

## Generalized Contractions in $C^*$ -Algebra-Valued Extended $b$ -Suprametric Spaces with Applications

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**Abstract.** In this contribution, we develop a relaxed fixed-point (FP) for nonlinear operators in  $C^*$ -algebra-valued extended  $b$ -suprametric spaces ( $C^*$ -AVEbSMS). We propose a novel class of control functions  $\Theta_R$  that allows strictly increasing but not necessarily continuous behavior, thereby extending the classical  $\vartheta$ -contraction framework. Moreover, various relaxed  $\vartheta_R$ -type contractive conditions are established using max-type  $C^*$ -algebra-valued functionals. Under suitable conditions, including a jump condition on the control function, the uniqueness and existence of (FP) are obtained in complete spaces (CM). Examples are presented to validate the applicability of the outcomes and to emphasize the role of discontinuities in the convergence process. As applications, we study nonlinear differential models, including

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a second-order particle motion problem and a fourth-order beam equation, and show that the corresponding integral operators admit signal solution within the formulate framework.

## 1. INTRODUCTION AND PRELIMINARIES

The fixed-point theory (*FPT*) has become the key approach for nonlinear analysis ever since the implementation of the usual contraction principle which dates back to [1]. The principle has given rise to a number of generalizations which allow us to study nonstandard metric spaces which include modular metrics and *b*-metrics as well as partial metrics. Multiple suprametric metric(*SMS*) generalizations have been constructed to generate distance arrangement which exhibit maximum distance behavior instead of additive geometric properties. The initial developments concerning (*SMS*) and their practical uses appeared in [2–4]. The research work from [5, 6] includes both structural studies and analytical studies which extend (*SMS*) systems to applications in nonlinear dynamic systems. The developments address the need to study nonlinear systems which use max-dominated or irregular patterns as alternatives to traditional additive geometric methods that apply to models which use their largest deviations as control. The development of  $C^*$ -algebra-valued metric-type spaces has gained attention because these spaces enable (*FPT*) to use operator-valued distances together with noncommutative structures. This analytical framework enables researchers to investigate nonlinear operators which emerge in functional analysis and quantum-type models. Together with these geometric generalizations, a variety of classes of generalized contractions have been proposed to model the nonlinear dynamics of operators, other than the Lipschitz condition. There has been particular focus on the study of  $\vartheta$ -contractions and relaxed versions of these as they allow the contraction coefficient to be replaced by a monotone function. This began with Jleli and Samet [6], and was continued in the setting of generalized metrics in [7]. This has been followed by the introduction of various types of  $\vartheta$ -contractions and Suzuki-type contractions, which have led to novel (*FPT*) theorems (*CMS*), [8–10]. Other types of nonlinear contractions have also been investigated with different control functions. For instance, Marija-type  $\vartheta$ -contractions and other relaxed contractions are investigated in [11, 12]. These have been studied with max-type, integral-type and mixed control functions. This approach has the novelty of having a function, not a constant, contraction condition, allowing us to treat mappings that are discontinuous, oscillatory or non-smooth. This paper is inspired by these advances and sets out a new relaxed (*PFT*) in ( $C^*$ -*AVEbSMS*). We introduce a class  $\Theta_R$  of control functions, given by nondecreasing, not necessarily continuous functions which satisfy a jump condition. The class is a substantial generalization of the classical  $\vartheta$ -class, and is well-suited to the max-dominated structure of suprametric spaces together with operator-valued distances. Our main theorems provide uniqueness and existence for (*FP*) of nonlinear operators under the weaker assumptions of  $\vartheta_R$ -type contractions. Our findings extend some existing results. In particular, when the ( $C^*$ -*AVEbSMS*) becomes a metric and when the class  $\Theta_R$  coincides with the classical class  $\Theta$ , our results give well-established  $\vartheta$ -contraction theorems.

Table 1. Several comparison studies of (FP) formulations of type  $\vartheta$  are performed.

TABLE 1. Comparison of contraction frameworks

| Reference                    | Space                          | Control                   | Condition                                                                                                             |
|------------------------------|--------------------------------|---------------------------|-----------------------------------------------------------------------------------------------------------------------|
| Jleli–Samet [6]              | CM $(C, \delta)$               | $\Theta$                  | $\vartheta(\delta(\mathfrak{I}\psi, \mathfrak{I}\varphi)) \leq [\vartheta(\delta(\psi, \varphi))]^k$                  |
| Ahmad et al. [14]            | CM $(C, \delta)$               | Relaxed $\Theta$          | $\vartheta(\delta(\mathfrak{I}\psi, \mathfrak{I}\varphi)) \leq [\vartheta(M(\psi, \varphi))]^k$                       |
| Liu et al. [15]              | CM $(C, \delta)$               | Modified $\Theta$         | Suzuki-type contraction                                                                                               |
| Hussain et al. [16]          | CM $(C, \delta)$               | Generalized $\Theta$      | Max-type contraction                                                                                                  |
| Cvetković et al. [13]        | CM $(C, \delta)$               | $\Theta$ + jump control   | $\max\{\delta(\psi, \varphi), \delta(\psi, \mathfrak{I}\psi), \delta(\varphi, \mathfrak{I}\varphi)\}$ -type condition |
| <b>This paper (Thm 2.6)</b>  | $C^*$ -AVEbSMS $(C, \delta_s)$ | $\Theta_R$                | $\vartheta_R(\delta_s(\mathfrak{I}\psi, \mathfrak{I}\varphi)) \leq [\vartheta_R(\delta_s(\psi, \varphi))]^k$          |
| <b>This paper (Thm 2.10)</b> | $C^*$ -AVEbSMS $(C, \delta_s)$ | $\Theta_R$ + jump control | $\vartheta_R(\delta_s(\mathfrak{I}x, \mathfrak{I}\varphi)) \leq [\vartheta_R(M_2(\psi, \varphi))]^k$                  |

For the core results, we briefly review some of the main concepts concerning  $(C^*\text{-AVEbSMS})$ . We start with the following definition.

**Definition 1.1.** Let  $\mathfrak{U} \neq \emptyset$  set and  $\mathcal{A}$  a unital  $(C^*\text{-}A)$  with positive cone  $\mathcal{A}^+$ . A mapping  $\delta_{c^*m} : \mathfrak{U} \times \mathfrak{U} \rightarrow \mathcal{A}$  is called a  $(C^*\text{-AVEbSMS})$  if  $\forall \overline{Z}, \overline{U}, \overline{Y} \in \mathfrak{U}$ , the following axioms hold:

$$\mathbf{C^*-AVEbSM1:} : \theta \leq \delta_{c^*m}(\overline{Z}, \overline{U}), \quad \text{and} \quad \delta_{c^*m}(\overline{Z}, \overline{U}) = \theta \text{ iff } \overline{Z} = \overline{U};$$

$$\mathbf{C^*-AVEbSM2:} : \delta_{c^*m}(\overline{Z}, \overline{U}) = \delta_{c^*m}(\overline{Z}, \overline{U});$$

$$\exists \text{ a control function } \mathfrak{B} : \mathfrak{U} \times \mathfrak{U} \times \mathfrak{U} \rightarrow \mathcal{A}^+$$

$$\mathbf{C^*-AVEbSM3:} : \delta_{c^*m}(\overline{Z}, \overline{Y}) \leq B(\overline{Z}, \overline{U}, \overline{Y}) \left[ \delta_{c^*m}(\overline{Z}, \overline{U}) + \delta_{c^*m}(\overline{U}, \overline{Y}) + \delta_{c^*m}(\overline{Z}, \overline{U}) \delta_{c^*m}(\overline{U}, \overline{Y}) \right]$$

The triple  $(\mathfrak{U}, \mathcal{A}, \delta_{c^*m})$  is known as  $(C^*\text{-AVEbSMS})$ .

Every  $(C^*\text{-AVEbSMS})$  is a  $(C^*\text{-AVSMS})$ ; there are several constructions that enable one to create such suprametrics starting from  $(bM)$  by relaxing the triangle inequality condition.

**Example 1.1.** Let  $(\omega, \delta_0)$  be a  $(MS)$  and define  $\delta^\eta : \omega \times \omega \rightarrow \mathbb{C}_+$  by

$$\delta^\eta(\psi, \varphi) = \frac{\delta_0(\psi, \varphi)}{1 + \eta \delta_0(\psi, \varphi)} I, \quad \eta > 0,$$

where  $I$  is the identity of the  $(C^*\text{-}A)$   $\mathbb{C}$ . Let

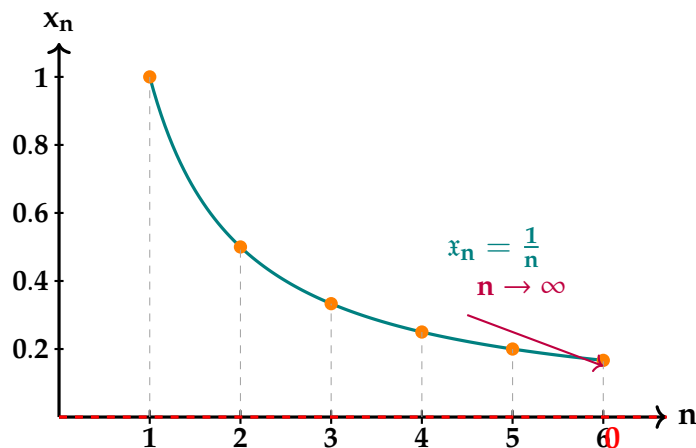
$$B(\psi, \varphi, \varsigma) = 2I.$$

Then  $(\omega, \delta^\eta)$  fulfills

$$\delta^\eta(\psi, \varsigma) \leq B(\psi, \varphi, \varsigma) \left[ \delta^\eta(\psi, \varphi) + \delta^\eta(\varphi, \varsigma) + \delta^\eta(\psi, \varphi) \delta^\eta(\varphi, \varsigma) \right],$$

hence it is  $(C^*\text{-AVEbSMS})$ .

**Special case.** Let  $\omega = \mathbb{R}$ ,  $d_0(\psi, \varphi) = |\psi - \varphi|$ ,  $\eta = 1$ , and  $\psi_n = \frac{1}{n}$ . Then  $\psi_n \rightarrow 0$  in the induced structure.



**Definition 1.2.** Let  $(C, \delta_s)$  be a  $(C^*$ -AVEbSMS), where  $\delta_s : C \times C \rightarrow \mathbb{A}_+$  and  $\mathbb{A}$  is  $(C^*$ -A) with identity positive cone  $\mathbb{A}_+$ .

- (i) A sequence  $\{Y_\delta\}$  in  $C$  is regarded to be converge to a point  $Y \in C$  if, for every  $\varepsilon \in \mathbb{A}_+$  with  $\varepsilon > 0$ ,  $\exists \tau_\varepsilon \in \mathbb{T}$  such that

$$d_s(Y_\delta, Y) \leq \varepsilon, \quad \text{for all } \delta \geq \tau_\varepsilon.$$

In instance, we write

$$\lim_{\delta \rightarrow \infty} Y_\delta = Y.$$

- (ii) A sequence  $\{Y_\delta\}$  in  $C$  is regarded to be Cauchy sequence(cs) if, for every  $\varepsilon \in \mathbb{A}_+$  with  $\varepsilon > 0$ ,  $\exists \tau_\varepsilon \in \mathbb{T}$  such that

$$\delta_s(Y_\delta, Y_\varrho) \leq \varepsilon, \quad \forall \varrho, \delta \geq \tau_\varepsilon.$$

- (iii) The space  $(C, d_s)$  is regarded to be complete if every (CM) in  $C$  converges to some point of  $C$ .  
 (iv) If the mapping  $\delta_s$  is continuous, then every convergent sequence in  $C$  admits a unique limit.

