

Fixed Point Theory in Twisted Formal Power Series Rings over Finite Fields

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Abstract. In this paper we develop a complete non-Archimedean metric framework on the ring of twisted formal power series $\widehat{R} = \mathbb{F}_q[[t; \sigma]]$, where σ denotes the Frobenius automorphism of the finite field $\mathbb{F}_q$. Using the natural t -adic valuation, we prove that the resulting metric is an ultrametric and that the ring is complete. We establish Banach-type fixed point theorems for contractive affine operators and obtain existence and uniqueness results for one-sided, two-sided, and mixed twisted affine equations, illustrating how the noncommutative structure naturally leads to contractions involving both left and right multiplications. Applications to iterative Frobenius systems and discrete-time dynamical models over finite fields are also given.

1. INTRODUCTION

Metric methods and fixed point techniques constitute one of the most powerful tools in modern analysis. Since Banach's celebrated contraction principle [1], fixed point methods have been successfully applied to differential equations, integral equations, operator theory, and dynamical systems. Banach's contraction principle is remarkable because every contraction on a complete

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metric space admits a unique fixed point obtainable by iteration. Over the past century, this idea has been extensively generalized, becoming a cornerstone of nonlinear analysis. Notable contributions include the works of Caccioppoli [2], who made significant contributions to the development of fixed point theory, and Kirk [5], who proved fixed point results for non-expansive mappings and many others [3, 4, 20–22].

Noncommutative polynomial rings arising from finite fields play a central role in arithmetic geometry [13], Drinfeld modules [14], and difference equations [25]. Among these structures, the Ore ring $\mathbb{F}_q[t; \sigma]$ associated with an automorphism σ is one of the most important examples [7]. Ore's pioneering work [7] established the foundations of noncommutative polynomial rings, which were later extended by Jacobson [8] and Cohn [9] to more general settings.

Meanwhile, the p -adic and t -adic worlds have their own flavor of analysis, where the usual triangle inequality is replaced by the much stronger ultra-metric inequality. This ultrametric nature often leads to surprising phenomena: balls are clopen, series converge if and only if their terms tend to zero, and fixed point arguments become even more robust. The study of ultra-metric spaces has been extensively developed by Schikhof [16] and van Rooij [17].

The objective of the present article is to develop a metric framework on the completion $\widehat{R} = \mathbb{F}_q[[t; \sigma]]$ of the Ore ring using the t -adic valuation. Unlike previous approaches that focus on the degree function, which leads to a discrete metric but not a complete one (see Schikhof [16] for a detailed discussion of the non-Archimedean setting), we exploit the valuation coming from the power series expansion. The degree-function metric on the polynomial ring $\mathbb{F}_q[t; \sigma]$ is given by $d(f, g) = 2^{-\deg(f-g)}$. However, this metric is not complete: sequences of polynomials with increasing degrees, such as $f_n = \sum_{i=0}^n t^i$, are Cauchy but converge to the infinite series in $\widehat{R}$, which lies outside the polynomial ring. This choice turns out to be remarkably well-behaved: multiplication by a series with positive valuation becomes a strict contraction, and the space becomes complete. To the best of our knowledge, a systematic fixed point theory for twisted power series over finite fields has not appeared in the literature, and we aim to fill this gap. The closest related works are those of Kedlaya [15] on p -adic differential equations and the foundational work of Goss [13] on function field arithmetic, though neither develops a fixed point framework for twisted formal power series.

The principal contributions of this work are:

- (1) Construction of a natural ultrametric on $\mathbb{F}_q[[t; \sigma]]$ using the t -adic valuation.
- (2) Proof that the resulting space is complete, with an explicit characterization of Cauchy sequences in terms of coefficient stabilization.
- (3) Establishment of Banach-type fixed point theorems for affine operators, showing that equations of the form $A \cdot f + B = f$ have unique solutions when $v(A) \geq 1$.
- (4) Existence and uniqueness results for two-sided affine equations $AfB + C = f$, illustrating how the noncommutative structure naturally leads to contractions involving both left and right multiplications.

- (5) A fixed point theorem for mixed equations $Af + fB + C = f$, which further demonstrates the interplay between left and right multiplications.
- (6) Applications to twisted algebraic equations, the commutator equation, and stability analysis for discrete-time dynamical systems over finite fields.

