Neumann algebra is unitary and the von Neumann algebra $\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))'$ is finite, therefore T must be unitary. Also,

$$\begin{aligned}\{U\eta : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\} &= \{UT\zeta : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\} \\ &= \{T(U\zeta) : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}.\end{aligned}$$

Since $\{U\zeta : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ is a Hilbert basis and T is unitary, it follows that $\{U\eta : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ is also a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$. Hence $\eta \in W(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. $\square$

In particular, if a complete NTF vector fails to be a complete wandering vector, then $W(\mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ must be empty.

We now characterize frame vectors corresponding to a unitary system.

Theorem 3.6. Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary system of right linear operators acting on a right QHS $\mathcal{H}^{rt}(\mathfrak{H})$ and $0 \neq \zeta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Then $\psi \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a frame with frame bounds A_1 and B_1 with respect to $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ if and only if $L\zeta = \psi$, where $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ is unique and $A_1\mathcal{P} \leq LL^* \leq B_1\mathcal{P}$, where $\mathcal{P}$ denotes the orthogonal projection.

Proof. Suppose $\psi \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a frame with bounds A_1 and B_1 for $\overline{\text{span}}\{U\psi : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ and let $X = \overline{\text{span}}\{U\psi : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$. Define a right linear operator $S : \mathcal{H}^{rt}(\mathfrak{H}) \rightarrow \mathcal{H}^{rt}(\mathfrak{H})$ as

$$\begin{aligned}Sz &= \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} U\zeta \langle U\psi | z \rangle \quad \text{for } z \in X \text{ and} \\ Sz &= 0 \quad \text{for } z \perp X\end{aligned}$$

S is well defined and bounded right linear operator on $\mathcal{H}^{rt}(\mathfrak{H})$. Then $L = S^*$ is the required right linear operator i.e., $L\zeta = \psi$ and $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Let $z \in X$ then

$$\begin{aligned}\langle \zeta | Sz \rangle &= \left\langle \zeta \left| \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} U\zeta \langle U\psi | z \rangle \right. \right\rangle \\ &= \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} \langle \zeta | U\zeta \rangle \langle U\psi | z \rangle \\ &= \langle \psi | z \rangle\end{aligned}$$

$\implies \langle S^*\zeta | z \rangle = \langle \psi | z \rangle$ This gives $L\zeta = \psi$. Further by definition of S we have $SU\psi = U\zeta$, $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ and for any $V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ we have $VL\zeta = LV\zeta$. Hence $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$.

If $\mathcal{P}$ is an orthogonal projection onto X then for any $z \in X$

$$A_1\|z\|^2 \leq \|L^*z\|^2 \leq B_1\|z\|^2,$$

and hence $A_1\mathcal{P} \leq LL^* \leq B_1\mathcal{P}$.

Conversely, suppose $L \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$ satisfies $L\zeta = \psi$ and $A_1\mathcal{P} \leq LL^* \leq B_1\mathcal{P}$. Then $U\psi = L(U\zeta)$ for all $U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, and for all $z \in X$,

$$\sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle z | U\psi \rangle|^2 = \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle L^*z | U\zeta \rangle|^2 = \|L^*z\|^2.$$

Therefore,

$$A_1 \|z\|^2 \leq \sum_{U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))} |\langle z | U\psi \rangle|^2 \leq B_1 \|z\|^2.$$

Hence ψ generates a frame for X with bounds A_1 and B_1 . $\square$

As a direct corollary of Theorem 3.6, we derive the following characterization of complete frame vectors.

Corollary 3.4. *Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary system of right linear operators acting on a right QHS $\mathcal{H}^{rt}(\mathfrak{H})$, and $0 \neq \zeta \in W(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$. Then $\psi \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a frame for $\mathcal{H}^{rt}(\mathfrak{H})$ associated with the unitary system $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ with bounds A_1 and B_1 if and only if $T\zeta = \psi$, where $T \in C_\zeta(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})))$, is a unique right linear operator and $A_1 I \leq TT^* \leq B_1 I$.*

Proof. If ψ generates a frame for $\mathcal{H}^{rt}(\mathfrak{H})$, then $\overline{\text{span}}\{U\psi : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\} = \mathcal{H}^{rt}(\mathfrak{H})$, and the projection $\mathcal{P}$ in Theorem 3.6 is the identity operator I . The result follows immediately.