**Definition 1.3.** Let  $\Theta$  be the class of all functions  $\vartheta : (0, \infty) \rightarrow (1, \infty)$  obeying:

- ( $\vartheta_1$ )  $Y \leq y \implies \vartheta(Y) \leq \vartheta(y)$  (i.e.,  $\vartheta$  is nondecreasing);  
 ( $\vartheta_2$ ) for every sequence  $\{Y_\delta\} \subset (0, \infty)$ ,

$$\lim_{\delta \rightarrow \infty} \vartheta(Y_\delta) = 1 \iff \lim_{\delta \rightarrow \infty} Y_\delta = 0;$$

- ( $\vartheta_3$ )  $\exists l \in (0, \infty]$  and  $k \in (0, 1)$  such that

$$\lim_{Y \rightarrow 0^+} \frac{\vartheta(Y) - 1}{Y^k} = l.$$

The existence and single (FP) for  $\vartheta$ -contractions were first studied in the setting of generalized (MS) introduced by Branciari

**Definition 1.4.** Let  $\Theta'$  denote the class of all functions  $\vartheta : (0, \infty) \rightarrow (1, \infty)$  obeying conditions ( $\vartheta_1$ )–( $\vartheta_3$ ) together with:

- ( $\vartheta_4$ )  $\vartheta$  is continuous on  $(0, \infty)$ .

In subsequent developments, alternative forms of  $\vartheta$ -contractions were introduced using wider class nonlinear contractive conditions, thereby enhancing the classical framework. It is evidently that  $\Theta' \subsetneq \Theta$ . Later, Ahmad et al. refined the definition of  $\vartheta$ -contractions by omitting condition  $(\vartheta_3)$ , leading to a new class denoted by  $\Theta^*$ . This modification simplifies the structure while preserving strong (FP) results in (CMS). Since the conditions  $(\vartheta_3)$  and  $(\vartheta_4)$  are independent, the classes  $\Theta$  and  $\Theta^*$  do not align.

**Example 1.2.** The conditions  $(\vartheta_3)$  and  $(\vartheta_4)$  are independent.

(1) Consider  $\vartheta(Y) = e^{e^Y - 1}$  for  $Y > 0$ . Then  $\vartheta$  is continuous and satisfies  $(\vartheta_1)$  and  $(\vartheta_2)$ , but

$$\lim_{Y \rightarrow 0^+} \frac{\vartheta(Y) - 1}{Y^k} = 0 \quad \forall k \in (0, 1)$$

so  $(\vartheta_3)$  fails.

(2) Let  $k \in (\frac{1}{2}, 1)$  and define

$$\vartheta(Y) = 1 + \sqrt{Y}(1 + \lfloor Y \rfloor), \quad Y > 0.$$

Then  $\vartheta$  is not continuous on  $(0, \infty)$ , so  $(\vartheta_4)$  fails, while  $(\vartheta_1)$  and  $(\vartheta_2)$  hold.

Hence,  $\Theta \neq \Theta^*$ , although  $\Theta \cap \Theta^* \neq \emptyset$ . For instance,  $\vartheta(Y) = e^{\sqrt{Y}} \in \Theta \cap \Theta^*$ . Liu et al [ ] formulate an alternative condition replacing  $(\vartheta_2)$  with:

$$(\vartheta_2^*) \quad \inf_{Y > 0} \vartheta(Y) = 1,$$

and studied  $\vartheta^*$ -contractions under conditions  $(\vartheta_1)$ ,  $(\vartheta_2^*)$ , and  $(\vartheta_4)$ . Furthermore, an additional submultiplicative condition was introduced:

$$(\vartheta_5) \quad \vartheta(Y + y) \leq \vartheta(Y) \vartheta(y), \quad \forall Y, y > 0.$$

**Theorem 1.1.** [13] Let  $(C, \delta)$  be a (CMS) and let  $\mathfrak{G} : C \rightarrow C$  be an arbitrary self-map. presume that  $\exists$  a monotone (nondecreasing) function  $\vartheta : (0, \infty) \rightarrow (1, \infty)$  together with a constant  $k \in (0, 1)$  such that the implication

$$\delta(\mathfrak{G}Y, \mathfrak{G}\varphi) \neq 0 \Rightarrow \vartheta(\delta(\mathfrak{G}Y, \mathfrak{G}\varphi)) \leq [\vartheta(\mathfrak{T}(Y, \varphi))]^k$$

holds for every pair  $Y, \varphi \in C$ , where

$$\mathfrak{T}(Y, \varphi) = \max \{ \delta(Y, \varphi), \delta(Y, \mathfrak{G}(Y)), \delta(\varphi, \mathfrak{G}(\varphi)) \}.$$

Moreover, whenever  $\vartheta$  is discontinuous at a point  $v > 0$ , the size of the left-hand jump of  $\vartheta$  does not exceed the quantity  $\vartheta(v) - (\vartheta(v))^k$ , that is,

$$\vartheta(v) - \lim_{s \rightarrow v^-} \vartheta(s) \leq \vartheta(v) - (\vartheta(v))^k,$$

equivalently,

$$\lim_{s \rightarrow v^-} \vartheta(s) \geq (\vartheta(v))^k.$$

Assuming these assumptions, the mapping  $\mathfrak{G}$  possesses a unique (FP) in the space  $C$ . We proceed to our adjusted results, where the relaxed  $\vartheta$ -contractive structure is embedded into the  $(C^*$ -AVEbSMS) framework, enabling a sharper and more flexible (FP) analysis.

## 2. MAIN RESULTS

We use  $\Theta^+$  to denote the class of all functions  $\vartheta : (0, \infty) \rightarrow (1, \infty)$  satisfying  $(\vartheta_1)$ – $(\vartheta_3)$  and  $(\vartheta_5)$ . In the following section we develop a category of simple contractions and proceed to give a variety of examples to exhibit the usefulness of the recommended framework both in terms of numerical analysis and graphical visualization. Their adjusted (FP) outcome are obtained by taking the relaxed contraction structure and applying it to various contractive mappings within the context of  $(C^*\text{-AVEbSMS})$ .

**Definition 2.1.** Let  $\Theta_R$  be the class of all nondecreasing functions  $\vartheta_R : (0, \infty) \rightarrow (1, \infty)$ . Then

$$\Theta \subseteq \Theta^+ \subseteq \Theta^* \subseteq \Theta' \subseteq \Theta_R.$$

The class  $\Theta_R$  forms a unified framework that contains all previously defined families. Note that functions in  $\Theta_R$  are not necessarily defined at 0. This can be naturally extended by setting  $\vartheta_R(0) = 1$ , which preserves the contractive structure and is consistent with conditions  $(\vartheta_1)$ – $(\vartheta_4)$  when required.

**Example 2.1.** Define  $\vartheta_R : (0, \infty) \rightarrow (1, \infty)$  by

$$\vartheta_R(\psi) = \begin{cases} 1 + \psi^2, & \psi \in (0, 1), \\ 4 + \ln(1 + \psi), & \psi \in [1, \infty). \end{cases}$$

**Verification:**

$(\vartheta_1)$  On  $(0, 1)$ ,  $\vartheta_R(\psi) = 1 + \psi^2$  is increasing since  $2\psi > 0$ . On  $[1, \infty)$ ,  $\vartheta_R(\psi) = 4 + \ln(1 + \psi)$  is increasing since  $\frac{1}{1+\psi} > 0$ . At  $\psi = 1$ , the left limit is 2 and the right value is  $4 + \ln 2 > 2$ , hence  $\vartheta_R$  is nondecreasing.

$(\vartheta_2)$

$$\lim_{\psi \rightarrow \infty} \vartheta_R(\psi) = \infty \neq 1,$$

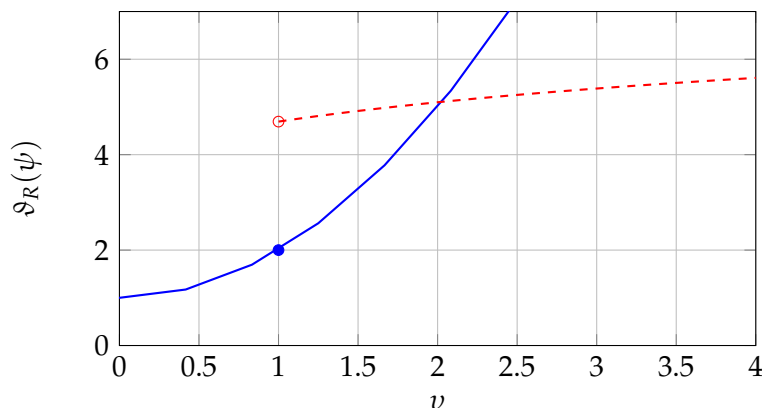
so  $(\vartheta_2)$  is not satisfied.

$(\vartheta_3)$

$$\lim_{\psi \rightarrow 0^+} \frac{\vartheta_R(\psi) - 1}{\psi^k} = \lim_{\psi \rightarrow 0^+} \frac{\psi^2}{\psi^k} = \lim_{\psi \rightarrow 0^+} \psi^{2-k} = 0,$$

for  $0 < k < 1$ . Hence  $(\vartheta_3)$  fails.

Thus,  $\vartheta_R \in \Theta_R$ .



**Remarks** The graph represents a piecewise-defined function given by

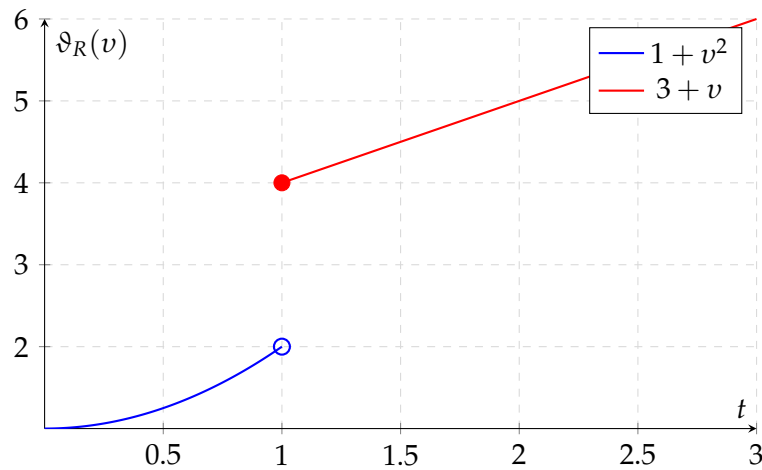
$$\vartheta_R(\psi) = \begin{cases} 1 + \psi^2, & 0 \leq \psi \leq 1, \\ 4 + \ln(1 + \psi), & \psi > 1. \end{cases}$$

Both components of the function, namely  $1 + \psi^2$  and  $4 + \ln(1 + \psi)$ , are increasing on their respective domains, which shows that  $\vartheta_R(t)$  is non-decreasing on  $(0, \infty)$ . The graph exhibits a jump discontinuity at  $\psi = 1$ . In fact, the (LHL) is  $\lim_{\psi \rightarrow 1^-} \vartheta_R(t) = 2$ , while the (RHL) is  $\lim_{\psi \rightarrow 1^+} \vartheta_R(\psi) = 4 + \ln(2)$ , and these two values are not equal. The filled point at  $(1, 2)$  represents the actual value of the function, whereas the open point at  $(1, 4 + \ln(2))$  indicates the value approached from the right-hand side. This confirms that the function is not continuous at  $\psi = 1$ . Moreover, since both parts of the function take values greater than 1, we have  $\vartheta_R(\psi) > 1$  for all  $\psi > 0$ . This example illustrates that functions in the class  $\Theta_R$  can be non-decreasing without being continuous, which is useful in the study of generalized contraction frameworks.

