

Soft Union Quasi-Interior Ideals of Semigroups**Fatıma Zehra Kocakaya¹, Aslıhan Sezgin², Emin Aygün³, Thiti Gaketem^{4,*}**¹*Graduate School of Natural and Applied Sciences, Amasya University, Amasya, Türkiye*²*Faculty of Education, Amasya University, Amasya, Türkiye*³*Faculty of Sciences, Erciyes University, Kayseri, Türkiye*⁴*Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand***Corresponding author: thiti.ga@up.ac.th*

ABSTRACT. The generalization of algebraic structure ideals has proven to be both intriguing and beneficial for mathematicians. In this context, the notion of the quasi-interior (QI) ideal was introduced as a generalization of both quasi-ideals and interior ideals in semigroups. This paper applies this concept to soft set theory and semigroups, proposing a new type of soft union (S-uni) ideal called the "soft union (S-uni) quasi-interior (QI) ideal." The primary goal of this study is to explore the relationships between S-uni QI ideals and other specific types of S-uni ideals in semigroups. It has been shown that every S-uni interior ideal of a semigroup is an S-uni QI ideal, and every S-uni ideal is also an S-uni QI ideal. Furthermore, the S-uni bi-ideal of a group is an S-uni QI ideal, the S-uni quasi-ideal of a regular group is an S-uni QI ideal, and the idempotent S-uni QI ideal is both an S-uni bi-quasi-ideal and an S-uni bi-interior ideal. Counterexamples are provided to show that the reverse of these statements is not always true. We demonstrate that for the converses to hold, the semigroup must either be a group or regular, or the S-uni QI ideal must be idempotent. Our main theorem shows that if a subsemigroup of a semigroup is a QI ideal, its soft characteristic function is an S-uni QI ideal, and vice versa. This theorem establishes a link between semigroup theory and soft set theory. Through this result, we illustrate how the concept relates to existing algebraic structures in classical semigroup theory. Additionally, we provide conceptual characterizations and an analysis of the concept in terms of soft set operations, such as soft anti image and soft inverse image, and support our claims with specific examples. Lastly, we establish and present the connection between a regular semigroup and the structure of S-uni QI ideals.

1. Introduction

Lorem Semigroups are fundamental in various areas of mathematics, as they provide the abstract algebraic foundation for "memoryless" systems, which reset with each iteration. In applied mathematics, semigroups-first formally studied in the early 1900s-serve as crucial models for linear time-invariant systems. Since finite semigroups are closely related to finite automata, their study is essential in theoretical computer science. Furthermore, in probability theory, there

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is a connection between semigroups and Markov processes. The concept of ideals is vital for understanding mathematical structures and their applications. Consequently, many mathematicians have focused much of their research on generalizing ideals within algebraic structures. In other words, advancing the exploration of algebraic structures requires the generalization of ideals within them. By utilizing the concept and properties of ideal generalizations in algebraic structures, numerous mathematicians have made significant discoveries and characterizations of these structures. Dedekind introduced the concept of ideals in the theory of algebraic numbers, and Noether extended it to associative rings. The idea of a one-sided ideal in any algebraic structure is an extension of the ideal concept, and both one-sided and two-sided ideals remain fundamental in ring theory.

In 1952, Good and Hughes [1] introduced the concept of bi-ideals for semigroups. Steinfeld [2] was the first to present the idea of quasi-ideals for semigroups, which was later extended to rings. A quasi-ideal is a generalization of both right and left ideals, while a bi-ideal is a further generalization of the quasi-ideal. The concept of interior ideals was first introduced by Lajos [3] and later studied in more detail by Szasz [4,5]. Interior ideals are a generalization of traditional ideals. Rao [6-9] developed several new types of ideals for semigroups, which are generalizations of existing ones, such as bi-interior ideals, bi-quasi-ideals, QI ideals, weak-interior ideals, and bi-quasi-interior ideals. Additionally, the concept of essential ideals in semigroups was introduced by Baupradist et al. [10]. As a more generalized concept of various types of ideals, the idea of "almost" ideals was proposed, and their characteristics, along with their relationships with related ideals, were thoroughly examined. In this context, the concept of almost ideals in semigroups was first introduced in [11]. A later paper [12] extended the concept of bi-ideals to almost bi-ideals in semigroups. The notion of almost quasi-ideals was first presented in [13], while the ideas of almost interior ideals and weakly almost interior ideals in semigroups were proposed in [14], expanding and studying the concepts of almost ideals and interior ideals in semigroups. The concepts of almost subsemigroups, almost bi-quasi-ideals, almost bi-interior ideals, and almost bi-quasi-interior ideals in semigroups were introduced by the authors in [15-18], respectively. Furthermore, in [13, 15-20], several fuzzy almost ideal types for semigroups were explored.