Conversely, if $A_1 I \leq TT^* \leq B_1 I$, then the associated frame spans $\mathcal{H}^{rt}(\mathfrak{H})$. Hence ψ generates a frame for the QHS $\mathcal{H}^{rt}(\mathfrak{H})$ associated with the unitary system $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$. $\square$

4. WANDERING VECTORS AND UNITARY REPRESENTATIONS

It is well established that any complete NTF in a complex Hilbert space admits a dilation to a Hilbert basis in a larger space. In this section, we extend this result to the setting of right QHSs and unitary systems. In particular, we prove that every complete NTF vector admits a dilation to a complete wandering vector whenever the underlying unitary system forms a group.

Theorem 4.1. *Suppose $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary group of right-linear operators on the right QHS $\mathcal{H}^{rt}(\mathfrak{H})$. Suppose $\eta \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a NTF for $\mathcal{H}^{rt}(\mathfrak{H})$ associated with $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$. Then there exists a right QHS $\mathcal{K}^{rt}(\mathfrak{H}) \supseteq \mathcal{H}^{rt}(\mathfrak{H})$ together with a unitary group $\mathfrak{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))$ on $\mathcal{K}^{rt}(\mathfrak{H})$ for which the following holds:*

- (i) $\mathfrak{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))$ admits a complete wandering vector (in particular, a Hilbert basis of the form $\{\lambda_U \zeta : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$),
 - (ii) $\mathcal{H}^{rt}(\mathfrak{H})$ is invariant under $\mathfrak{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))$,
 - (iii) $\mathfrak{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))|_{\mathcal{H}^{rt}(\mathfrak{H})} = \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$,
 - (iv) the map $U \mapsto \lambda_U$ defines a group isomorphism from $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ onto $\mathfrak{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))$,
- and the pair $(\mathcal{K}^{rt}(\mathfrak{H}), \mathfrak{U}_1(\mathcal{K}^{rt}(\mathfrak{H})))$ is unique up to unitary equivalence.

Proof. Let $\mathcal{K}^{rt}(\mathfrak{H}) = \ell^2(\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})), \mathfrak{H})$, the right QHS of square-summable $\mathfrak{H}$ -valued functions on $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$, with canonical Hilbert basis $\{\chi_U : U \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$. Define λ_U to be the left regular representation of $\mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ on $\mathcal{K}^{rt}(\mathfrak{H})$ by

$$\lambda_U \chi_V = \chi_{UV}, \quad U, V \in \mathfrak{U}(\mathcal{H}^{rt}(\mathfrak{H})).$$

Then each right-linear operator λ_U on $\mathcal{K}^{rt}(\mathfrak{H})$ is a unitary and

$$\mathcal{U}_1(\mathcal{K}^{rt}(\mathfrak{H})) := \{\lambda_U : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$$

is a unitary group on $\mathcal{K}^{rt}(\mathfrak{H})$. Let $W : \mathcal{H}^{rt}(\mathfrak{H}) \rightarrow \mathcal{K}^{rt}(\mathfrak{H})$ be the analysis operator defined by

$$W(z) = \sum_{U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))} \chi_U \langle z | U\eta \rangle, \quad z \in \mathcal{H}^{rt}(\mathfrak{H}).$$

Since $\{U\eta : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ is a complete NTF for $\mathcal{H}^{rt}(\mathfrak{H})$, W is a right-linear isometry. For any $U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))$ and $z \in \mathcal{H}^{rt}(\mathfrak{H})$, we obtain

$$\lambda_U W(z) = \sum_{V \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))} \chi_{UV} \langle z | V\eta \rangle = \sum_{V \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))} \chi_V \langle z | U^* V\eta \rangle.$$

Using the identity $\langle z | U^* w \rangle = \langle Uz | w \rangle$, valid for right-linear unitary operators in QHSs, we obtain

$$\lambda_U W(z) = W(Uz),$$

and hence

$$\lambda_U W = WU.$$

We now verify each assertion:

(i) Since $\{\chi_U : U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H}))\}$ is a Hilbert basis for $\mathcal{K}^{rt}(\mathfrak{H})$, and

$$\lambda_V \chi_I = \chi_V, \quad V \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})),$$

it follows that χ_I is a complete wandering vector for $\mathcal{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))$.