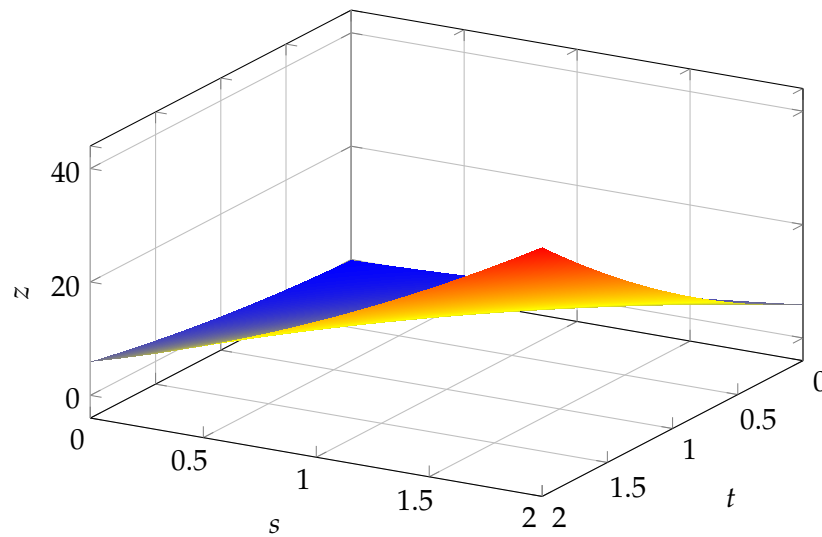
**Example 2.2.** Define  $\vartheta_R : (0, \infty) \rightarrow (1, \infty)$  such that

$$\vartheta_R(v) = \begin{cases} 1 + v^\alpha, & v \in (0, 1), \\ 3 + v, & v \in [1, \infty), \end{cases}$$

where  $\alpha > 1$ . Let  $(\sigma, \mathcal{A}, \Delta)$  be a  $(C^*$ -AVEbSMS) and set  $v = \|\Delta(\psi, \varphi)\|$ .

**Figure 1:** Graph of  $\vartheta_R(v)$ **Figure 2:** Surface of the inequality

$$\|\delta(\psi, \varsigma)\| \leq (1 + v + s)(v + s + vs)$$



**Remark.** The graph of  $\vartheta_R$  shows a jump discontinuity at  $v = 1$ . The surface illustrates the nonlinear growth of the  $(C^*\text{-AVEbSMS})$  inequality.

**Example 2.3.** Let  $(\sigma, \mathcal{A}, \delta)$  be a  $(C^*\text{-AVEbSMS})$ , where  $\sigma = (0, \infty)$ ,  $\mathcal{A} = M_2(\mathbb{C})$ , and define

$$\delta(Y, \varphi) = \begin{pmatrix} |Y - \varphi| & 0 \\ 0 & |Y - \varphi| \end{pmatrix}.$$

Let  $B(Y, \varphi, \varsigma) = 2I$ , where  $I$  is the identity matrix. For all  $Y, \varphi, \varsigma \in \sigma$ , we have

$$|Y - \varsigma| \leq |Y - \varphi| + |\varphi - \varsigma| + |Y - \varphi||\varphi - \varsigma|.$$

Thus,

$$\delta(Y, \varsigma) \leq 2 \left[ \delta(Y, \varphi) + \delta(\varphi, \varsigma) + \delta(Y, \varphi) \delta(\varphi, \varsigma) \right].$$

Hence, the space satisfies the extended  $b$ -suprametric inequality.

**Definition of  $\vartheta_R$ .** Define  $\vartheta_R : (0, \infty) \rightarrow (1, \infty)$  by

$$\vartheta_R(v) = \begin{cases} 1 + v^3, & v \in (0, 1), \\ 3 + \ln(v), & v \in [1, \infty). \end{cases}$$

**Discontinuity at  $v = 1$ .**

$$\lim_{v \rightarrow 1^-} \vartheta_R(v) = 2, \quad \vartheta_R(1) = 3.$$

Thus,  $\vartheta_R$  is discontinuous at  $v = 1$ .

**Verification of Conditions.**  $(\vartheta_1)$ : The function is non-decreasing since

$$\frac{\delta}{\delta v}(1 + v^3) = 3v^2 > 0, \quad \frac{\delta}{\delta v}(3 + \ln v) = \frac{1}{v} > 0.$$

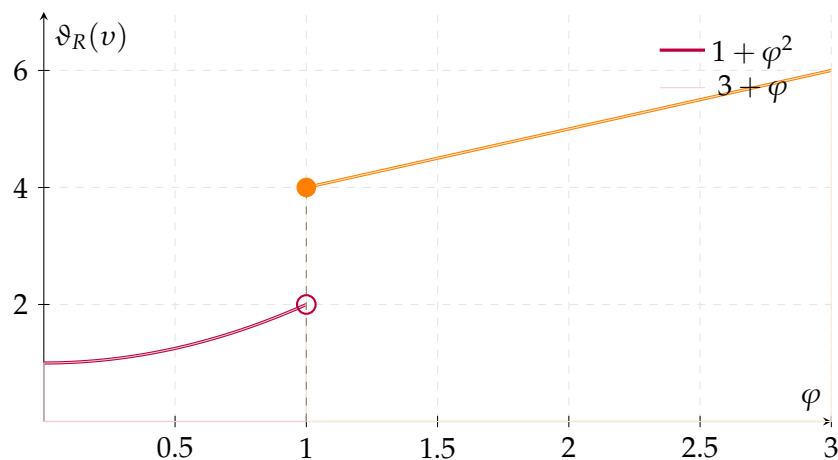
$(\vartheta_2)$ :

$$\lim_{v \rightarrow \infty} \vartheta_R(v) = \infty \neq 1,$$

so it is not obeyed.  $(\vartheta_3)$ :

$$\lim_{v \rightarrow 0^+} \frac{\vartheta_R(v) - 1}{v^k} = \lim_{v \rightarrow 0^+} v^{3-k} = 0,$$

so it is not obeyed. Hence,  $\vartheta_R \in \Theta_R$ .



**Example 2.4.** Let  $(\sigma, \mathcal{A}, \Delta)$  be a  $(C^*$ -AVEbSMS), where  $\sigma = (0, \infty)$ ,  $\mathcal{A} = M_2(\mathbb{C})$ , and define

$$\Delta(Y, \varphi) = \begin{pmatrix} |Y - \varphi|^2 & 0 \\ 0 & |Y - \varphi|^2 \end{pmatrix}.$$

Let

$$B(Y, \varphi, \varsigma) = 3I, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Triangle Inequality.** For all  $Y, \varphi, \varsigma \in \sigma$ , we have

$$|Y - \varsigma|^2 \leq 3(|Y - \varphi|^2 + |\varphi - \varsigma|^2 + |Y - \varphi|^2 |\varphi - \varsigma|^2).$$

Thus,

$$\Delta(Y, \varsigma) \leq 3[\Delta(Y, \varphi) + \Delta(\varphi, \varsigma) + \Delta(Y, \varphi)\Delta(\varphi, \varsigma)].$$

Hence, the space satisfies the required inequality.

**Definition of  $\vartheta_R$ .** Define  $\vartheta_R : (0, \infty) \rightarrow (1, \infty)$  by

$$\vartheta_R(v) = \begin{cases} 1 + v^{1/2}, & v \in (0, 1), \\ 4 + \ln(v), & v \in [1, \infty). \end{cases}$$

**Discontinuity at  $v = 1$ .**

$$\lim_{v \rightarrow 1^-} \vartheta_R(v) = 1 + 1^{1/2} = 2, \quad \vartheta_R(1) = 4 + \ln(1) = 4.$$

Thus,  $\vartheta_R$  is discontinuous at  $v = 1$  (jump from 2 to 4).

**Verification of Conditions.**  $(\vartheta_1)$ : On  $(0, 1)$ ,

$$\frac{\partial}{\partial v}(1 + v^{1/2}) = \frac{1}{2}v^{-1/2} > 0.$$

On  $[1, \infty)$ ,

$$\frac{\partial}{\partial v}(4 + \ln v) = \frac{1}{v} > 0.$$

Thus,  $\vartheta_R$  is non-decreasing on  $(0, \infty)$ .

$(\vartheta_2)$ :

$$\lim_{v \rightarrow \infty} \vartheta_R(v) = \infty \neq 1,$$

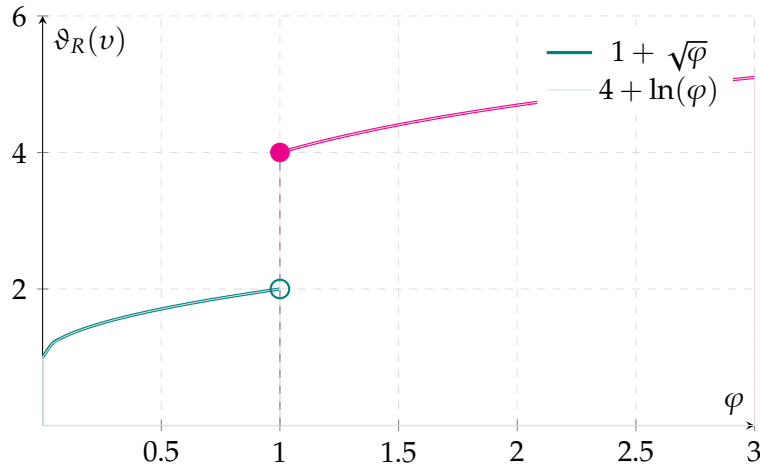
so it is not satisfied.

$(\vartheta_3)$ :

$$\lim_{v \rightarrow 0^+} \frac{\vartheta_R(v) - 1}{v^k} = \lim_{v \rightarrow 0^+} \frac{v^{1/2}}{v^k} = \lim_{v \rightarrow 0^+} v^{\frac{1}{2}-k}.$$

For  $0 < k < 1$ , this limit equals 0 (for  $k < 1/2$ ) or diverges, hence condition fails.

Consequently,  $\vartheta_R(v)$  obeys  $(\vartheta_1)$  but not  $(\vartheta_2)$  and  $(\vartheta_3)$ , so  $\vartheta_R \in \Theta_R$ .

FIGURE 1. visualization of the discontinuity of  $\vartheta_R(v)$  at  $v = 1$ 

**Theorem 2.1.** Let  $\mathfrak{G} : \mathfrak{w} \rightarrow \mathfrak{w}$  be a mapping on a complete  $(C^*\text{-avebsms}) (\mathfrak{w}, \mathfrak{d}_s)$ . Assume that  $\exists$  a nondecreasing function  $\vartheta_R : (0, \infty) \rightarrow (1, \infty)$  and a constant  $k \in (0, 1)$  such that

$$\vartheta_R(\mathfrak{d}_s(\mathfrak{G}Y, \mathfrak{G}v)) \leq \left( \vartheta_R(\mathfrak{d}_s(Y, v)) \right)^k, \quad (2.1)$$

$\forall Y, v \in \mathfrak{w}$  with  $\mathfrak{G}Y \neq \mathfrak{G}v$ . Moreover, at every discontinuity point  $t > 0$  of  $\vartheta_R$ , the left-jump satisfies

$$\vartheta_R(v) - \lim_{s \rightarrow v^-} \vartheta_R(s) \leq \vartheta_R(v) - (\vartheta_R(v))^k.$$

Then  $\mathfrak{G}$  admits a unique (FP)  $t$  in  $\mathfrak{w}$ .

**Proof** Let  $Y_0 \in \mathfrak{w}$  be arbitrary and define a sequence  $(Y_\delta)$  by

$$Y_\delta = \mathfrak{G}Y_{\delta-1}, \quad \delta \in \mathbb{N}.$$

If  $\exists \delta$  such that  $Y_\delta = Y_{\delta-1}$ , then  $Y_{\delta-1}$  is a (fp) of  $\mathfrak{G}$ . Hence, assume that  $Y_\delta \neq Y_{\delta-1} \forall \delta$ , we obtain

$$\vartheta_R(\mathfrak{d}_s(Y_\delta, Y_{\delta+1})) = \vartheta_R(\mathfrak{d}_s(\mathfrak{G}Y_{\delta-1}, \mathfrak{G}Y_\delta)) \leq \left( \vartheta_R(\mathfrak{d}_s(Y_{\delta-1}, Y_\delta)) \right)^k.$$

Proceeding inductively, we get

$$\vartheta_R(\mathfrak{d}_s(Y_\delta, Y_{\delta+1})) \leq \left( \vartheta_R(\mathfrak{d}_s(Y_0, Y_1)) \right)^{k^\delta}.$$

Taking the limit as  $\delta \rightarrow \infty$ , we obtain

$$1 \leq \lim_{\delta \rightarrow \infty} \vartheta_R(\mathfrak{d}_s(Y_\delta, Y_{\delta+1})) \leq \lim_{\delta \rightarrow \infty} \left( \vartheta_R(\mathfrak{d}_s(Y_0, Y_1)) \right)^{k^\delta} = 1,$$

which implies

$$\lim_{\delta \rightarrow \infty} \mathfrak{d}_s(Y_\delta, Y_{\delta+1}) = 0. \quad (2.2)$$

Moreover, we have

$$\vartheta_R(\mathfrak{d}_s(Y_\delta, Y_{\delta+1})) < \vartheta_R(\mathfrak{d}_s(Y_{\delta-1}, Y_\delta)),$$

which implies

$$\check{\delta}_s(Y_\delta, Y_{\delta+1}) < \check{\delta}_s(Y_{\delta-1}, Y_\delta).$$

Thus,  $\{\check{\delta}_s(Y_{\delta-1}, Y_\delta)\}$  is a decreasing sequence, hence convergent. Let

$$a = \lim_{\delta \rightarrow \infty} \check{\delta}_s(Y_{\delta-1}, Y_\delta).$$

If  $a > 0$ , then by continuity from the left,

$$\vartheta_R(a) \leq \lim_{\delta \rightarrow \infty} \vartheta_R(\check{\delta}_s(Y_\delta, Y_{\delta+1})) = 1,$$

which contradicts  $\vartheta_R(t) > 1$ . Hence  $a = 0$  and (2.2) holds.