Molodtsov [21] introduced "Soft Set Theory" in 1999 to address problems involving uncertainty and to find suitable solutions for them. Since then, numerous significant studies have been conducted on concepts related to soft sets, particularly regarding soft set operations. Maji et al. [22] provided definitions related to soft sets and defined specific operations on them. Pei and Miao [23] and Ali et al. [24] introduced various operations on soft sets. Sezgin and Atagün [25] also contributed to the study of soft set operations. For more detailed information on soft set operations, which have become popular since their inception, we refer to [26-37]. The concept and

operations of soft sets were further modified by Çağman and Enginoğlu [38]. Çağman et al. [39] introduced the concept of soft int-groups, which led to the exploration of various soft algebraic systems. Sezgin [40], by applying soft sets to semigroup theory, defined soft union (S-uni) semigroups, left (right/two-sided) ideals, and bi-ideals of semigroups. Sezgin et al. [41] introduced S-uni interior ideals, quasi-ideals, and generalized bi-ideals of semigroups, thoroughly examining their fundamental properties. In terms of the S-uni substructures of semigroups, Sezer et al. [42] defined and classified certain types of semigroups. In [43], specific regularities of semigroups were characterized by soft union quasi-ideals, soft union (generalized) bi-ideals, and soft union semiprime ideals. As a generalization of various types of soft intersection ideals, soft intersection almost ideals were proposed and studied in [44-55]. The soft forms of different algebraic structures have been explored in [56-69].

Rao [70] defined and studied QI ideals and fuzzy QI ideals in semirings, exploring their relationships. Rao [71] also examined the structure of QI ideals in semirings. Rao [72, 73] defined and investigated the relationships and properties of QI ideals in semirings, QI ideals in Γ -semirings, and fuzzy QI ideals. Rao [9, 70] introduced the concepts of QI ideals and weak-interior ideals in semigroups and defined the structures of QI ideals and fuzzy QI ideals in semigroups. Rao [74] studied fuzzy soft QI ideals in Γ -semirings and investigated their properties. Saeid, Rao, Kona, and Rafi [75] introduced the concept of fuzzy (soft) QI ideals in semirings and characterized regular semirings in terms of their fuzzy (soft) QI ideals. Rao [76] defined and exemplified QI ideals and their properties in Γ -semirings, presenting them with theorems and proofs. Srikanth and Shobhalatha [77] studied QI ideals and fuzzy soft QI ideals in triple semirings and explored their properties. Rao [78] introduced the concepts of QI ideals and fuzzy QI ideals of Γ -semigroups into the literature. Thus, the structure of QI ideals and their relationships in both classical and fuzzy algebra have been thoroughly studied.

In this paper, we apply the concept to soft set theory and semigroups by introducing "S-uni QI ideals of semigroups." We investigate the relationships between S-uni QI ideals and other types of S-uni ideals in a semigroup. It has been shown that every S-uni interior ideal of a semigroup is an S-uni QI ideal, every S-uni ideal is an S-uni QI ideal, the S-uni bi-ideal of a group is an S-uni QI ideal, the S-uni quasi-ideal of a regular group is an S-uni QI ideal, the idempotent S-uni QI ideal is an S-uni bi-quasi-ideal, and an S-uni bi-interior ideal. Counterexamples are provided to show that the converses of these statements are not always valid. It has also been proven that the validity of the converses depends on additional conditions. We have derived that the S-uni interior ideal of a group is an S-uni QI ideal, every idempotent soft set S-uni QI ideal is an S-uni ideal, the S-uni ideal of a regular semigroup is an S-uni QI ideal, the S-uni bi-ideal of a group is an S-uni QI ideal, every idempotent soft set S-uni QI ideal is an S-uni bi-ideal, and the S-uni quasi-ideal of a regular group is an S-uni QI ideal. Our main theorem, which asserts that if a