(ii) From $\lambda_U W = WU$, we have

$$\lambda_U(W(\mathcal{H}^{rt}(\mathfrak{H}))) \subseteq W(\mathcal{H}^{rt}(\mathfrak{H})), \quad U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})),$$

and hence $W(\mathcal{H}^{rt}(\mathfrak{H}))$ is invariant under $\mathcal{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))$. Identifying $\mathcal{H}^{rt}(\mathfrak{H})$ with $W(\mathcal{H}^{rt}(\mathfrak{H}))$, we conclude that $\mathcal{H}^{rt}(\mathfrak{H})$ is invariant under $\mathcal{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))$.

(iii) From the intertwining relation, we obtain

$$W^* \lambda_U W = U, \quad U \in \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})),$$

which implies that

$$\mathcal{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))|_{\mathcal{H}^{rt}(\mathfrak{H})} = \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})).$$

(iv) The mapping $\Phi : \mathcal{U}(\mathcal{H}^{rt}(\mathfrak{H})) \rightarrow \mathcal{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))$ defined by $\Phi(U) = \lambda_U$ is a group homomorphism since

$$\lambda_U \lambda_V = \lambda_{UV}, \quad \lambda_I = I, \quad \lambda_U^{-1} = \lambda_{U^{-1}}.$$

It is injective since $\lambda_U = \lambda_V$ implies $\chi_U = \chi_V$, and hence $U = V$. It is surjective by definition of $\mathcal{U}_1(\mathcal{K}^{rt}(\mathfrak{H}))$. Thus Φ is a group isomorphism.

Finally, the uniqueness up to unitary equivalence follows from the uniqueness of the left regular representation. $\square$

We now restate the dilation theorem in the context of unitary group representations on right quaternionic Hilbert spaces.

Theorem 4.2. *Suppose $\mathcal{U}$ is a discrete group and $\Pi : \mathcal{U} \rightarrow \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$ is a unitary representation of $\mathcal{U}$ on a separable right QHS $\mathcal{H}^{rt}(\mathfrak{H})$. Suppose $\eta \in \mathcal{H}^{rt}(\mathfrak{H})$ generates a NTF for $\mathcal{H}^{rt}(\mathfrak{H})$. Then there exist a separable right QHS $\mathcal{H}_1^{rt}(\mathfrak{H})$, a unitary representation $\Pi_1 : \mathcal{U} \rightarrow \mathfrak{B}(\mathcal{H}_1^{rt}(\mathfrak{H}))$, and a vector $\eta_1 \in \mathcal{H}_1^{rt}(\mathfrak{H})$ such that:*

- (i) $\eta \oplus \eta_1$ is a complete wandering vector for the representation $(\Pi \oplus \Pi_1)$ on $\mathcal{H}^{rt}(\mathfrak{H}) \oplus \mathcal{H}_1^{rt}(\mathfrak{H})$,
- (ii) $\{(\Pi \oplus \Pi_1)(s)(\eta \oplus \eta_1) : s \in \mathcal{U}\}$ is a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H}) \oplus \mathcal{H}_1^{rt}(\mathfrak{H})$,
- (iii) $(\Pi_1, \mathcal{H}_1^{rt}(\mathfrak{H}), \eta_1)$ is unique up to unitary equivalence.