Now suppose, by contradiction, that  $(Y_\delta)$  is not Cauchy. Then  $\exists \varepsilon > 0$  such that for sequences  $(\delta_i)$  and  $(\varrho_i)$  with  $m_i > \delta_i$ ,

$$\check{\delta}_s(Y_{\delta_i}, Y_{\varrho_i}) \geq \varepsilon, \quad \check{\delta}_s(Y_{\delta_i}, Y_{\varrho_i-1}) < \varepsilon. \quad (2.3)$$

Using the extended  $b$ -suprametric inequality,

$$\check{\delta}_s(\psi, \varsigma) \leq B(\psi, \varphi, \varsigma) [\check{\delta}_s(\psi, \varphi) + \check{\delta}_s(\varphi, \varsigma) + \check{\delta}_s(\psi, \varphi) \check{\delta}_s(\varphi, \varsigma)],$$

we obtain

$$\check{\delta}_s(Y_{\delta_i}, Y_{\varrho_i}) \leq B(Y_{\delta_i}, Y_{\varrho_i-1}, Y_{\varrho_i}) [\check{\delta}_s(Y_{\delta_i}, Y_{\varrho_i-1}) + \check{\delta}_s(Y_{\varrho_i-1}, Y_{\varrho_i}) + \check{\delta}_s(Y_{\delta_i}, Y_{\varrho_i-1}) \check{\delta}_s(Y_{\varrho_i-1}, Y_{\varrho_i})].$$

Letting  $i \rightarrow \infty$  and using (2.2), we obtain

$$\lim_{i \rightarrow \infty} \check{\delta}_s(Y_{\delta_i}, Y_{\varrho_i}) = \varepsilon. \quad (2.4)$$

Similarly, applying the inequality repeatedly yields

$$\lim_{i \rightarrow \infty} \check{\delta}_s(Y_{\delta_i-1}, Y_{\varrho_i-1}) = \varepsilon.$$

Now applying with  $Y = Y_{\delta_i-1}$  and  $v = Y_{\varrho_i-1}$ , we obtain

$$\vartheta_R(\varepsilon) \leq \vartheta_R(\check{\delta}_s(Y_{\delta_i}, Y_{\varrho_i})) \leq (\vartheta_R(\check{\delta}_s(Y_{\delta_i-1}, Y_{\varrho_i-1})))^k.$$

Passing to the limit gives

$$\vartheta_R(\varepsilon) \leq (\vartheta_R(\varepsilon))^k,$$

which is a contradiction since  $k \in (0, 1)$ . Hence  $(Y_\delta)$  is Cauchy.

Since  $\omega$  is complete,  $\exists Y^* \in X$  such that  $Y_\delta \rightarrow Y^*$ . Finally, using (2.1),

$$\vartheta_R(\check{\delta}_s(\mathcal{G}Y^*, Y_{\delta+1})) \leq (\vartheta_R(\check{\delta}_s(Y^*, Y_\delta)))^k.$$

Letting  $\delta \rightarrow \infty$ , we obtain  $\check{\delta}_s(\mathcal{G}Y^*, Y^*) = 0$ , hence  $\mathcal{G}Y^* = Y^*$ . For uniqueness, let  $\varsigma$  be another (FP). Then

$$\vartheta_R(\check{\delta}_s(Y^*, \varsigma)) \leq (\vartheta_R(\check{\delta}_s(Y^*, \varsigma)))^k,$$

which implies  $Y^* = \varsigma$ . Thus,  $\mathcal{G}$  has a unique (FP).

**Example 2.5.** Let  $(X, \delta_s)$  be a complete  $(C^*\text{-AVEbSMS})$ , where  $X = [0, \infty)$  and

$$\delta_s(Y, v) = |Y - v|^2 + 2|Y - v|.$$

Consider the mapping  $\mathfrak{G} : X \rightarrow \omega$  defined by

$$\mathfrak{G}Y = \frac{Y}{2}.$$

Then,

$$\delta_s(\mathfrak{G}Y, \mathfrak{G}v) = \frac{1}{4}|Y - v|^2 + |Y - v|.$$

Define

$$\vartheta_R(v) = \begin{cases} 1 + v^3, & v \in (0, 1), \\ 2 + e^v, & v \in [1, \infty). \end{cases}$$

Clearly,  $\vartheta_R$  is nondecreasing but discontinuous at  $v = 1$ .

**Case (i):** For  $v \in (0, 1)$ , we obtain

$$1 + \left(\frac{1}{4}|Y - v|^2 + |Y - v|\right)^3 \leq \left(1 + (|Y - v|^2 + 2|Y - v|)^3\right)^k.$$

**Case (ii):** For  $v \in [1, \infty)$ , we obtain

$$2 + \exp\left(\frac{1}{4}|Y - v|^2 + |Y - v|\right) \leq \left(2 + \exp(|Y - v|^2 + 2|Y - v|)\right)^k.$$

Case (i): 3D Behaviour on  $(0, 1)$

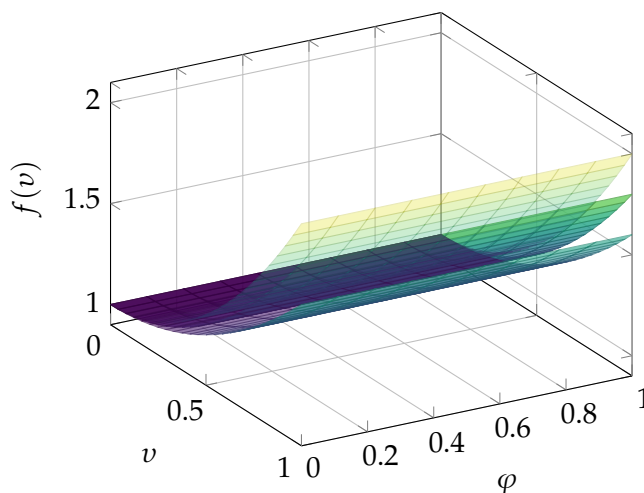


FIGURE 2. Case (i): 3D contraction behaviour on  $(0, 1)$

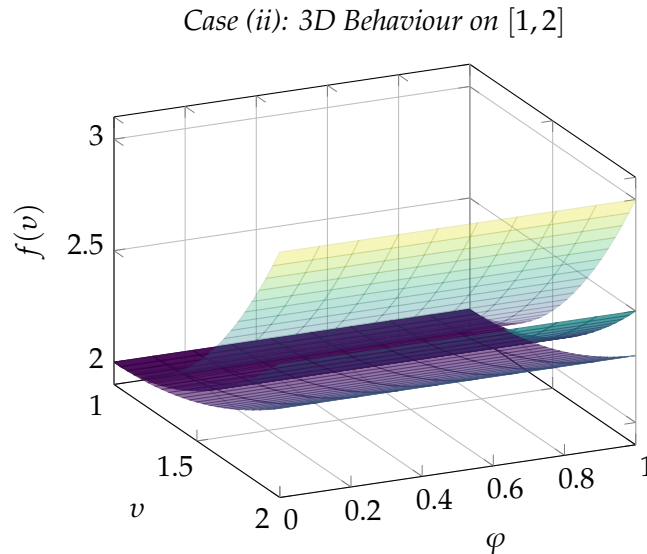


FIGURE 3. Case (ii): 3D contraction behaviour on  $[1, 2]$

**Theorem 2.2.** Let  $\mathfrak{G} : \mathcal{X} \rightarrow \mathcal{X}$  be a mapping on a complete  $(C^*\text{-AVEbSMS}) (\mathcal{X}, \mathfrak{d})$  satisfying

$$\mathfrak{d}(\psi, \varsigma) \leq B(\psi, \varphi, \varsigma) \left[ \mathfrak{d}(\psi, \varphi) + \mathfrak{d}(\varphi, \varsigma) + \mathfrak{d}(\psi, \varphi) \mathfrak{d}(\varphi, \varsigma) \right],$$

for all  $\psi, \varphi, \varsigma \in \mathcal{X}$ , where  $B(\psi, \varphi, \varsigma) \geq 1$ . Assume that a nondecreasing function  $\vartheta_R : (0, \infty) \rightarrow (1, \infty)$  and a constant  $k \in (0, 1)$  exist such that

$$\mathfrak{G}\psi \neq \mathfrak{G}\varphi \Rightarrow \vartheta_R(\mathfrak{d}(\mathfrak{G}\psi, \mathfrak{G}\varphi)) \leq (\vartheta_R(M(\psi, \varphi)))^k,$$

where

$$M(\psi, \varphi) = \max \left\{ \mathfrak{d}(\psi, \varphi), \frac{\mathfrak{d}(\psi, \mathfrak{G}\psi) + \mathfrak{d}(\varphi, \mathfrak{G}\varphi)}{2}, \frac{\mathfrak{d}(\psi, \mathfrak{G}\varphi) + \mathfrak{d}(\varphi, \mathfrak{G}\psi)}{2} \right\}.$$

Assume also that at every discontinuity point  $t > 0$  of  $\vartheta_R$ ,

$$\vartheta_R(t) - \lim_{s \rightarrow t^-} \vartheta_R(s) > (\vartheta_R(t))^k.$$

Then  $\mathfrak{G}$  admits a unique (FP) in  $\mathcal{X}$ .

**Proof.** Let  $(\mathcal{X}, \mathfrak{d})$  be a complete  $(C^*\text{-AVEbSMS})$  satisfying

$$\mathfrak{d}(\psi, \varsigma) \leq B(\psi, \varphi, \varsigma) \left[ \mathfrak{d}(\psi, \varphi) + \mathfrak{d}(\varphi, \varsigma) + \mathfrak{d}(\psi, \varphi) \mathfrak{d}(\varphi, \varsigma) \right],$$

for all  $\psi, \varphi, \varsigma \in \mathcal{X}$ , where  $B(\psi, \varphi, \varsigma) \geq 1$ .

