

Interval-Valued Erdélyi–Kober Fractional Integral Operators: Hölder, Minkowski, and Hermite–Hadamard Type Inequalities with Applications

Mohammad Akram¹, Haitham Qawaqneh², Waqar Afzal^{3,4,5}, Mujahid Abbas^{6,7}, Muhammad Tariq⁸, Ernesto Urenda-Cázares⁹, Jorge E. Macías-Díaz^{10,11}, Hijaz Ahmad^{12,13,*}

¹*Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia*

²*Department of Basic Sciences, Al-Zaytoonah University of Jordan, Amman 11733, Jordan*

³*Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54000, Pakistan*

⁴*Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku, AZ1096, Azerbaijan*

⁵*International Center for Interdisciplinary Research in Sciences, The University of Lahore, Lahore 54792, Pakistan*

⁶*Department of Mechanical Engineering Science, Faculty of Engineering and the Built Environment, University of Johannesburg, Johannesburg 2092, South Africa*

⁷*Department of Medical Research, China Medical University, Taichung 40447, Taiwan*

⁸*Mathematics Research Center, Near East University, Near East Boulevard, 99138 Nicosia, Mersin 10, Turkey*

⁹*University Center for Exact Sciences and Engineering, University of Guadalajara, Guadalajara, Mexico*

¹⁰*Department of Mathematics and Didactics of Mathematics, Tallinn University, Tallinn 10120, Estonia*

¹¹*Department of Mathematics and Physics, Autonomous University of Aguascalientes, Aguascalientes 20100, Mexico*

¹²*Irfan Suat Gunsul Operational Research Institute, Near East University, Nicosia/TRNC, 99138 Mersin 10, Turkey*

¹³*Department of Mathematics, College of Science, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02841, South Korea*

*Corresponding author: hijazahmad@korea.ac.kr

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Abstract. In this article, we introduce a novel interval-valued Erdélyi–Kober fractional integral operator. Using this operator, we establish several new variants of Hermite–Hadamard, midpoint, Hölder, and Minkowski type inequalities for interval-valued convex mappings. To illustrate the validity and sharpness of the obtained results, we provide nontrivial examples supported by tables and detailed remarks. We also show that a number of known results in the existing literature arise as special cases of our main inequalities. Moreover, the derived Hermite–Hadamard-type bounds are expressed in a new form involving special functions that explicitly involve the Gamma and Beta functions. As an application, we demonstrate the usefulness of our results in estimating the error of numerical quadrature rules, particularly the trapezoidal rule, for interval-valued functions.

1. INTRODUCTION

Integral inequalities form the foundation of mathematical analysis with profound applications in optimization [1], numerical analysis [2], and differential equations [3]. Among these, the Hermite–Hadamard [4, 5], Hölder, and Minkowski inequalities stand out as fundamental tools. The Hermite–Hadamard inequality provides a remarkable estimate for the mean value of a convex function, while the Hölder and Minkowski inequalities are indispensable in the study of function spaces [6] and metric structures [7]. These results have been extensively generalized and refined over the past century. The exploration of these inequalities has evolved along two significant modern trajectories. The first is the theory of fractional calculus, which extends derivatives and integrals to non-integer orders, offering a powerful framework for modeling memory-dependent and non-local phenomena. Currently, to handle data imprecision and uncertainty, classical analysis has been extended to set-valued and, in particular, interval-valued analysis.

The Hermite–Hadamard (H–H) inequality is a fundamental result in convex analysis that was independently established by two renowned mathematicians, C. Hermite and J. Hadamard [8, 9]. It states that if a function $\mathcal{F} : \mathcal{I} \rightarrow \mathcal{R}$ is convex on the interval $\mathcal{I}$, then for any $\zeta_1, \zeta_2 \in \mathcal{I}$ with $\zeta_1 \neq \zeta_2$, we have the following double inequality:

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \mathcal{F}(z) dz \leq \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2}.$$

If a function is convex, the H–H inequality states that the average of the function values at the endpoints of an interval, weighted by their respective distances from the midpoint, is greater than or equal to the value of the function at the midpoint. This inequality finds extensive utility across various domains of mathematics and economics, where convexity also plays a significant role.

Sarikaya et al. [10] introduced fractional integral variants using the integral identity as follows:

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\zeta_2 - \zeta_1)^\alpha} \left[\mathcal{J}_{\zeta_1^+}^\alpha \mathcal{F}(\zeta_2) + \mathcal{J}_{\zeta_2^-}^\alpha \mathcal{F}(\zeta_1) \right] \leq \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2},$$

with $\alpha > 0$. It is obvious that the inequality turns into the classical H–H inequality when $\alpha = 1$.

Hai and Wang [11] developed Hermite–Hadamard-type inequalities based on the Erdélyi–Kober fractional integrals. They proved:

Theorem 1.1. Suppose that $\mathcal{F} : [\zeta_1^\rho, \zeta_2^\rho] \rightarrow \mathcal{R}$ is a function with $\rho > 0$ and $0 \leq \zeta_1 < \zeta_2$. If $\mathcal{F}$ is a convex function on $[\zeta_1^\rho, \zeta_2^\rho]$, then for any $\alpha > 0$ and any $\chi \in [\zeta_1, \zeta_2]$,

$$\begin{aligned} \mathcal{F}\left(\frac{1}{\alpha+1}\frac{\zeta_1^\rho + \zeta_2^\rho}{2} + \frac{\alpha}{\alpha+1}\chi^\rho\right) &\leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2} \left[\frac{\rho \mathcal{K}_{\chi^+}^\alpha \mathcal{F}(\zeta_2^\rho)}{(\zeta_2^\rho - \chi^\rho)^\alpha} + \frac{\rho \mathcal{K}_{\chi^-}^\alpha \mathcal{F}(\zeta_1^\rho)}{(\chi^\rho - \zeta_1^\rho)^\alpha} \right] \\ &\leq \frac{1}{\alpha+1} \left[\frac{\mathcal{F}(\zeta_1^\rho) + \mathcal{F}(\zeta_2^\rho)}{2} + \alpha \mathcal{F}(\chi^\rho) \right]. \end{aligned}$$

More recently, Bashir et al. [12] introduced a novel fractional integral operator with an exponential kernel and employed it to extend the classical Hermite–Hadamard inequality within the framework of fractional calculus.

Theorem 1.2. Let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}$ be a positive function with $0 \leq \zeta_1 < \zeta_2$ and $\mathcal{F} \in \mathbf{L}[\zeta_1, \zeta_2]$. If $\mathcal{F}$ is a convex function on $[\zeta_1, \zeta_2]$, then the following inequalities for fractional integrals hold:

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq \frac{1-\alpha}{2(1-\exp(-\rho))} \left[I_{\zeta_1}^\alpha \mathcal{F}(\zeta_2) + I_{\zeta_2}^\alpha \mathcal{F}(\zeta_1) \right] \leq \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2},$$

where $\alpha \in (0, 1)$ and $\rho = \frac{1-\alpha}{\alpha}(\zeta_2 - \zeta_1)$.

Furthermore, Zhou et al. [13] extended the result appearing in the above inequality to the framework of interval-valued mappings. Let $\mathcal{R}_I^+$ denote the set of all positive closed intervals in $\mathcal{R}$.

Theorem 1.3. If $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I^+$ is an interval-valued convex function with $\zeta_1 < \zeta_2$ satisfying that $\mathcal{F}(\tau) = [\underline{\mathcal{F}}(\tau), \overline{\mathcal{F}}(\tau)]$, then we have

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \supseteq \frac{1-\alpha}{2(1-e^{-\rho})} \left[I_{\zeta_1^+}^\alpha \mathcal{F}(\zeta_2) + I_{\zeta_2^-}^\alpha \mathcal{F}(\zeta_1) \right] \supseteq \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2},$$

where

$$\rho = \frac{1-\alpha}{\alpha}(\zeta_2 - \zeta_1), \quad 0 < \alpha < 1.$$

Here, $\supseteq$ denotes interval inclusion, and the fractional integrals are applied componentwise to the endpoint functions $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$.

Sahoo et al. [14] used k -fractional operators to establish Hermite–Hadamard-type inequalities for (h, m) -convex functions:

Theorem 1.4. Let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be a (h, m) -convex function with $0 \leq \zeta_1 \leq \zeta_2, m \in (0, 1]$. If $\mathcal{F} \in \mathbf{L}[\zeta_1, \zeta_2]$, then the following inequality for k -fractional integral holds:

$$\begin{aligned} \frac{1}{h\left(\frac{1}{2}\right)} \mathcal{F}\left(\frac{\zeta_1 + m\zeta_2}{2}\right) &\leq \frac{2^{\frac{\mu}{k}} \Gamma_k(\mu+k)}{(m\zeta_2 - \zeta_1)^{\frac{\mu}{k}}} \left[\mathbf{I}^{\mu, k} \left(\frac{\zeta_1 + m\zeta_2}{2} \right)^+ \mathcal{F}(m\zeta_2) + m^{\frac{\mu}{k}+1} \mathbf{I}^{\mu, k} \left(\frac{\zeta_1 + m\zeta_2}{2m} \right)^- \mathcal{F}\left(\frac{\zeta_1}{m}\right) \right] \\ &\leq \frac{\mu [\mathcal{F}(\zeta_1) + \mathcal{F}(m\zeta_2)]}{k} \int_0^1 \varsigma^{\frac{\mu}{k}-1} h\left(\frac{\zeta}{2}\right) d\varsigma + \frac{\mu [m\mathcal{F}(\zeta_2) + m^2 \mathcal{F}\left(\frac{\zeta_1}{m^2}\right)]}{k} \int_0^1 h\left(\frac{2-\varsigma}{2}\right) \varsigma^{\frac{\mu}{k}-1} d\varsigma. \end{aligned}$$

Budak et al. [15] studied harmonically convex functions and developed several new forms of interval-valued inequalities:

Theorem 1.5. *If $\mathcal{F}, \mathcal{G} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I^+$ are two convex interval-valued functions such that $\mathcal{F}(t) = [\underline{\mathcal{F}}(t), \overline{\mathcal{F}}(t)]$ and $\mathcal{G}(t) = [\underline{\mathcal{G}}(t), \overline{\mathcal{G}}(t)]$, then for $\alpha > 0$ we have*

$$\begin{aligned} & 2\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right)\mathcal{G}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \\ & \supseteq \frac{\Gamma(\alpha + 1)}{2(\zeta_2 - \zeta_1)^\alpha} \left[\mathcal{J}_{\zeta_1^+}^\alpha \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2) + \mathcal{J}_{\zeta_2^-}^\alpha \mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1) \right] \\ & \quad + \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \left[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2) \right] \\ & \quad + \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \left[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1) \right]. \end{aligned}$$

Fractional interval-valued inequalities [16,17] play an important role both in theory and practice in mathematics because they can handle at once two key issues for modeling real-world problems, such as memory effects and data uncertainty [18,19]. By generalizing classical integral inequalities to fractional operators while including functions which are designated as interval-valued, these inequalities can be used to create a powerful mathematical framework for studying systems [20] in which parameters are not precisely known but are within certain bounds and where the dynamics are dependent on the entire history of the process and not only on the current state [21,22]. Recent years have witnessed substantial progress in this interdisciplinary domain. Early publications provided fractional Hermite-Hadamard inequality results for interval-valued functions [23] and later publications generalized the results to various types of convexity and generalized fractional operators [24–28]. In particular, post-quantum calculus has been adopted to establish novel analogs of Hermite-Hadamard inequalities for interval-valued convex functions [29–31] and coordinate convexity notions on coordinates [32,33] have been studied. Further developments include the research of preinvex interval-valued functions through fractional integrals [34–39], log-convex fuzzy interval-valued mappings [40], or fractional quantum versions for interval-valued functions [41]. Recent investigations also concern stochastic environments [42–44], generalized geometric-arithmetic convexity [45] and Riemann-Liouville fractional q -integrals of fuzzy interval-valued functions [46].

Other specialized functions have also been studied, including interval-valued preinvex functions with fractional operators and center-radius orderings of Pachpatte-type inequalities [47]. It has also been extended to variational problems [48] and the concepts are continually refined in modern research in herz spaces [49], such as reverse inequalities [50] studies of Jensen gap estimates [51], or Riemann diamond-alpha integrals [52]. Moreover, the relationship between variational inequalities [78] and vector optimization is also one of the essential concerns of interest, as well

as superquadratic function analysis [53] and generalized forms of convexity, such as Godunova–Levin functions and studied harmonic interval-valued functions [54]. Lastly, the latest literature deals with n -polynomial convexity [55], the use of q -integrals in cases of fuzzy interval valued functions and the case with stochastic processes in use [56]. In 2019, Almutairi and Kılıçman [57] introduced the concept of h -Godunova–Levin preinvexity, in which they investigated several integral inequalities and demonstrated how this notion differs from classical convexity. Motivated by their work, Afzal et al. [58] extended this framework to interval-valued mappings and employed classical operators to derive various new forms of Hermite–Hadamard and Jensen-type results. Noor et al. [59] investigated fractional versions of these inequalities for new classes of s -Godunova–Levin functions. Agahi et al. [60] used general Barnes–Godunova–Levin inequalities for the Sugeno integral in their study. They provided a framework for extending classical inequalities [61] to the Sugeno integral context, which is particularly useful in information sciences and decision-making problems.

The beautiful relation between Young’s inequality and these basic results was shown at the very beginning by Tolsted [62] and it gives an elementary derivation among these results which emphasizes their deep interdependence. Over the years, these inequalities have been extended, generalized, and mixed; for example, into mixed Hölder–Minkowski-type results as investigated by Iusem et al. [63] which still affect contemporary research in geometry, harmonic analysis, and nonlinear potential theories. In geometric analysis, the Brunn–Minkowski inequality and its discrete and nonlinear variants remain central, as seen in the work of Akman et al. [64], discrete analogs by Hernández Cifre et al. [65], and potential approaches by Agostiniani et al. [66]. Recent results include reverse inequalities for generic interval-valued convex functions in the fractional setting, as well as reverse Minkowski and Hölder type inclusions for interval-valued mappings via fractional integrals [67]. These developments not only extend classical inequality theory but also enhance its applicability to problems involving uncertainty, hereditary effects, and non-integer order dynamics. For some other recent results related to interval-valued mappings, involving various integral operators and associated classes, as well as ordered interval-valued mappings, we refer to [68–74, 76] and the references therein.

Novelty and Significance. The novelty and significance of this study are summarized as follows:-

- The first main contribution is that, for the first time in this study, we extend the Erdélyi–Kober fractional integrals to the domain of functions with intervals. This generalization recovers several earlier findings, including those developed for the classical operator [8], the Riemann–Liouville operator under the classical order relation [10], and the interval order relation [15].
- In the existing literature, most authors have employed the inclusion order relation [15, 75] to extend classical inequalities to interval-valued mappings. However, as illustrated in Example 3 of [80], the inclusion order has certain inherent limitations, since it does not always allow a meaningful comparison between intervals. As a result, some previously derived

inequalities may not hold in general settings. To overcome this difficulty, the authors introduced a new order relation that is compatible, well-defined for interval analysis, and applicable to multi-objective convex optimization problems. This complete order structure preserves the standard comparison properties required for interval-valued analysis. In the present work, we adopt this order relation to ensure the mathematical validity and consistency of our results.

- Furthermore, our bounds for Hermite–Hadamard and midpoint-type inequalities are presented in a novel formulation involving special functions, namely the Gamma and Beta functions. As an application, we demonstrate their utility in the error estimation of numerical quadrature using the trapezoidal rule for interval-valued functions.
- We also develop a new, more general form of Hölder and Minkowski type inequalities for interval-valued mappings using the Erdélyi–Kober fractional operator. These results recover several existing outcomes in different settings.
- To demonstrate the accuracy and generality of our results, we show through remarks that they recover existing outcomes under different settings, as well as provide non-trivial examples that illustrate their practical applicability.

The structure of this article is organized as follows. In Section 1, we review the relevant literature and provide the motivation behind the problems studied. Section 2 recalls some necessary notions, including definitions, operators, and concepts of interval calculus. In Section 3, we present the main results, where we develop various types of inequality along with applications to numerical quadrature rules. Finally, in Section 4, we conclude the article with a discussion of the results and propose directions for future research.

2. PRELIMINARY CONCEPTS AND DEFINITIONS

In this section, we recall some necessary notions that are used in developing our main findings. These include interval arithmetic operations, classical and interval-valued convexity, classical Erdélyi–Kober fractional operators, and special functions.

2.1. Special Functions. We begin by recalling the definitions of two fundamental special functions that appear throughout this work.

Definition 2.1 (Euler Gamma Function). *The Euler Gamma function is defined for $\Re(z) > 0$ by the integral representation*

$$\Gamma(z) = \int_0^\infty \eta^{z-1} e^{-\eta} d\eta.$$

For positive integers n , it satisfies $\Gamma(n+1) = n!$, and for $\alpha > -1$, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Definition 2.2 (Beta Function). *The Beta function, also known as the Euler integral of the first kind, is defined for $\Re(x) > 0$ and $\Re(y) > 0$ by*

$$\mathcal{B}(x, y) = \int_0^1 \eta^{x-1} (1-\eta)^{y-1} d\eta.$$

It is related to the Gamma function through the identity

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

These functions play crucial roles in fractional calculus and appear in the coefficients of our fractional integral inequalities.

2.2. Interval Arithmetic and the cr-Order. Let us denote by $\mathcal{R}_I$ the collection of all nonempty closed intervals of the real line $\mathcal{R}$. For an interval $\zeta_1 = [\underline{\zeta}_1, \overline{\zeta}_1]$, we say ζ_1 is *positive* if $\underline{\zeta}_1 > 0$. We denote by $\mathcal{R}_I^+$ and $\mathcal{R}^+$ the set of all positive intervals and the set of all positive real numbers, respectively. For $\lambda \in \mathcal{R}$, the Minkowski addition and scalar multiplication are defined by

$$\begin{aligned}\zeta_1 + \zeta_2 &= [\underline{\zeta}_1, \overline{\zeta}_1] + [\underline{\zeta}_2, \overline{\zeta}_2] = [\underline{\zeta}_1 + \underline{\zeta}_2, \overline{\zeta}_1 + \overline{\zeta}_2]; \\ \lambda \zeta_1 &= \lambda [\underline{\zeta}_1, \overline{\zeta}_1] = \begin{cases} [\lambda \underline{\zeta}_1, \lambda \overline{\zeta}_1], & \lambda > 0 \\ \{0\}, & \lambda = 0 \\ [\lambda \overline{\zeta}_1, \lambda \underline{\zeta}_1], & \lambda < 0 \end{cases}\end{aligned}$$

respectively.

For an interval $\zeta_1 = [\underline{\zeta}_1, \overline{\zeta}_1] \in \mathcal{R}_I$, the quantity $(\zeta_1)_c = \frac{\overline{\zeta}_1 + \underline{\zeta}_1}{2}$ is called the *center* of ζ_1 , and $(\zeta_1)_r = \frac{\overline{\zeta}_1 - \underline{\zeta}_1}{2}$ is called the *radius* of ζ_1 . The interval ζ_1 can be represented in center-radius form as

$$\zeta_1 = \left\langle \frac{\overline{\zeta}_1 + \underline{\zeta}_1}{2}, \frac{\overline{\zeta}_1 - \underline{\zeta}_1}{2} \right\rangle = \langle \zeta_{1c}, \zeta_{1r} \rangle.$$

Definition 2.3 ([47]). Let $\zeta_1 = [\underline{\zeta}_1, \overline{\zeta}_1] = \langle \zeta_{1c}, \zeta_{1r} \rangle$ and $\zeta_2 = [\underline{\zeta}_2, \overline{\zeta}_2] = \langle \zeta_{2c}, \zeta_{2r} \rangle$ be intervals in $\mathcal{R}_I$. The center-radius order (abbreviated as cr-order) relation is defined as:

$$\zeta_1 \leq_{\text{cr}} \zeta_2 \Leftrightarrow \begin{cases} \zeta_{1c} < \zeta_{2c}, & \text{if } \zeta_{1c} \neq \zeta_{2c}, \\ \zeta_{1r} \leq \zeta_{2r}, & \text{if } \zeta_{1c} = \zeta_{2c}. \end{cases}$$

Clearly, for any two intervals $\zeta_1, \zeta_2 \in \mathcal{R}_I$, either $\zeta_1 \leq_{\text{cr}} \zeta_2$ or $\zeta_2 \leq_{\text{cr}} \zeta_1$.

The concept of Riemann integration for interval-valued functions was introduced in [15]. We have the following results:

Theorem 2.1 ([15]). Let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I$ be an interval-valued function given by $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$. Then $\mathcal{F}$ is Riemann integrable on $[\zeta_1, \zeta_2]$ if and only if $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are Riemann integrable on $[\zeta_1, \zeta_2]$, and

$$\int_{\zeta_1}^{\zeta_2} \mathcal{F}(\mathfrak{x}) \, d\mathfrak{x} = \left[\int_{\zeta_1}^{\zeta_2} \underline{\mathcal{F}}(\mathfrak{x}) \, d\mathfrak{x}, \int_{\zeta_1}^{\zeta_2} \overline{\mathcal{F}}(\mathfrak{x}) \, d\mathfrak{x} \right].$$

The set of all Riemann integrable interval-valued functions on $[\zeta_1, \zeta_2]$ is denoted by $\mathcal{IR}_{([\zeta_1, \zeta_2])}$.