Proof. Since $\{\Pi(s)\eta : s \in \mathcal{U}\}$ is a complete NTF for $\mathcal{H}^{rt}(\mathfrak{H})$, by Theorem 4.1, there exist a separable right QHS $\mathcal{K}^{rt}(\mathfrak{H}) \supseteq \mathcal{H}^{rt}(\mathfrak{H})$, a unitary representation $\Pi' : \mathcal{U} \rightarrow \mathfrak{B}(\mathcal{K}^{rt}(\mathfrak{H}))$, and a complete wandering vector $\zeta \in \mathcal{K}^{rt}(\mathfrak{H})$ such that $\{\Pi'(s)\zeta : s \in \mathcal{U}\}$ forms a Hilbert basis for $\mathcal{K}^{rt}(\mathfrak{H})$, and $\mathcal{H}^{rt}(\mathfrak{H})$ is invariant under Π' with

$$\Pi'(s)|_{\mathcal{H}^{rt}(\mathfrak{H})} = \Pi(s), \quad s \in \mathcal{U}.$$

Let

$$\mathcal{H}_1^{rt}(\mathfrak{H}) = (I - P)\mathcal{K}^{rt}(\mathfrak{H}).$$

where, P is the orthogonal projection of $\mathcal{K}^{rt}(\mathfrak{H})$ onto $\mathcal{H}^{rt}(\mathfrak{H})$. Then $\mathcal{K}^{rt}(\mathfrak{H}) = \mathcal{H}^{rt}(\mathfrak{H}) \oplus \mathcal{H}_1^{rt}(\mathfrak{H})$, and $\mathcal{H}_1^{rt}(\mathfrak{H})$ is invariant under Π' . Define $\Pi_1 : \mathcal{U} \rightarrow \mathfrak{B}(\mathcal{H}_1^{rt}(\mathfrak{H}))$ by

$$\Pi_1(s) = \Pi'(s)|_{\mathcal{H}_1^{rt}(\mathfrak{H})}, \quad s \in \mathcal{U}. \quad (4.1)$$

Then $\Pi' = \Pi \oplus \Pi_1$ on $\mathcal{H}^{rt}(\mathfrak{H}) \oplus \mathcal{H}_1^{rt}(\mathfrak{H})$. Write $\zeta = \eta \oplus \eta_1$ with $\eta_1 \in \mathcal{H}_1^{rt}(\mathfrak{H})$. Then for each $s \in \mathcal{U}$,

$$(\Pi \oplus \Pi_1)(s)(\eta \oplus \eta_1) = \Pi'(s)\zeta.$$

We now verify each assertion:

(i) Since $\{\Pi'(s)\zeta : s \in \mathcal{U}\}$ is a Hilbert basis for $\mathcal{K}^{rt}(\mathfrak{H})$, it follows that $\eta \oplus \eta_1 = \zeta$ constitutes a complete wandering vector associated with $(\Pi \oplus \Pi_1)(\mathcal{U})$.

(ii) Equation (4.1) gives

$$\{(\Pi \oplus \Pi_1)(s)(\eta \oplus \eta_1) : s \in \mathcal{U}\} = \{\Pi'(s)\zeta : s \in \mathcal{U}\},$$

which is a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H}) \oplus \mathcal{H}_1^{rt}(\mathfrak{H})$.

(iii) The representation $(\Pi_1, \mathcal{H}_1^{rt}(\mathfrak{H}), \eta_1)$ is unique, since the dilation $(\mathcal{K}^{rt}(\mathfrak{H}), \Pi', \zeta)$ is unique up to unitary equivalence. $\square$

Let $\mathcal{U}$ be a group and let $\mathcal{H}^{rt}(\mathfrak{H})$ be a separable right QHS. We say a representation $(\Pi, \mathcal{U}, \mathcal{H}^{rt}(\mathfrak{H}))$, a *quaternionic frame representation* provided that it admits a complete NTF vector. Two complete NTF vectors $\eta_1, \eta_2 \in \mathcal{H}^{rt}(\mathfrak{H})$ for a representation Π are said to be *equivalent* if the frames

$$\{\Pi(s)\eta_1 : s \in \mathcal{U}\} \quad \text{and} \quad \{\Pi(s)\eta_2 : s \in \mathcal{U}\}$$

are unitarily equivalent. Two quaternionic frame representations $(\Pi_1, \mathcal{U}, \mathcal{H}_1^{rt}(\mathfrak{H}))$ and $(\Pi_2, \mathcal{U}, \mathcal{H}_2^{rt}(\mathfrak{H}))$ are said to be *equivalent* whenever a unitary operator $\theta : \mathcal{H}_1^{rt}(\mathfrak{H}) \rightarrow \mathcal{H}_2^{rt}(\mathfrak{H})$ exists such that

$$\theta \Pi_1(s) \theta^* = \Pi_2(s), \quad s \in \mathcal{U}.$$

We now relate the unitary equivalence of frame systems to the equivalence of the corresponding representations.