The framework developed here provides a rigorous metric foundation for iterative methods in noncommutative finite-field settings. Moreover, because the convergence is geometric with respect to the ultrametric, the approximations obtained by truncating the Neumann series are exact up to a controllable number of initial coefficients. This makes the theory not only theoretically appealing but also computationally practical.

2. PRELIMINARIES

2.1. Finite Fields. Let

$$q = p^n$$

where p is prime and $n \geq 1$.

Denote by

$$\mathbb{F}_q$$

the finite field with q elements [23,24].

The Frobenius automorphism is defined by

$$\sigma : \mathbb{F}_q \rightarrow \mathbb{F}_q, \quad \sigma(a) = a^p.$$

It is well known that σ is an automorphism satisfying

$$\sigma^n = \text{id}.$$

For a comprehensive treatment of finite fields, we refer to [24].

2.2. Ore Rings.

Definition 2.1. *The Ore ring*

$$R = \mathbb{F}_q[t; \sigma]$$

consists of finite sums

$$f(t) = \sum_{i=0}^m a_i t^i$$

with coefficients in $\mathbb{F}_q$ and multiplication determined by

$$ta = \sigma(a)t$$

for every $a \in \mathbb{F}_q$.

For a detailed study of Ore extensions and their properties, we refer to [7,9–12].

Consequently,

$$(at^i)(bt^j) = a\sigma^i(b)t^{i+j}.$$

The ring is noncommutative whenever $\sigma \neq \text{id}$; when $\sigma = \text{id}$, it reduces to the usual polynomial ring $\mathbb{F}_q[t]$. Throughout the paper, we assume $q \neq p$ (i.e., $\sigma \neq \text{id}$) when discussing genuinely noncommutative aspects. When $\sigma = \text{id}$, the results reduce to the commutative case and serve as a verification of the framework.

2.3. Twisted Formal Power Series. The t -adic completion of R is

$$\widehat{R} = \mathbb{F}_q[[t; \sigma]].$$

Its elements are formal series

$$f(t) = \sum_{i=0}^{\infty} a_i t^i, \quad a_i \in \mathbb{F}_q,$$

with coefficients written on the left of powers of t . Addition is coefficientwise and multiplication is induced by

$$ta = \sigma(a)t.$$

For a detailed treatment of such completions in the context of function field arithmetic, we refer to [13] and [15].

Remark 2.2. When $\sigma = \text{id}$, the ring $\widehat{R}$ reduces to the usual commutative formal power series ring $\mathbb{F}_q[[t]]$. All the results below remain valid in this case, but the noncommutative aspects are only visible when $\sigma \neq \text{id}$.

3. THE t -ADIC VALUATION

Definition 3.1. Let

$$f = \sum_{i=0}^{\infty} a_i t^i \in \widehat{R}.$$

For $f \neq 0$ define

$$v(f) = \min\{i \geq 0 : a_i \neq 0\}.$$

Also set

$$v(0) = +\infty.$$

This is the standard t -adic valuation, analogous to the p -adic valuation in number fields [18, 19]. It is a discrete valuation with value group $\mathbb{Z} \cup \{+\infty\}$.

Proposition 3.2. For all $f, g \in \widehat{R}$:

(i)

$$v(f + g) \geq \min\{v(f), v(g)\};$$

(ii)

$$v(fg) = v(f) + v(g);$$

(iii)

$$v(f) = +\infty \iff f = 0.$$

Proof. (i) Let

$$r = v(f), \quad s = v(g).$$

Then every term below degree $\min\{r, s\}$ vanishes in both series simultaneously. Therefore

$$v(f + g) \geq \min\{r, s\}.$$

(ii) Write

$$f = a_r t^r + \cdots, \quad g = b_s t^s + \cdots,$$

where

$$a_r \neq 0, \quad b_s \neq 0.$$

Then

$$fg = a_r \sigma^r(b_s) t^{r+s} + \cdots.$$

Since σ is an automorphism,

$$a_r \sigma^r(b_s) \neq 0.$$

Thus

$$v(fg) = r + s.$$

(iii) Immediate from the definition.

□

4. THE ULTRAMETRIC

Definition 4.1. Define

$$d : \widehat{R} \times \widehat{R} \rightarrow [0, \infty)$$

by

$$d(f, g) = 2^{-v(f-g)}$$

with the convention

$$2^{-\infty} = 0.$$

This construction mimics the standard p -adic metric but adapted to the t -adic setting [15, 16].