subsemigroup of a semigroup is a QI ideal, its soft characteristic function is an S-uni QI ideal, and vice versa, bridges the gap between semigroup theory and soft set theory. Using this theorem, we demonstrate how this concept connects to the existing algebraic structures in classical semigroup theory. We also provide characterizations of S-uni QI ideals under soft set operations, such as soft anti image and inverse image. Additionally, the relationship between a regular semigroup and the structure of S-uni QI ideals is proven and presented. Section 1 offers a general overview of the topic, while Section 2 explores the fundamental concepts of semigroups and soft set ideals, along with their relevant definitions and implications. In Section 3, we introduce the concept of S-uni QI ideals and examine its properties, as well as their relationships with other types of S-uni ideals through concrete examples. Section 4 provides a summary of our findings and discusses.

2. Preliminaries

Throughout this paper, S denotes a semigroup. A nonempty subset C of S is called a subsemigroup of S if $CC \subseteq C$, is called a bi-ideal of S if $CC \subseteq C$ and $CSC \subseteq C$, is called an interior ideal of S if $SCS \subseteq C$, and is called a quasi-ideal of S if $CS \cap SC \subseteq C$. A subsemigroup C of S is called a left (L-) QI ideal of S if $SCSC \subseteq C$, is called a right (R-) QI ideal of S if $CSCS \subseteq C$, and is called a QI ideal of S if it is both L-QI ideal of S and R-QI ideal of S [9]. If S is a regular semigroup, then for all $\eta \in S$, there exist an element $z \in S$ such that $\eta = \eta z \eta$. A semigroup S is called L-simple (R-simple) if it contains no proper L- (R-) ideal of S and is called simple if it contains no proper ideal.

Theorem 2.1 [79, 80]. Let S be a semigroup. Then,

- (1) S is L-simple (R-simple) if and only if (iff) $Su = S$ ($uS = S$) for all $u \in S$. That is, for every $u, b \in S$, there exists $m \in S$ such that $bu = mu$ ($b = um$)
- (2) S is both L-simple and R-simple iff S is a group.

Definition 2.2 [21, 38]. Let E be the parameter set, U be the universal set, $P(U)$ be the power set of U , and $\beta \subseteq E$. The soft set (SS) f_β over U is a function such that $f_\beta: E \rightarrow P(U)$, where for all $u \notin \beta$, $f_\beta(u) = \emptyset$. That is, $f_\beta = \{(u, f_\beta(u)): u \in E, f_\beta(u) \in P(U)\}$.

The set of all SSs over U is designated by $S_E(U)$ throughout this paper.

Definition 2.3 [38]. Let $f_\beta \in S_E(U)$. If $f_\beta(s) = \emptyset$ for all $s \in E$, then f_β is called a null SS and indicated by $\emptyset_E$.

Definition 2.4 [38]. Let $f_c, f_d \in S_E(U)$. If $f_c(a) \subseteq f_d(a)$, for all $a \in E$, then f_c is a soft subset of f_d and indicated by $f_c \subseteq f_d$. If $f_c(a) \supseteq f_d(a)$, for all $a \in E$, then f_c is a soft superset of f_d and indicated by $f_c \supseteq f_d$, and if $f_c(a) = f_d(a)$, for all $a \in E$, then f_c is called soft equal to f_d and denoted by $f_c = f_d$.

Definition 2.5 [38]. Let $f_c, f_D \in S_E(U)$. The union (intersection) of f_c and f_D is the SS $f_c \tilde{\cup} f_D$ ($f_c \tilde{\cap} f_D$), where $(f_c \tilde{\cup} f_D)(r) = f_c(r) \cup f_D(r)$ ($(f_c \tilde{\cap} f_D)(r) = f_c(r) \cap f_D(r)$), for all $r \in E$, respectively.