Theorem 4.3. Suppose $\mathcal{U}$ is a discrete group and $(\Pi_1, \mathcal{U}, \mathcal{H}_1^{rt}(\mathfrak{H}))$ and $(\Pi_2, \mathcal{U}, \mathcal{H}_2^{rt}(\mathfrak{H}))$ are quaternionic frame representations with $\eta_1 \in \mathcal{H}_1^{rt}(\mathfrak{H})$ and $\eta_2 \in \mathcal{H}_2^{rt}(\mathfrak{H})$ as the complete NTF vectors for $\mathcal{H}_1^{rt}(\mathfrak{H})$ and $\mathcal{H}_2^{rt}(\mathfrak{H})$ respectively. The representations Π_1 and Π_2 are unitarily equivalent provided the corresponding frames $\{\Pi_1(s)\eta_1 : s \in \mathcal{U}\}$ and $\{\Pi_2(s)\eta_2 : s \in \mathcal{U}\}$ are unitarily equivalent.

Proof. Since the frames $\{\Pi_1(s)\eta_1 : s \in \mathcal{U}\}$ and $\{\Pi_2(s)\eta_2 : s \in \mathcal{U}\}$ are unitarily equivalent, there exists a unitary operator $U : \mathcal{H}_1^{rt}(\mathfrak{H}) \rightarrow \mathcal{H}_2^{rt}(\mathfrak{H})$ such that

$$U(\Pi_1(s)\eta_1) = \Pi_2(s)\eta_2, \quad s \in \mathcal{U}.$$

Let $s_1, s_2 \in \mathcal{U}$. Then

$$\begin{aligned} U\Pi_1(s_1)\Pi_1(s_2)\eta_1 &= U\Pi_1(s_1s_2)\eta_1 = \Pi_2(s_1s_2)\eta_2 = \Pi_2(s_1)\Pi_2(s_2)\eta_2 \\ &= \Pi_2(s_1)U\Pi_1(s_2)\eta_1. \end{aligned}$$

Since $\{\Pi_1(s)\eta_1 : s \in \mathcal{U}\}$ is a complete frame for $\mathcal{H}_1^{rt}(\mathfrak{H})$, its linear span is dense in $\mathcal{H}_1^{rt}(\mathfrak{H})$. Therefore, for all $z \in \mathcal{H}_1^{rt}(\mathfrak{H})$,

$$U\Pi_1(s_1)z = \Pi_2(s_1)Uz.$$

Thus,

$$U\Pi_1(s) = \Pi_2(s)U, \quad s \in \mathcal{U},$$

which implies

$$U\Pi_1(s)U^* = \Pi_2(s), \quad s \in \mathcal{U}.$$

Hence, Π_1 and Π_2 are unitarily equivalent representations. $\square$

We illustrate the above theory with a canonical example arising from the left regular representation of a discrete group.

Example 4.1. Suppose $\mathcal{U}$ is a discrete group and let $\mathcal{H}^{rt}(\mathfrak{H}) = \ell^2(\mathcal{U}, \mathfrak{H})$ be the separable right QHS of square-summable $\mathfrak{H}$ -valued functions on $\mathcal{U}$. The left regular representation, $\lambda^L : \mathcal{U} \rightarrow \mathfrak{B}(\mathcal{H}^{rt}(\mathfrak{H}))$ is defined by

$$(\lambda^L(s)z)(t) = z(s^{-1}t), \quad s, t \in \mathcal{U}, z \in \mathcal{H}^{rt}(\mathfrak{H}).$$