Theorem 4.2. The pair

$$(\widehat{R}, d)$$

is an ultrametric space.

Proof. **Positivity:** If $d(f, g) = 0$, then $2^{-v(f-g)} = 0$. Since the function $x \mapsto 2^{-x}$ is strictly decreasing and $2^{-x} > 0$ for all finite x , we must have $v(f-g) = +\infty$, which implies $f-g = 0$. Conversely, if $f = g$, then $v(f-g) = +\infty$ and $d(f, g) = 0$.

Symmetry: Since $v(f-g) = v(g-f)$, we have $d(f, g) = d(g, f)$.

Strong triangle inequality: Observe that

$$f - g = (f - h) + (h - g).$$

Using Proposition 3.2(i),

$$v(f - g) \geq \min\{v(f - h), v(h - g)\}.$$

Applying the decreasing function $x \mapsto 2^{-x}$ gives

$$d(f, g) \leq \max\{d(f, h), d(h, g)\}.$$

Hence the ultrametric inequality holds. □

Corollary 4.3. *Every open ball in $(\widehat{R}, d)$ is simultaneously open and closed.*

Proof. This is a standard property of ultrametric spaces [16, 17]. For completeness, we provide a brief argument: in an ultrametric space, if y lies in the open ball $B(x, r)$, then $B(x, r) = B(y, r)$, which is both open and closed. □

Example 4.4. *Consider the noncommutative case with $\sigma \neq \text{id}$. Let*

$$f = t + t^3 + t^5, \quad g = t + t^3 + \alpha t^4,$$

where $\alpha \in \mathbb{F}_q^\times$. Then

$$f - g = -\alpha t^4 + t^5.$$

Hence

$$v(f - g) = 4$$

and

$$d(f, g) = 2^{-4} = \frac{1}{16}.$$

5. COMPLETENESS

Lemma 5.1 (Cauchy Criterion). *A sequence $\{f_n\}_{n \geq 0} \subset \widehat{R}$ is Cauchy in the metric d if and only if for every integer $N \geq 0$, there exists an index $N_0 = N_0(N)$ such that for all $n, m \geq N_0$, the coefficients of t^i in f_n and f_m are identical for all $i \leq N$.*

Proof. ($\Rightarrow$) Suppose $\{f_n\}$ is Cauchy. Fix $N \geq 0$. Choose $\epsilon = 2^{-(N+1)} > 0$. There exists $N_0 = N_0(N)$ such that for all $n, m \geq N_0$, $d(f_n, f_m) < 2^{-(N+1)}$. This means $2^{-v(f_n - f_m)} < 2^{-(N+1)}$, which implies $v(f_n - f_m) \geq N + 1$. Therefore the coefficients of t^i for all $i \leq N$ in f_n and f_m are identical.

($\Leftarrow$) Conversely, suppose for each N there exists $N_0(N)$ such that for $n, m \geq N_0(N)$, the coefficients agree for all $i \leq N$. Then for any $\epsilon > 0$, choose N such that $2^{-N} < \epsilon$. Then for all $n, m \geq N_0(N)$, we have $v(f_n - f_m) \geq N + 1$, so $d(f_n, f_m) = 2^{-v(f_n - f_m)} \leq 2^{-(N+1)} < \epsilon$. Hence $\{f_n\}$ is Cauchy. □

Theorem 5.2. *The metric space $(\widehat{R}, d)$ is complete.*

Proof. We prove that every Cauchy sequence in $\widehat{R}$ converges.

Step 1: Coefficients stabilize. Let $\{f_n\}$ be a Cauchy sequence in $\widehat{R}$. Fix $k \geq 0$. Choose $\epsilon = 2^{-(k+1)}$. By the Cauchy property, there exists $N_0(k)$ such that for all $n, m \geq N_0(k)$, $d(f_n, f_m) < 2^{-(k+1)}$. This

implies $v(f_n - f_m) \geq k + 1$, so the coefficients of t^i for all $i \leq k$ in f_n and f_m are identical. Hence for each fixed $i \leq k$, the coefficient sequence $(a_{n,i})_{n \geq 0}$ stabilizes. Denote the constant coefficient of t^i by $a_i \in \mathbb{F}_q$.