Definition 2.6 [38]. Let $f_c, f_D \in S_E(U)$. Then, v-product ($\wedge$ -product) of f_c and f_D , denoted by $f_c \vee f_D$ ($f_c \wedge f_D$) is defined by $(f_c \vee f_D)(r, z) = f_c(r) \cup f_D(z)$ ($(f_c \wedge f_D)(r, z) = f_c(r) \cap f_D(z)$) for all $(r, z) \in E \times E$, respectively.

Definition 2.7 [39]. Let $f_{\mathfrak{R}}, f_{\mathfrak{U}} \in S_E(U)$ and ζ , be a function from $\mathfrak{R}$ to $\mathfrak{U}$. Then, soft anti image of $f_{\mathfrak{R}}$ under ζ , and soft pre-image (or soft inverse image) of $f_{\mathfrak{U}}$ under ζ are the SSs $\zeta^*(f_{\mathfrak{R}})$ and $\zeta^{-1}(f_{\mathfrak{U}})$ such that

$$(\zeta^*(f_{\mathfrak{R}}))(\zeta) = \begin{cases} \bigcup \{f_{\mathfrak{R}}(\check{v}) \mid \check{v} \in \mathfrak{R} \text{ and } \zeta(\check{v}) = \zeta\}, & \text{if } \zeta^{-1}(\zeta) \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $\zeta \in \mathfrak{U}$ and $(\zeta^{-1}(f_{\mathfrak{U}}))(\check{v}) = f_{\mathfrak{U}}(\zeta(\check{v}))$ for all $\check{v} \in \mathfrak{R}$.

Definition 2.8 [40]. Let $f_B \in S_E(U)$ and $N \subseteq U$. Then, lower N -inclusion of f_B , denoted by $\mathcal{L}(f_B; N)$, is defined as $\mathcal{L}(f_B; N) = \{x \in B \mid f_B(x) \subseteq N\}$.

Definition 2.9 [40]. Let $f_S, y_S \in S_S(U)$. S-uni product $f_S * y_S$ is defined by

$$(f_S * y_S)(d) = \begin{cases} \bigcap_{d=xz} \{f_S(x) \cup y_S(z)\}, & \text{if } \exists x, z \in S \text{ such that } d = xz \\ U, & \text{otherwise} \end{cases}$$

Theorem 2.10 [40]. Let $f_S, y_S, \rho_S \in S_S(U)$. Then,

- i) $(f_S * y_S) * \rho_S = f_S * (y_S * \rho_S)$
- ii) $f_S * y_S \neq y_S * f_S$, generally.
- iii) $f_S * (y_S \tilde{\cup} \rho_S) = (f_S * y_S) \tilde{\cup} (f_S * \rho_S)$ and $(f_S \tilde{\cup} y_S) * \rho_S = (f_S * \rho_S) \tilde{\cup} (y_S * \rho_S)$
- iv) $f_S * (y_S \tilde{\cap} \rho_S) = (f_S * y_S) \tilde{\cap} (f_S * \rho_S)$ and $(f_S \tilde{\cap} y_S) * \rho_S = (f_S * \rho_S) \tilde{\cap} (y_S * \rho_S)$
- v) If $f_S \subseteq y_S$, then $f_S * \rho_S \subseteq y_S * \rho_S$ and $\rho_S * f_S \subseteq \rho_S * y_S$
- vi) If $n_S, h_S \in S_S(U)$ such that $n_S \subseteq f_S$ and $h_S \subseteq y_S$, then $n_S * h_S \subseteq f_S * y_S$.