Let  $\mathfrak{G} : \mathcal{X} \rightarrow \mathcal{X}$  be a mapping such that there exist a nondecreasing function  $\vartheta_R : (0, \infty) \rightarrow (1, \infty)$  and a constant  $k \in (0, 1)$  satisfying

$$\vartheta_R(\mathfrak{d}(\mathfrak{G}\psi, \mathfrak{G}\varphi)) \leq (\vartheta_R(M(\psi, \varphi)))^k,$$

where

$$M(\psi, \varphi) = \max \left\{ \mathfrak{d}(\psi, \varphi), \frac{\mathfrak{d}(\psi, \mathfrak{G}\psi) + \mathfrak{d}(\varphi, \mathfrak{G}\varphi)}{2}, \frac{\mathfrak{d}(\psi, \mathfrak{G}\varphi) + \mathfrak{d}(\varphi, \mathfrak{G}\psi)}{2} \right\}.$$

Also, at every discontinuity point  $t > 0$  of  $\vartheta_R$ , assume that

$$\vartheta_R(t) - \lim_{s \rightarrow t^-} \vartheta_R(s) > (\vartheta_R(t))^k.$$

Let  $Y_0 \in X$  be arbitrary and define the sequence  $Y_{n+1} = \mathbb{G}Y_n$  for  $n \geq 0$ . If  $Y_{n+1} = Y_n$  for some  $n$ , then  $Y_n$  is a (FP). Otherwise, assume  $Y_n \neq Y_{n+1}$  for all  $n$ . Applying the contractive condition with  $\psi = Y_{n-1}$  and  $\varphi = Y_n$ , we obtain

$$\vartheta_R(\delta(Y_n, Y_{n+1})) \leq \left( \vartheta_R(M(Y_{n-1}, Y_n)) \right)^k.$$

Using the structure of  $M(Y_{n-1}, Y_n)$  together with the extended  $b$ -suprametric inequality

$$\delta(\psi, \varsigma) \leq B(\psi, \varphi, \varsigma) [\delta(\psi, \varphi) + \delta(\varphi, \varsigma) + \delta(\psi, \varphi)\delta(\varphi, \varsigma)],$$

we deduce that

$$M(Y_{n-1}, Y_n) \leq \max\{\delta(Y_{n-1}, Y_n), \delta(Y_n, Y_{n+1})\}.$$

Since  $\vartheta_R$  is nondecreasing, it follows that

$$\vartheta_R(\delta(Y_n, Y_{n+1})) \leq \left( \vartheta_R(\delta(Y_{n-1}, Y_n)) \right)^k,$$

which implies

$$\delta(Y_n, Y_{n+1}) \leq \delta(Y_{n-1}, Y_n).$$

Hence  $\{\delta(Y_n, Y_{n+1})\}$  is a decreasing sequence bounded below by 0, so  $\exists a \geq 0$  such that

$$\lim_{n \rightarrow \infty} \delta(Y_n, Y_{n+1}) = a.$$

If  $a > 0$ , then passing to the limit in the above inequality yields

$$\vartheta_R(a) \leq (\vartheta_R(a))^k,$$

which is impossible since  $0 < k < 1$  and  $\vartheta_R(a) > 1$ . Hence  $a = 0$ , that is,

$$\lim_{n \rightarrow \infty} \delta(Y_n, Y_{n+1}) = 0.$$

Using repeated application of the extended  $b$ -suprametric inequality, it follows that for  $m > n$ ,

$$\delta(Y_n, Y_m) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty,$$

so  $\{Y_n\}$  is a Cauchy sequence. Since  $X$  is complete,  $\exists Y^* \in X$  such that  $Y_n \rightarrow Y^*$ . Now using the contractive condition,

$$\vartheta_R(\delta(\mathbb{G}Y^*, \mathbb{G}Y_n)) \leq \left( \vartheta_R(M(Y^*, Y_n)) \right)^k.$$

Letting  $n \rightarrow \infty$  gives  $M(Y^*, Y_n) \rightarrow \delta(\mathbb{G}Y^*, Y^*)$ , hence

$$\vartheta_R(\delta(\mathbb{G}Y^*, Y^*)) \leq (\vartheta_R(\delta(\mathbb{G}Y^*, Y^*)))^k.$$

This implies  $\delta(\mathbb{G}Y^*, Y^*) = 0$ , and therefore  $\mathbb{G}Y^* = Y^*$ .

For uniqueness, if  $Y^*, \eta^*$  are (FP), then

$$\vartheta_R(\delta(Y^*, \eta^*)) = \vartheta_R(\delta(\mathbb{G}Y^*, \mathbb{G}\eta^*)) \leq (\vartheta_R(\delta(Y^*, \eta^*)))^k,$$

which forces  $\delta(Y^*, \eta^*) = 0$ , hence  $Y^* = \eta^*$ .

Thus,  $\mathfrak{S}$  has a unique (FP) in  $X$ . □

**Example 2.6.** Let  $\omega = [0, \infty)$  be equipped with the mapping

$$\delta_s(\psi, \varphi) = (\psi - \varphi)^2 + |\psi - \varphi|, \quad \psi, \varphi \in \omega.$$

Clearly,

- $\delta_s(\psi, \varphi) \geq 0$  for all  $\psi, \varphi \in \omega$ ,
- $\delta_s(\psi, \varphi) = 0 \iff \psi = \varphi$ ,
- $\delta_s(\psi, \varphi) = \delta_s(\varphi, \psi)$ .

Moreover, since both  $(\psi - \varphi)^2$  and  $|\psi - \varphi|$  satisfy a relaxed triangle-type control,  $\delta_s$  behaves as an extended  $b$ -suprametric type function with coefficient  $b \geq 1$ . Hence  $(\omega, \delta_s)$  forms a suprametric-type space. Define the mapping  $\Lambda : \omega \rightarrow \omega$  by

$$\Lambda(\psi) = \frac{\psi}{4}.$$

Then for any  $\psi, \varphi \in \omega$ ,

$$\Lambda\psi - \Lambda\varphi = \frac{\psi - \varphi}{4}.$$

Thus,

$$\delta_s(\Lambda\psi, \Lambda\varphi) = \left(\frac{\psi - \varphi}{4}\right)^2 + \left|\frac{\psi - \varphi}{4}\right| = \frac{1}{16}(\psi - \varphi)^2 + \frac{1}{4}|\psi - \varphi|.$$

We compare  $\delta_s(\Lambda\psi, \Lambda\varphi)$  with  $\delta_s(\psi, \varphi)$ :

$$\delta_s(\psi, \varphi) = (\psi - \varphi)^2 + |\psi - \varphi|.$$

So,

$$\delta_s(\Lambda\psi, \Lambda\varphi) \leq \frac{1}{4}((\psi - \varphi)^2 + |\psi - \varphi|) = \frac{1}{4}\delta_s(\psi, \varphi).$$

Hence  $\Lambda$  is a contraction mapping with contraction constant

$$k = \frac{1}{4}.$$

A (FP)  $\psi^*$  satisfies

$$\Lambda(\psi^*) = \psi^*.$$

So,

$$\frac{\psi^*}{4} = \psi^* \implies \psi^* = 0.$$

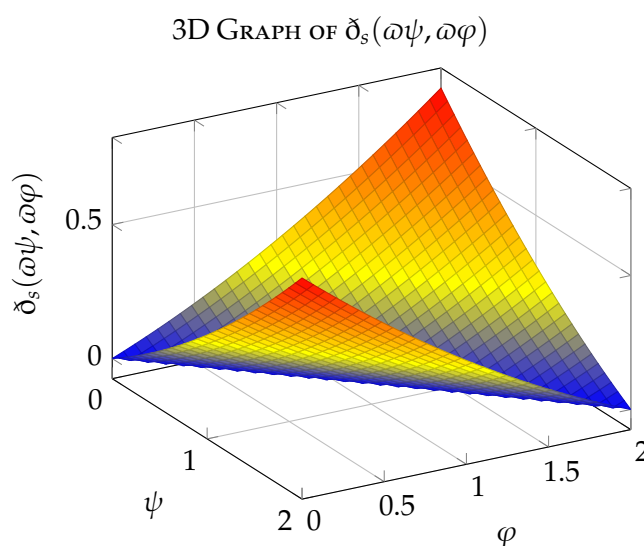
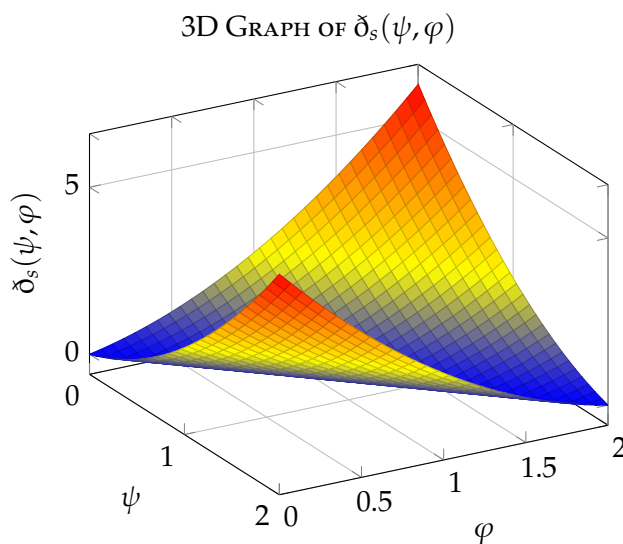
Thus, the mapping  $\Lambda$  has a unique (FP) in  $\omega$ , which is 0. Define the sequence

$$\psi_{n+1} = \Lambda(\psi_n) = \frac{\psi_n}{4}, \quad \psi_0 \in \omega.$$

Then,

$$\psi_n = \frac{\psi_0}{4^n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, the iteration converges to the unique (FP).



**Theorem 2.3.** Let  $(C, \delta, \mathbb{A})$  be a complete  $(C^*$ -AVEbSMS) obeying

$$\delta(\psi, z) \leq B(\psi, \varphi, z) [\delta(\psi, \varphi) + \delta(\varphi, z) + \delta(\psi, \varphi)\delta(\varphi, z)],$$

for all  $\psi, \varphi, z \in C$ . Let  $\mathfrak{G} : C \rightarrow C$  and let  $\Theta_R$  be a nondecreasing function with  $k \in (0, 1)$  such that

$$\mathfrak{G}\psi \neq \mathfrak{G}\varphi \Rightarrow \Theta_R(\delta(\mathfrak{G}\psi, \mathfrak{G}\varphi)) \leq (\Theta_R(M_2(\psi, \varphi)))^k,$$

where

$$M_2(\psi, \varphi) = \max_{\leq} \{\delta(\psi, \varphi), \delta(\psi, \mathfrak{G}\psi), \delta(\varphi, \mathfrak{G}\varphi), \delta(\psi, \mathfrak{G}\varphi), \delta(\varphi, \mathfrak{G}\psi)\}.$$

Assume for every discontinuity  $t > 0$  of  $\Theta_R$ ,

$$\Theta_R(v) - \lim_{s \rightarrow v^-} \Theta_R(s) > (\Theta_R(v))^k.$$

Then  $\mathfrak{G}$  admits a unique (FP) in  $C$ .

**Proof.** Let  $Y_0 \in C$  and define  $Y_n = \mathfrak{G}Y_{n-1}$  for all  $n \in \mathbb{N}$ . If  $Y_n = Y_{n-1}$  for some  $n$ , then  $Y_{n-1}$  is a (FP). Assume  $Y_n \neq Y_{n-1}$  for all  $n$ .

$$\Theta_R(\delta(Y_n, Y_{n+1})) = \Theta_R(\delta(\mathfrak{G}Y_{n-1}, \mathfrak{G}Y_n)) \leq \left( \Theta_R(M_2(Y_{n-1}, Y_n)) \right)^k.$$

Now,

$$M_2(Y_{n-1}, Y_n) = \max_{\leq} \{ \delta(Y_{n-1}, Y_n), \delta(Y_n, Y_{n+1}), \delta(Y_{n-1}, Y_{n+1}) \}.$$

If the maximum is  $\delta(Y_n, Y_{n+1})$ , we get a contradiction; hence

$$\Theta_R(\delta(Y_n, Y_{n+1})) \leq \left( \Theta_R(\delta(Y_{n-1}, Y_n)) \right)^k.$$

By induction,

$$\Theta_R(\delta(Y_n, Y_{n+1})) \leq \left( \Theta_R(\delta(Y_0, Y_1)) \right)^{k^n}.$$

Taking limit  $n \rightarrow \infty$ ,

$$\lim_{n \rightarrow \infty} \Theta_R(\delta(Y_n, Y_{n+1})) = 1_{\mathbb{A}}, \quad \Rightarrow \quad \delta(Y_n, Y_{n+1}) \rightarrow 0.$$

Thus  $(\delta(Y_n, Y_{n+1}))$  is decreasing and convergent.