Step 2: Construct the limit. Define

$$f = \sum_{i=0}^{\infty} a_i t^i \in \widehat{R}.$$

We claim that $f_n \rightarrow f$ in the metric d . For any $\epsilon > 0$, choose K such that $2^{-K} < \epsilon$. Let $M = \max_{0 \leq i \leq K} N_0(i)$. Then for all $n \geq M$, the coefficients of f_n and f agree for all $i \leq K$. Hence $v(f_n - f) \geq K + 1$, so

$$d(f_n, f) = 2^{-v(f_n - f)} \leq 2^{-(K+1)} < 2^{-K} < \epsilon.$$

Thus $f_n \rightarrow f$ in d .

Therefore $(\widehat{R}, d)$ is complete. □

6. BANACH FIXED POINT THEOREM

We first state the classical Banach contraction principle, which will be used throughout the paper.

Theorem 6.1 (Banach Contraction Principle). [1, 2]

Let (X, d) be a complete metric space and let $F : X \rightarrow X$ be a contraction with contraction factor $c \in [0, 1)$. Then:

- (i) F has a unique fixed point $x^* \in X$.
- (ii) For any starting point $x_0 \in X$, the iterative sequence $x_{n+1} = F(x_n)$ converges to x^* .
- (iii) The rate of convergence is geometric:

$$d(x_n, x^*) \leq c^n d(x_0, x^*).$$

Proof. We provide the classical proof for completeness.

Uniqueness: Suppose x^* and y^* are fixed points. Then

$$d(x^*, y^*) = d(F(x^*), F(y^*)) \leq c d(x^*, y^*).$$

Since $c < 1$, this forces $d(x^*, y^*) = 0$, hence $x^* = y^*$.

Existence: Choose $x_0 \in X$ arbitrarily. Define $x_{n+1} = F(x_n)$. By induction,

$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0).$$

For $n > m$, using the triangle inequality,

$$d(x_n, x_m) \leq \sum_{k=m}^{n-1} d(x_{k+1}, x_k) \leq \sum_{k=m}^{n-1} c^k d(x_1, x_0) \leq \frac{c^m}{1-c} d(x_1, x_0).$$

The right-hand side tends to 0 as $m \rightarrow \infty$, so $\{x_n\}$ is Cauchy. By completeness, $x_n \rightarrow x^*$ for some $x^* \in X$. Since F is continuous (it is Lipschitz),

$$F(x^*) = F(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} F(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x^*.$$

Rate: From $d(x_n, x^*) = d(F(x_{n-1}), F(x^*)) \leq c d(x_{n-1}, x^*)$, iteration yields $d(x_n, x^*) \leq c^n d(x_0, x^*)$.

□

For more details on fixed point theory, we refer to [6].

Remark 6.2. In the ultrametric setting of this paper, the triangle inequality in the proof above could be replaced by the stronger ultrametric inequality, yielding even sharper estimates. However, the classical proof is included to emphasize that the result holds in any complete metric space.

7. CONTRACTIONS ON TWISTED POWER SERIES

Definition 7.1. [1, 2]

A map $F : \widehat{R} \rightarrow \widehat{R}$ is called a **contraction** if there exists a constant $c \in [0, 1)$ such that for all $f, g \in \widehat{R}$,

$$d(F(f), F(g)) \leq c \cdot d(f, g).$$

The constant c is the **contraction factor**.

Lemma 7.2 (Left Multiplication). Let $A \in \widehat{R}$ be a nonzero twisted power series with $v(A) = m \geq 0$. Define the left multiplication map

$$L_A(f) = A \cdot f.$$

Then L_A is a contraction with factor $c = 2^{-m}$ whenever $v(A) \geq 1$. If $v(A) = 0$, then $c = 1$, which is not a contraction.

Proof. For any $f, g \in \widehat{R}$,

$$L_A(f) - L_A(g) = A \cdot (f - g).$$

By Proposition 3.2(ii),

$$v(A \cdot (f - g)) = v(A) + v(f - g) = m + v(f - g).$$

Therefore

$$d(L_A(f), L_A(g)) = 2^{-(m+v(f-g))} = 2^{-m} \cdot 2^{-v(f-g)} = 2^{-m} \cdot d(f, g).$$

Thus $c = 2^{-m} < 1$ whenever $m > 0$. If $m = 0$, then A is a unit in $\widehat{R}$ and $c = 1$, which is not a contraction. Hence we require $v(A) \geq 1$ for a strict contraction.