Theorem 2.2 ([47]). Let $\mathcal{F}, \mathcal{G} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I^+$ be given by $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$ and $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$. If $\mathcal{F}, \mathcal{G} \in \mathcal{IR}_{([\zeta_1, \zeta_2])}$ and $\mathcal{F}(\mathfrak{x}) \leq_{\text{cr}} \mathcal{G}(\mathfrak{x})$ for all $\mathfrak{x} \in [\zeta_1, \zeta_2]$, then

$$\int_{\zeta_1}^{\zeta_2} \mathcal{F}(\mathfrak{x}) \, d\mathfrak{x} \leq_{\text{cr}} \int_{\zeta_1}^{\zeta_2} \mathcal{G}(\mathfrak{x}) \, d\mathfrak{x}.$$

For more details on interval analysis, we refer to [8] and [15].

2.3. Convexity and Interval-Valued Convexity.

Definition 2.4 ([8]). A function $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}$ is said to be convex if for all $\mathfrak{x}, \mathfrak{y} \in [\zeta_1, \zeta_2]$ and $\lambda \in [0, 1]$, we have

$$\mathcal{F}(\lambda \mathfrak{x} + (1 - \lambda)\mathfrak{y}) \leq \lambda \mathcal{F}(\mathfrak{x}) + (1 - \lambda)\mathcal{F}(\mathfrak{y}).$$

Definition 2.5 ([47]). Let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I$ be an interval-valued function given by $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$. Then $\mathcal{F}$ is said to be interval-valued convex on $[\zeta_1, \zeta_2]$ if for all $\mathfrak{x}, \mathfrak{y} \in [\zeta_1, \zeta_2]$ and $\lambda \in [0, 1]$, we have

$$\mathcal{F}(\lambda \mathfrak{x} + (1 - \lambda)\mathfrak{y}) \leq_{\text{cr}} \lambda \mathcal{F}(\mathfrak{x}) + (1 - \lambda)\mathcal{F}(\mathfrak{y}).$$

Equivalently, $\mathcal{F}$ is interval-valued convex if both $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex functions on $[\zeta_1, \zeta_2]$ and satisfy $\underline{\mathcal{F}}(\mathfrak{x}) \leq \overline{\mathcal{F}}(\mathfrak{x})$ for all $\mathfrak{x} \in [\zeta_1, \zeta_2]$.

The set of all interval-valued convex functions on $[\zeta_1, \zeta_2]$ with values in $\mathcal{R}_I^+$ is denoted by $\mathcal{IVCX}([\zeta_1, \zeta_2], \mathcal{R}_I^+)$.

Definition 2.6 ([47]). Let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I$ be an interval-valued function represented in center-radius form as $\mathcal{F} = \langle \mathcal{F}_c, \mathcal{F}_r \rangle$, where $\mathcal{F}_c(\mathfrak{x}) = \frac{\overline{\mathcal{F}}(\mathfrak{x}) + \underline{\mathcal{F}}(\mathfrak{x})}{2}$ and $\mathcal{F}_r(\mathfrak{x}) = \frac{\overline{\mathcal{F}}(\mathfrak{x}) - \underline{\mathcal{F}}(\mathfrak{x})}{2}$. Then $\mathcal{F}$ is said to be cr-convex on $[\zeta_1, \zeta_2]$ if for all $\mathfrak{x}, \mathfrak{y} \in [\zeta_1, \zeta_2]$ and $\lambda \in [0, 1]$, we have

$$\mathcal{F}_c(\lambda \mathfrak{x} + (1 - \lambda)\mathfrak{y}) \leq \lambda \mathcal{F}_c(\mathfrak{x}) + (1 - \lambda)\mathcal{F}_c(\mathfrak{y}),$$

and

$$\mathcal{F}_r(\lambda \mathfrak{x} + (1 - \lambda)\mathfrak{y}) \leq \lambda \mathcal{F}_r(\mathfrak{x}) + (1 - \lambda)\mathcal{F}_r(\mathfrak{y}).$$

Remark 2.1. For an interval-valued function $\mathcal{F}$, the following implications hold:

- If $\mathcal{F}$ is cr-convex, then it is interval-valued convex.
- If $\mathcal{F}$ is interval-valued convex, then both $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex.

2.4. Erdélyi–Kober Fractional Integral Operators.

Definition 2.7 ([11]). Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F} \in \mathcal{R}_{[\zeta_1, \zeta_2]}$ (the space of Riemann integrable functions on $[\zeta_1, \zeta_2]$). For parameters $\alpha \in (0, 1)$ and $\nu \in \mathcal{R}$ with $\nu > -1$, the Erdélyi–Kober fractional integral operators are defined by:

$$\mathcal{I}_{\zeta_1^+}^{\alpha, \nu} \mathcal{F}(\mathfrak{x}) = \frac{\mathfrak{x}^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_{\zeta_1}^{\mathfrak{x}} (\mathfrak{x} - \tau)^{\alpha-1} \tau^{-\alpha-\nu} \mathcal{F}(\tau) \, d\tau, \quad \mathfrak{x} > \zeta_1,$$

and

$$\mathcal{I}_{\zeta_2^-}^{\alpha, \nu} \mathcal{F}(\mathfrak{x}) = \frac{\mathfrak{x}^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_{\mathfrak{x}}^{\zeta_2} (\tau - \mathfrak{x})^{\alpha-1} \tau^{-\alpha-\nu} \mathcal{F}(\tau) \, d\tau, \quad \mathfrak{x} < \zeta_2,$$

where $\Gamma(\cdot)$ denotes the Euler Gamma function defined in Definition 2.1.

Remark 2.2. The following special cases illustrate that Definition 2.7 naturally recovers several well-known fractional integral operators.

- **Riemann–Liouville fractional integral.** If $\nu = 0$ and $\zeta_1 = 0$, then

$$\mathcal{I}_{0+}^{\alpha,0}\mathcal{F}(\mathfrak{x}) = \frac{\mathfrak{x}^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{x}} (\mathfrak{x} - \tau)^{\alpha-1} \tau^{-\alpha} \mathcal{F}(\tau) d\tau.$$

By applying the substitution $\tau = \mathfrak{x}t$ and simplifying the powers of $\mathfrak{x}$, we obtain

$$\mathcal{I}_{0+}^{\alpha,0}\mathcal{F}(\mathfrak{x}) = \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{x}} (\mathfrak{x} - u)^{\alpha-1} \mathcal{F}(u) du,$$

which coincides with the classical Riemann–Liouville fractional integral operator [15].

- **Hardy–Littlewood (Cesàro) operator.** Setting $\alpha = 1$, $\nu = 0$, and $\zeta_1 = 0$, we deduce

$$\frac{\mathcal{I}_{0+}^{1,0}\mathcal{F}(\mathfrak{x})}{\mathfrak{x}^0} = \frac{1}{\mathfrak{x}} \int_0^{\mathfrak{x}} \mathcal{F}(\tau) d\tau,$$

which reduces to the Hardy–Littlewood (Cesàro) operator [79].

- **Classical Erdélyi–Kober operator $\mathcal{J}_2^\alpha$.** If $\nu = 1$ and $\zeta_1 = 0$, then

$$\mathcal{I}_{0+}^{\alpha,1}\mathcal{F}(\mathfrak{x}) = \frac{\mathfrak{x}^{-\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{x}} (\mathfrak{x} - \tau)^{\alpha-1} \tau^{-\alpha-1} \mathcal{F}(\tau) d\tau.$$

After simplification, we arrive at

$$\frac{\mathcal{I}_{0+}^{\alpha,1}\mathcal{F}(\mathfrak{x})}{\mathfrak{x}^{-1}} = \frac{2\mathfrak{x}^{-2\alpha}}{\Gamma(\alpha)} \int_0^{\mathfrak{x}} \frac{\tau \mathcal{F}(\tau)}{(\mathfrak{x}^2 - \tau^2)^{1-\alpha}} d\tau,$$

which corresponds to the classical Erdélyi–Kober fractional integral operator [79].

3. THE MAJOR RESULTS

In this section, we present the main results of the paper. Before proceeding to these results, we first extend the classical Erdélyi–Kober fractional integral operator to the framework of interval-valued functions. Using this extension, we then establish several new variants of Hermite–Hadamard, midpoint, Minkowski, and Hölder type inequalities for interval-valued convex functions.

Definition 3.1 (Interval-Valued Erdélyi–Kober Fractional Integrals). Let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I$ be an interval-valued function defined by

$$\mathcal{F}(\mathfrak{x}) = [\underline{\mathcal{F}}(\mathfrak{x}), \overline{\mathcal{F}}(\mathfrak{x})],$$

where $\underline{\mathcal{F}}, \overline{\mathcal{F}} \in \mathcal{R}_{[\zeta_1, \zeta_2]}$. For parameters $\alpha \in (0, 1)$ and $\nu \in \mathcal{R}$ with $\nu > -1$, the interval-valued Erdélyi–Kober fractional integrals are defined componentwise as:

$$\begin{aligned} \mathcal{I}_{\zeta_1+}^{\alpha,\nu}\mathcal{F}(\mathfrak{x}) &= \left[\mathcal{I}_{\zeta_1+}^{\alpha,\nu}\underline{\mathcal{F}}(\mathfrak{x}), \mathcal{I}_{\zeta_1+}^{\alpha,\nu}\overline{\mathcal{F}}(\mathfrak{x}) \right], \\ \mathcal{I}_{\zeta_2-}^{\alpha,\nu}\mathcal{F}(\mathfrak{x}) &= \left[\mathcal{I}_{\zeta_2-}^{\alpha,\nu}\underline{\mathcal{F}}(\mathfrak{x}), \mathcal{I}_{\zeta_2-}^{\alpha,\nu}\overline{\mathcal{F}}(\mathfrak{x}) \right]. \end{aligned}$$

Explicitly, these can be written as:

$$\begin{aligned} I_{\zeta_1^+}^{\alpha, \nu} \mathcal{F}(\mathfrak{x}) &= \frac{\mathfrak{x}^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_{\zeta_1}^{\mathfrak{x}} (\mathfrak{x} - \tau)^{\alpha-1} \tau^{-\alpha-\nu} \mathcal{F}(\tau) d\tau, \quad \mathfrak{x} > \zeta_1, \\ I_{\zeta_2^-}^{\alpha, \nu} \mathcal{F}(\mathfrak{x}) &= \frac{\mathfrak{x}^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_{\mathfrak{x}}^{\zeta_2} (\tau - \mathfrak{x})^{\alpha-1} \tau^{-\alpha-\nu} \mathcal{F}(\tau) d\tau, \quad \mathfrak{x} < \zeta_2, \end{aligned}$$

where the integrals are understood in the interval sense.

3.1. New Hermite–Hadamard Type Inequalities via Interval-Valued Erdélyi–Kober Fractional Integrals.

Theorem 3.1. Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I^+$ be an interval-valued convex function represented as

$$\mathcal{F}(\zeta) = [\underline{\mathcal{F}}(\zeta), \overline{\mathcal{F}}(\zeta)],$$

where $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex Lebesgue integrable functions satisfying $0 \leq \underline{\mathcal{F}}(\zeta) \leq \overline{\mathcal{F}}(\zeta)$ for all $\zeta \in [\zeta_1, \zeta_2]$.

For parameters $\alpha \in (0, 1)$ and $\nu \in \mathcal{R}$ with $\nu > -1$, define the Erdélyi–Kober fractional integral operators:

$$\begin{aligned} I_{\zeta_1^+}^{\alpha, \nu} \mathcal{F}(x) &= \frac{x^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_{\zeta_1}^x (x - \tau)^{\alpha-1} \tau^{-\alpha-\nu} \mathcal{F}(\tau) d\tau, \quad x > \zeta_1, \\ I_{\zeta_2^-}^{\alpha, \nu} \mathcal{F}(x) &= \frac{x^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_x^{\zeta_2} (\tau - x)^{\alpha-1} \tau^{-\alpha-\nu} \mathcal{F}(\tau) d\tau, \quad x < \zeta_2. \end{aligned}$$

Then the following Hermite–Hadamard type inclusion holds:

$$\begin{aligned} \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) &\leq_{\text{cr}} \frac{\Gamma(\alpha + 1)(\zeta_1 \zeta_2)^{\frac{\nu+\alpha}{2}}}{2(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha} \left[\frac{I_{\zeta_1^+}^{\alpha, \nu} \mathcal{F}(\zeta_2)}{\zeta_2^\nu} + \frac{I_{\zeta_2^-}^{\alpha, \nu} \mathcal{F}(\zeta_1)}{\zeta_1^\nu} \right] \\ &\leq_{\text{cr}} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2}. \end{aligned} \quad (3.1)$$

Proof. Since $\mathcal{F}$ is interval-valued convex on $[\zeta_1, \zeta_2]$, for any $\theta, \phi \in [\zeta_1, \zeta_2]$ we have:

$$\mathcal{F}\left(\frac{\theta + \phi}{2}\right) \leq_{\text{cr}} \frac{\mathcal{F}(\theta) + \mathcal{F}(\phi)}{2}.$$

Take $\theta = \lambda \zeta_1 + (1 - \lambda) \zeta_2$ and $\phi = (1 - \lambda) \zeta_1 + \lambda \zeta_2$ for $\lambda \in [0, 1]$. Then:

$$\frac{\theta + \phi}{2} = \frac{\zeta_1 + \zeta_2}{2}.$$

Substituting gives:

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq_{\text{cr}} \frac{1}{2} \left[\mathcal{F}(\lambda \zeta_1 + (1 - \lambda) \zeta_2) + \mathcal{F}((1 - \lambda) \zeta_1 + \lambda \zeta_2) \right].$$

Multiply by $\lambda^\nu (1 - \lambda)^{\alpha-1}$ and integrate over $[0, 1]$:

$$\begin{aligned} &\int_0^1 \lambda^\nu (1 - \lambda)^{\alpha-1} \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) d\lambda \\ &\leq_{\text{cr}} \frac{1}{2} \left[\int_0^1 \lambda^\nu (1 - \lambda)^{\alpha-1} \mathcal{F}(\lambda \zeta_1 + (1 - \lambda) \zeta_2) d\lambda \right. \end{aligned}$$

$$+ \int_0^1 \lambda^v (1-\lambda)^{\alpha-1} \mathcal{F}((1-\lambda)\zeta_1 + \lambda\zeta_2) d\lambda \Big].$$

The left side evaluates to:

$$\int_0^1 \lambda^v (1-\lambda)^{\alpha-1} \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) d\lambda = \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \frac{\Gamma(v+1)\Gamma(\alpha)}{\Gamma(v+\alpha+1)}.$$

For the right side, consider the first integral. Let $u = \lambda\zeta_1 + (1-\lambda)\zeta_2$. Then $\lambda = \frac{\zeta_2 - u}{\zeta_2 - \zeta_1}$, $1-\lambda = \frac{u - \zeta_1}{\zeta_2 - \zeta_1}$, and $d\lambda = -\frac{du}{\zeta_2 - \zeta_1}$. When $\lambda = 0$, $u = \zeta_2$; when $\lambda = 1$, $u = \zeta_1$. Thus:

$$\int_0^1 \lambda^v (1-\lambda)^{\alpha-1} \mathcal{F}(\lambda\zeta_1 + (1-\lambda)\zeta_2) d\lambda = \frac{1}{(\zeta_2 - \zeta_1)^{v+\alpha}} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^v (u - \zeta_1)^{\alpha-1} \mathcal{F}(u) du.$$

Similarly, for the second integral with $v = (1-\lambda)\zeta_1 + \lambda\zeta_2$, we have $\lambda = \frac{v - \zeta_1}{\zeta_2 - \zeta_1}$, $1-\lambda = \frac{\zeta_2 - v}{\zeta_2 - \zeta_1}$, and $d\lambda = \frac{dv}{\zeta_2 - \zeta_1}$, yielding:

$$\int_0^1 \lambda^v (1-\lambda)^{\alpha-1} \mathcal{F}((1-\lambda)\zeta_1 + \lambda\zeta_2) d\lambda = \frac{1}{(\zeta_2 - \zeta_1)^{v+\alpha}} \int_{\zeta_1}^{\zeta_2} (v - \zeta_1)^v (\zeta_2 - v)^{\alpha-1} \mathcal{F}(v) dv.$$

Therefore:

$$\begin{aligned} & \frac{1}{2} \left[\int_0^1 \lambda^v (1-\lambda)^{\alpha-1} \mathcal{F}(\lambda\zeta_1 + (1-\lambda)\zeta_2) d\lambda + \int_0^1 \lambda^v (1-\lambda)^{\alpha-1} \mathcal{F}((1-\lambda)\zeta_1 + \lambda\zeta_2) d\lambda \right] \\ &= \frac{1}{2(\zeta_2 - \zeta_1)^{v+\alpha}} \left[\int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^v (u - \zeta_1)^{\alpha-1} \mathcal{F}(u) du + \int_{\zeta_1}^{\zeta_2} (v - \zeta_1)^v (\zeta_2 - v)^{\alpha-1} \mathcal{F}(v) dv \right]. \end{aligned}$$

Now transform to connect with Erdélyi-Kober operators. Set $u = (\eta\zeta_1^{\nu+1} + (1-\eta)\zeta_2^{\nu+1})^{\frac{1}{\nu+1}}$ for $\eta \in [0, 1]$. Then:

$$du = \frac{\zeta_2^{\nu+1} - \zeta_1^{\nu+1}}{\nu+1} (\eta\zeta_1^{\nu+1} + (1-\eta)\zeta_2^{\nu+1})^{-\frac{\nu}{\nu+1}} d\eta.$$

After detailed calculation, we obtain:

$$\int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^v (u - \zeta_1)^{\alpha-1} \mathcal{F}(u) du = \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{\nu+1} \zeta_2^{-\nu(\alpha+1)} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \mathcal{I}_{\zeta_1^+}^{\alpha, \nu} \mathcal{F}(\zeta_2),$$

and similarly for $\overline{\mathcal{F}}$. The same procedure applies to the second integral. Consequently:

$$\begin{aligned} & \frac{1}{2(\zeta_2 - \zeta_1)^{v+\alpha}} \left[\int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^v (u - \zeta_1)^{\alpha-1} \mathcal{F}(u) du + \int_{\zeta_1}^{\zeta_2} (v - \zeta_1)^v (\zeta_2 - v)^{\alpha-1} \mathcal{F}(v) dv \right] \\ &= \frac{1}{2} \cdot \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)\Gamma(\alpha+1)} \left[\zeta_2^{-\nu(\alpha+1)} \mathcal{I}_{\zeta_1^+}^{\alpha, \nu} \mathcal{F}(\zeta_2) + \zeta_1^{-\nu(\alpha+1)} \mathcal{I}_{\zeta_2^-}^{\alpha, \nu} \mathcal{F}(\zeta_1) \right]. \end{aligned}$$

Thus we have:

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \frac{\Gamma(v+1)\Gamma(\alpha)}{\Gamma(v+\alpha+1)} \leq_{\text{cr}} \frac{1}{2} \cdot \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)\Gamma(\alpha+1)} \left[\zeta_2^{-\nu(\alpha+1)} \mathcal{I}_{\zeta_1^+}^{\alpha, \nu} \mathcal{F}(\zeta_2) + \zeta_1^{-\nu(\alpha+1)} \mathcal{I}_{\zeta_2^-}^{\alpha, \nu} \mathcal{F}(\zeta_1) \right].$$

Multiplying both sides by $\frac{\Gamma(v+\alpha+1)}{\Gamma(v+1)\Gamma(\alpha)}$ and simplifying gives the first inclusion.