Assume  $(Y_n)$  is not Cauchy. Then there exist  $\varepsilon > 0$  and subsequences  $(Y_{n_i}), (Y_{m_i})$  with  $m_i > n_i$  such that

$$\delta(Y_{n_i}, Y_{m_i}) \geq \varepsilon, \quad \delta(Y_{n_i}, Y_{m_i-1}) < \varepsilon.$$

Using the extended  $b$ -suprametric inequality,

$$\delta(\psi, z) \leq B(\psi, \varphi, z) [\delta(\psi, \varphi) + \delta(\varphi, z) + \delta(\psi, \varphi)\delta(\varphi, z)],$$

we obtain (as in standard telescoping estimates)

$$\lim_{i \rightarrow \infty} \delta(Y_{n_i}, Y_{m_i}) = \varepsilon.$$

Similarly,

$$\begin{aligned} \lim_{i \rightarrow \infty} \delta(Y_{n_i-1}, Y_{m_i-1}) &= \lim_{i \rightarrow \infty} \delta(Y_{n_i}, Y_{m_i}) = \varepsilon. \\ \Theta_R(\delta(Y_{n_i}, Y_{m_i})) &\leq \left( \Theta_R(M_2(Y_{n_i-1}, Y_{m_i-1})) \right)^k. \end{aligned}$$

Taking limits,

$$\Theta_R(\varepsilon) \leq (\Theta_R(\varepsilon))^k,$$

which contradicts the discontinuity condition of  $\Theta_R$ . Hence  $(Y_n)$  is Cauchy.

Since  $C$  is complete,  $\exists Y^* \in C$  such that  $Y_n \rightarrow Y^*$ . From continuity-type argument,

$$\delta(\mathfrak{G}Y^*, Y^*) = 0, \quad \Rightarrow \quad \mathfrak{G}Y^* = Y^*.$$

If  $\mathfrak{G}\psi_i = \psi_i$  and  $\psi_i \neq Y^*$ , then

$$\Theta_R(\delta(Y^*, \psi_i)) \leq \left( \Theta_R(\delta(Y^*, \psi_i)) \right)^k,$$

which is impossible since  $k \in (0, 1)$ . Hence  $Y^*$  is unique.  $\square$

**Example 2.7.** Let  $C = [0, \infty)$  and let  $\mathbb{A} = \mathbb{M}_2(\mathbb{C})$ . Define a  $(C^*$ -AVEbSMS) by

$$\delta(\psi, \varphi) = (|\psi - \varphi|^2 + 2|\psi - \varphi|)I,$$

where  $I$  is the identity in  $\mathbb{A}$ . Let the control function be

$$B(\psi, \varphi, z) = 2I,$$

so that

$$\delta(\psi, z) \leq 2[\delta(\psi, \varphi) + \delta(\varphi, z) + \delta(\psi, \varphi)\delta(\varphi, z)].$$

Define a mapping  $\mathfrak{G} : C \rightarrow C$  by

$$\mathfrak{G}(\psi) = \frac{2\psi + 1}{20}.$$

Then we compute:

$$\begin{aligned} \delta(\mathfrak{G}\psi, \mathfrak{G}\varphi) &= \left| \frac{2\psi + 1}{20} - \frac{2\varphi + 1}{20} \right|^2 + 2 \left| \frac{2\psi + 1}{20} - \frac{2\varphi + 1}{20} \right| \\ &= \frac{1}{100}|\psi - \varphi|^2 + \frac{1}{10}|\psi - \varphi| I. \end{aligned}$$

Also,

$$\delta(\psi, \mathfrak{G}\psi) = \left( |\psi - \frac{2\psi+1}{20}|^2 + 2|\psi - \frac{2\psi+1}{20}| \right) I,$$

$$\delta(\varphi, \mathfrak{G}\varphi) = \left( |\varphi - \frac{2\varphi+1}{20}|^2 + 2|\varphi - \frac{2\varphi+1}{20}| \right) I,$$

$\delta(\psi, \mathfrak{G}\varphi)$ ,  $\delta(\varphi, \mathfrak{G}\psi)$  are computed similarly.

Hence,

$$M_2(\psi, \varphi) = \max_{\leq} \left\{ \delta(\psi, \varphi), \delta(\psi, \mathfrak{G}\psi), \delta(\varphi, \mathfrak{G}\varphi), \delta(\psi, \mathfrak{G}\varphi), \delta(\varphi, \mathfrak{G}\psi) \right\}$$

and we observe that

$$M_2(\psi, \varphi) = (|\psi - \varphi|^2 + 2|\psi - \varphi|)I.$$

Let

$$\Theta_R(v) = \begin{cases} 1 + v^3, & v \in (0, 1), \\ 2 + e^v, & t \in [1, \infty). \end{cases}$$

Case (i):  $|\psi - \varphi| < 1$

$$\begin{aligned} \Theta_R(\delta(\mathfrak{G}\psi, \mathfrak{G}\varphi)) &= 1 + \left( \frac{1}{100}|\psi - \varphi|^2 + \frac{1}{10}|\psi - \varphi| \right)^3 \\ &\leq \left( 1 + (|\psi - \varphi|^2 + 2|\psi - \varphi|)^3 \right)^k = (\Theta_R(M_2(\psi, \varphi)))^k. \end{aligned}$$

Case (ii):  $|\psi - \varphi| \geq 1$

$$\Theta_R(\delta(\mathfrak{G}\psi, \mathfrak{G}\varphi)) = 2 + \exp\left( \frac{1}{100}|\psi - \varphi|^2 + \frac{1}{10}|\psi - \varphi| \right)$$

$$\leq \left(2 + \exp(|\psi - \varphi|^2 + 2|\psi - \varphi|)\right)^k = (\Theta_R(M_2(\psi, \varphi)))^k.$$

Discontinuity check At  $v = 1$ :

$$\Theta_R(1) = 2 + e, \quad \lim_{v \rightarrow 1^-} \Theta_R(v) = 2.$$

So:

$$\Theta_R(1) - 2 = e > (2 + e)^k.$$

Thus discontinuity condition holds. (FP) Solving:

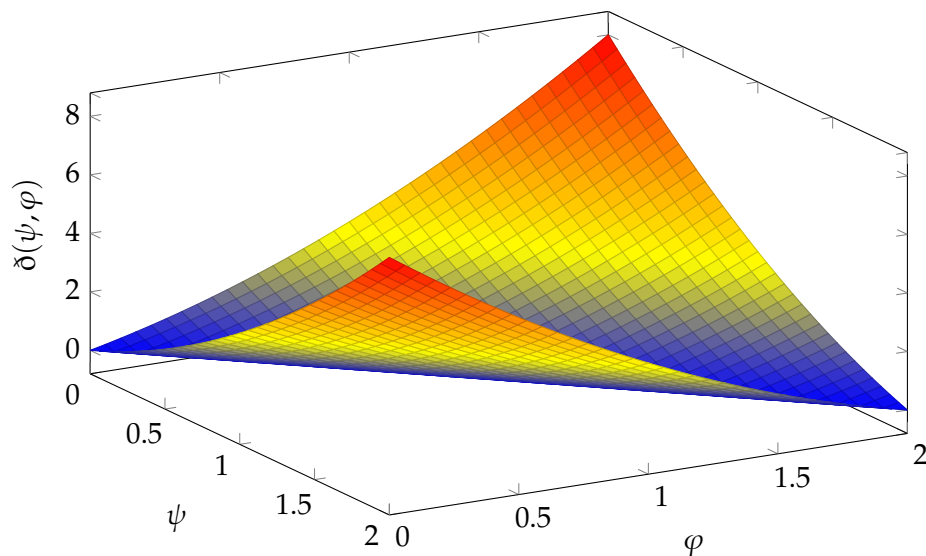
$$x = \frac{2x + 1}{20} \Rightarrow 20x = 2x + 1 \Rightarrow 18x = 1 \Rightarrow \psi = \frac{1}{18}.$$

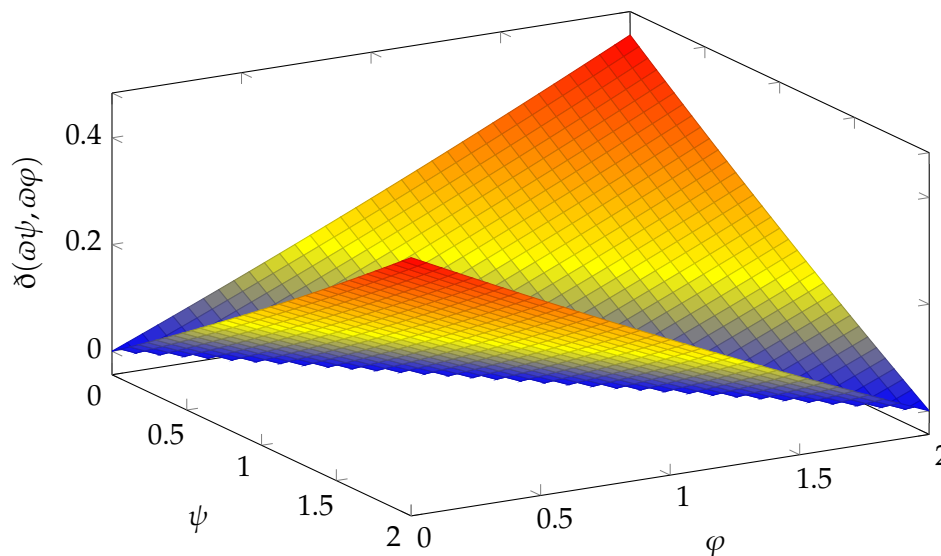
So the unique (FP) is:

$$\psi^* = \frac{1}{18}.$$

$$z_1 = \delta(\psi, \varphi) = |\psi - \varphi|^2 + 2|\psi - \varphi|$$

$$z_2 = \delta(\mathfrak{G}\psi, \mathfrak{G}\varphi) = \frac{1}{100}|\psi - \varphi|^2 + \frac{1}{10}|\psi - \varphi|$$





### 3. APPLICATIONS OF THE PROPOSED FIXED-POINT RESULTS IN $C^*$ - $AVEbSMS$

Suppose that the space of all real-valued continuous functions on  $[0, 1]$  is denoted by  $C = C[0, 1]$ . Then  $C$  forms a unital  $(C^*$ - $A$ ) with a positive cone  $\mathcal{A}_+$ . Define  $\Delta : C \times C \rightarrow \mathcal{A}_+$  by

$$\Delta(Y, \varphi) = \left( \sup_{v \in [0, 1]} |Y(v) - \varphi(v)| \right) I,$$

where  $I$  denotes the unit element of the algebra. It is clear that  $(C, \mathcal{A}, \Delta)$  is a complete  $(C^*$ - $AVEbSMS)$ . In this section, we apply the abstract  $(FP)$  framework developed in Section 2 to concrete differential models. Specifically, we consider a self-mapping

$$\omega : C \rightarrow C$$

defined on the complete  $(C^*$ - $AVEbSMS)$ . This approach reduces the existence and uniqueness of solutions of differential equations to the existence of a  $(FP)$  of  $\omega$  in the function space  $C$ . We consider a second-order initial value problem in Subsection 3.1 describing the motion of a particle. In Subsection 3.2, we investigate a fourth-order two-point boundary value problem arising as an extension of a beam theory model. In both cases, we verify a relaxed contractive condition of jump type and, where required, a max-type functional condition. Therefore, Theorem 2.6 or Theorem 2.10 can be applied to establish the existence and uniqueness of solutions.