□

Lemma 7.3 (Right Multiplication). Let $B \in \widehat{R}$ be a nonzero twisted power series with $v(B) = n \geq 0$. Define the right multiplication map

$$R_B(f) = f \cdot B.$$

Then R_B is a contraction with factor $c = 2^{-n}$ whenever $v(B) \geq 1$. If $v(B) = 0$, then $c = 1$, which is not a contraction.

Proof. For any $f, g \in \widehat{R}$,

$$R_B(f) - R_B(g) = (f - g) \cdot B.$$

Since $v((f - g) \cdot B) = v(f - g) + v(B) = v(f - g) + n$, we obtain

$$d(R_B(f), R_B(g)) = 2^{-(v(f-g)+n)} = 2^{-n} \cdot d(f, g).$$

Thus $c = 2^{-n} < 1$ whenever $n > 0$. If $n = 0$, then B is a unit in $\widehat{R}$ and $c = 1$, which is not a contraction. Hence we require $v(B) \geq 1$ for a strict contraction. $\square$

Corollary 7.4 (Affine Maps). *For any $A \in \widehat{R}$ with $v(A) \geq 1$ and any $B \in \widehat{R}$, the affine map*

$$F(f) = A \cdot f + B$$

is a contraction with factor $2^{-v(A)}$.

Proof. $F(f) - F(g) = A \cdot (f - g)$, so the calculation follows Lemma 7.2. $\square$

Corollary 7.5. *For any $A \in \widehat{R}$ with $v(A) \geq 1$ and any $B \in \widehat{R}$, the twisted affine equation*

$$A \cdot f + B = f$$

has a unique solution $f^* \in \widehat{R}$. Moreover, the solution is given by the Neumann series

$$f^* = \sum_{k=0}^{\infty} A^k B,$$

where the series converges in the metric d .

Proof. Rewrite the equation as $f = A \cdot f + B = F(f)$. By Corollary 7.4, F is a contraction with factor $2^{-v(A)} < 1$. Since $(\widehat{R}, d)$ is complete (Theorem 5.2), Theorem 6.1 guarantees a unique fixed point. Starting from $f_0 = 0$, iteration gives

$$f_1 = B, \quad f_2 = A \cdot B + B, \quad f_3 = A^2 \cdot B + A \cdot B + B, \quad \dots$$

Hence $f^* = \sum_{k=0}^{\infty} A^k B$. The convergence follows from $d(A^k B, 0) = 2^{-(v(A^k)+v(B))}$, which decays geometrically since $v(A^k) = k v(A)$. $\square$

Theorem 7.6 (Two-Sided Affine Equations). *Let $A, B \in \widehat{R}$ with $v(A) \geq 1$ and $v(B) \geq 1$, and let $C \in \widehat{R}$. Then the two-sided twisted affine equation*

$$A f B + C = f$$

has a unique solution $f^* \in \widehat{R}$.

Moreover, the solution is given by the Neumann series

$$f^* = \sum_{k=0}^{\infty} A^k C B^k,$$

where the series converges in the metric d .

Proof. Define the map

$$F(f) = AfB + C.$$

For any $f, g \in \widehat{R}$,

$$F(f) - F(g) = A(f - g)B.$$

By Proposition 3.2(ii),

$$v(F(f) - F(g)) = v(A) + v(f - g) + v(B).$$

Therefore

$$d(F(f), F(g)) = 2^{-(v(A)+v(B))} d(f, g).$$

Since $v(A) \geq 1$ and $v(B) \geq 1$, the contraction factor is

$$c = 2^{-(v(A)+v(B))} < 1.$$

Thus F is a contraction on the complete ultrametric space $(\widehat{R}, d)$. By Theorem 6.1, F has a unique fixed point f^* .

Starting from $f_0 = 0$, iteration gives

$$f_1 = C, \quad f_2 = ACB + C, \quad f_3 = A^2CB^2 + ACB + C, \quad \dots$$

Hence

$$f^* = \sum_{k=0}^{\infty} A^k CB^k.$$

The convergence follows from

$$d(A^k CB^k, 0) = 2^{-(v(A^k)+v(C)+v(B^k))} = 2^{-k(v(A)+v(B))-v(C)},$$

which decays geometrically since $v(A) + v(B) \geq 2$.

□

Remark 7.7. The two-sided equation $AfB + C = f$ uses the noncommutativity of $\widehat{R}$. In the commutative setting, this would reduce to the one-sided equation $(AB)f + C = f$. In the noncommutative setting, the equation cannot generally be rewritten as a scalar one-sided multiplication equation, and the left and right multiplications interact in an essential way.