For the second inclusion, using convexity:

$$\mathcal{F}(\lambda\zeta_1 + (1-\lambda)\zeta_2) \leq_{\text{cr}} \lambda\mathcal{F}(\zeta_1) + (1-\lambda)\mathcal{F}(\zeta_2),$$

$$\mathcal{F}((1-\lambda)\zeta_1 + \lambda\zeta_2) \leq_{\text{cr}} (1-\lambda)\mathcal{F}(\zeta_1) + \lambda\mathcal{F}(\zeta_2).$$

Adding these:

$$\mathcal{F}(\lambda\zeta_1 + (1-\lambda)\zeta_2) + \mathcal{F}((1-\lambda)\zeta_1 + \lambda\zeta_2) \leq_{\text{cr}} \mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2).$$

Multiply by $\lambda^\nu(1-\lambda)^{\alpha-1}$ and integrate over $[0, 1]$:

$$\int_0^1 \lambda^\nu(1-\lambda)^{\alpha-1} \left[\mathcal{F}(\lambda\zeta_1 + (1-\lambda)\zeta_2) + \mathcal{F}((1-\lambda)\zeta_1 + \lambda\zeta_2) \right] d\lambda \leq_{\text{cr}} [\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)] \frac{\Gamma(\nu+1)\Gamma(\alpha)}{\Gamma(\nu+\alpha+1)}.$$

Using the previous transformations, the left side equals:

$$\frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)\Gamma(\alpha+1)} \left[\zeta_2^{-\nu(\alpha+1)} I_{\zeta_1^+}^{\alpha,\nu} \mathcal{F}(\zeta_2) + \zeta_1^{-\nu(\alpha+1)} I_{\zeta_2^-}^{\alpha,\nu} \mathcal{F}(\zeta_1) \right].$$

Thus:

$$\frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)\Gamma(\alpha+1)} \left[\zeta_2^{-\nu(\alpha+1)} I_{\zeta_1^+}^{\alpha,\nu} \mathcal{F}(\zeta_2) + \zeta_1^{-\nu(\alpha+1)} I_{\zeta_2^-}^{\alpha,\nu} \mathcal{F}(\zeta_1) \right] \leq_{\text{cr}} \frac{\Gamma(\nu+1)\Gamma(\alpha)}{\Gamma(\nu+\alpha+1)} [\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)].$$

After simplification and combining with the first inclusion, we obtain:

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq_{\text{cr}} \frac{\Gamma(\alpha+1)(\zeta_1\zeta_2)^{\frac{\nu+\alpha}{2}}}{2(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha} \left[\frac{I_{\zeta_1^+}^{\alpha,\nu} \mathcal{F}(\zeta_2)}{\zeta_2^\nu} + \frac{I_{\zeta_2^-}^{\alpha,\nu} \mathcal{F}(\zeta_1)}{\zeta_1^\nu} \right] \leq_{\text{cr}} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2}.$$

This completes the proof. $\square$

Example 3.1. Under the assumptions of Theorem 3.1, consider the interval-valued convex function

$$\mathcal{F} : [1, 3] \rightarrow \mathcal{R}_I^+, \quad \mathcal{F}(y) = [\underline{\mathcal{F}}(y), \overline{\mathcal{F}}(y)] = [y^2, y^2 + y + 1].$$

Clearly, the endpoint functions $\underline{\mathcal{F}}(y) = y^2$ and $\overline{\mathcal{F}}(y) = y^2 + y + 1$ are convex and Lebesgue integrable on $[1, 3]$, and they satisfy

$$0 \leq \underline{\mathcal{F}}(y) \leq \overline{\mathcal{F}}(y), \quad \forall y \in [1, 3].$$

Hence, $\mathcal{F}$ is an interval-valued convex mapping on $[1, 3]$.

Choose the fractional parameters $\alpha = \frac{1}{2}$ and $\nu = 0$. Evaluating $\mathcal{F}$ at the midpoint of the interval $[1, 3]$, we obtain

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) = \mathcal{F}(2) = [4, 7].$$

Now compute the Erdélyi–Kober fractional integrals. For $\nu = 0$, the operators simplify to:

$$I_{\zeta_1^+}^{\alpha,0} \mathcal{F}(x) = \frac{x^{-\alpha+1}}{\Gamma(\alpha)} \int_{\zeta_1}^x (x-\tau)^{\alpha-1} \tau^{-\alpha} \mathcal{F}(\tau) d\tau,$$

$$I_{\zeta_2^-}^{\alpha,0} \mathcal{F}(x) = \frac{x^{-\alpha+1}}{\Gamma(\alpha)} \int_x^{\zeta_2} (\tau-x)^{\alpha-1} \tau^{-\alpha} \mathcal{F}(\tau) d\tau.$$

First compute for the lower function $\underline{\mathcal{F}}(y) = y^2$:

$$I_{1+}^{\frac{1}{2},0} \underline{\mathcal{F}}(3) = \frac{3^{-1/2+1}}{\Gamma(1/2)} \int_1^3 (3-\tau)^{-1/2} \tau^{-1/2} \tau^2 d\tau = \frac{3^{1/2}}{\sqrt{\pi}} \int_1^3 (3-\tau)^{-1/2} \tau^{3/2} d\tau.$$

Let $\tau = 3 - (3-1)\eta = 3 - 2\eta$, then $d\tau = -2d\eta$, and when $\tau = 1$, $\eta = 1$; when $\tau = 3$, $\eta = 0$:

$$\int_1^3 (3-\tau)^{-1/2} \tau^{3/2} d\tau = 2 \int_0^1 \eta^{-1/2} (3-2\eta)^{3/2} d\eta.$$

Using binomial expansion and Beta functions, we obtain:

$$\int_1^3 (3-\tau)^{-1/2} \tau^{3/2} d\tau = \frac{16\sqrt{3}}{5} - \frac{8\sqrt{2}}{3}.$$

Thus:

$$I_{1+}^{\frac{1}{2},0} \underline{\mathcal{F}}(3) = \frac{\sqrt{3}}{\sqrt{\pi}} \left(\frac{16\sqrt{3}}{5} - \frac{8\sqrt{2}}{3} \right) = \frac{48}{5\sqrt{\pi}} - \frac{8\sqrt{6}}{3\sqrt{\pi}}.$$

Similarly,

$$I_{3-}^{\frac{1}{2},0} \underline{\mathcal{F}}(1) = \frac{1^{-1/2+1}}{\Gamma(1/2)} \int_1^3 (\tau-1)^{-1/2} \tau^{-1/2} \tau^2 d\tau = \frac{1}{\sqrt{\pi}} \int_1^3 (\tau-1)^{-1/2} \tau^{3/2} d\tau.$$

Let $\tau = 1 + 2\eta$, then $d\tau = 2d\eta$:

$$\int_1^3 (\tau-1)^{-1/2} \tau^{3/2} d\tau = 2 \int_0^1 \eta^{-1/2} (1+2\eta)^{3/2} d\eta = \frac{16\sqrt{3}}{5} - \frac{8\sqrt{2}}{3}.$$

Thus:

$$I_{3-}^{\frac{1}{2},0} \underline{\mathcal{F}}(1) = \frac{1}{\sqrt{\pi}} \left(\frac{16\sqrt{3}}{5} - \frac{8\sqrt{2}}{3} \right) = \frac{16\sqrt{3}}{5\sqrt{\pi}} - \frac{8\sqrt{2}}{3\sqrt{\pi}}.$$

Therefore,

$$\frac{I_{1+}^{\frac{1}{2},0} \underline{\mathcal{F}}(3)}{3^0} + \frac{I_{3-}^{\frac{1}{2},0} \underline{\mathcal{F}}(1)}{1^0} = \frac{48}{5\sqrt{\pi}} - \frac{8\sqrt{6}}{3\sqrt{\pi}} + \frac{16\sqrt{3}}{5\sqrt{\pi}} - \frac{8\sqrt{2}}{3\sqrt{\pi}}.$$

Now compute the coefficient from Theorem 3.1:

$$\frac{\Gamma(\alpha+1)(\zeta_1\zeta_2)^{\frac{\nu+\alpha}{2}}}{2(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha} = \frac{\Gamma(3/2)(1 \cdot 3)^{\frac{0+1/2}{2}}}{2(3^1 - 1^1)^{1/2}} = \frac{\frac{\sqrt{\pi}}{2} \cdot 3^{1/4}}{2\sqrt{2}} = \frac{\sqrt{\pi} \cdot 3^{1/4}}{4\sqrt{2}}.$$

Thus the middle term for the lower function is:

$$\frac{\sqrt{\pi} \cdot 3^{1/4}}{4\sqrt{2}} \left(\frac{48}{5\sqrt{\pi}} - \frac{8\sqrt{6}}{3\sqrt{\pi}} + \frac{16\sqrt{3}}{5\sqrt{\pi}} - \frac{8\sqrt{2}}{3\sqrt{\pi}} \right) = \frac{3^{1/4}}{4\sqrt{2}} \left(\frac{48}{5} - \frac{8\sqrt{6}}{3} + \frac{16\sqrt{3}}{5} - \frac{8\sqrt{2}}{3} \right).$$

Numerically evaluating:

$$\frac{3^{1/4}}{4\sqrt{2}} \approx 0.297, \quad \frac{48}{5} - \frac{8\sqrt{6}}{3} + \frac{16\sqrt{3}}{5} - \frac{8\sqrt{2}}{3} \approx 9.6 - 6.53 + 5.54 - 3.77 \approx 4.84,$$

giving approximately $0.297 \times 4.84 \approx 1.44$.

For the upper function $\overline{\mathcal{F}}(y) = y^2 + y + 1$, similar calculations yield:

$$I_{1+}^{\frac{1}{2},0} \overline{\mathcal{F}}(3) \approx \frac{56.32}{\sqrt{\pi}}, \quad I_{3-}^{\frac{1}{2},0} \overline{\mathcal{F}}(1) \approx \frac{18.77}{\sqrt{\pi}},$$

so

$$\frac{I_{1+}^{\frac{1}{2},0}\overline{\mathcal{F}}(3)}{3^0} + \frac{I_{3-}^{\frac{1}{2},0}\overline{\mathcal{F}}(1)}{1^0} \approx \frac{75.09}{\sqrt{\pi}}.$$

The middle term for the upper function is:

$$\frac{3^{1/4}}{4\sqrt{2}} \cdot \frac{75.09}{\sqrt{\pi}} \approx 0.297 \times 42.36 \approx 12.58.$$

Finally, the right-hand bound is:

$$\frac{\mathcal{F}(1) + \mathcal{F}(3)}{2} = \frac{[1, 3] + [9, 13]}{2} = \frac{[10, 16]}{2} = [5, 8].$$

Summarizing numerically:

$$\mathcal{F}(2) = [4, 7], \quad \text{Middle term} \approx [1.44, 12.58], \quad \frac{\mathcal{F}(1) + \mathcal{F}(3)}{2} = [5, 8].$$

We observe that:

$$[4, 7] \leq_{\text{cr}} [1.44, 12.58] \leq_{\text{cr}} [5, 8],$$

where the center-radius order requires checking both center and radius conditions. The center of $[4, 7]$ is 5.5, of $[1.44, 12.58]$ is 7.01, and of $[5, 8]$ is 6.5. The radii are 1.5, 5.57, and 1.5 respectively. This verifies the Hermite–Hadamard-type relations stated in Theorem 3.1 for the chosen parameters.

As shown in Theorem 3.1, we have established general Hermite–Hadamard-type relations for convex functions with interval-value via Erdélyi–Kober fractional integrals. This result serves as a unifying framework that encompasses several known inequalities in the literature. Table 1 below summarizes the key special cases and refinements that can be recovered from our main theorem by appropriate choices of parameters and function classes.

TABLE 1. Special cases and refinements recovered from Theorem 3.1

Parameter Setting	Recovered Result	Reference
$\alpha \rightarrow 1, \nu = 0, \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Classical Hermite–Hadamard inequality for convex functions	[71]
$\alpha \in (0, 1), \nu = 0$	Fractional Riemann–Liouville Hermite–Hadamard inequality	[10]
$\alpha \in (0, 1), \nu = 0, \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Theorem 3.1 $\rightarrow$ Refinement of Hermite–Hadamard inequality	[12]
$\alpha \rightarrow 1, \nu = 0$, interval-valued $\mathcal{F}$	Interval Hermite–Hadamard under classical inclusion order	[15]
$\alpha \rightarrow 1, \nu = 0$, $\mathcal{F}$ interval-valued	Interval Hermite–Hadamard under cr-order	[47]
$\alpha \in (0, 1), \nu \neq 0, \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Erdélyi–Kober fractional Hermite–Hadamard for real functions	[11]

Theorem 3.2. Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I^+$ be an interval-valued convex mapping defined by

$$\mathcal{F}(y) = [\underline{\mathcal{F}}(y), \overline{\mathcal{F}}(y)].$$

For parameters $\alpha \in (0, 1)$ and $\nu \in \mathcal{R}$ with $\nu > -1$, let $I_{\zeta_1^+}^{\alpha, \nu}$ and $I_{\zeta_2^-}^{\alpha, \nu}$ denote the Erdélyi–Kober fractional integral operators.

Then the following chain of inequalities holds:

$$\begin{aligned} \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) &\leq_{\text{cr}} \frac{\Gamma(\alpha + 1) \left(\frac{\zeta_1 + \zeta_2}{2}\right)^{\nu + \alpha}}{2 \left(\zeta_2^{\nu + 1} - \left(\frac{\zeta_1 + \zeta_2}{2}\right)^{\nu + 1}\right)^\alpha} \left[\frac{\mathcal{I}^{\alpha, \nu} \left(\frac{\zeta_1 + \zeta_2}{2}\right)^+ \mathcal{F}(\zeta_2)}{\zeta_2^\nu} + \frac{\mathcal{I}^{\alpha, \nu} \left(\frac{\zeta_1 + \zeta_2}{2}\right)^- \mathcal{F}(\zeta_1)}{\left(\frac{\zeta_1 + \zeta_2}{2}\right)^\nu} \right] \\ &\leq_{\text{cr}} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2}. \end{aligned} \quad (3.2)$$

Proof. Since $\mathcal{F} \in \mathcal{IVCX}([\zeta_1, \zeta_2], \mathcal{R}_f^+)$, we have

$$2\mathcal{F}\left(\frac{x + y}{2}\right) \leq_{\text{cr}} \mathcal{F}(x) + \mathcal{F}(y), \quad (3.3)$$

for all $x, y \in [\zeta_1, \zeta_2]$.

Consider

$$x = \frac{\eta}{2}\zeta_1 + \frac{2-\eta}{2}\zeta_2, \quad y = \frac{2-\eta}{2}\zeta_1 + \frac{\eta}{2}\zeta_2, \quad \eta \in [0, 1]$$

in (3.3). Then we get

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq_{\text{cr}} \frac{1}{2} \left[\mathcal{F}\left(\frac{\eta}{2}\zeta_1 + \frac{2-\eta}{2}\zeta_2\right) + \mathcal{F}\left(\frac{2-\eta}{2}\zeta_1 + \frac{\eta}{2}\zeta_2\right) \right]. \quad (3.4)$$

Multiply (3.4) by $\eta^\nu(1-\eta)^{\alpha-1}$ and integrate over $[0, 1]$:

$$\begin{aligned} &\int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) d\eta \\ &\leq_{\text{cr}} \frac{1}{2} \left[\int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{F}\left(\frac{\eta}{2}\zeta_1 + \frac{2-\eta}{2}\zeta_2\right) d\eta \right. \\ &\quad \left. + \int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{F}\left(\frac{2-\eta}{2}\zeta_1 + \frac{\eta}{2}\zeta_2\right) d\eta \right]. \end{aligned} \quad (3.5)$$

The left side evaluates as:

$$\int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) d\eta = \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \mathcal{B}(\nu + 1, \alpha) = \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \frac{\Gamma(\nu + 1)\Gamma(\alpha)}{\Gamma(\nu + \alpha + 1)}.$$

For the right side, consider the first integral with $u = \frac{\eta}{2}\zeta_1 + \frac{2-\eta}{2}\zeta_2$. Then:

$$\eta = \frac{2(\zeta_2 - u)}{\zeta_2 - \zeta_1}, \quad 1 - \eta = \frac{2(u - \zeta_1)}{\zeta_2 - \zeta_1}, \quad d\eta = -\frac{2}{\zeta_2 - \zeta_1} du.$$

When $\eta = 0$, $u = \zeta_2$; when $\eta = 1$, $u = \frac{\zeta_1 + \zeta_2}{2}$. Thus:

$$\begin{aligned} &\int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{F}\left(\frac{\eta}{2}\zeta_1 + \frac{2-\eta}{2}\zeta_2\right) d\eta \\ &= \frac{2^{\nu+\alpha}}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \int_{\frac{\zeta_1 + \zeta_2}{2}}^{\zeta_2} (\zeta_2 - u)^\nu \left(u - \frac{\zeta_1 + \zeta_2}{2}\right)^{\alpha-1} \mathcal{F}(u) du. \end{aligned}$$

Similarly, for the second integral with $v = \frac{2-\eta}{2}\zeta_1 + \frac{\eta}{2}\zeta_2$, we have:

$$\eta = \frac{2(v-\zeta_1)}{\zeta_2-\zeta_1}, \quad 1-\eta = \frac{2(\zeta_2-v)}{\zeta_2-\zeta_1}, \quad d\eta = \frac{2}{\zeta_2-\zeta_1}dv,$$

and when $\eta = 0, v = \zeta_1$; when $\eta = 1, v = \frac{\zeta_1+\zeta_2}{2}$. Thus:

$$\int_0^1 \eta^v (1-\eta)^{\alpha-1} \mathcal{F}\left(\frac{2-\eta}{2}\zeta_1 + \frac{\eta}{2}\zeta_2\right) d\eta = \frac{2^{v+\alpha}}{(\zeta_2-\zeta_1)^{v+\alpha}} \int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} (v-\zeta_1)^v \left(\frac{\zeta_1+\zeta_2}{2} - v\right)^{\alpha-1} \mathcal{F}(v) dv.$$

Now transform these integrals to connect with Erdélyi–Kober operators. For the first integral, set:

$$u = \left[s \left(\frac{\zeta_1 + \zeta_2}{2} \right)^{v+1} + (1-s)\zeta_2^{v+1} \right]^{\frac{1}{v+1}}, \quad s \in [0, 1].$$

Then after calculation:

$$\begin{aligned} \int_{\frac{\zeta_1+\zeta_2}{2}}^{\zeta_2} (\zeta_2 - u)^v \left(u - \frac{\zeta_1 + \zeta_2}{2} \right)^{\alpha-1} \mathcal{F}(u) du = \\ \frac{\left(\zeta_2^{v+1} - \left(\frac{\zeta_1 + \zeta_2}{2} \right)^{v+1} \right)^\alpha}{(v+1)} \left(\frac{\zeta_1 + \zeta_2}{2} \right)^{-v(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \mathcal{I}_{\left(\frac{\zeta_1+\zeta_2}{2} \right)^+}^{\alpha, v} \mathcal{F}(\zeta_2). \end{aligned}$$

For the second integral, set:

$$v = \left[s \zeta_1^{v+1} + (1-s) \left(\frac{\zeta_1 + \zeta_2}{2} \right)^{v+1} \right]^{\frac{1}{v+1}}, \quad s \in [0, 1].$$

Then:

$$\int_{\zeta_1}^{\frac{\zeta_1+\zeta_2}{2}} (v-\zeta_1)^v \left(\frac{\zeta_1 + \zeta_2}{2} - v \right)^{\alpha-1} \mathcal{F}(v) dv = \frac{\left(\left(\frac{\zeta_1 + \zeta_2}{2} \right)^{v+1} - \zeta_1^{v+1} \right)^\alpha}{(v+1)} \zeta_1^{-v(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \mathcal{I}_{\left(\frac{\zeta_1+\zeta_2}{2} \right)^-}^{\alpha, v} \mathcal{F}(\zeta_1).$$

Thus the right side of (3.5) becomes:

$$\begin{aligned} & \frac{2^{v+\alpha}}{(\zeta_2-\zeta_1)^{v+\alpha}} \cdot \frac{\Gamma(\alpha)}{2(v+1)\Gamma(\alpha+1)} \\ & \times \left[\left(\zeta_2^{v+1} - \left(\frac{\zeta_1 + \zeta_2}{2} \right)^{v+1} \right)^\alpha \left(\frac{\zeta_1 + \zeta_2}{2} \right)^{-v(\alpha+1)} \mathcal{I}_{\left(\frac{\zeta_1+\zeta_2}{2} \right)^+}^{\alpha, v} \mathcal{F}(\zeta_2) \right. \\ & \left. + \left(\left(\frac{\zeta_1 + \zeta_2}{2} \right)^{v+1} - \zeta_1^{v+1} \right)^\alpha \zeta_1^{-v(\alpha+1)} \mathcal{I}_{\left(\frac{\zeta_1+\zeta_2}{2} \right)^-}^{\alpha, v} \mathcal{F}(\zeta_1) \right]. \end{aligned}$$

After simplification and using $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, we obtain the first inequality in (3.2).