Definition 2.11 [40]. Let $B \subseteq S$. We denote by ζ_{B^c} the soft characteristic function (SCF) of the complement B and it is defined as

$$\zeta_{B^c}(\zeta) = \begin{cases} U, & \text{if } \zeta \in S \setminus B \\ \emptyset, & \text{if } \zeta \in B \end{cases}$$

Theorem 2.12 [40]. Let $\emptyset \neq B, R \subseteq S$. Then,

- i) If $B \subseteq R$, then $\zeta_{B^c} \subseteq \zeta_{R^c}$
- ii) $\zeta_{B^c} \tilde{\cap} \zeta_{R^c} = \zeta_{B^c \cap R^c}$ and $\zeta_{B^c} \tilde{\cup} \zeta_{R^c} = \zeta_{B^c \cup R^c}$.

Definition 2.13 [40]. An SS q_S over U is called an S-uni subsemigroup of S if $q_S(e\epsilon) \subseteq q_S(e) \cup q_S(\epsilon)$ for all $e, \epsilon \in S$.

Here note that in [40], the definition of “S-uni subsemigroup of S ” is given as “S-uni semigroup of S ”; however in this paper, without loss of generality, we prefer to use “S-uni subsemigroup of S ”.

Definition 2.14 [40, 41]. An SS l_S over U is called an S-uni L- (R-) ideal of S if $l_S(m\eta) \subseteq l_S(\eta)$ ($l_S(m\eta) \subseteq l_S(\eta)$) for all $m, \eta \in S$, and is called an S-uni two-sided ideal (S-uni ideal) of S if it is both S-uni L-ideal of S over U and S-uni R-ideal of S over U . An S-uni subsemigroup l_S is called an S-uni bi-ideal of S if $l_S(\eta\eta) \subseteq l_S(\eta) \cup l_S(\eta)$ for all $\eta, \eta \in S$. An SS l_S over U is called an S-uni interior ideal of S if $l_S(\eta\eta) \subseteq l_S(\eta)$ for all $\eta, \eta \in S$.

It is easy to see that if $l_S(\alpha) = \emptyset$ for all $\alpha \in S$, then l_S is an S-uni subsemigroup (L-ideal, R-ideal, ideal, bi-ideal, interior ideal). We denote such a kind of S-uni subsemigroup (L-ideal, R-ideal, ideal, bi-ideal, interior ideal) by $\tilde{0}$. It is obvious that $\tilde{0} = \zeta_{S^c}$, that is, $\tilde{0}(\alpha) = \emptyset$ for all $\alpha \in S$ [40, 41].

Definition 2.15 [41, 81, 82]. An SS l_S over U is called an S-uni quasi-ideal of S over U if $(l_S * \tilde{0}) \cup (\tilde{0} * l_S) \cong l_S$. An SS l_S over U is called an S-uni L- (R-) bi-quasi ideal of S if $(\tilde{0} * l_S) \cup (l_S * \tilde{0}) \cong l_S$ and is called an S-uni bi-quasi ideal of S if it is both S-uni L-bi-quasi ideal of S over U and S-uni R-bi-quasi ideal of S over U . An SS l_S over U is called an S-uni bi-interior ideal of S over U if $(\tilde{0} * l_S * \tilde{0}) \cup (l_S * \tilde{0} * l_S) \cong l_S$.

Theorem 2.16 [40]. Let $g_S \in S_S(U)$. Then,

- i) $\tilde{0} * \tilde{0} \cong \tilde{0}$
- ii) $\tilde{0} * g_S \cong \tilde{0}$ and $g_S * \tilde{0} \cong \tilde{0}$
- iii) $g_S \cap \tilde{0} = \tilde{0}$ and $g_S \cup \tilde{0} = g_S$.

Theorem 2.17 [40, 41]. Let $l_S \in S_S(U)$. Then,

- (1) l_S is an S-uni subsemigroup iff $(l_S * l_S) \cong l_S$,
- (2) l_S is an S-uni L- (R-) ideal iff $(\tilde{0} * l_S) \cong l_S$ and $(l_S * \tilde{0}) \cong l_S$,
- (3) l_S is an S-uni bi-ideal iff $(l_S * l_S) \cong l_S$ and $(l_S * \tilde{0} * l_S) \cong l_S$,
- (4) l_S is an S-uni interior ideal iff $(\tilde{0} * l_S * \tilde{0}) \cong l_S$,

Theorem 2.18 [40, 41]. Let S be a semigroup. Then,

- (1) Every S-uni L- (R- /two-sided) ideal is an S-uni subsemigroup (S-uni bi-ideal/S-uni quasi-ideal).
- (2) Every S-uni quasi-ideal is an S-uni subsemigroup (S-uni bi-ideal).