Theorem 7.8 (Mixed Affine Equations). Let $A, B \in \widehat{R}$ with $v(A) \geq 1$ and $v(B) \geq 1$, and let $C \in \widehat{R}$. Then the mixed twisted affine equation

$$Af + fB + C = f$$

has a unique solution $f^* \in \widehat{R}$.

Proof. Define the map

$$F(f) = Af + fB + C.$$

For any $f, g \in \widehat{R}$,

$$F(f) - F(g) = A(f - g) + (f - g)B.$$

By Proposition 3.2(i),

$$v(F(f) - F(g)) \geq \min\{v(A(f - g)), v((f - g)B)\}.$$

Using Proposition 3.2(ii),

$$v(A(f - g)) = v(A) + v(f - g),$$

and

$$v((f - g)B) = v(f - g) + v(B).$$

Therefore

$$v(F(f) - F(g)) \geq v(f - g) + \min\{v(A), v(B)\}.$$

Applying the decreasing function $x \mapsto 2^{-x}$ gives

$$d(F(f), F(g)) \leq 2^{-\min\{v(A), v(B)\}} d(f, g).$$

Since $v(A) \geq 1$ and $v(B) \geq 1$, the contraction factor is

$$c = 2^{-\min\{v(A), v(B)\}} < 1.$$

Thus F is a contraction on the complete ultrametric space $(\widehat{R}, d)$. By Theorem 6.1, F has a unique fixed point f^* . □

Remark 7.9. The mixed equation $Af + fB + C = f$ involves both left and right multiplications simultaneously. Unlike the two-sided equation $AfB + C = f$, the mixed equation cannot be solved by a simple Neumann series of the form $\sum A^k C B^k$. However, Banach's theorem still guarantees existence and uniqueness.

8. APPLICATIONS

8.1. Solving Twisted Linear Equations. Consider the twisted linear equation

$$t \cdot f + a_0 = f,$$

where $a_0 \in \mathbb{F}_q$. This is a special case of Corollary 7.5 with $A = t$ and $B = a_0$.

Example 8.1. Take $p = 2$ and $\mathbb{F}_q = \mathbb{F}_4 = \{0, 1, \alpha, \alpha + 1\}$ with $\alpha^2 = \alpha + 1$. Let $a_0 = \alpha$. Starting from $f_0 = 0$:

$$\begin{aligned}
 f_1 &= \alpha, \\
 f_2 &= t \cdot \alpha + \alpha \\
 &= \sigma(\alpha)t + \alpha \\
 &= \alpha^2 t + \alpha \\
 &= (\alpha + 1)t + \alpha.
 \end{aligned}$$

Continuing,

$$f_3 = \alpha t^2 + (\alpha + 1)t + \alpha.$$

With the left-coefficient convention (coefficients written on the left of powers of t), the fixed point is

$$f^*(t) = \sum_{i=0}^{\infty} \sigma^i(\alpha) t^i.$$

To verify: since coefficients are written on the left, $t \cdot (\sigma^i(\alpha) t^i) = \sigma^{i+1}(\alpha) t^{i+1}$. Hence

$$t \cdot f^* + \alpha = \sum_{i=1}^{\infty} \sigma^i(\alpha) t^i + \alpha = f^*.$$

8.2. The Commutator Equation. Define the commutator map

$$C(f) = t \cdot f - f \cdot t.$$

If $f = \sum a_i t^i$, then

$$t \cdot f = \sum \sigma(a_i) t^{i+1}, \quad f \cdot t = \sum a_i t^{i+1},$$

so

$$C(f) = \sum (\sigma(a_i) - a_i) t^{i+1}.$$

Thus, for all $f, g \in \widehat{R}$,

$$v(C(f) - C(g)) \geq 1 + v(f - g).$$

Note that equality may fail if the leading coefficient is fixed by σ , but the inequality always holds.

Therefore

$$d(C(f), C(g)) \leq \frac{1}{2} d(f, g).$$

Hence C is a contraction with factor at most $1/2$.

Corollary 8.2. *The commutator equation*

$$tf - ft = f$$

has the unique solution $f = 0$ in $\widehat{R}$.