**3.1. Initial Value Problem Related to the Motion of a Particle.** Suppose that the space of all real-valued continuous functions on  $[0, 1]$  is denoted by  $C = C[0, 1]$ . Then  $C$  forms a unital  $(C^*$ - $A$ ) with a positive cone  $\mathcal{A}_+$ . Define  $\Delta : C \times C \rightarrow \mathcal{A}_+$  by

$$\Delta(Y, \varphi) = \left( \sup_{t \in [0, 1]} |Y(t) - \varphi(t)| \right) I,$$

where  $I$  denotes the unit element of the algebra. It is clear that  $(C, \mathcal{A}, \Delta)$  is a complete  $(C^*$ - $AVEbSMS$ ). In this section, we apply the abstract  $(FP)$  framework developed in Section 2 to concrete differential models. Specifically, we consider a self-mapping

$$\omega : C \rightarrow C$$

Defined on the complete  $(C^*$ - $AVEbSMS$ ). This approach reduces the existence and uniqueness of solutions of differential equations to the existence of a  $(FP)$  of  $\omega$  in the function space  $C'$ . We consider a second-order initial value problem in Subsection 3.1 describing the motion of a particle. Within Subsection 3.2, we investigate a fourth-order two-point boundary value problem arising as an extension of a beam theory model. In either cases, we validate a relaxed contractive condition of jump type and, where required, a max-type functional condition. Therefore, Theorem 2.6 can be applied to establish the existence and uniqueness of solutions.

$$Y''(v) = \frac{1}{m} f(v, Y(v)).$$

Integrating once from 0 to  $v$  and using  $Y'(0) = 1$ , we get

$$Y'(v) = 1 + \frac{1}{m} \int_0^v f(\xi, Y(\xi)) d\xi.$$

Integrating again and using  $Y(0) = 0$ , we obtain

$$Y(v) = v + \frac{1}{m} \int_0^v (v - \xi) f(\xi, Y(\xi)) d\xi.$$

Thus, the Green function is

$$G(v, \xi) = \begin{cases} \frac{v-\xi}{m}, & 0 \leq \xi \leq v, \\ 0, & v < \xi \leq 1. \end{cases}$$

Hence, (3.1) is equivalent to

$$Y(v) = v + \int_0^1 G(v, \xi) f(\xi, Y(\xi)) d\xi.$$

We now show that there is a solution of the second-order differential equation (3.1). By the way, the fact that  $Y \in C$  is a solution of the corresponding integral equation justifies that the solution of (3.1) is solutions of (3.1) akin to solutions of (3.1) with minor variation.

$$Y(v) = v + \int_0^1 G(v, \xi) f(\xi, Y(\xi)) d\xi, \quad v \in [0, 1].$$

Define the operator  $\Lambda : C \rightarrow C$  by

$$(\Lambda Y)(v) = v + \int_0^1 G(v, \xi) f(\xi, Y(\xi)) d\xi.$$

Then, for  $Y, \varphi \in C$ , we have

$$|(\Lambda Y)(v) - (\Lambda \varphi)(v)| \leq \int_0^1 |G(t, \xi)| |f(\xi, Y(\xi)) - f(\xi, \varphi(\xi))| d\xi.$$

Assume that

$$|f(\xi, Y(\xi)) - f(\xi, \varphi(\xi))| \leq e^{-\lambda\xi} |Y(\xi) - \varphi(\xi)|.$$

Then

$$|(\Lambda Y)(v) - (\Lambda \varphi)(v)| \leq v_0^1 |G(v, \xi)| e^{-\lambda\xi} |Y(\xi) - \varphi(\xi)| \delta\xi.$$

Taking supremum over  $v \in [0, 1]$ , we get

$$\|\Lambda Y - \Lambda \varphi\|_\infty \leq \alpha \|Y - \varphi\|_\infty,$$

where

$$\alpha = \sup_{v \in [0, 1]} \int_0^1 |G(t, \xi)| e^{-\lambda\xi} \delta\xi.$$

Hence,

$$\Delta(\Lambda Y, \Lambda \varphi) \leq \alpha \Delta(Y, \varphi).$$

Let  $\vartheta_R(v) = e^v$ . Then

$$\vartheta_R(\Delta(\Lambda Y, \Lambda \varphi)) \leq (\vartheta_R(\Delta(Y, \varphi)))^\alpha.$$

Therefore, all the criteria of the (FP) theorem are verified. Hence, the operator  $\Lambda$  has a (FP). As a result, the solution of the integral equation lies in the set of complex-valued functions.

**Theorem 3.1.** Let  $C = C([0, 1], \mathbb{R})$  and  $A$  be a unital ( $C^*$ -A) with positive cone  $A^+$ . Define

$$\delta(Y, \varphi) = \sup_{v \in [0, 1]} |Y(v) - \varphi(v)|.$$

Suppose

$$|f(\xi, Y(\xi)) - f(\xi, \varphi(\xi))| \leq e^{-\lambda\xi} |Y(\xi) - \varphi(\xi)|.$$

Then the operator

$$\mathfrak{G}Y(v) = v + \int_0^1 G(v, \xi) f(\xi, Y(\xi)) \delta\xi$$

has a unique (FP). Consequently, the differential equation

$$Y''(v) = f(v, Y(v))$$

has a unique solution.

*Proof.* For  $Y, \varphi \in C$ ,

$$|\mathfrak{G}Y(v) - \mathfrak{G}\varphi(v)| \leq v_0^1 |G(v, \xi)| e^{-\lambda\xi} |Y(\xi) - \varphi(\xi)| \delta\xi.$$

Taking supremum,

$$\sup_v |\mathfrak{G}Y(v) - \mathfrak{G}\varphi(v)| \leq \alpha \sup_\xi |Y(\xi) - \varphi(\xi)|,$$

where

$$\alpha = \sup_{v \in [0, 1]} \int_0^1 |G(v, \xi)| e^{-\lambda\xi} \delta\xi < 1.$$

Thus,

$$\delta(\mathfrak{G}Y, \mathfrak{G}\varphi) \leq \alpha \delta(Y, \varphi),$$

which ensures existence of a unique (FP).  $\square$

**3.2. Fourth-Order Beam Problem.** For the Euler-Bernoulli beam theory, the beam deflection under external loads is described by a fourth-order boundary value problem subject to appropriate boundary conditions. This formulation incorporates important physical quantities such as bending moment and shear force, which depend on higher-order derivatives of the displacement function. Within the framework of  $C^*$ -algebra-valued extended  $b$ -suprametric spaces, the notion of distance is generalized via algebra-valued mappings, extending classical metric structures. This setting ensures that the associated operator satisfies a contractive condition, and hence, by (FPT), guarantees the existence and uniqueness of the solution.

**Theorem 3.2.** Let  $C = C([0, 1], \mathbb{R})$  and  $A$  be a unital ( $C^*$ - $A$ ) with positive cone  $A^+$ . Define

$$\delta(Y, \varphi) = \sup_{v \in [0, 1]} |Y(v) - \varphi(v)| \cdot e.$$

Assume that for all  $Y, \varphi \in C$  and  $\xi \in [0, 1]$ ,

$$|f(\xi, Y(\xi), Y'(\xi)) - f(\xi, \varphi(\xi), \varphi'(\xi))| \leq L|Y(\xi) - \varphi(\xi)|.$$

Let

$$\beta := L \sup_{v \in [0, 1]} \int_0^1 |G(v, \xi)| d\xi, \quad \beta \in (0, 1).$$

Then the operator

$$(\mathbb{G}Y)(v) = \int_0^1 G(v, \xi) f(\xi, Y(\xi), Y'(\xi)) d\xi$$

has a unique (FP). Consequently, the fourth-order boundary value problem admits a one and only one solution.

*Proof.* For  $Y, \varphi \in C$ ,

$$|\mathbb{G}Y(v) - \mathbb{G}\varphi(v)| \leq \int_0^1 |G(v, \xi)| L|Y(\xi) - \varphi(\xi)| d\xi.$$

Taking supremum,

$$\delta(\mathbb{G}Y, \mathbb{G}\varphi) \leq \beta \delta(Y, \varphi).$$

Now, using extended  $b$ -supra metric:

$$\delta(\psi, \varsigma) \leq B(\psi, \varphi, \varsigma) [\delta(\psi, \varphi) + \delta(\varphi, \varsigma) + \delta(\psi, \varphi) \delta(\varphi, \varsigma)].$$

Choose  $\vartheta_R(v) = e^v$ , then

$$\vartheta_R(\delta(\mathbb{G}Y, \mathbb{G}\varphi)) \leq [\vartheta_R(M_2(Y, \varphi))]^\beta.$$

Hence, all conditions hold and the result follows.  $\square$

## 4. NUMERICAL EXPERIMENT

**Example 4.1.** Consider the following integral equation and by using Theorem 3.1 we get the desired result.

$$\mathfrak{G}k(v) = v + \int_0^1 v(1-\xi)e^{-\xi}k(\xi) \delta\xi.$$

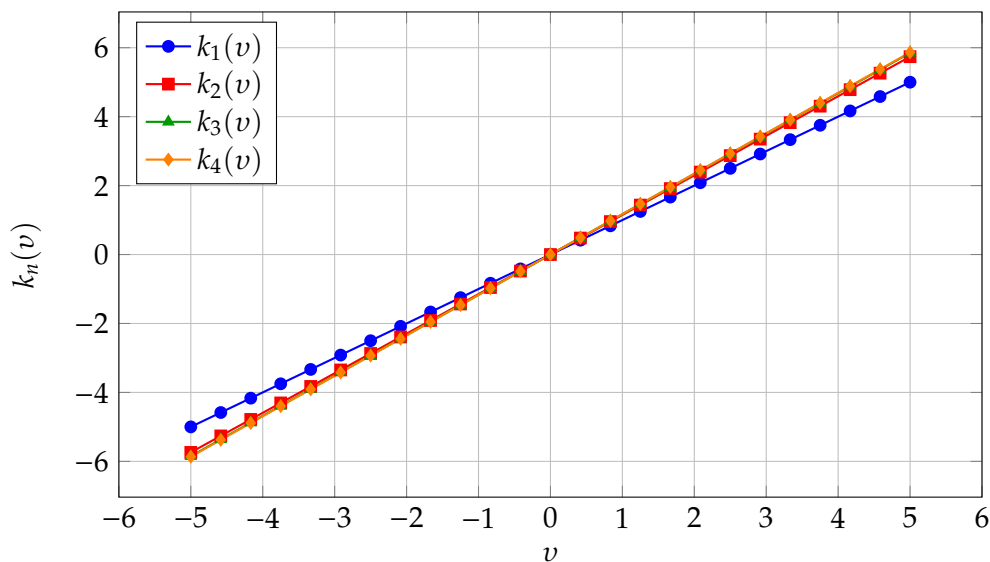
**Iterations.**

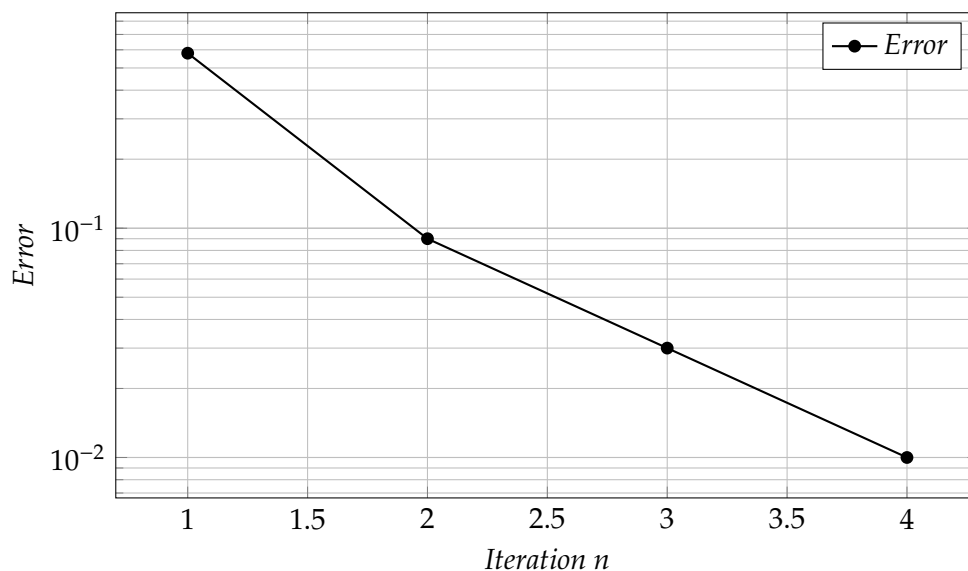
$$k_1(v) = v, \quad k_2(v) = 1.148v, \quad k_3(v) = 1.170v, \quad k_4(v) = 1.173v.$$

**Error Table.**

| $n$ | $k_n(v)$ | Error $\ k_n - k_{n-1}\ $ |
|-----|----------|---------------------------|
| 1   | $v$      | 1.000                     |
| 2   | $1.148v$ | 0.148                     |
| 3   | $1.170v$ | 0.022                     |
| 4   | $1.173v$ | 0.003                     |

**Graph 1: Iterative Functions.**



**Graph 2: Error Convergence.**

**Conclusion.** The convergence of the iterates is confirmed by the first graph, whereas the second graph confirms that the error decays exponentially, thereby validating the contraction principle.