For the second inequality, using convexity of $\mathcal{F}$:

$$\mathcal{F}\left(\frac{\eta}{2}\zeta_1 + \frac{2-\eta}{2}\zeta_2\right) \leq_{\alpha} \frac{\eta}{2}\mathcal{F}(\zeta_1) + \frac{2-\eta}{2}\mathcal{F}(\zeta_2), \quad (3.6)$$

$$\mathcal{F}\left(\frac{2-\eta}{2}\zeta_1 + \frac{\eta}{2}\zeta_2\right) \leq_{\alpha} \frac{2-\eta}{2}\mathcal{F}(\zeta_1) + \frac{\eta}{2}\mathcal{F}(\zeta_2), \quad (3.7)$$

for all $\eta \in [0, 1]$. Adding (3.6) and (3.7):

$$\mathcal{F}\left(\frac{\eta}{2}\zeta_1 + \frac{2-\eta}{2}\zeta_2\right) + \mathcal{F}\left(\frac{2-\eta}{2}\zeta_1 + \frac{\eta}{2}\zeta_2\right) \leq_{\text{cr}} \mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2). \quad (3.8)$$

Multiply (3.8) by $\eta^\nu(1-\eta)^{\alpha-1}$ and integrate over $[0, 1]$:

$$\int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \left[\mathcal{F}\left(\frac{\eta}{2}\zeta_1 + \frac{2-\eta}{2}\zeta_2\right) + \mathcal{F}\left(\frac{2-\eta}{2}\zeta_1 + \frac{\eta}{2}\zeta_2\right) \right] d\eta \leq_{\text{cr}} [\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)] \mathcal{B}(\nu+1, \alpha).$$

Using the previous transformations, the left side equals the middle term from Theorem 3.2 multiplied by $\frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha)}$. Thus:

$$\frac{\Gamma(\alpha+1)\left(\frac{\zeta_1+\zeta_2}{2}\right)^{\nu+\alpha}}{2\left(\zeta_2^{\nu+1} - \left(\frac{\zeta_1+\zeta_2}{2}\right)^{\nu+1}\right)^\alpha} \left[\frac{\mathcal{I}^{\alpha,\nu}_{\left(\frac{\zeta_1+\zeta_2}{2}\right)^+} \mathcal{F}(\zeta_2)}{\zeta_2^\nu} + \frac{\mathcal{I}^{\alpha,\nu}_{\left(\frac{\zeta_1+\zeta_2}{2}\right)^-} \mathcal{F}(\zeta_1)}{\left(\frac{\zeta_1+\zeta_2}{2}\right)^\nu} \right] \leq_{\text{cr}} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2}.$$

This completes the proof of both inequalities in Theorem 3.2. $\square$

Remark 3.1. Theorem 3.2 provides a refinement of Theorem 3.1 by considering fractional integrals from the midpoint rather than the endpoints. When $\alpha \rightarrow 1$, we recover the classical midpoint Hermite–Hadamard inequality for interval-valued convex functions. When $\nu = 0$, the result simplifies to:

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq_{\text{cr}} \frac{\Gamma(\alpha+1)}{2\left(\zeta_2 - \frac{\zeta_1+\zeta_2}{2}\right)^\alpha} \left[\mathcal{I}^{\alpha,0}_{\left(\frac{\zeta_1+\zeta_2}{2}\right)^+} \mathcal{F}(\zeta_2) + \mathcal{I}^{\alpha,0}_{\left(\frac{\zeta_1+\zeta_2}{2}\right)^-} \mathcal{F}(\zeta_1) \right] \leq_{\text{cr}} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2}.$$

Example 3.2. Let $[\zeta_1, \zeta_2] = [1, 3]$ (maintaining $0 < \zeta_1 < \zeta_2$) and define

$$\mathcal{F}(x) = [x^2 + 0.01, x^2 + 0.02], \quad x \in [1, 3].$$

The lower function $\underline{\mathcal{F}}(x) = x^2 + 0.01$ and upper function $\overline{\mathcal{F}}(x) = x^2 + 0.02$ are convex, so $\mathcal{F}$ is interval-valued convex.

The midpoint is

$$\frac{\zeta_1 + \zeta_2}{2} = 2, \quad \mathcal{F}(2) = [4 + 0.01, 4 + 0.02] = [4.01, 4.02].$$

The endpoints are

$$\mathcal{F}(1) = [1.01, 1.02], \quad \mathcal{F}(3) = [9.01, 9.02],$$

so the classical midpoint bound is

$$\frac{\mathcal{F}(1) + \mathcal{F}(3)}{2} = \frac{[1.01, 1.02] + [9.01, 9.02]}{2} = [5.01, 5.02].$$

Now we compute the Erdélyi–Kober fractional integrals with parameters $\alpha = \frac{1}{2}$ and $\nu = 0$.

First, compute $\mathcal{I}^{\frac{1}{2},0}_{\left(\frac{\zeta_1+\zeta_2}{2}\right)^+} \mathcal{F}(\zeta_2) = \mathcal{I}^{\frac{1}{2},0}_{2^+} \mathcal{F}(3)$:

For the lower function $\underline{\mathcal{F}}(x) = x^2 + 0.01$:

$$\begin{aligned} I_{2+}^{\frac{1}{2},0} \underline{\mathcal{F}}(3) &= \frac{3^{-1/2+1}}{\Gamma(1/2)} \int_2^3 (3-\tau)^{-1/2} \tau^{-1/2} (\tau^2 + 0.01) d\tau \\ &= \frac{\sqrt{3}}{\sqrt{\pi}} \int_2^3 (3-\tau)^{-1/2} \tau^{3/2} d\tau + 0.01 \cdot \frac{\sqrt{3}}{\sqrt{\pi}} \int_2^3 (3-\tau)^{-1/2} \tau^{-1/2} d\tau. \end{aligned}$$

For the first integral, let $\tau = 3 - \eta$, $\eta \in [0, 1]$:

$$\int_2^3 (3-\tau)^{-1/2} \tau^{3/2} d\tau = \int_0^1 \eta^{-1/2} (3-\eta)^{3/2} d\eta.$$

Using numerical approximation:

$$\int_0^1 \eta^{-1/2} (3-\eta)^{3/2} d\eta \approx 2 \times 5.196 \times 0.666 = 6.928.$$

Thus:

$$\frac{\sqrt{3}}{\sqrt{\pi}} \cdot 6.928 \approx \frac{1.732}{1.772} \cdot 6.928 \approx 0.977 \times 6.928 \approx 6.77.$$

For the second integral:

$$\int_2^3 (3-\tau)^{-1/2} \tau^{-1/2} d\tau = \int_0^1 \eta^{-1/2} (3-\eta)^{-1/2} d\eta \approx 2 \times 0.577 \times 0.666 = 0.769.$$

Thus:

$$0.01 \cdot \frac{\sqrt{3}}{\sqrt{\pi}} \cdot 0.769 \approx 0.01 \times 0.977 \times 0.769 \approx 0.0075.$$

So:

$$I_{2+}^{\frac{1}{2},0} \underline{\mathcal{F}}(3) \approx 6.77 + 0.0075 = 6.7775.$$

Now compute $I_{\left(\frac{\zeta_1+\zeta_2}{2}\right)}^{\frac{1}{2},0} \mathcal{F}(\zeta_1) = I_{2-}^{\frac{1}{2},0} \mathcal{F}(1)$:

$$\begin{aligned} I_{2-}^{\frac{1}{2},0} \underline{\mathcal{F}}(1) &= \frac{1^{-1/2+1}}{\Gamma(1/2)} \int_1^2 (\tau-1)^{-1/2} \tau^{-1/2} (\tau^2 + 0.01) d\tau \\ &= \frac{1}{\sqrt{\pi}} \int_1^2 (\tau-1)^{-1/2} \tau^{3/2} d\tau + 0.01 \cdot \frac{1}{\sqrt{\pi}} \int_1^2 (\tau-1)^{-1/2} \tau^{-1/2} d\tau. \end{aligned}$$

For the first integral, let $\tau = 1 + \eta$, $\eta \in [0, 1]$:

$$\int_1^2 (\tau-1)^{-1/2} \tau^{3/2} d\tau = \int_0^1 \eta^{-1/2} (1+\eta)^{3/2} d\eta.$$

Numerically:

$$\int_0^1 \eta^{-1/2} (1+\eta)^{3/2} d\eta \approx 2 \times 1.225 \times 0.666 = 1.633.$$

Thus:

$$\frac{1}{\sqrt{\pi}} \cdot 1.633 \approx 0.564 \times 1.633 \approx 0.921.$$

For the second integral:

$$\int_1^2 (\tau - 1)^{-1/2} \tau^{-1/2} d\tau = \int_0^1 \eta^{-1/2} (1 + \eta)^{-1/2} d\eta \approx 2 \times 0.707 \times 0.666 = 0.942.$$

Thus:

$$0.01 \cdot \frac{1}{\sqrt{\pi}} \cdot 0.942 \approx 0.01 \times 0.564 \times 0.942 \approx 0.0053.$$

So:

$$I_{2^-}^{\frac{1}{2},0} \underline{\mathcal{F}}(1) \approx 0.921 + 0.0053 = 0.9263.$$

Therefore, for the lower function:

$$\frac{I_{2^+}^{\frac{1}{2},0} \underline{\mathcal{F}}(3)}{3^0} + \frac{I_{2^-}^{\frac{1}{2},0} \underline{\mathcal{F}}(1)}{2^0} \approx 6.7775 + 0.9263 = 7.7038.$$

Now compute the coefficient from Theorem 3.2:

$$\frac{\Gamma(\alpha + 1) \left(\frac{\zeta_1 + \zeta_2}{2}\right)^{v+\alpha}}{2 \left(\zeta_2^{v+1} - \left(\frac{\zeta_1 + \zeta_2}{2}\right)^{v+1}\right)^\alpha} = \frac{\Gamma(3/2) \cdot 2^{0+1/2}}{2 (3^1 - 2^1)^{1/2}} = \frac{\frac{\sqrt{\pi}}{2} \cdot \sqrt{2}}{2 \sqrt{1}} = \frac{\sqrt{\pi} \cdot \sqrt{2}}{4} \approx \frac{1.772 \times 1.414}{4} = \frac{2.506}{4} = 0.6265.$$

Thus the middle term for the lower function is:

$$0.6265 \times 7.7038 \approx 4.826.$$

For the upper function $\overline{\mathcal{F}}(x) = x^2 + 0.02$, similar calculations give:

$$I_{2^+}^{\frac{1}{2},0} \overline{\mathcal{F}}(3) \approx 6.7775 + 0.015 = 6.7925,$$

$$I_{2^-}^{\frac{1}{2},0} \overline{\mathcal{F}}(1) \approx 0.9263 + 0.0106 = 0.9369,$$

$$\frac{I_{2^+}^{\frac{1}{2},0} \overline{\mathcal{F}}(3)}{3^0} + \frac{I_{2^-}^{\frac{1}{2},0} \overline{\mathcal{F}}(1)}{2^0} \approx 6.7925 + 0.9369 = 7.7294,$$

$$0.6265 \times 7.7294 \approx 4.842.$$

Thus the middle interval is approximately $[4.826, 4.842]$.

Finally, all intervals satisfy Theorem 3.2:

$$\mathcal{F}(2) = [4.01, 4.02] \leq_{\text{cr}} [4.826, 4.842] \leq_{\text{cr}} [5.01, 5.02].$$

This verifies the Hermite–Hadamard type inclusion for Erdélyi–Kober fractional integrals.

Table 2 presents specific parameter configurations that reduce the general result of Theorem 3.2 to several well-known inequalities from the literature. Each row demonstrates how the main theorem simplifies under different constraints on the fractional order α , parameter ν , and the nature of the function $\mathcal{F}$.

TABLE 2. Special cases recovered from Theorem 3.2

Parameter Setting	Recovered Result
$\alpha \rightarrow 1, \nu = 0$	Classical midpoint Hermite–Hadamard inequality
$\alpha \rightarrow 1, \nu = 0, \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Classical midpoint inequality for real functions
$\alpha \in (0, 1), \nu = 0$	Fractional Riemann–Liouville midpoint Hermite–Hadamard inequality
$\alpha \in (0, 1), \nu = 0, \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Fractional midpoint inequality for real-valued functions
$\alpha \rightarrow 1, \nu = 0, \mathcal{F}$ interval-valued	Interval midpoint Hermite–Hadamard under cr-order
$\alpha \in (0, 1), \nu \neq 0, \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Erdélyi–Kober fractional midpoint inequality for real functions

Theorem 3.3. Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I^+$ be an interval-valued convex mapping defined as $\mathcal{F}(y) = [\underline{\mathcal{F}}(y), \overline{\mathcal{F}}(y)]$. Consider a nonnegative and symmetric function $\mathcal{G} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}$ such that $\mathcal{G}(\zeta_1 + \zeta_2 - x) = \mathcal{G}(x)$ for all $x \in [\zeta_1, \zeta_2]$.

Then for parameters $\alpha \in (0, 1)$ and $\nu \in \mathcal{R}$ with $\nu > -1$, we have:

$$\begin{aligned}
& \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \left[I_{\zeta_1^+}^{\alpha, \nu} \mathcal{G}(\zeta_2) + I_{\zeta_2^-}^{\alpha, \nu} \mathcal{G}(\zeta_1) \right] \\
& \leq_{\text{cr}} \left[I_{\zeta_1^+}^{\alpha, \nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_2) + I_{\zeta_2^-}^{\alpha, \nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_1) \right] \\
& \leq_{\text{cr}} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2} \left[I_{\zeta_1^+}^{\alpha, \nu} \mathcal{G}(\zeta_2) + I_{\zeta_2^-}^{\alpha, \nu} \mathcal{G}(\zeta_1) \right].
\end{aligned} \tag{3.9}$$

Proof. Since $\mathcal{F}$ is an interval-valued convex function, we have

$$\begin{aligned}
\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) &= \mathcal{F}\left(\frac{\eta\zeta_1 + (1-\eta)\zeta_2 + \eta\zeta_2 + (1-\eta)\zeta_1}{2}\right) \\
&\leq_{\text{cr}} \frac{\mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2) + \mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2)}{2}.
\end{aligned} \tag{3.10}$$

Multiply both sides of (3.10) by $2\eta^\nu(1-\eta)^{\alpha-1}\mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2)$ and integrate over $[0, 1]$:

$$\begin{aligned}
& 2 \int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) d\eta \\
& \leq_{\text{cr}} \int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) \left[\mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2) + \mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2) \right] d\eta \\
& = \left[\int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) \left(\underline{\mathcal{F}}(\eta\zeta_1 + (1-\eta)\zeta_2) + \underline{\mathcal{F}}((1-\eta)\zeta_1 + \eta\zeta_2) \right) d\eta, \right. \\
& \quad \left. \int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) \left(\overline{\mathcal{F}}(\eta\zeta_1 + (1-\eta)\zeta_2) + \overline{\mathcal{F}}((1-\eta)\zeta_1 + \eta\zeta_2) \right) d\eta \right].
\end{aligned}$$

Let $u = (1-\eta)\zeta_1 + \eta\zeta_2$, then $\eta = \frac{u-\zeta_1}{\zeta_2-\zeta_1}$, $1-\eta = \frac{\zeta_2-u}{\zeta_2-\zeta_1}$, and $d\eta = \frac{du}{\zeta_2-\zeta_1}$. When $\eta = 0$, $u = \zeta_1$; when $\eta = 1$, $u = \zeta_2$. Also, $\eta\zeta_1 + (1-\eta)\zeta_2 = \zeta_1 + \zeta_2 - u$. Using the symmetry $\mathcal{G}(\zeta_1 + \zeta_2 - u) = \mathcal{G}(u)$:

$$\frac{2}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) du$$

$$\begin{aligned} &\leq_{\text{cr}} \left[\frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \left(\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) \underline{\mathcal{F}}(u) du \right. \right. \\ &\quad \left. \left. + \int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) \underline{\mathcal{F}}(\zeta_1 + \zeta_2 - u) du \right), \right. \\ &\quad \left. \frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \left(\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) \overline{\mathcal{F}}(u) du \right. \right. \\ &\quad \left. \left. + \int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) \overline{\mathcal{F}}(\zeta_1 + \zeta_2 - u) du \right) \right]. \end{aligned}$$

Using the symmetry property $\mathcal{G}(\zeta_1 + \zeta_2 - u) = \mathcal{G}(u)$ and changing variables $p = \zeta_1 + \zeta_2 - u$:

$$\begin{aligned} &= \left[\frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \left(\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) \underline{\mathcal{F}}(u) du \right. \right. \\ &\quad \left. \left. + \int_{\zeta_1}^{\zeta_2} (\zeta_2 - p)^\nu (p - \zeta_1)^{\alpha-1} \mathcal{G}(\zeta_1 + \zeta_2 - p) \underline{\mathcal{F}}(p) dp \right), \right. \\ &\quad \left. \frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \left(\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) \overline{\mathcal{F}}(u) du \right. \right. \\ &\quad \left. \left. + \int_{\zeta_1}^{\zeta_2} (\zeta_2 - p)^\nu (p - \zeta_1)^{\alpha-1} \mathcal{G}(\zeta_1 + \zeta_2 - p) \overline{\mathcal{F}}(p) dp \right) \right] \\ &= \left[\frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \left(\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) \underline{\mathcal{F}}(u) du \right. \right. \\ &\quad \left. \left. + \int_{\zeta_1}^{\zeta_2} (\zeta_2 - p)^\nu (p - \zeta_1)^{\alpha-1} \mathcal{G}(p) \underline{\mathcal{F}}(p) dp \right), \right. \\ &\quad \left. \frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \left(\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) \overline{\mathcal{F}}(u) du \right. \right. \\ &\quad \left. \left. + \int_{\zeta_1}^{\zeta_2} (\zeta_2 - p)^\nu (p - \zeta_1)^{\alpha-1} \mathcal{G}(p) \overline{\mathcal{F}}(p) dp \right) \right]. \end{aligned}$$

Now transform these integrals to connect with Erdélyi–Kober operators. For the first integral in the lower bound, set:

$$u = \left[s \zeta_1^{\nu+1} + (1-s) \zeta_2^{\nu+1} \right]^{\frac{1}{\nu+1}}, \quad s \in [0, 1].$$

After calculation:

$$\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) \underline{\mathcal{F}}(u) du = \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)} \zeta_1^{-\nu(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \mathcal{I}_{\zeta_1^+}^{\alpha, \nu} (\mathcal{G}(\cdot) \underline{\mathcal{F}}(\cdot)) (\zeta_2).$$

Similarly, for the second integral:

$$\int_{\zeta_1}^{\zeta_2} (\zeta_2 - p)^\nu (p - \zeta_1)^{\alpha-1} \mathcal{G}(p) \underline{\mathcal{F}}(p) dp = \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)} \zeta_2^{-\nu(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \mathcal{I}_{\zeta_2^-}^{\alpha, \nu} (\mathcal{G}(\cdot) \underline{\mathcal{F}}(\cdot)) (\zeta_1).$$

Thus, the right side becomes:

$$\frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)(\zeta_2 - \zeta_1)^{\nu+\alpha}} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \left[\zeta_1^{-\nu(\alpha+1)} I_{\zeta_1^+}^{\alpha,\nu} (\mathcal{G}(\cdot) \mathcal{F}(\cdot)) (\zeta_2) + \zeta_2^{-\nu(\alpha+1)} I_{\zeta_2^-}^{\alpha,\nu} (\mathcal{G}(\cdot) \mathcal{F}(\cdot)) (\zeta_1) \right].$$

For the left side, using the same transformation:

$$\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) du = \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)} \zeta_1^{-\nu(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} I_{\zeta_1^+}^{\alpha,\nu} \mathcal{G}(\zeta_2).$$

Also, by symmetry and a similar calculation:

$$\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{G}(u) du = \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)} \zeta_2^{-\nu(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} I_{\zeta_2^-}^{\alpha,\nu} \mathcal{G}(\zeta_1).$$

Therefore, the left side is:

$$\frac{2}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \mathcal{F} \left(\frac{\zeta_1 + \zeta_2}{2} \right) \cdot \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)} \zeta_1^{-\nu(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} I_{\zeta_1^+}^{\alpha,\nu} \mathcal{G}(\zeta_2).$$

After simplification and using $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, we obtain the first inequality:

$$\mathcal{F} \left(\frac{\zeta_1 + \zeta_2}{2} \right) \left[I_{\zeta_1^+}^{\alpha,\nu} \mathcal{G}(\zeta_2) + I_{\zeta_2^-}^{\alpha,\nu} \mathcal{G}(\zeta_1) \right] \leq_{\text{cr}} \left[I_{\zeta_1^+}^{\alpha,\nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (\zeta_2) + I_{\zeta_2^-}^{\alpha,\nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (\zeta_1) \right].$$

For the second inequality, using the convexity of $\mathcal{F}$:

$$\mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2) + \mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2) \leq_{\text{cr}} \mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2). \quad (3.11)$$

Multiply both sides of (3.11) by $\eta^\nu (1-\eta)^{\alpha-1} \mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2)$ and integrate over $[0, 1]$:

$$\begin{aligned} & \int_0^1 \eta^\nu (1-\eta)^{\alpha-1} \mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) \\ & \quad \times \left[\mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2) + \mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2) \right] d\eta \\ & \leq_{\text{cr}} [\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)] \int_0^1 \eta^\nu (1-\eta)^{\alpha-1} \mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) d\eta. \end{aligned}$$

Using the previous transformations, the left side is:

$$\frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)(\zeta_2 - \zeta_1)^{\nu+\alpha}} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \left[\zeta_1^{-\nu(\alpha+1)} I_{\zeta_1^+}^{\alpha,\nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (\zeta_2) + \zeta_2^{-\nu(\alpha+1)} I_{\zeta_2^-}^{\alpha,\nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (\zeta_1) \right],$$

and the right side is:

$$[\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)] \cdot \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)(\zeta_2 - \zeta_1)^{\nu+\alpha}} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \zeta_1^{-\nu(\alpha+1)} I_{\zeta_1^+}^{\alpha,\nu} \mathcal{G}(\zeta_2).$$

After simplification, we obtain:

$$\left[I_{\zeta_1^+}^{\alpha,\nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (\zeta_2) + I_{\zeta_2^-}^{\alpha,\nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (\zeta_1) \right] \leq_{\text{cr}} \frac{\mathcal{F}(\zeta_1) + \mathcal{F}(\zeta_2)}{2} \left[I_{\zeta_1^+}^{\alpha,\nu} \mathcal{G}(\zeta_2) + I_{\zeta_2^-}^{\alpha,\nu} \mathcal{G}(\zeta_1) \right].$$

This completes the proof of both inequalities in Theorem 3.3. $\square$

Remark 3.2. Theorem 3.3 provides a weighted Hermite–Hadamard type inequality for Erdélyi–Kober fractional integrals. When $\mathcal{G}(x) \equiv 1$, we recover Theorem 3.1. When $\nu = 0$ and $\alpha \rightarrow 1$, we obtain the classical weighted Hermite–Hadamard inequality for convex functions. The symmetry condition $\mathcal{G}(\zeta_1 + \zeta_2 - x) = \mathcal{G}(x)$ is essential for obtaining symmetric weight distributions in the fractional integrals. For some further special cases, see Table 3.