Proof. The map $F(f) = tf - ft$ is a contraction with factor $1/2$ by the computation above. The zero series is clearly a fixed point. By uniqueness from Theorem 6.1, it is the only fixed point. $\square$

8.3. Discrete-Time Dynamical Systems. Consider the discrete-time system

$$f_{n+1} = A \cdot f_n + u_n,$$

where $A \in \widehat{R}$ with $v(A) \geq 1$ and $u_n \in \mathbb{F}_q \hookrightarrow \widehat{R}$ is a control input (embedded as constant series).

Theorem 8.3 (Stability). *For constant input $u_n = u$, the system converges to the unique fixed point f^* of $f = A \cdot f + u$ exponentially fast:*

$$d(f_n, f^*) \leq (2^{-v(A)})^n d(f_0, f^*).$$

Proof. If $f_n = f^*$, the estimate is trivial. Otherwise, Proposition 3.2(ii) yields

$$v(f_{n+1} - f^*) = v(A) + v(f_n - f^*)$$

by subtracting the fixed point equation $f^* = A \cdot f^* + u$ from $f_{n+1} = A \cdot f_n + u$.

Therefore

$$d(f_{n+1}, f^*) = 2^{-v(A)} d(f_n, f^*).$$

Iteration yields the result. $\square$

Example 8.4. Let $A = t$, so $v(A) = 1$ and the contraction factor is $1/2$. For $u = 1$, the fixed point satisfies $f = tf + 1$, i.e., $f = \sum_{i=0}^{\infty} t^i$. Starting from $f_0 = 0$:

$$f_1 = 1, \quad f_2 = t + 1, \quad f_3 = t^2 + t + 1, \quad \dots$$

These partial sums converge in the ultrametric d to the geometric series.

9. COMPARISON WITH CLASSICAL RESULTS

The fixed point theory developed in this paper differs from classical Banach fixed point theory in several fundamental aspects. While the underlying principle remains the same, the non-Archimedean nature of our setting introduces distinctive features that merit comparison.

A key advantage of the ultrametric setting is that convergence is particularly rapid: if two series agree up to degree N , their distance is at most $2^{-(N+1)}$. This makes the theory highly suitable for computational applications where finite truncations provide exact information up to a controllable precision.

Moreover, the noncommutativity of the twisted power series ring allows for genuinely two-sided fixed point problems of the form $AfB + C = f$ and mixed problems of the form $Af + fB + C = f$, which cannot generally be rewritten as scalar one-sided multiplication equations.

Classical Banach Fixed Point	Our Twisted Power Series Framework
Complete metric space	Complete ultrametric space (stronger condition)
Contraction factor $c < 1$	Explicit factor $c = 2^{-v(A)}$ for multiplication by A
Unique fixed point	Unique solution of $A \cdot f + B = f$
Applications: ODEs, integral equations	Twisted algebraic equations, finite-field dynamics
Linear operators: bounded	Left multiplication by twisted series
Triangle inequality	Ultrametric inequality (stronger)
Convergence: exponential in iteration count	Coefficientwise convergence controlled by valuation growth

TABLE 1. Comparison of classical and twisted fixed point theories.

This demonstrates that the fixed point theory developed here adapts classical Banach fixed point methods to the algebraic structure of twisted power series rings and highlights features arising from noncommutativity.

10. CONCLUSION AND FUTURE DIRECTIONS

We have constructed a complete ultrametric on the twisted formal power series ring $\mathbb{F}_q[[t; \sigma]]$ using the t -adic valuation. This metric structure turns left multiplication by series with positive valuation into contractions, enabling a Banach-type fixed point theorem.

The main results establish existence and uniqueness of solutions to affine twisted equations, analyze the commutator equation, and provide stability estimates for discrete-time dynamical systems over finite fields. A key contribution is the result for two-sided affine equations $AfB + C = f$ and mixed equations $Af + fB + C = f$, which illustrate how the noncommutative structure naturally leads to contractions involving both left and right multiplications.

Possible future directions include extensions to multivariate twisted series rings

$$\mathbb{F}_q[[t_1, \dots, t_m; \sigma_1, \dots, \sigma_m]]$$

and investigation of the Hausdorff dimension of subsets of the ultrametric space $(\widehat{R}, d)$ defined by valuation conditions. Other directions include nonlinear equations of the form

$$Af + \Phi(f) = f,$$

where Φ is a nonlinear contraction, matrix-valued twisted series, and applications to coding theory and cryptography, particularly Reed-Solomon-like codes over twisted rings.

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