Let $\{e_s\}_{s \in \mathcal{U}}$ denote the canonical Hilbert basis of $\mathcal{H}^{rt}(\mathfrak{H})$, where

$$e_s(t) = \delta_{s,t}, \quad t \in \mathcal{U}.$$

Then, for any $s, t \in \mathcal{U}$, we have

$$\lambda^L(t)e_s = e_{ts}.$$

Therefore, the set $\{\lambda^L(t)e_s : t \in \mathcal{U}\} = \{e_{ts} : t \in \mathcal{U}\}$ forms a Hilbert basis for $\mathcal{H}^{rt}(\mathfrak{H})$. Hence $\lambda^L(\mathcal{U})$ admits e_s as a complete wandering vector e_s , and therefore

$$e_s \in W(\lambda^L(\mathcal{U})).$$

Since $\lambda^L(\mathcal{U})$ is a unitary group, its commutant $\lambda^L(\mathcal{U})'$ is a von Neumann algebra. Moreover, since $W(\lambda^L(\mathcal{U})) \neq \emptyset$, we obtain

$$C_{e_t}(\lambda^L(\mathcal{U})) = \lambda^L(\mathcal{U})'.$$

We now present the final result, which provides a canonical realization of quaternionic frame representations.

Theorem 4.4. Suppose $\mathcal{U}$ is a discrete group and $\mathcal{H}^{rt}(\mathfrak{H})$ is a right QHS. Then every quaternionic frame representation of a discrete group $\mathcal{U}$ is unitarily equivalent to a subrepresentation of λ^L , the left regular representation on $\ell^2(\mathcal{U}, \mathfrak{H})$.

Proof. Suppose $(\Pi, \mathcal{U}, \mathcal{H}^{rt}(\mathfrak{H}))$ is a quaternionic frame representation having a complete NTF vector $\eta \in \mathcal{H}^{rt}(\mathfrak{H})$. Then $\{\Pi(g)\eta : g \in \mathcal{U}\}$ is a NTF for $\mathcal{H}^{rt}(\mathfrak{H})$. Define the analysis operator $W : \mathcal{H}^{rt}(\mathfrak{H}) \rightarrow \ell^2(\mathcal{U}, \mathfrak{H})$ by

$$W(z) = \sum_{g \in \mathcal{U}} e_g \langle z | \Pi(g)\eta \rangle, \quad z \in \mathcal{H}^{rt}(\mathfrak{H}),$$

where $\{e_g\}_{g \in \mathcal{U}}$ is the canonical Hilbert basis of $\ell^2(\mathcal{U}, \mathfrak{H})$. Since $\{\Pi(g)\eta\}$ is a NTF for $\mathcal{H}^{rt}(\mathfrak{H})$, W is a right-linear isometry. The left regular representation λ^L of $\mathcal{U}$ on $\ell^2(\mathcal{U}, \mathfrak{H})$ is defined by

$$(\lambda^L(a)f)(g) = f(a^{-1}g), \quad a, g \in \mathcal{U}.$$

For any $a \in \mathcal{U}$ and $z \in \mathcal{H}^{rt}(\mathfrak{H})$, we compute

$$\lambda^L(a)W(z) = \sum_{g \in \mathcal{U}} e_{ag} \langle z | \Pi(g)\eta \rangle = \sum_{g \in \mathcal{U}} e_g \langle z | \Pi(a^{-1}g)\eta \rangle.$$

Using the identity $\langle z | \Pi(a^{-1})w \rangle = \langle \Pi(a)z | w \rangle$, we obtain

$$\lambda^L(a)W(z) = \sum_{g \in \mathcal{U}} e_g \langle \Pi(a)z | \Pi(g)\eta \rangle = W(\Pi(a)z).$$

Therefore,

$$\lambda^L(a)W = W\Pi(a), \quad a \in \mathcal{U}.$$

Thus, $W(\mathcal{H}^{rt}(\mathfrak{H}))$ is an invariant subspace of $\ell^2(\mathcal{U}, \mathfrak{H})$ under λ^L , and

$$W\Pi(a)W^* = \lambda^L(a)|_{W(\mathcal{H}^{rt}(\mathfrak{H}))}.$$

Hence, Π is unitarily equivalent to the subrepresentation $\lambda^L|_{W(\mathcal{H}^{rt}(\mathfrak{H}))}$ of the left regular representation. $\square$

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