Proposition 2.19 [40]. Let $l_S \in S_S(U)$, α be a subset of U , $Im(l_S)$ be the image of l_S such that $\alpha \in Im(l_S)$. If l_S is an S-uni subsemigroup of S , then $\mathcal{L}(l_S; \alpha)$ is a subsemigroup of S .

Theorem 2.20 [40]. $\emptyset \neq \mathcal{Z} \subseteq S$ is a subsemigroup iff the SS $\mathcal{F}_S$ defined by

$$\mathcal{F}_S(\zeta) = \begin{cases} \alpha, & \text{if } \zeta \in S \setminus \mathcal{Z} \\ \beta, & \text{if } \zeta \in \mathcal{Z} \end{cases}$$

is an S-uni subsemigroup, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

3. Soft Union Quasi-interior Ideals of Semigroups

In this section, we presented the concept of soft union quasi-interior ideals of semigroups, provided its examples, thoroughly examined its relationships with other soft union ideals, and analyzed them in terms of certain SS concepts and operations.

Definition 3.1. A soft set f_S over U is called a soft union left quasi-interior ideal of S over U if $f_S(xyzt) \subseteq f_S(y) \cup f_S(t)$, and is called a soft union right quasi-interior ideal of S over U if $f_S(xyzt) \subseteq f_S(x) \cup f_S(z)$ for all $x, y, z, t \in S$. A soft set over U is called a soft union quasi-interior ideal of S if it is both soft union left-quasi-interior ideal and soft union right quasi-interior ideal of S over U .

For the sake of brevity, soft union left (right) quasi-interior ideal of S over U is abbreviated by S-uni L-QI (R-QI), and soft union quasi-interior ideal is abbreviated by S-uni QI ideal.

Example 3.2. Consider the semigroup $S = \{\alpha, \tilde{\omega}, \zeta\}$ defined by the following table:

.	α	$\tilde{\omega}$	ζ
α	α	α	α
$\tilde{\omega}$	α	$\tilde{\omega}$	$\tilde{\omega}$
ζ	α	$\tilde{\omega}$	$\tilde{\omega}$

Let f_S , l_S and g_S be SSs over $U = \mathbb{Z}$ as follows: $f_S = \{(\alpha, \{1,5\}), (\tilde{\omega}, \{1,5,7\}), (\zeta, \{1,5,7,11\})\}$, $l_S = \{(\alpha, \{9\}), (\tilde{\omega}, \{3,9\}), (\zeta, \{3,8,9\})\}$ and $g_S = \{(\alpha, \{6,12\}), (\tilde{\omega}, \{2,4,6,10\}), (\zeta, \{13\})\}$.

Then, one can easily show that f_S and l_S is an S-uni QI ideal of S . Here, we find it appropriate to give a few concrete examples of elements for ease of illustration in order to be more understandable. In fact, $f_S(\alpha\tilde{\omega}\zeta\alpha) = f_S(\alpha) \subseteq f_S(\tilde{\omega}) \cup f_S(\alpha)$, $f_S(\tilde{\omega}\zeta\tilde{\omega}\alpha) = f_S(\alpha) \subseteq f_S(\zeta) \cup f_S(\alpha)$ and $f_S(\zeta\zeta\zeta) = f_S(\tilde{\omega}) \subseteq f_S(\zeta) \cup f_S(\zeta)$.

It can be easily shown that the SS f_S satisfies the S-uni L-QI ideal condition for all other element combinations of the set S . Similarly, $f_S(\zeta\zeta\tilde{\omega}\alpha) = f_S(\alpha) \subseteq f_S(\zeta) \cup f_S(\tilde{\omega})$, $f_S(\alpha\zeta\alpha\tilde{\omega}) = f_S(\alpha) \subseteq f_S(\alpha) \cup f_S(\alpha)$ and $f_S(\tilde{\omega}\tilde{\omega}\zeta\alpha) = f_S(\alpha) \subseteq f_S(\tilde{\omega}) \cup f_S(\zeta)$.