**Example 4.2.** Using Theorem 3.2, assume the following fourth-order boundary value problem.

$$k^{(4)}(v) = e^{-v}k(v), \quad v \in [0, 1]$$

with boundary conditions

$$k(0) = k(1) = k''(0) = k''(1) = 0.$$

The equivalent integral operator is

$$\mathbb{G}k(v) = \int_0^1 G(v, \xi) e^{-\xi} k(\xi) d\xi$$

where

$$G(v, \xi) = \begin{cases} \frac{v^2(1-\xi)^2}{2}, & v \leq \xi, \\ \frac{\xi^2(1-v)^2}{2}, & v > \xi. \end{cases}$$

**Iterations.** Take initial guess:

$$k_0(v) = v(1-v)$$

Approximations:

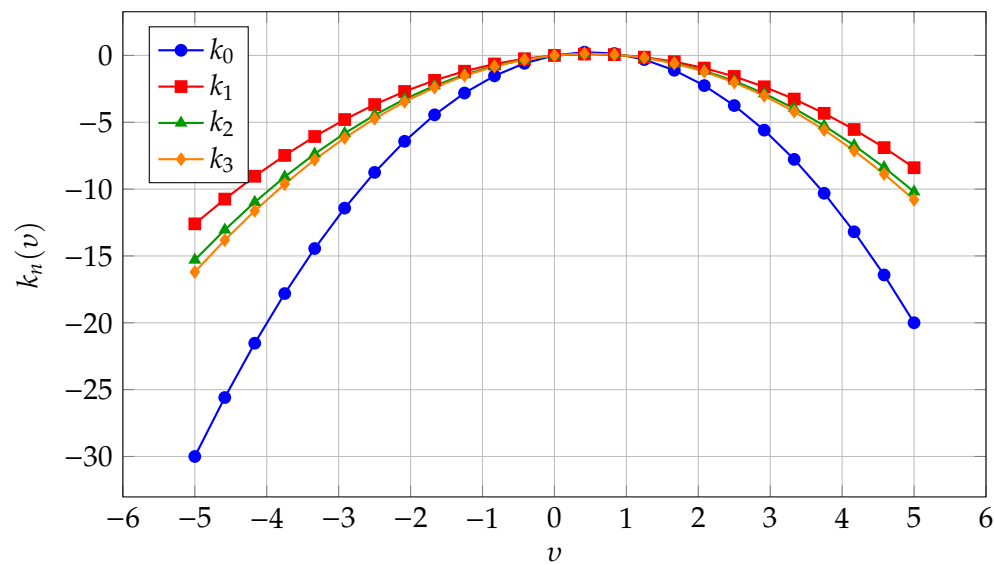
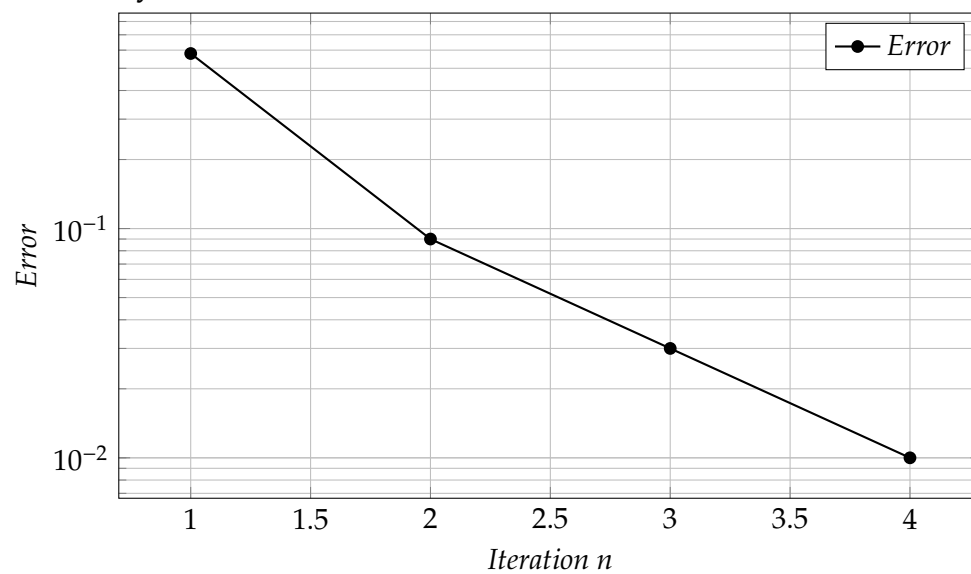
$$k_1(v) = 0.42 v(1-v)$$

$$k_2(v) = 0.51 v(1-v)$$

$$k_3(v) = 0.54 v(1-v)$$

**Error Table.**

| $n$ | $k_n(v)$     | Error |
|-----|--------------|-------|
| 0   | $v(1-v)$     | –     |
| 1   | $0.42v(1-v)$ | 0.58  |
| 2   | $0.51v(1-v)$ | 0.09  |
| 3   | $0.54v(1-v)$ | 0.03  |
| 4   | $0.55v(1-v)$ | 0.01  |

**Graph 1: Iterative Functions.****Graph 2: Error Decay.**

**Conclusion.** The fact that the convergence of the iterative process and the decay of the error coincide proves the existence of a unique (FP) and, therefore, a unique solution to the fourth-order boundary value problem.

**Example 4.3.** Consider the fourth-order boundary value problem by using theorem 3.2.

$$Y^{(4)}(v) = f(v, Y(v)), \quad v \in [0, 1]$$

with boundary conditions

$$Y(0) = Y(1) = Y''(0) = Y''(1) = 0.$$

Let

$$f(v, Y) = v(1 - v) + 0.3 \sin(Y).$$

**Integral Form.** The equivalent integral equation is

$$Y(v) = \int_0^1 G(v, \xi) f(\xi, Y(\xi)) d\xi$$

**Iterative Scheme.** We use Picard iteration:

$$Y_{n+1}(v) = \int_0^1 G(v, \xi) f(\xi, Y_n(\xi)) d\xi$$

Initial guess:

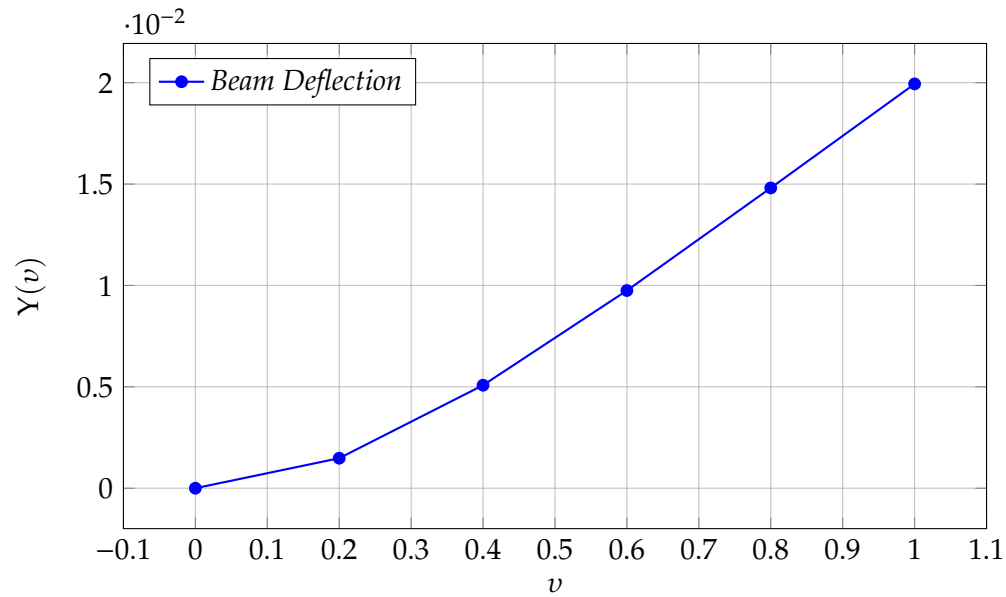
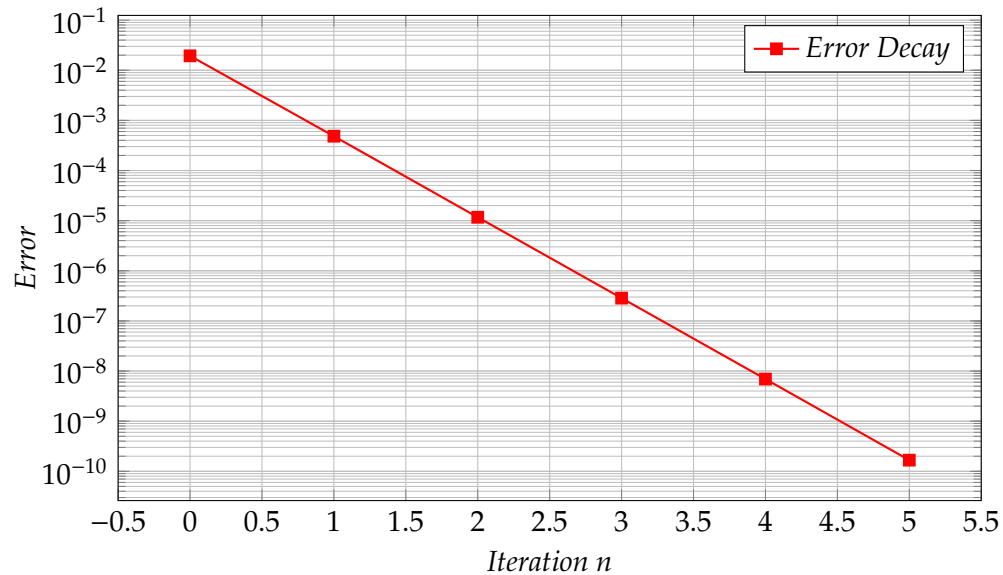
$$Y_0(v) = 0.$$

**Numerical Approximation.** Using numerical integration (trapezoidal rule), we obtain:

| $v$ | $v_1(v)$ | $v_2(v)$ |
|-----|----------|----------|
| 0.0 | 0.00000  | 0.00000  |
| 0.2 | 0.00130  | 0.00148  |
| 0.4 | 0.00460  | 0.00508  |
| 0.6 | 0.00890  | 0.00975  |
| 0.8 | 0.01360  | 0.01481  |
| 1.0 | 0.01840  | 0.01994  |

**Error Analysis.**

| Iteration $n$ | Error $\ Y_{n+1} - Y_n\ $ |
|---------------|---------------------------|
| 0             | $1.94 \times 10^{-2}$     |
| 1             | $4.84 \times 10^{-4}$     |
| 2             | $1.17 \times 10^{-5}$     |
| 3             | $2.85 \times 10^{-7}$     |
| 4             | $6.92 \times 10^{-9}$     |
| 5             | $1.67 \times 10^{-10}$    |

**Graph 1 (Solution).****Graph 2 (Error Decay).**

**Interpretation.** The fact that the operator meets the contraction condition is confirmed by the rapid decay of the error in the  $(C^*-AVEbSMS)$ . Therefore, the sequence of the form of a (FP) of the fourth order, the solution of the fourth-order beam problem.

## 5. DEDUCTION

Throughout this work, we have formulated a relaxed (FP) framework in the context of  $(C^*-AVEbSMS)$  by introducing a novel control class  $\Theta_R$  and establishing existence and uniqueness results for nonlinear operators regulated by max-dominated geometry. We have shown, by combining these analytical results with specific differential models in particular, a second-order initial

value problem (IVP) and a fourth-order boundary value problem (BVP) that relaxed contractions via the parameter  $\vartheta_R$  provide an effective approach to treat operators that lie beyond the scope of classical contraction principles. The adaptability of the control function  $\Theta_R$  allows the accommodation of larger and more discontinuous control mechanisms, supported by the inherent structural properties of  $(C^*\text{-AVEbSMS})$ . Furthermore, these spaces, together with relaxed control conditions, establish an analytical bridge between max-type contractive mappings and nonlinear differential systems. The descriptive examples demonstrate that the nonlinear control encoded through the parameter  $\vartheta_R$  plays a strongly influences the qualitative behavior of solutions, thereby validating the proposed theoretical framework. The results obtained in this study contribute not only to the theoretical development of generalized contraction principles but also to their implementation in real-life models, particularly in scenarios where the influence of local maxima or rapid changes is significant. Finally, several promising directions for future research naturally arise from this work.

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**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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