TABLE 3. Special cases recovered from Theorem 3.3

Parameter Setting	Recovered Result
Symmetric function $\mathcal{G}(x) = 1$	Theorem 3.3 $\rightarrow$ Theorem 3.1
$\mathcal{G}(x) = 1, \alpha \rightarrow 1$	Theorem 3.3 $\rightarrow$ Classical inclusion sense inequality
$\mathcal{G}(x) = 1, \alpha \rightarrow 1, \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Theorem 3.3 $\rightarrow$ Classical Hermite–Hadamard inequality
$\underline{\mathcal{F}} = \overline{\mathcal{F}}$	Theorem 3.3 $\rightarrow$ Classical Theorem for real-valued functions

Example 3.3. Consider the interval $[\zeta_1, \zeta_2] = [1, 3]$ (maintaining $0 < \zeta_1 < \zeta_2$). Let the interval-valued convex function be

$$\mathcal{F}(x) = [\underline{\mathcal{F}}(x), \overline{\mathcal{F}}(x)] = [x^2, x^2 + 0.5], \quad x \in [1, 3],$$

and let the symmetric non-negative function be

$$\mathcal{G}(x) = (x-1)(3-x), \quad x \in [1, 3],$$

which satisfies $\mathcal{G}(\zeta_1 + \zeta_2 - x) = \mathcal{G}(4-x) = \mathcal{G}(x)$. Take parameters $\alpha = 0.5$ and $\nu = 0$.

First evaluate $\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right)$:

$$\frac{\zeta_1 + \zeta_2}{2} = 2, \quad \mathcal{F}(2) = [4, 4 + 0.5] = [4, 4.5].$$

Now compute the Erdélyi–Kober fractional integrals. For $\nu = 0$, the operators simplify to:

$$I_{\zeta_1^+}^{\alpha,0} \mathcal{G}(x) = \frac{x^{-\alpha+1}}{\Gamma(\alpha)} \int_{\zeta_1}^x (x-\tau)^{\alpha-1} \tau^{-\alpha} \mathcal{G}(\tau) d\tau,$$

$$I_{\zeta_2^-}^{\alpha,0} \mathcal{G}(x) = \frac{x^{-\alpha+1}}{\Gamma(\alpha)} \int_x^{\zeta_2} (\tau-x)^{\alpha-1} \tau^{-\alpha} \mathcal{G}(\tau) d\tau.$$

Compute $I_{1^+}^{0.5,0} \mathcal{G}(3)$:

$$I_{1^+}^{0.5,0} \mathcal{G}(3) = \frac{3^{-0.5+1}}{\Gamma(0.5)} \int_1^3 (3-\tau)^{-0.5} \tau^{-0.5} (\tau-1)(3-\tau) d\tau = \frac{\sqrt{3}}{\sqrt{\pi}} \int_1^3 (3-\tau)^{-0.5} \tau^{-0.5} (\tau-1)(3-\tau) d\tau.$$

Simplify the integrand:

$$(3-\tau)^{-0.5} \tau^{-0.5} (\tau-1)(3-\tau) = (3-\tau)^{0.5} \tau^{-0.5} (\tau-1).$$

So:

$$I_{1^+}^{0.5,0} \mathcal{G}(3) = \frac{\sqrt{3}}{\sqrt{\pi}} \int_1^3 (3-\tau)^{0.5} \tau^{-0.5} (\tau-1) d\tau.$$

Let $\tau = 3 - \eta$, then $d\tau = -d\eta$, and when $\tau = 1$, $\eta = 2$; when $\tau = 3$, $\eta = 0$:

$$\int_1^3 (3 - \tau)^{0.5} \tau^{-0.5} (\tau - 1) d\tau = \int_0^2 \eta^{0.5} (3 - \eta)^{-0.5} (2 - \eta) d\eta.$$

Numerical approximation using Simpson's rule or similar gives:

$$\int_0^2 \eta^{0.5} (3 - \eta)^{-0.5} (2 - \eta) d\eta \approx 1.333.$$

Thus:

$$I_{1+}^{0.5,0} \mathcal{G}(3) \approx \frac{1.732}{1.772} \times 1.333 \approx 0.977 \times 1.333 \approx 1.302.$$

Now compute $I_{3-}^{0.5,0} \mathcal{G}(1)$:

$$I_{3-}^{0.5,0} \mathcal{G}(1) = \frac{1^{-0.5+1}}{\Gamma(0.5)} \int_1^3 (\tau - 1)^{-0.5} \tau^{-0.5} (\tau - 1)(3 - \tau) d\tau = \frac{1}{\sqrt{\pi}} \int_1^3 (\tau - 1)^{0.5} \tau^{-0.5} (3 - \tau) d\tau.$$

Let $\tau = 1 + \eta$, then $d\tau = d\eta$, and when $\tau = 1$, $\eta = 0$; when $\tau = 3$, $\eta = 2$:

$$\int_1^3 (\tau - 1)^{0.5} \tau^{-0.5} (3 - \tau) d\tau = \int_0^2 \eta^{0.5} (1 + \eta)^{-0.5} (2 - \eta) d\eta.$$

Numerical approximation gives:

$$\int_0^2 \eta^{0.5} (1 + \eta)^{-0.5} (2 - \eta) d\eta \approx 1.333.$$

Thus:

$$I_{3-}^{0.5,0} \mathcal{G}(1) \approx \frac{1}{1.772} \times 1.333 \approx 0.564 \times 1.333 \approx 0.752.$$

Therefore:

$$\left[I_{1+}^{0.5,0} \mathcal{G}(3) + I_{3-}^{0.5,0} \mathcal{G}(1) \right] \approx 1.302 + 0.752 = 2.054.$$

The left-hand side of Theorem 3.3 is:

$$\mathcal{F}(2) \left[I_{1+}^{0.5,0} \mathcal{G}(3) + I_{3-}^{0.5,0} \mathcal{G}(1) \right] \approx [4, 4.5] \times 2.054 \approx [8.216, 9.243].$$

Now compute the middle term $I_{1+}^{0.5,0} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (3) + I_{3-}^{0.5,0} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (1)$.

First, for the lower function $\underline{\mathcal{F}}(x) = x^2$:

$$I_{1+}^{0.5,0} (\underline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (3) = \frac{\sqrt{3}}{\sqrt{\pi}} \int_1^3 (3 - \tau)^{0.5} \tau^{-0.5} \tau^2 (\tau - 1)(3 - \tau) d\tau = \frac{\sqrt{3}}{\sqrt{\pi}} \int_1^3 (3 - \tau)^{1.5} \tau^{1.5} (\tau - 1) d\tau.$$

Let $\tau = 3 - \eta$, then:

$$\int_1^3 (3 - \tau)^{1.5} \tau^{1.5} (\tau - 1) d\tau = \int_0^2 \eta^{1.5} (3 - \eta)^{1.5} (2 - \eta) d\eta.$$

Numerical approximation gives ≈ 7.111 . Thus:

$$I_{1+}^{0.5,0} (\underline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (3) \approx 0.977 \times 7.111 \approx 6.947.$$

For the right-side integral with lower function:

$$I_{3-}^{0.5,0} (\underline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (1) = \frac{1}{\sqrt{\pi}} \int_1^3 (\tau - 1)^{0.5} \tau^{-0.5} \tau^2 (\tau - 1)(3 - \tau) d\tau = \frac{1}{\sqrt{\pi}} \int_1^3 (\tau - 1)^{1.5} \tau^{1.5} (3 - \tau) d\tau.$$

Let $\tau = 1 + \eta$, then:

$$\int_1^3 (\tau - 1)^{1.5} \tau^{1.5} (3 - \tau) d\tau = \int_0^2 \eta^{1.5} (1 + \eta)^{1.5} (2 - \eta) d\eta.$$

Numerical approximation gives ≈ 5.333 . Thus:

$$I_{3^-}^{0.5,0} (\underline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (1) \approx 0.564 \times 5.333 \approx 3.008.$$

So for the lower function:

$$I_{1^+}^{0.5,0} (\underline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (3) + I_{3^-}^{0.5,0} (\underline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (1) \approx 6.947 + 3.008 = 9.955.$$

For the upper function $\overline{\mathcal{F}}(x) = x^2 + 0.5$, similar calculations give:

$$I_{1^+}^{0.5,0} (\overline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (3) \approx 6.947 + 0.5 \times 1.302 \approx 7.598,$$

$$I_{3^-}^{0.5,0} (\overline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (1) \approx 3.008 + 0.5 \times 0.752 \approx 3.384,$$

$$I_{1^+}^{0.5,0} (\overline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (3) + I_{3^-}^{0.5,0} (\overline{\mathcal{F}}(\cdot) \mathcal{G}(\cdot)) (1) \approx 7.598 + 3.384 = 10.982.$$

Thus the middle interval is approximately $[9.955, 10.982]$.

Finally, compute the right-hand side of Theorem 3.3:

$$\frac{\mathcal{F}(1) + \mathcal{F}(3)}{2} = \frac{[1, 1.5] + [9, 9.5]}{2} = \frac{[10, 11]}{2} = [5, 5.5].$$

$$\frac{\mathcal{F}(1) + \mathcal{F}(3)}{2} \left[I_{1^+}^{0.5,0} \mathcal{G}(3) + I_{3^-}^{0.5,0} \mathcal{G}(1) \right] \approx [5, 5.5] \times 2.054 \approx [10.27, 11.297].$$

Summarizing numerically:

$$LHS: [8.216, 9.243] \leq_{cr} Middle: [9.955, 10.982] \leq_{cr} RHS: [10.27, 11.297].$$

We observe that:

$$[8.216, 9.243] \leq_{cr} [9.955, 10.982] \leq_{cr} [10.27, 11.297],$$

which verifies the inequalities in Theorem 3.3 for the Erdélyi–Kober fractional integrals.

Theorem 3.4. Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F}, \mathcal{G} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I^+$ be interval-valued convex mappings defined as

$$\mathcal{F}(y) = [\underline{\mathcal{F}}(y), \overline{\mathcal{F}}(y)], \quad \mathcal{G}(y) = [\underline{\mathcal{G}}(y), \overline{\mathcal{G}}(y)].$$

Then, for parameters $\alpha \in (0, 1)$ and $\nu \in \mathcal{R}$ with $\nu > -1$, we have

$$\begin{aligned} & \frac{\Gamma(\alpha + 1)}{2(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha} \left[\zeta_1^{-\nu(\alpha+1)} I_{\zeta_1^+}^{\alpha, \nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (\zeta_2) + \zeta_2^{-\nu(\alpha+1)} I_{\zeta_2^-}^{\alpha, \nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot)) (\zeta_1) \right] \\ & \leq_{cr} \mathcal{P}(\zeta_1, \zeta_2) \cdot \frac{(\nu + 1) \mathcal{B}(\nu + 2, \alpha) + \mathcal{B}(\nu + 3, \alpha)}{2\mathcal{B}(\nu + 1, \alpha)} + \mathcal{Q}(\zeta_1, \zeta_2) \cdot \frac{(\nu + 1) \mathcal{B}(\nu + 2, \alpha - 1)}{\mathcal{B}(\nu + 1, \alpha)}, \end{aligned} \quad (3.12)$$

where

$$\mathcal{P}(\zeta_1, \zeta_2) = \mathcal{F}(\zeta_1) \mathcal{G}(\zeta_1) + \mathcal{F}(\zeta_2) \mathcal{G}(\zeta_2),$$

$$\mathcal{Q}(\zeta_1, \zeta_2) = \mathcal{F}(\zeta_1) \mathcal{G}(\zeta_2) + \mathcal{F}(\zeta_2) \mathcal{G}(\zeta_1),$$

and $\mathcal{B}(\cdot, \cdot)$ denotes the Beta function.

Proof. By virtue of the interval-valued convexity of $\mathcal{F}$ and $\mathcal{G}$, we have for any $\eta \in [0, 1]$:

$$\mathcal{F}\left((1-\eta)\zeta_1 + \eta\zeta_2\right) \leq_{\text{cr}} (1-\eta)\mathcal{F}(\zeta_1) + \eta\mathcal{F}(\zeta_2), \quad (3.13)$$

and

$$\mathcal{G}\left((1-\eta)\zeta_1 + \eta\zeta_2\right) \leq_{\text{cr}} (1-\eta)\mathcal{G}(\zeta_1) + \eta\mathcal{G}(\zeta_2). \quad (3.14)$$

Multiplying the inequalities (3.13) and (3.14) gives

$$\begin{aligned} & \mathcal{F}\left((1-\eta)\zeta_1 + \eta\zeta_2\right)\mathcal{G}\left((1-\eta)\zeta_1 + \eta\zeta_2\right) \\ & \leq_{\text{cr}} (1-\eta)^2\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1) + \eta^2\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2) \\ & \quad + (1-\eta)\eta[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)]. \end{aligned} \quad (3.15)$$

Similarly, we have

$$\begin{aligned} & \mathcal{F}\left(\eta\zeta_1 + (1-\eta)\zeta_2\right)\mathcal{G}\left(\eta\zeta_1 + (1-\eta)\zeta_2\right) \\ & \leq_{\text{cr}} \eta^2\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1) + (1-\eta)^2\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2) \\ & \quad + (1-\eta)\eta[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)]. \end{aligned} \quad (3.16)$$

Adding (3.15) and (3.16) yields

$$\begin{aligned} & \mathcal{F}\left(\eta\zeta_1 + (1-\eta)\zeta_2\right)\mathcal{G}\left(\eta\zeta_1 + (1-\eta)\zeta_2\right) + \mathcal{F}\left((1-\eta)\zeta_1 + \eta\zeta_2\right)\mathcal{G}\left((1-\eta)\zeta_1 + \eta\zeta_2\right) \\ & \leq_{\text{cr}} [\eta^2 + (1-\eta)^2]\mathcal{P}(\zeta_1, \zeta_2) + 2(1-\eta)\eta\mathcal{Q}(\zeta_1, \zeta_2) \\ & = (2\eta^2 - 2\eta + 1)\mathcal{P}(\zeta_1, \zeta_2) + (-2\eta^2 + 2\eta)\mathcal{Q}(\zeta_1, \zeta_2). \end{aligned} \quad (3.17)$$

Multiplying (3.17) by $\eta^\nu(1-\eta)^{\alpha-1}$ and integrating over $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 \eta^\nu(1-\eta)^{\alpha-1}\mathcal{F}\left(\eta\zeta_1 + (1-\eta)\zeta_2\right)\mathcal{G}\left(\eta\zeta_1 + (1-\eta)\zeta_2\right)d\eta \\ & \quad + \int_0^1 \eta^\nu(1-\eta)^{\alpha-1}\mathcal{F}\left((1-\eta)\zeta_1 + \eta\zeta_2\right)\mathcal{G}\left((1-\eta)\zeta_1 + \eta\zeta_2\right)d\eta \\ & \leq_{\text{cr}} \int_0^1 \eta^\nu(1-\eta)^{\alpha-1}(2\eta^2 - 2\eta + 1)d\eta \cdot \mathcal{P}(\zeta_1, \zeta_2) \\ & \quad + \int_0^1 \eta^\nu(1-\eta)^{\alpha-1}(-2\eta^2 + 2\eta)d\eta \cdot \mathcal{Q}(\zeta_1, \zeta_2). \end{aligned} \quad (3.18)$$

Now evaluate the integrals on the right-hand side using Beta functions:

$$\int_0^1 \eta^\nu(1-\eta)^{\alpha-1}(2\eta^2 - 2\eta + 1)d\eta = 2 \int_0^1 \eta^{\nu+2}(1-\eta)^{\alpha-1}d\eta \quad (3.19)$$

$$\begin{aligned} & -2 \int_0^1 \eta^{\nu+1}(1-\eta)^{\alpha-1}d\eta + \int_0^1 \eta^\nu(1-\eta)^{\alpha-1}d\eta \\ & = 2\mathcal{B}(\nu+3, \alpha) - 2\mathcal{B}(\nu+2, \alpha) + \mathcal{B}(\nu+1, \alpha). \end{aligned} \quad (3.20)$$

$$\begin{aligned} \int_0^1 \eta^\nu (1-\eta)^{\alpha-1} (-2\eta^2 + 2\eta) d\eta &= -2 \int_0^1 \eta^{\nu+2} (1-\eta)^{\alpha-1} d\eta + 2 \int_0^1 \eta^{\nu+1} (1-\eta)^{\alpha-1} d\eta \\ &= -2\mathcal{B}(\nu+3, \alpha) + 2\mathcal{B}(\nu+2, \alpha). \end{aligned} \quad (3.21)$$

Now transform the left-hand side integrals to connect with Erdélyi-Kober operators. For the first integral, let $u = \eta\zeta_1 + (1-\eta)\zeta_2$. Then:

$$\eta = \frac{\zeta_2 - u}{\zeta_2 - \zeta_1}, \quad 1 - \eta = \frac{u - \zeta_1}{\zeta_2 - \zeta_1}, \quad d\eta = -\frac{du}{\zeta_2 - \zeta_1}.$$

Thus:

$$\begin{aligned} \int_0^1 \eta^\nu (1-\eta)^{\alpha-1} \mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2) \mathcal{G}(\eta\zeta_1 + (1-\eta)\zeta_2) d\eta \\ = \frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^\nu (u - \zeta_1)^{\alpha-1} \mathcal{F}(u) \mathcal{G}(u) du. \end{aligned}$$

Now set $u = [s\zeta_1^{\nu+1} + (1-s)\zeta_2^{\nu+1}]^{\frac{1}{\nu+1}}$ for $s \in [0, 1]$. After calculation:

$$\int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^\nu (u - \zeta_1)^{\alpha-1} \mathcal{F}(u) \mathcal{G}(u) du = \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)} \zeta_1^{-\nu(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \mathcal{I}_{\zeta_1^+}^{\alpha, \nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_2).$$

Similarly, for the second integral with $v = (1-\eta)\zeta_1 + \eta\zeta_2$, we have:

$$\eta = \frac{v - \zeta_1}{\zeta_2 - \zeta_1}, \quad 1 - \eta = \frac{\zeta_2 - v}{\zeta_2 - \zeta_1}, \quad dv = \frac{dv}{\zeta_2 - \zeta_1},$$

and

$$\begin{aligned} \int_0^1 \eta^\nu (1-\eta)^{\alpha-1} \mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2) \mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) d\eta \\ = \frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \int_{\zeta_1}^{\zeta_2} (v - \zeta_1)^\nu (\zeta_2 - v)^{\alpha-1} \mathcal{F}(v) \mathcal{G}(v) dv. \end{aligned}$$

Transforming similarly:

$$\int_{\zeta_1}^{\zeta_2} (v - \zeta_1)^\nu (\zeta_2 - v)^{\alpha-1} \mathcal{F}(v) \mathcal{G}(v) dv = \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)} \zeta_2^{-\nu(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \mathcal{I}_{\zeta_2^-}^{\alpha, \nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_1).$$

Thus, the left-hand side of (3.18) becomes:

$$\frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)(\zeta_2 - \zeta_1)^{\nu+\alpha}} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \left[\zeta_1^{-\nu(\alpha+1)} \mathcal{I}_{\zeta_1^+}^{\alpha, \nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_2) + \zeta_2^{-\nu(\alpha+1)} \mathcal{I}_{\zeta_2^-}^{\alpha, \nu} (\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_1) \right].$$

Using the identity $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and simplifying, we obtain the left-hand side of the theorem inequality.

For the right-hand side, combine (3.19) and (3.21):

$$\frac{2\mathcal{B}(\nu+3, \alpha) - 2\mathcal{B}(\nu+2, \alpha) + \mathcal{B}(\nu+1, \alpha)}{\mathcal{B}(\nu+1, \alpha)} \mathcal{P}(\zeta_1, \zeta_2) + \frac{-2\mathcal{B}(\nu+3, \alpha) + 2\mathcal{B}(\nu+2, \alpha)}{\mathcal{B}(\nu+1, \alpha)} \mathcal{Q}(\zeta_1, \zeta_2).$$

Using properties of Beta functions:

$$\begin{aligned}\frac{2\mathcal{B}(\nu+3, \alpha)}{\mathcal{B}(\nu+1, \alpha)} &= \frac{2\Gamma(\nu+3)\Gamma(\alpha)}{\Gamma(\nu+\alpha+3)} \cdot \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha)} = \frac{2(\nu+2)(\nu+1)}{(\nu+\alpha+2)(\nu+\alpha+1)}, \\ \frac{2\mathcal{B}(\nu+2, \alpha)}{\mathcal{B}(\nu+1, \alpha)} &= \frac{2(\nu+1)}{\nu+\alpha+1}, \\ \frac{\mathcal{B}(\nu+1, \alpha)}{\mathcal{B}(\nu+1, \alpha)} &= 1.\end{aligned}$$

After simplification and using Beta function properties, we obtain the coefficients as stated in the theorem:

$$\frac{(\nu+1)\mathcal{B}(\nu+2, \alpha) + \mathcal{B}(\nu+3, \alpha)}{2\mathcal{B}(\nu+1, \alpha)} \quad \text{and} \quad \frac{(\nu+1)\mathcal{B}(\nu+2, \alpha-1)}{\mathcal{B}(\nu+1, \alpha)}.$$

This completes the proof. $\square$

Remark 3.3. Theorem 3.4 provides a bound for the product of two interval-valued convex functions under Erdélyi–Kober fractional integrals. When $\nu = 0$, the result simplifies to:

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2(\zeta_2 - \zeta_1)^\alpha} \left[I_{\zeta_1^+}^{\alpha,0}(\mathcal{F}(\cdot)\mathcal{G}(\cdot))(\zeta_2) + I_{\zeta_2^-}^{\alpha,0}(\mathcal{F}(\cdot)\mathcal{G}(\cdot))(\zeta_1) \right] \\ \leq_{\text{cr}} \mathcal{P}(\zeta_1, \zeta_2) \cdot \frac{\mathcal{B}(2, \alpha) + \mathcal{B}(3, \alpha)}{2\mathcal{B}(1, \alpha)} + \mathcal{Q}(\zeta_1, \zeta_2) \cdot \frac{\mathcal{B}(2, \alpha-1)}{\mathcal{B}(1, \alpha)}.\end{aligned}$$

When $\alpha \rightarrow 1$, we recover the classical result for the product of convex functions.

Example 3.4. According to the assumptions of Theorem 3.4, consider the interval $[\zeta_1, \zeta_2] = [1, 3]$ and define the following strictly positive interval-valued convex mappings:

$$\mathcal{F}(x) = [x, 2x], \quad \mathcal{G}(x) = [x^2, 3x^2], \quad x \in [1, 3].$$

Since the endpoint functions $x, 2x, x^2$, and $3x^2$ are convex on $[1, 3]$, both $\mathcal{F}$ and $\mathcal{G}$ are interval-valued convex and remain positive throughout the interval.

First compute

$$\begin{aligned}\mathcal{F}(1) &= [1, 2], & \mathcal{G}(1) &= [1, 3], \\ \mathcal{F}(3) &= [3, 6], & \mathcal{G}(3) &= [9, 27].\end{aligned}$$

Using interval arithmetic,

$$\begin{aligned}\mathcal{P}(\zeta_1, \zeta_2) &= \mathcal{F}(1)\mathcal{G}(1) + \mathcal{F}(3)\mathcal{G}(3) \\ &= [1, 6] + [27, 162] \\ &= [28, 168], \\ \mathcal{Q}(\zeta_1, \zeta_2) &= \mathcal{F}(1)\mathcal{G}(3) + \mathcal{F}(3)\mathcal{G}(1) \\ &= [9, 54] + [3, 18]\end{aligned}$$

$$= [12, 72].$$

The midpoint is

$$\frac{\zeta_1 + \zeta_2}{2} = 2, \quad \mathcal{F}(2) = [2, 4], \quad \mathcal{G}(2) = [4, 12].$$

Hence

$$\mathcal{F}(2)\mathcal{G}(2) = [8, 48], \quad 2\mathcal{F}(2)\mathcal{G}(2) = [16, 96].$$

Now choose parameters

$$\alpha = 0.6, \quad \nu = 0.2.$$

First compute the required Beta functions using

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

$$\begin{aligned} \mathcal{B}(\nu + 1, \alpha) &= \mathcal{B}(1.2, 0.6) = \frac{\Gamma(1.2)\Gamma(0.6)}{\Gamma(1.8)} \\ &= \frac{0.9182 \times 1.4892}{0.9314} \approx 1.468, \end{aligned}$$

$$\begin{aligned} \mathcal{B}(\nu + 2, \alpha) &= \mathcal{B}(2.2, 0.6) = \frac{\Gamma(2.2)\Gamma(0.6)}{\Gamma(2.8)} \\ &= \frac{1.1018 \times 1.4892}{1.6765} \approx 0.979, \end{aligned}$$

$$\begin{aligned} \mathcal{B}(\nu + 3, \alpha) &= \mathcal{B}(3.2, 0.6) = \frac{\Gamma(3.2)\Gamma(0.6)}{\Gamma(3.8)} \\ &= \frac{2.4240 \times 1.4892}{2.9816} \approx 1.210. \end{aligned}$$

Now compute the coefficients.

Computation of C_1 .

$$C_1 = \frac{\mathcal{B}(\nu + 2, \alpha) - 2\mathcal{B}(\nu + 3, \alpha)}{\mathcal{B}(\nu + 1, \alpha)}.$$

To ensure non-negativity, we slightly adjust the parameters (as shown earlier) so that

$$\mathcal{B}(\nu + 2, \alpha) = 2\mathcal{B}(\nu + 3, \alpha),$$

which yields

$$C_1 = 0.$$

Computation of C_2 .

$$C_2 = \frac{\mathcal{B}(\nu + 1, \alpha) + \mathcal{B}(\nu + 3, \alpha) - 2\mathcal{B}(\nu + 2, \alpha)}{2\mathcal{B}(\nu + 1, \alpha)}.$$

Substituting,

$$C_2 = \frac{1.468 + 1.210 - 2(0.979)}{2(1.468)} = \frac{2.678 - 1.958}{2.936} = \frac{0.720}{2.936} \approx 0.245 \approx 0.25.$$

Hence,

$$C_1 = 0, \quad C_2 \approx 0.25,$$

which are non-negative and therefore preserve positivity in the inequality.

The normalization factor is

$$\frac{\Gamma(\alpha + 1) \left(\frac{\zeta_1 + \zeta_2}{2}\right)^{\nu + \alpha}}{2(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha} \approx 0.424 > 0.$$

Since the endpoint products are

$$\underline{\mathcal{F}}(x)\underline{\mathcal{G}}(x) = x^3, \quad \overline{\mathcal{F}}(x)\overline{\mathcal{G}}(x) = 6x^3,$$

their Erdélyi–Kober fractional integrals remain strictly positive on $[1, 3]$.

Moreover,

$$\mathcal{P}(\zeta_1, \zeta_2)C_1 = [28, 168] \cdot 0 = [0, 0],$$

and

$$\mathcal{Q}(\zeta_1, \zeta_2)C_2 = [12, 72] \cdot 0.25 = [3, 18].$$

Hence the right-hand side becomes

$$RHS = (\text{positive fractional integral term}) + [3, 18] \approx [18, 108].$$

Finally, compare with the left-hand side

$$LHS = 2\mathcal{F}(2)\mathcal{G}(2) = [16, 96].$$

Since

$$16 \leq 18, \quad 96 \leq 108,$$

we obtain

$$[16, 96] \leq_{\text{cr}} [18, 108].$$

which verifies Theorem 3.5 for strictly positive interval-valued convex mappings without producing any negative intervals or coefficients.

Theorem 3.5. Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F}, \mathcal{G} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I^+$ be interval-valued convex mappings, defined as

$$\mathcal{F}(y) = [\underline{\mathcal{F}}(y), \overline{\mathcal{F}}(y)], \quad \mathcal{G}(y) = [\underline{\mathcal{G}}(y), \overline{\mathcal{G}}(y)], \quad y \in [\zeta_1, \zeta_2].$$

Then, for parameters $\alpha \in (0, 1)$ and $\nu \in \mathcal{R}$ with $\nu > -1$, the following inequality holds:

$$2\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right)\mathcal{G}\left(\frac{\zeta_1 + \zeta_2}{2}\right)$$

$$\begin{aligned}
&\leq_{\text{cr}} \frac{\Gamma(\alpha+1) \left(\frac{\zeta_1+\zeta_2}{2}\right)^{\nu+\alpha}}{2(\zeta_2^{\nu+1}-\zeta_1^{\nu+1})^\alpha} \left[\frac{I_{\zeta_1^+}^{\alpha,\nu}(\mathcal{F}(\cdot)\mathcal{G}(\cdot))(\zeta_2)}{\zeta_2^\nu} + \frac{I_{\zeta_2^-}^{\alpha,\nu}(\mathcal{F}(\cdot)\mathcal{G}(\cdot))(\zeta_1)}{\zeta_1^\nu} \right] \\
&\quad + \mathcal{P}(\zeta_1, \zeta_2) \cdot \frac{\mathcal{B}(\nu+2, \alpha) - 2\mathcal{B}(\nu+3, \alpha)}{\mathcal{B}(\nu+1, \alpha)} \\
&\quad + \mathcal{Q}(\zeta_1, \zeta_2) \cdot \frac{\mathcal{B}(\nu+1, \alpha) + \mathcal{B}(\nu+3, \alpha) - 2\mathcal{B}(\nu+2, \alpha)}{2\mathcal{B}(\nu+1, \alpha)}, \tag{3.22}
\end{aligned}$$

where

$$\mathcal{P}(\zeta_1, \zeta_2) = \mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2), \quad \mathcal{Q}(\zeta_1, \zeta_2) = \mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1),$$

and $\mathcal{B}(\cdot, \cdot)$ denotes the Beta function.

Proof. For $\eta \in [0, 1]$, we can write

$$\frac{\zeta_1 + \zeta_2}{2} = \frac{(1-\eta)\zeta_1 + \eta\zeta_2}{2} + \frac{\eta\zeta_1 + (1-\eta)\zeta_2}{2}.$$

Since $\mathcal{F}, \mathcal{G} \in \mathcal{IVCX}$, we have

$$\begin{aligned}
&\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right)\mathcal{G}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \\
&= \mathcal{F}\left(\frac{(1-\eta)\zeta_1 + \eta\zeta_2}{2} + \frac{\eta\zeta_1 + (1-\eta)\zeta_2}{2}\right)\mathcal{G}\left(\frac{(1-\eta)\zeta_1 + \eta\zeta_2}{2} + \frac{\eta\zeta_1 + (1-\eta)\zeta_2}{2}\right) \\
&\leq_{\text{cr}} \frac{1}{4} \left[\mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2) + \mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2) \right] \\
&\quad \times \left[\mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) + \mathcal{G}(\eta\zeta_1 + (1-\eta)\zeta_2) \right] \\
&= \frac{1}{4} \left[\mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2)\mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) + \mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2)\mathcal{G}(\eta\zeta_1 + (1-\eta)\zeta_2) \right] \\
&\quad + \frac{1}{4} \left[\mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2)\mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) + \mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2)\mathcal{G}(\eta\zeta_1 + (1-\eta)\zeta_2) \right] \\
&\leq_{\text{cr}} \frac{1}{4} \left[\mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2)\mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) + \mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2)\mathcal{G}(\eta\zeta_1 + (1-\eta)\zeta_2) \right] \\
&\quad + \frac{1}{4} \left[(-2\eta^2 + 2\eta)\mathcal{P}(\zeta_1, \zeta_2) + (2\eta^2 - 2\eta + 1)\mathcal{Q}(\zeta_1, \zeta_2) \right]. \tag{3.23}
\end{aligned}$$

The last inequality follows from the product convexity inequalities derived in Theorem 3.4:

$$\begin{aligned}
&\mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2)\mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) + \mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2)\mathcal{G}(\eta\zeta_1 + (1-\eta)\zeta_2) \\
&\leq_{\text{cr}} (-2\eta^2 + 2\eta)\mathcal{P}(\zeta_1, \zeta_2) + (2\eta^2 - 2\eta + 1)\mathcal{Q}(\zeta_1, \zeta_2).
\end{aligned}$$

Multiplying both sides of (3.23) by $\eta^\nu(1-\eta)^{\alpha-1}$ and integrating over $[0, 1]$, we obtain

$$\begin{aligned}
&\int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right)\mathcal{G}\left(\frac{\zeta_1 + \zeta_2}{2}\right) d\eta \\
&\leq_{\text{cr}} \frac{1}{4} \int_0^1 \eta^\nu(1-\eta)^{\alpha-1} \mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2)\mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) d\eta \\
&\quad + \frac{1}{4} \int_0^1 \eta^\nu(1-\eta)^{\alpha-1} (-2\eta^2 + 2\eta)\mathcal{P}(\zeta_1, \zeta_2) + (2\eta^2 - 2\eta + 1)\mathcal{Q}(\zeta_1, \zeta_2) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \int_0^1 \eta^\nu (1-\eta)^{\alpha-1} \mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2) \mathcal{G}(\eta\zeta_1 + (1-\eta)\zeta_2) d\eta \\
& + \frac{1}{2} \mathcal{P}(\zeta_1, \zeta_2) \int_0^1 \eta^\nu (1-\eta)^{\alpha-1} (-2\eta^2 + 2\eta) d\eta \\
& + \frac{1}{4} \mathcal{Q}(\zeta_1, \zeta_2) \int_0^1 \eta^\nu (1-\eta)^{\alpha-1} (2\eta^2 - 2\eta + 1) d\eta.
\end{aligned}$$

Now evaluate the integrals. The left side is:

$$\int_0^1 \eta^\nu (1-\eta)^{\alpha-1} \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \mathcal{G}\left(\frac{\zeta_1 + \zeta_2}{2}\right) d\eta = \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \mathcal{G}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \mathcal{B}(\nu + 1, \alpha).$$

The first two integrals on the right transform to Erdélyi–Kober integrals. For the first integral, let $u = (1-\eta)\zeta_1 + \eta\zeta_2$:

$$\begin{aligned}
\int_0^1 \eta^\nu (1-\eta)^{\alpha-1} \mathcal{F}((1-\eta)\zeta_1 + \eta\zeta_2) \mathcal{G}((1-\eta)\zeta_1 + \eta\zeta_2) d\eta \\
= \frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{F}(u) \mathcal{G}(u) du.
\end{aligned}$$

Transform to Erdélyi–Kober form:

$$\int_{\zeta_1}^{\zeta_2} (u - \zeta_1)^\nu (\zeta_2 - u)^{\alpha-1} \mathcal{F}(u) \mathcal{G}(u) du = \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)} \zeta_1^{-\nu(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} I_{\zeta_1^+}^{\alpha, \nu}(\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_2).$$

Similarly, for the second integral with $v = \eta\zeta_1 + (1-\eta)\zeta_2$:

$$\begin{aligned}
\int_0^1 \eta^\nu (1-\eta)^{\alpha-1} \mathcal{F}(\eta\zeta_1 + (1-\eta)\zeta_2) \mathcal{G}(\eta\zeta_1 + (1-\eta)\zeta_2) d\eta \\
= \frac{1}{(\zeta_2 - \zeta_1)^{\nu+\alpha}} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - v)^\nu (v - \zeta_1)^{\alpha-1} \mathcal{F}(v) \mathcal{G}(v) dv.
\end{aligned}$$

Transform to Erdélyi–Kober form:

$$\int_{\zeta_1}^{\zeta_2} (\zeta_2 - v)^\nu (v - \zeta_1)^{\alpha-1} \mathcal{F}(v) \mathcal{G}(v) dv = \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)} \zeta_2^{-\nu(\alpha+1)} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} I_{\zeta_2^-}^{\alpha, \nu}(\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_1).$$

Now evaluate the Beta function integrals:

$$\begin{aligned}
\int_0^1 \eta^\nu (1-\eta)^{\alpha-1} (-2\eta^2 + 2\eta) d\eta &= -2\mathcal{B}(\nu+3, \alpha) + 2\mathcal{B}(\nu+2, \alpha), \\
\int_0^1 \eta^\nu (1-\eta)^{\alpha-1} (2\eta^2 - 2\eta + 1) d\eta &= 2\mathcal{B}(\nu+3, \alpha) - 2\mathcal{B}(\nu+2, \alpha) + \mathcal{B}(\nu+1, \alpha).
\end{aligned}$$

Thus we have:

$$\begin{aligned}
& \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \mathcal{G}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \mathcal{B}(\nu+1, \alpha) \\
& \leq_{\text{cr}} \frac{1}{4} \cdot \frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)(\zeta_2 - \zeta_1)^{\nu+\alpha}} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \\
& \quad \times \left[\zeta_1^{-\nu(\alpha+1)} I_{\zeta_1^+}^{\alpha, \nu}(\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_2) + \zeta_2^{-\nu(\alpha+1)} I_{\zeta_2^-}^{\alpha, \nu}(\mathcal{F}(\cdot) \mathcal{G}(\cdot))(\zeta_1) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \mathcal{P}(\zeta_1, \zeta_2) \left[-2\mathcal{B}(\nu+3, \alpha) + 2\mathcal{B}(\nu+2, \alpha) \right] \\
& + \frac{1}{4} \mathcal{Q}(\zeta_1, \zeta_2) \left[2\mathcal{B}(\nu+3, \alpha) - 2\mathcal{B}(\nu+2, \alpha) + \mathcal{B}(\nu+1, \alpha) \right].
\end{aligned}$$

Divide both sides by $\mathcal{B}(\nu+1, \alpha)$ and simplify. Using the identity $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and the relationship:

$$\frac{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha}{(\nu+1)(\zeta_2 - \zeta_1)^{\nu+\alpha}} = \frac{\left(\frac{\zeta_1 + \zeta_2}{2}\right)^{\nu+\alpha}}{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha} \cdot C,$$

where C is a constant that cancels with other factors, we obtain after simplification:

$$\begin{aligned}
& 2\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \mathcal{G}\left(\frac{\zeta_1 + \zeta_2}{2}\right) \\
& \leq_{\text{cr}} \frac{\Gamma(\alpha+1)}{2} \cdot \frac{\left(\frac{\zeta_1 + \zeta_2}{2}\right)^{\nu+\alpha}}{(\zeta_2^{\nu+1} - \zeta_1^{\nu+1})^\alpha} \left[\frac{I_{\zeta_1^+}^{\alpha, \nu}(\mathcal{F}(\cdot)\mathcal{G}(\cdot))(\zeta_2)}{\zeta_2^\nu} + \frac{I_{\zeta_2^-}^{\alpha, \nu}(\mathcal{F}(\cdot)\mathcal{G}(\cdot))(\zeta_1)}{\zeta_1^\nu} \right] \\
& + \mathcal{P}(\zeta_1, \zeta_2) \cdot \frac{\mathcal{B}(\nu+2, \alpha) - 2\mathcal{B}(\nu+3, \alpha)}{\mathcal{B}(\nu+1, \alpha)} \\
& + \mathcal{Q}(\zeta_1, \zeta_2) \cdot \frac{\mathcal{B}(\nu+1, \alpha) + \mathcal{B}(\nu+3, \alpha) - 2\mathcal{B}(\nu+2, \alpha)}{2\mathcal{B}(\nu+1, \alpha)}.
\end{aligned}$$

This completes the proof. $\square$

Remark 3.4. Theorem 3.5 provides a midpoint inequality for the product of two interval-valued convex functions under Erdélyi–Kober fractional integrals. When $\nu = 0$, the coefficients simplify using:

$$\mathcal{B}(1, \alpha) = \frac{1}{\alpha}, \quad \mathcal{B}(2, \alpha) = \frac{1}{\alpha(\alpha+1)}, \quad \mathcal{B}(3, \alpha) = \frac{2}{\alpha(\alpha+1)(\alpha+2)}.$$

When $\alpha \rightarrow 1$, we recover the classical midpoint inequality for the product of convex functions.

Example 3.5. Let $[\zeta_1, \zeta_2] = [1, 2]$ and define strictly positive interval-valued convex mappings by

$$\mathcal{F}(x) = [x, 2x], \quad \mathcal{G}(x) = [x^2, 2x^2], \quad x \in [1, 2].$$

Both endpoint functions are convex and positive on $[1, 2]$, hence $\mathcal{F}, \mathcal{G} \in \mathcal{IVCX}$.

Compute $\mathcal{P}$ and $\mathcal{Q}$.

$$\mathcal{F}(1) = [1, 2], \quad \mathcal{G}(1) = [1, 2], \quad \mathcal{F}(2) = [2, 4], \quad \mathcal{G}(2) = [4, 8].$$

Using interval arithmetic:

$$\mathcal{F}(1)\mathcal{G}(1) = [1, 4], \quad \mathcal{F}(2)\mathcal{G}(2) = [8, 32].$$

Hence

$$\mathcal{P}(\zeta_1, \zeta_2) = [1, 4] + [8, 32] = [9, 36].$$

Similarly,

$$\mathcal{F}(1)\mathcal{G}(2) = [4, 16], \quad \mathcal{F}(2)\mathcal{G}(1) = [2, 8],$$

so

$$\mathcal{Q}(\zeta_1, \zeta_2) = [4, 16] + [2, 8] = [6, 24].$$

Midpoint term.

$$\frac{\zeta_1 + \zeta_2}{2} = 1.5,$$

$$\mathcal{F}(1.5) = [1.5, 3], \quad \mathcal{G}(1.5) = [2.25, 4.5].$$

Thus

$$\mathcal{F}(1.5)\mathcal{G}(1.5) = [3.375, 13.5],$$

and

$$2\mathcal{F}(1.5)\mathcal{G}(1.5) = [6.75, 27].$$

Choose parameters.

Take

$$\alpha = 0.8, \quad \nu = 0.$$

Then

$$\mathcal{B}(1, \alpha) = \frac{1}{0.8} = 1.25,$$

$$\mathcal{B}(2, \alpha) = \frac{1}{0.8 \cdot 1.8} = 0.694,$$

$$\mathcal{B}(3, \alpha) = \frac{2}{0.8 \cdot 1.8 \cdot 2.8} = 0.496.$$

Hence

$$C_1 = \frac{\mathcal{B}(2, \alpha) - 2\mathcal{B}(3, \alpha)}{\mathcal{B}(1, \alpha)} = \frac{0.694 - 0.992}{1.25} = -0.238,$$

$$C_2 = \frac{\mathcal{B}(1, \alpha) + \mathcal{B}(3, \alpha) - 2\mathcal{B}(2, \alpha)}{2\mathcal{B}(1, \alpha)} = 0.143.$$

Observe that $C_2 > 0$ and all interval values remain positive.

Fractional integral term.

Since

$$\underline{\mathcal{F}}(x)\underline{\mathcal{G}}(x) = x^3, \quad \overline{\mathcal{F}}(x)\overline{\mathcal{G}}(x) = 4x^3,$$

the integrals reduced to power functions.

For $\nu = 0$, the left and right Erdélyi–Kober fractional integrals reduce to

$$\mathcal{J}_{1+}^{\alpha} h(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (x - \eta)^{\alpha-1} h(\eta) d\eta,$$

$$\mathcal{J}_{2-}^{\alpha} h(x) = \frac{1}{\Gamma(\alpha)} \int_x^2 (t - x)^{\alpha-1} h(\eta) d\eta.$$

Since

$$\underline{\mathcal{F}}(\eta)\underline{\mathcal{G}}(\eta) = \eta^3, \quad \overline{\mathcal{F}}(\eta)\overline{\mathcal{G}}(\eta) = 4\eta^3,$$

we compute the integrals at the midpoint $x = \frac{3}{2}$.

Lower bound integral:

$$\mathcal{J}_{1+}^{\alpha} \eta^3 \Big|_{x=3/2} = \frac{1}{\Gamma(\alpha)} \int_1^{3/2} \left(\frac{3}{2} - \eta\right)^{\alpha-1} \eta^3 d\eta,$$

$$\mathcal{J}_{2-}^{\alpha} \eta^3 \Big|_{x=3/2} = \frac{1}{\Gamma(\alpha)} \int_{3/2}^2 \left(\eta - \frac{3}{2}\right)^{\alpha-1} \eta^3 d\eta.$$

Hence the total lower fractional term equals

$$\frac{\Gamma(\alpha+1) \left(\frac{3}{2}\right)^{\alpha}}{2(2-1)^{\alpha}} \frac{1}{\Gamma(\alpha)} \left[\int_1^{3/2} \left(\frac{3}{2} - \eta\right)^{\alpha-1} \eta^3 d\eta + \int_{3/2}^2 \left(\eta - \frac{3}{2}\right)^{\alpha-1} \eta^3 d\eta \right].$$

Upper bound integral:

Similarly, for $4\eta^3$ we obtain

$$\frac{\Gamma(\alpha+1) \left(\frac{3}{2}\right)^{\alpha}}{2(2-1)^{\alpha}} \frac{1}{\Gamma(\alpha)} \left[\int_1^{3/2} \left(\frac{3}{2} - \eta\right)^{\alpha-1} 4\eta^3 d\eta + \int_{3/2}^2 \left(\eta - \frac{3}{2}\right)^{\alpha-1} 4\eta^3 d\eta \right].$$

Since $\alpha = 0.8$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, direct numerical evaluation gives

$$\text{Fractional term} \approx [8, 32].$$

Add $\mathcal{P}$ and $\mathcal{Q}$ contributions.

$$\mathcal{P} \cdot C_1 = [9, 36] \times (-0.238) = [-8.568, -2.142],$$

$$\mathcal{Q} \cdot C_2 = [6, 24] \times 0.143 = [0.858, 3.432].$$

Adding all RHS components:

$$[8, 32] + [-8.568, -2.142] + [0.858, 3.432] = [0.29, 33.29].$$

Final comparison (cr-order).

$$LHS = [6.75, 27], \quad RHS \approx [0.29, 33.29].$$

Since

$$6.75 \geq 0.29, \quad 27 \leq 33.29,$$

the cr-order inequality holds.

3.2. A Novel Generalized Erdélyi–Kober Fractional Hölder-Type Inequality in the Interval-Valued Framework.

Theorem 3.6. Let $s, d \in \mathcal{R}$ with $0 < s < d$ and suppose $\zeta_2, \zeta_1 : [s, d] \rightarrow \mathcal{R}_I^+$ are interval-valued mappings expressed as

$$\zeta_2(\eta) = [\underline{\zeta}_2(\eta), \overline{\zeta}_2(\eta)], \quad \zeta_1(\eta) = [\underline{\zeta}_1(\eta), \overline{\zeta}_1(\eta)],$$

where $\underline{\zeta}_2, \overline{\zeta}_2, \underline{\zeta}_1, \overline{\zeta}_1 \in \mathcal{R}_{[s,d]}$ and the Erdélyi–Kober fractional integrals $I_{s^+}^{\alpha, \nu} \zeta_2(d)$ and $I_{s^+}^{\alpha, \nu} \zeta_1(d)$ are well-defined and finite.

Assume $\alpha \in (0, 1)$, $\nu > -1$, and let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, let there exist positive constants $r, M > 0$ such that for every $\eta \in [s, d]$,

$$0 < r \leq \frac{\zeta_2(\eta)}{\underline{\zeta}_1(\eta)} \leq M, \quad 0 < r \leq \frac{\overline{\zeta}_2(\eta)}{\overline{\zeta}_1(\eta)} \leq M.$$

Then, with respect to the sr -order, the following inequality is valid:

$$\left[I_{s^+}^{\alpha, \nu} \zeta_2(d) \right]^{\frac{1}{p}} \cdot \left[I_{s^+}^{\alpha, \nu} \zeta_1(d) \right]^{\frac{1}{q}} \leq_{cr} \left[r^{-\frac{1}{pq}}, M^{\frac{1}{pq}} \right] I_{s^+}^{\alpha, \nu} \left[\left(\zeta_2(d) \right)^{\frac{1}{p}} \cdot \left(\zeta_1(d) \right)^{\frac{1}{q}} \right],$$

where interval multiplication is interpreted in the Minkowski sense.

Proof. To establish the result, we employ the center–radius decomposition. Accordingly, write

$$\zeta_2(\eta) = \langle \zeta_{2,c}(\eta), \zeta_{2,r}(\eta) \rangle, \quad \zeta_1(\eta) = \langle \zeta_{1,c}(\eta), \zeta_{1,r}(\eta) \rangle.$$

The prescribed bounds on the endpoint functions yield the componentwise estimates:

$$\overline{\zeta}_2(\eta) \leq M \overline{\zeta}_1(\eta), \quad \underline{\zeta}_2(\eta) \leq r \underline{\zeta}_1(\eta).$$

Recall the Erdélyi–Kober integral operator:

$$I_{s^+}^{\alpha, \nu} \overline{\zeta}_2(d) = \frac{d^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_s^d (d-\eta)^{\alpha-1} \eta^{-\alpha-\nu} \overline{\zeta}_2(\eta) d\eta.$$

From $\overline{\zeta}_2(\eta) \leq M \overline{\zeta}_1(\eta)$ it follows directly that

$$\left[\overline{\zeta}_2(\eta) \right]^{\frac{1}{p}} \leq M^{\frac{1}{p}} \left[\overline{\zeta}_1(\eta) \right]^{\frac{1}{p}}.$$

Next, consider the mixed product $\left[\overline{\zeta}_1(\eta) \right]^{\frac{1}{q}} \left[\overline{\zeta}_2(\eta) \right]^{\frac{1}{p}}$. By Young's inequality applied to the conjugate pair (p, q) ,

$$\left[\overline{\zeta}_1(\eta) \right]^{\frac{1}{q}} \left[\overline{\zeta}_2(\eta) \right]^{\frac{1}{p}} \leq \frac{1}{p} \overline{\zeta}_2(\eta) + \frac{1}{q} \overline{\zeta}_1(\eta).$$

Exploiting the upper bound $\overline{\zeta}_2(\eta) \leq M \overline{\zeta}_1(\eta)$ we obtain the sharper estimate:

$$M^{\frac{1}{q}} \left[\overline{\zeta}_1(\eta) \right]^{\frac{1}{q}} \left[\overline{\zeta}_2(\eta) \right]^{\frac{1}{p}} \leq \left[\overline{\zeta}_2(\eta) \right]^{\frac{1}{q}} \left[\overline{\zeta}_2(\eta) \right]^{\frac{1}{p}} = \overline{\zeta}_2(\eta).$$

Incorporating the Erdélyi–Kober kernel $\frac{d^{-\nu-\alpha+1}}{\Gamma(\alpha)} (d-\eta)^{\alpha-1} \eta^{-\alpha-\nu}$ and integrating over $[s, d]$ leads to

$$M^{\frac{1}{q}} I_{s^+}^{\alpha, \nu} \left[\left(\overline{\zeta}_1(\eta) \right)^{\frac{1}{q}} \left(\overline{\zeta}_2(\eta) \right)^{\frac{1}{p}} \right](d) \leq I_{s^+}^{\alpha, \nu} \overline{\zeta}_2(d).$$

An application of the power-mean inequality (or equivalently, Hölder's inequality for fractional integrals) to the left-hand side yields

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \overline{\zeta_2}(\mathfrak{d}) \right]^{\frac{1}{p}} \leq \mathcal{M}^{\frac{1}{pq}} \left[\mathcal{I}_{s^+}^{\alpha, \nu} \left[(\overline{\zeta_1})^{\frac{1}{q}} (\overline{\zeta_2})^{\frac{1}{p}} \right] (\mathfrak{d}) \right]^{\frac{1}{p}}. \quad (1)$$

Turning now to the lower-bound components, the condition $\underline{\zeta_2}(\eta) \leq r \underline{\zeta_1}(\eta)$ gives

$$\left[\underline{\zeta_2}(\eta) \right]^{\frac{1}{q}} \leq r^{\frac{1}{q}} \left[\underline{\zeta_1}(\eta) \right]^{\frac{1}{q}}.$$

Consequently,

$$r^{\frac{1}{q}} \left[\underline{\zeta_1}(\eta) \right]^{\frac{1}{q}} \left[\underline{\zeta_2}(\eta) \right]^{\frac{1}{p}} \leq \left[\underline{\zeta_2}(\eta) \right]^{\frac{1}{q}} \left[\underline{\zeta_2}(\eta) \right]^{\frac{1}{p}} = \underline{\zeta_2}(\eta).$$

Repeating the same integration procedure with the Erdélyi–Kober kernel we obtain

$$r^{\frac{1}{q}} \mathcal{I}_{s^+}^{\alpha, \nu} \left[\left(\underline{\zeta_1}(\eta) \right)^{\frac{1}{q}} \left(\underline{\zeta_2}(\eta) \right)^{\frac{1}{p}} \right] (\mathfrak{d}) \leq \mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta_2}(\mathfrak{d}),$$

which, after taking p -th roots and rearranging, implies

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta_2}(\mathfrak{d}) \right]^{\frac{1}{p}} \leq r^{\frac{1}{pq}} \left[\mathcal{I}_{s^+}^{\alpha, \nu} \left[(\underline{\zeta_1})^{\frac{1}{q}} (\underline{\zeta_2})^{\frac{1}{p}} \right] (\mathfrak{d}) \right]^{\frac{1}{p}}. \quad (2)$$

Now, by construction of the $\mathfrak{s}\mathfrak{r}$ -order, the interval-valued fractional integral satisfies

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_2(\mathfrak{d}) \right]^{\frac{1}{p}} = \left\langle \frac{\left[\mathcal{I}_{s^+}^{\alpha, \nu} \overline{\zeta_2}(\mathfrak{d}) \right]^{\frac{1}{p}} + \left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta_2}(\mathfrak{d}) \right]^{\frac{1}{p}}}{2}, \frac{\left[\mathcal{I}_{s^+}^{\alpha, \nu} \overline{\zeta_2}(\mathfrak{d}) \right]^{\frac{1}{p}} - \left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta_2}(\mathfrak{d}) \right]^{\frac{1}{p}}}{2} \right\rangle.$$

Combining (1) and (2) provides the following componentwise majorizations:

$$\begin{aligned} \left[\mathcal{I}_{s^+}^{\alpha, \nu} \overline{\zeta_2}(\mathfrak{d}) \right]^{\frac{1}{p}} &\leq \mathcal{M}^{\frac{1}{pq}} \mathcal{I}_{s^+}^{\alpha, \nu} \left[(\overline{\zeta_1})^{\frac{1}{q}} (\overline{\zeta_2})^{\frac{1}{p}} \right] (\mathfrak{d}), \\ \left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta_2}(\mathfrak{d}) \right]^{\frac{1}{p}} &\leq r^{\frac{1}{pq}} \mathcal{I}_{s^+}^{\alpha, \nu} \left[(\underline{\zeta_1})^{\frac{1}{q}} (\underline{\zeta_2})^{\frac{1}{p}} \right] (\mathfrak{d}). \end{aligned}$$

In terms of the $\mathfrak{s}\mathfrak{r}$ -ordering, this translates to

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_2(\mathfrak{d}) \right]^{\frac{1}{p}} \leq_{\mathfrak{cr}} \left[r^{\frac{1}{pq}}, \mathcal{M}^{\frac{1}{pq}} \right] \mathcal{I}_{s^+}^{\alpha, \nu} \left[(\zeta_1)^{\frac{1}{q}} (\zeta_2)^{\frac{1}{p}} \right] (\mathfrak{d}). \quad (3)$$

A parallel argument applied to ζ_1 —now invoking the reciprocal bounds $\frac{1}{\mathcal{M}} \underline{\zeta_2}(\eta) \leq \underline{\zeta_1}(\eta) \leq \frac{1}{r} \overline{\zeta_2}(\eta)$ produces the companion estimate

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_1(\mathfrak{d}) \right]^{\frac{1}{q}} \leq_{\mathfrak{cr}} \left[\frac{1}{\mathcal{M}^{\frac{1}{pq}}}, \frac{1}{r^{\frac{1}{pq}}} \right] \mathcal{I}_{s^+}^{\alpha, \nu} \left[(\zeta_1)^{\frac{1}{q}} (\zeta_2)^{\frac{1}{p}} \right] (\mathfrak{d}). \quad (4)$$

Finally, performing Minkowski-type multiplication of the interval relations (3) and (4) and invoking the monotonicity property of the $\mathfrak{s}\mathfrak{r}$ -order under multiplication by positive intervals, we arrive at

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_2(\mathfrak{d}) \right]^{\frac{1}{p}} \cdot \left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_1(\mathfrak{d}) \right]^{\frac{1}{q}} \leq_{\mathfrak{cr}} \left[r^{-\frac{1}{pq}}, \mathcal{M}^{\frac{1}{pq}} \right] \mathcal{I}_{s^+}^{\alpha, \nu} \left[\left(\zeta_2(\mathfrak{d}) \right)^{\frac{1}{p}} \cdot \left(\zeta_1(\mathfrak{d}) \right)^{\frac{1}{q}} \right].$$

Thus the desired inequality is established. $\square$

$\square$

Remark 3.5. Theorem 3.6 offers an innovative extension of the classical Hölder inequality within the framework of interval-valued functions and Erdélyi–Kober fractional operators. Its novelty lies in the following aspects:

- It incorporates fractional-order parameters into the interval-valued setting, thereby generalizing the results of Zhou et al. [67].
- In the special case where the Erdélyi–Kober integral reduces to the Riemann–Liouville fractional integral, the theorem recovers and even refines the inequalities reported in [77].

3.3. A Novel Erdélyi–Kober Fractional Minkowski-Type Inequality for Interval-Valued Functions.

Theorem 3.7. Let $s, d \in \mathcal{R}$ with $0 < s < d$ and consider $\zeta_2, \zeta_1 : [s, d] \rightarrow \mathcal{R}_I^+$ given in the form

$$\zeta_2(\eta) = [\underline{\zeta}_2(\eta), \overline{\zeta}_2(\eta)], \quad \zeta_1(\eta) = [\underline{\zeta}_1(\eta), \overline{\zeta}_1(\eta)],$$

for which the Erdélyi–Kober fractional integrals $I_{s^+}^{\alpha, \nu} \zeta_2^p(\eta)$ and $I_{s^+}^{\alpha, \nu} \zeta_1^p(\eta)$ are assumed to exist and remain finite. Let $\alpha \in (0, 1)$, $\nu > -1$, and let $p \leq 1$.

Assume further the existence of positive constants $r, \mathcal{M} > 0$ such that for all $\eta \in [s, d]$,

$$0 < r \leq \frac{\underline{\zeta}_2(\eta)}{\underline{\zeta}_1(\eta)} \leq \mathcal{M}, \quad 0 < r \leq \frac{\overline{\zeta}_2(\eta)}{\overline{\zeta}_1(\eta)} \leq \mathcal{M}.$$

Then, in the sense of the α -order, the following inequality holds:

$$\left[I_{s^+}^{\alpha, \nu} \zeta_2^p(d) \right]^{\frac{1}{p}} + \left[I_{s^+}^{\alpha, \nu} \zeta_1^p(d) \right]^{\frac{1}{p}} \leq_{cr} \left[\frac{1 + r(\mathcal{M} + 2)}{(1 + r)(1 + \mathcal{M})}, \frac{1 + \mathcal{M}(r + 2)}{(1 + r)(1 + \mathcal{M})} \right] \left[I_{s^+}^{\alpha, \nu} (\zeta_2 + \zeta_1)^p(d) \right]^{\frac{1}{p}}.$$

Proof. We again adopt the center–radius parametrization:

$$\zeta_2(\eta) = \langle \zeta_{2,c}(\eta), \zeta_{2,r}(\eta) \rangle, \quad \zeta_1(\eta) = \langle \zeta_{1,c}(\eta), \zeta_{1,r}(\eta) \rangle.$$

From the assumption $\frac{\overline{\zeta}_2(\eta)}{\overline{\zeta}_1(\eta)} \leq \mathcal{M}$ we deduce the pointwise estimate

$$(\mathcal{M} + 1)^p \overline{\zeta}_2^p(\eta) \leq \mathcal{M}^p (\overline{\zeta}_2 + \overline{\zeta}_1)^p(\eta). \quad (5)$$

Multiplying (5) by the Erdélyi–Kober kernel $\frac{d^{-\nu-\alpha+1}}{\Gamma(\alpha)} (d - \eta)^{\alpha-1} \eta^{-\alpha-\nu}$ and integrating over $[s, d]$ gives

$$\begin{aligned} & (\mathcal{M} + 1)^p \frac{d^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_s^d (d - \eta)^{\alpha-1} \eta^{-\alpha-\nu} \overline{\zeta}_2^p(\eta) d\eta \\ & \leq \mathcal{M}^p \frac{d^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_s^d (d - \eta)^{\alpha-1} \eta^{-\alpha-\nu} (\overline{\zeta}_2 + \overline{\zeta}_1)^p(\eta) d\eta, \end{aligned}$$

which, upon taking the p -th root, simplifies to

$$\left[I_{s^+}^{\alpha, \nu} \overline{\zeta}_2^p(d) \right]^{\frac{1}{p}} \leq \frac{\mathcal{M}}{\mathcal{M} + 1} \left[I_{s^+}^{\alpha, \nu} (\overline{\zeta}_2 + \overline{\zeta}_1)^p(d) \right]^{\frac{1}{p}}. \quad (6)$$

Analogously, the condition $\frac{\underline{\zeta}_2(\eta)}{\underline{\zeta}_1(\eta)} \leq r$ yields

$$(\mathfrak{r} + 1)^p \underline{\zeta}_2^p(\eta) \leq \mathfrak{r}^p (\underline{\zeta}_2 + \underline{\zeta}_1)^p(\eta). \quad (7)$$

Applying the same integral transformation to (7) we obtain

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta}_2^p(\mathfrak{d}) \right]^{\frac{1}{p}} \leq \frac{\mathfrak{r}}{\mathfrak{r} + 1} \left[\mathcal{I}_{s^+}^{\alpha, \nu} (\underline{\zeta}_2 + \underline{\zeta}_1)^p(\mathfrak{d}) \right]^{\frac{1}{p}}. \quad (8)$$

By the definition of the α -order for the interval-valued integral we have

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_2^p(\mathfrak{d}) \right]^{\frac{1}{p}} = \left\langle \frac{\left[\mathcal{I}_{s^+}^{\alpha, \nu} \overline{\zeta}_2^p(\mathfrak{d}) \right]^{\frac{1}{p}} + \left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta}_2^p(\mathfrak{d}) \right]^{\frac{1}{p}}}{2}, \frac{\left[\mathcal{I}_{s^+}^{\alpha, \nu} \overline{\zeta}_2^p(\mathfrak{d}) \right]^{\frac{1}{p}} - \left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta}_2^p(\mathfrak{d}) \right]^{\frac{1}{p}}}{2} \right\rangle.$$

Thus, the componentwise bounds (6) and (8) lead to the α -order relation

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_2^p(\mathfrak{d}) \right]^{\frac{1}{p}} \leq_{\alpha} \left[\frac{\mathfrak{r}}{\mathfrak{r} + 1}, \frac{\mathcal{M}}{\mathcal{M} + 1} \right] \left[\mathcal{I}_{s^+}^{\alpha, \nu} (\zeta_2 + \zeta_1)^p(\mathfrak{d}) \right]^{\frac{1}{p}}. \quad (9)$$

We now proceed to derive a symmetric estimate for ζ_1 . From $\frac{\overline{\zeta}_2(\eta)}{\overline{\zeta}_1(\eta)} \leq \mathfrak{r}$ we infer

$$\left(1 + \frac{1}{\mathfrak{r}} \right)^p \overline{\zeta}_1^p(\eta) \leq \frac{1}{\mathfrak{r}^p} (\overline{\zeta}_2 + \overline{\zeta}_1)^p(\eta). \quad (10)$$

Integrating (10) with the Erdélyi–Kober kernel yields

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \overline{\zeta}_1^p(\mathfrak{d}) \right]^{\frac{1}{p}} \leq \frac{1}{\mathfrak{r} + 1} \left[\mathcal{I}_{s^+}^{\alpha, \nu} (\overline{\zeta}_2 + \overline{\zeta}_1)^p(\mathfrak{d}) \right]^{\frac{1}{p}}. \quad (11)$$

Similarly, employing $\frac{\underline{\zeta}_2(\eta)}{\underline{\zeta}_1(\eta)} \leq \mathcal{M}$ gives

$$\left(1 + \frac{1}{\mathcal{M}} \right)^p \underline{\zeta}_1^p(\eta) \leq \frac{1}{\mathcal{M}^p} (\underline{\zeta}_2 + \underline{\zeta}_1)^p(\eta), \quad (12)$$

which, after integration, becomes

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta}_1^p(\mathfrak{d}) \right]^{\frac{1}{p}} \leq \frac{1}{\mathcal{M} + 1} \left[\mathcal{I}_{s^+}^{\alpha, \nu} (\underline{\zeta}_2 + \underline{\zeta}_1)^p(\mathfrak{d}) \right]^{\frac{1}{p}}. \quad (13)$$

Again, invoking the α -order representation for ζ_1 :

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_1^p(\mathfrak{d}) \right]^{\frac{1}{p}} = \left\langle \frac{\left[\mathcal{I}_{s^+}^{\alpha, \nu} \overline{\zeta}_1^p(\mathfrak{d}) \right]^{\frac{1}{p}} + \left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta}_1^p(\mathfrak{d}) \right]^{\frac{1}{p}}}{2}, \frac{\left[\mathcal{I}_{s^+}^{\alpha, \nu} \overline{\zeta}_1^p(\mathfrak{d}) \right]^{\frac{1}{p}} - \left[\mathcal{I}_{s^+}^{\alpha, \nu} \underline{\zeta}_1^p(\mathfrak{d}) \right]^{\frac{1}{p}}}{2} \right\rangle.$$

Combining (11) and (13) we obtain the α -order estimate

$$\left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_1^p(\mathfrak{d}) \right]^{\frac{1}{p}} \leq_{\alpha} \left[\frac{1}{\mathcal{M} + 1}, \frac{1}{\mathfrak{r} + 1} \right] \left[\mathcal{I}_{s^+}^{\alpha, \nu} (\zeta_2 + \zeta_1)^p(\mathfrak{d}) \right]^{\frac{1}{p}}. \quad (14)$$

Finally, adding the two α -ordered relations (9) and (14) and exploiting the linearity of the α -order with respect to interval addition, we arrive at

$$\begin{aligned}
\left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_2^p(\mathfrak{d}) \right]^{\frac{1}{p}} + \left[\mathcal{I}_{s^+}^{\alpha, \nu} \zeta_1^p(\mathfrak{d}) \right]^{\frac{1}{p}} &\leq_{\text{cr}} \left[\frac{r}{r+1} + \frac{1}{\mathcal{M}+1}, \frac{\mathcal{M}}{\mathcal{M}+1} + \frac{1}{r+1} \right] \\
&\quad \times \left[\mathcal{I}_{s^+}^{\alpha, \nu} (\zeta_2 + \zeta_1)^p(\mathfrak{d}) \right]^{\frac{1}{p}} \\
&= \left[\frac{1+r(\mathcal{M}+2)}{(1+r)(1+\mathcal{M})}, \frac{1+\mathcal{M}(r+2)}{(1+r)(1+\mathcal{M})} \right] \\
&\quad \times \left[\mathcal{I}_{s^+}^{\alpha, \nu} (\zeta_2 + \zeta_1)^p(\mathfrak{d}) \right]^{\frac{1}{p}}.
\end{aligned}$$

This chain of inequalities completes the proof. $\square$

Remark 3.6. A limiting configuration of Theorem 3.7 is attained by taking $\nu = 0$ together with $\alpha \rightarrow 1$, while additionally requiring that the endpoint functions coincide, i.e., $\underline{\zeta}_2 = \overline{\zeta}_2$ and $\underline{\zeta}_1 = \overline{\zeta}_1$. Under these specifications, the interval-valued functions ζ_2, ζ_1 collapse to ordinary real-valued mappings and the Erdélyi–Kober operator $\mathcal{I}_{s^+}^{\alpha, \nu}$ reduces exactly to the classical Riemann integral over $[s, \mathfrak{d}]$.

In this reduced setting, the α -order inequality established in Theorem 3.7 simplifies to a pair of scalar estimates that sandwich the sum of the p -th power integrals:

$$\frac{1+r(\mathcal{M}+2)}{(1+r)(1+\mathcal{M})} \left(\int_s^{\mathfrak{d}} (\zeta_2(\eta) + \zeta_1(\eta))^p d\eta \right)^{\frac{1}{p}} \leq \left(\int_s^{\mathfrak{d}} \zeta_2^p(\eta) d\eta \right)^{\frac{1}{p}} + \left(\int_s^{\mathfrak{d}} \zeta_1^p(\eta) d\eta \right)^{\frac{1}{p}}$$

and

$$\left(\int_s^{\mathfrak{d}} \zeta_2^p(\eta) d\eta \right)^{\frac{1}{p}} + \left(\int_s^{\mathfrak{d}} \zeta_1^p(\eta) d\eta \right)^{\frac{1}{p}} \leq \frac{1+\mathcal{M}(r+2)}{(1+r)(1+\mathcal{M})} \left(\int_s^{\mathfrak{d}} (\zeta_2(\eta) + \zeta_1(\eta))^p \eta d\eta \right)^{\frac{1}{p}}.$$

3.4. Applications. The Hermite–Hadamard inequality serves as a fundamental tool not only in theoretical convex analysis but also in numerical integration, providing intrinsic error estimates for quadrature rules. When the integrand is interval-valued, classical midpoint and trapezoidal approximations must be interpreted in terms of inclusion relations, leading to guaranteed enclosures of the exact integral. The Erdélyi–Kober fractional operators, due to their power-law weight structure, arise naturally in problems exhibiting radial symmetry and nonlocal diffusion phenomena.

Let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_I^+$ be an interval-valued convex function represented as

$$\mathcal{F}(\zeta) = [\underline{\mathcal{F}}(\zeta), \overline{\mathcal{F}}(\zeta)],$$

with $\underline{\mathcal{F}}, \overline{\mathcal{F}}$ convex Lebesgue integrable functions satisfying $0 \leq \underline{\mathcal{F}}(\zeta) \leq \overline{\mathcal{F}}(\zeta)$ for all $\zeta \in [\zeta_1, \zeta_2]$. Consider a partition $\mathcal{P} : \zeta_1 = v_0 < v_1 < \dots < v_{n-1} < v_n = \zeta_2$ of the interval $[\zeta_1, \zeta_2]$.

For numerical integration, we consider the Erdélyi–Kober weighted trapezoidal rule defined as follows: At each subinterval $[v_i, v_{i+1}]$, the approximation accounts for the singular kernel characteristic of the Erdélyi–Kober operator. Define the change of variable motivated by the transformation structure in Theorem 3.1:

$$\eta = \left(\frac{v^{\nu+1} - v_i^{\nu+1}}{v_{i+1}^{\nu+1} - v_i^{\nu+1}} \right), \quad v \in [v_i, v_{i+1}], \quad \eta \in [0, 1].$$

Then the Erdélyi–Kober fractional integral over $[v_i, v_{i+1}]$ can be expressed as:

$$\mathcal{I}_{v_i^+}^{\alpha, \nu} \mathcal{F}(v_{i+1}) = \frac{v_{i+1}^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_{v_i}^{v_{i+1}} (v_{i+1} - v)^{\alpha-1} v^{-\alpha-\nu} \mathcal{F}(v) dv.$$

For computational purposes, we approximate $\mathcal{F}(v)$ on $[v_i, v_{i+1}]$ by its linear interpolation:

$$\mathcal{L}_i(v) = \mathcal{F}(v_i) + \frac{\mathcal{F}(v_{i+1}) - \mathcal{F}(v_i)}{v_{i+1} - v_i} (v - v_i).$$

The Erdélyi–Kober weighted trapezoidal approximation for $\mathcal{I}_{v_i^+}^{\alpha, \nu} \mathcal{F}(v_{i+1})$ is then:

$$\mathcal{T}_i^{\text{EK}}(\mathcal{F}) = \frac{v_{i+1}^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_{v_i}^{v_{i+1}} (v_{i+1} - v)^{\alpha-1} v^{-\alpha-\nu} \mathcal{L}_i(v) dv.$$

After simplification using the substitution $v = (\eta v_i^{\nu+1} + (1-\eta)v_{i+1}^{\nu+1})^{\frac{1}{\nu+1}}$, we obtain the closed form:

$$\mathcal{T}_i^{\text{EK}}(\mathcal{F}) = \frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} \cdot \frac{\Gamma(\alpha+1)}{(\nu+1)^\alpha} (v_{i+1}^{\nu+1} - v_i^{\nu+1})^\alpha v_{i+1}^{-\nu(\alpha+1)} + \mathcal{R}_i,$$

where $\mathcal{R}_i$ denotes higher-order remainder terms involving $\mathcal{F}(v_{i+1}) - \mathcal{F}(v_i)$. However, for our error analysis, it is more natural to work directly with the transformed representation.

The inclusion error on the i -th subinterval is defined as:

$$\mathcal{E}_i(\mathcal{F}) = \mathcal{I}_{v_i^+}^{\alpha, \nu} \mathcal{F}(v_{i+1}) - \frac{v_{i+1}^{-\nu-\alpha+1}}{\Gamma(\alpha)} \frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} \int_{v_i}^{v_{i+1}} (v_{i+1} - v)^{\alpha-1} v^{-\alpha-\nu} dv.$$

Now we establish the crucial connection. From Theorem 3.1, for the interval $[v_i, v_{i+1}]$ we have the Hermite–Hadamard inclusion in the Erdélyi–Kober fractional context:

$$\begin{aligned} \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) &\leq_{\text{cr}} \frac{\Gamma(\alpha+1)(v_i v_{i+1})^{\frac{\nu+\alpha}{2}}}{2(v_{i+1}^{\nu+1} - v_i^{\nu+1})^\alpha} \left[\frac{\mathcal{I}_{v_i^+}^{\alpha, \nu} \mathcal{F}(v_{i+1})}{v_{i+1}^\nu} + \frac{\mathcal{I}_{v_{i+1}^-}^{\alpha, \nu} \mathcal{F}(v_i)}{v_i^\nu} \right] \\ &\leq_{\text{cr}} \frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2}. \end{aligned} \quad (3.24)$$

This double inclusion provides both lower and upper bounds for the fractional integral combination in terms of the midpoint and endpoint values, respectively.

Theorem 3.8. Let $\mathcal{F} : [\zeta_1, \zeta_2] \rightarrow \mathcal{R}_J^+$ be an interval-valued convex function with $0 < \zeta_1 < \zeta_2$, represented as $\mathcal{F}(\zeta) = [\underline{\mathcal{F}}(\zeta), \overline{\mathcal{F}}(\zeta)]$ where $\underline{\mathcal{F}}, \overline{\mathcal{F}}$ are convex Lebesgue integrable functions. For any partition $\mathcal{P}$ of $[\zeta_1, \zeta_2]$ and parameters $\alpha \in (0, 1)$, $\nu > -1$, the Erdélyi–Kober weighted quadrature error satisfies the inclusion bound:

$$|\mathcal{E}(\mathcal{F}, \mathcal{P})| \leq_{\text{cr}} \sum_{i=0}^{n-1} \mathcal{E}_i(\alpha, \nu),$$

where the subinterval error bound is:

$$\mathcal{E}_i(\alpha, v) = \frac{v_{i+1}^{-v-\alpha+1}}{\Gamma(\alpha)} \left\{ \frac{v_{i+1}^v (v_{i+1}^{v+1} - v_i^{v+1})^\alpha}{\Gamma(\alpha+1)(v_i v_{i+1})^{\frac{v+\alpha}{2}}} \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) - \frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} \int_{v_i}^{v_{i+1}} (v_{i+1} - v)^{\alpha-1} v^{-\alpha-v} dv \right\}.$$

Moreover, an alternative inclusion bound in terms of the endpoint values is:

$$|\mathcal{E}(\mathcal{F}, \mathcal{P})| \leq_{\text{cr}} \frac{1}{2} \sum_{i=0}^{n-1} \left(\frac{\Gamma(\alpha+1)(v_i v_{i+1})^{\frac{v+\alpha}{2}}}{2(v_{i+1}^{v+1} - v_i^{v+1})^\alpha} \right)^{-1} \left[\frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} - \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) \right].$$

Proof. We prove the first bound. From the definition of the Erdélyi–Kober fractional integral and the quadrature error decomposition, we have:

$$\begin{aligned} \mathcal{E}_i(\mathcal{F}) &= \frac{v_{i+1}^{-v-\alpha+1}}{\Gamma(\alpha)} \int_{v_i}^{v_{i+1}} (v_{i+1} - v)^{\alpha-1} v^{-\alpha-v} \mathcal{F}(v) dv \\ &\quad - \frac{v_{i+1}^{-v-\alpha+1}}{\Gamma(\alpha)} \frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} \int_{v_i}^{v_{i+1}} (v_{i+1} - v)^{\alpha-1} v^{-\alpha-v} dv. \end{aligned}$$

Now, from the Hermite–Hadamard inclusion (3.24) established in Theorem 3.1, we isolate the term involving $\mathcal{I}_{v_i^+}^{\alpha, v} \mathcal{F}(v_{i+1})$:

$$\frac{\mathcal{I}_{v_i^+}^{\alpha, v} \mathcal{F}(v_{i+1})}{v_{i+1}^v} \leq_{\text{cr}} \frac{2(v_{i+1}^{v+1} - v_i^{v+1})^\alpha}{\Gamma(\alpha+1)(v_i v_{i+1})^{\frac{v+\alpha}{2}}} \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) - \frac{\mathcal{I}_{v_{i+1}^-}^{\alpha, v} \mathcal{F}(v_i)}{v_i^v}.$$

Since $\mathcal{F}$ is convex and thus nonnegative, the right-sided Erdélyi–Kober operator $\mathcal{I}_{v_{i+1}^-}^{\alpha, v} \mathcal{F}(v_i)$ is nonnegative in the interval-valued sense. Therefore, we obtain the lower bound:

$$\mathcal{I}_{v_i^+}^{\alpha, v} \mathcal{F}(v_{i+1}) \leq_{\text{cr}} \frac{2v_{i+1}^v (v_{i+1}^{v+1} - v_i^{v+1})^\alpha}{\Gamma(\alpha+1)(v_i v_{i+1})^{\frac{v+\alpha}{2}}} \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right).$$

Substituting this lower bound into the error expression yields:

$$\begin{aligned} \mathcal{E}_i(\mathcal{F}) &\leq_{\text{cr}} \frac{v_{i+1}^{-v-\alpha+1}}{\Gamma(\alpha)} \left\{ \frac{2v_{i+1}^v (v_{i+1}^{v+1} - v_i^{v+1})^\alpha}{\Gamma(\alpha+1)(v_i v_{i+1})^{\frac{v+\alpha}{2}}} \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) - \frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} \int_{v_i}^{v_{i+1}} (v_{i+1} - v)^{\alpha-1} v^{-\alpha-v} dv \right\}. \end{aligned}$$

The upper bound follows similarly using the right-hand side of (3.24). Taking the absolute value in the sense of interval width and summing over all subintervals gives the desired result. $\square$

Corollary 3.1 (Special Case: $v = 0$). *Taking $v = 0$, the Erdélyi–Kober fractional operator reduces to the Riemann–Liouville fractional integral (up to normalization). The error bound simplifies to:*

$$|\mathcal{E}(\mathcal{F}, \mathcal{P})| \leq_{\text{cr}} \sum_{i=0}^{n-1} \frac{v_{i+1}^{1-\alpha}}{\Gamma(\alpha)} \left\{ \frac{2v_{i+1}^0 (v_{i+1} - v_i)^\alpha}{\Gamma(\alpha+1)(v_i v_{i+1})^{\frac{\alpha}{2}}} \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) - \frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} \int_{v_i}^{v_{i+1}} (v_{i+1} - v)^{\alpha-1} v^{-\alpha} dv \right\}. \quad (3.25)$$

For the classical case $\alpha = 1$, this further reduces to the standard trapezoidal rule error bound for interval-valued convex functions:

$$|\mathcal{E}(\mathcal{F}, \mathcal{P})| \leq_{\text{cr}} \sum_{i=0}^{n-1} (v_{i+1} - v_i) \left[\frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} - \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) \right].$$

Corollary 3.2 (Uniform Partition Case). For a uniform partition with $v_{i+1} - v_i = \frac{\zeta_2 - \zeta_1}{n} = h$, we have the asymptotic inclusion estimate:

$$|\mathcal{E}(\mathcal{F}, \mathcal{P})| \leq_{\text{cr}} O(h^{\alpha+1}) \sum_{i=0}^{n-1} \left[\frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} - \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) \right].$$

In particular, if $\mathcal{F}$ is bounded by $M_I = [\underline{M}, \overline{M}]$ on $[\zeta_1, \zeta_2]$ in the interval-valued sense, then:

$$|\mathcal{E}(\mathcal{F}, \mathcal{P})| \leq_{\text{cr}} \frac{M_I(\zeta_2^{v+1} - \zeta_1^{v+1})^\alpha}{2n^\alpha} \Phi(\alpha, v),$$

where $\Phi(\alpha, v)$ is a constant depending only on the fractional parameters.

Corollary 3.3 (Asymptotic Behavior as $\alpha \rightarrow 1$). When $\alpha \rightarrow 1$, the Erdélyi–Kober fractional integral reduces to the weighted integral with classical quadrature. Taking the limit $\alpha \rightarrow 1$ in the inclusion bound, we recover:

$$\lim_{\alpha \rightarrow 1} |\mathcal{E}(\mathcal{F}, \mathcal{P})| \leq_{\text{cr}} \frac{1}{2} \sum_{i=0}^{n-1} \left(\frac{(v_i v_{i+1})^{\frac{v+1}{2}}}{v_{i+1}^{v+1} - v_i^{v+1}} \right) \left[\frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} - \mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) \right].$$

Corollary 3.4 (Asymptotic Behavior as $\alpha \rightarrow 0$). When $\alpha \rightarrow 0$, the Erdélyi–Kober fractional integral reduces to the function evaluation at the endpoint (up to normalization). The limiting error bound becomes:

$$\lim_{\alpha \rightarrow 0} |\mathcal{E}(\mathcal{F}, \mathcal{P})| \leq_{\text{cr}} \sum_{i=0}^{n-1} \frac{v_{i+1}^{-v+1}}{1+v} \left[\mathcal{F}\left(\frac{v_i + v_{i+1}}{2}\right) - \frac{\mathcal{F}(v_i) + \mathcal{F}(v_{i+1})}{2} \cdot \frac{v_{i+1}^{v+1} - v_i^{v+1}}{(v+1)v_{i+1}^{v+1}} \right].$$

This provides a weighted bound that depends only on the function values at the endpoints and midpoint of each subinterval, scaled by the power-law weight characteristic of the Erdélyi–Kober operator.

Remark 3.7. The quadrature error bounds established in Theorem 3.8 are sharp in the sense that equality is attained for affine functions $\mathcal{F}(\zeta) = [c\zeta + d, c\zeta + d]$ with $c, d \geq 0$. Moreover, these inclusion bounds provide guaranteed numerical enclosures for fractional integrals of interval-valued data, which is particularly valuable in uncertainty quantification and validated numerics for fractional differential equations.

4. CONCLUSION

In this study, we have extended the Erdélyi–Kober fractional integrals to interval-valued functions for the first time, providing a unifying framework that recovers several known results from the classical and Riemann–Liouville settings under both classical and interval order relations. We have established new bounds for Hermite–Hadamard and midpoint-type inequalities, explicitly expressed using Gamma and Beta functions. Additionally, we derived generalized Hölder and Minkowski type inequalities for interval-valued mappings via the Erdélyi–Kober operator. The accuracy and applicability of our results are demonstrated through non-trivial examples and remarks, showing that existing inequalities in the literature emerge as special cases of our findings.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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