It can be easily shown that the SS f_S satisfies the S-uni R-QI ideal condition for all other element combinations of the set S , thus f_S is an S-uni QI ideal.

Similarly, $l_S(\zeta\alpha\zeta\alpha) = l_S(\alpha) \subseteq l_S(\alpha) \cup l_S(\alpha)$, $l_S(\alpha\alpha\tilde{\omega}\tilde{\omega}) = l_S(\alpha) \subseteq l_S(\alpha) \cup l_S(\tilde{\omega})$ and $l_S(\zeta\tilde{\omega}\tilde{\omega}\tilde{\omega}) = l_S(\tilde{\omega}) \subseteq l_S(\tilde{\omega}) \cup l_S(\tilde{\omega})$.

It can be easily shown that the SS l_S satisfies the S-uni L-QI ideal condition for all other element combinations of the set S . Similarly, $l_S(\tilde{\omega}\tilde{\omega}\tilde{\omega}\tilde{\omega}) = l_S(\tilde{\omega}) \subseteq l_S(\tilde{\omega}) \cup l_S(\tilde{\omega})$, $l_S(\tilde{\omega}\alpha\alpha\tilde{\omega}) = l_S(\alpha) \subseteq l_S(\tilde{\omega}) \cup l_S(\alpha)$ and $l_S(\alpha\zeta\tilde{\omega}\alpha) = l_S(\alpha) \subseteq l_S(\alpha) \cup l_S(\tilde{\omega})$.

It can be easily shown that the SS l_S satisfies the S-uni R-QI ideal condition for all other element combinations of the set S , thus l_S is an S-uni QI ideal. However, since $g_S(\alpha\tilde{\omega}\alpha\tilde{\omega}) = g_S(\alpha) \not\subseteq g_S(\tilde{\omega}) \cup g_S(\tilde{\omega})$, g_S is not an S-uni QI ideal.

It is well known that a subsemigroup Z of a semigroup S is an L- (R-) QI ideal of S if $SZS \subseteq Z$ ($SZS \subseteq Z$). It is natural to extend this property to semigroup theory with Proposition 3.3, Proposition 3.4 and Theorem 3.5.

Proposition 3.3. Let $f_S \in S_S(U)$. Then, f_S is an S-uni L-QI ideal iff $\tilde{O} *_{f_S} \tilde{O} *_{f_S} \tilde{O} \cong f_S$.

Proof: Suppose that f_S is an S-uni L-QI ideal and $a \in S$. If $(\tilde{O} *_{f_S} \tilde{O} *_{f_S})(a) = U$, then $\tilde{O} *_{f_S} \tilde{O} *_{f_S} \cong f_S$. Otherwise, there exist elements $x, y, h, n, r, s \in S$ such that $a = xy$, $x = hn$ and $n = rs$, for $a \in S$. Since f_S is an S-uni L-QI ideal, $f_S(a) = f_S(xy) = f_S((hn)y) = f_S(h(rs)y) \subseteq f_S(r) \cup f_S(y)$. Therefore,

$$\begin{aligned} (\tilde{O} *_{f_S} \tilde{O} *_{f_S})(a) &= [(\tilde{O} *_{f_S} \tilde{O}) *_{f_S}](a) \\ &= \bigcap_{a=xy} \{(\tilde{O} *_{f_S} \tilde{O})(x) \cup f_S(y)\} \\ &= \bigcap_{a=xy} \left\{ \bigcap_{x=hn} \{\tilde{O}(h) \cup (f_S * \tilde{O})(n)\} \cup f_S(y) \right\} \\ &= \bigcap_{a=xy} \left\{ \bigcap_{x=hn} \left\{ \tilde{O}(h) \cup \bigcap_{n=rs} \{f_S(r) \cup \tilde{O}(s)\} \right\} \cup f_S(y) \right\} \\ &= \bigcap_{a=hrsy} \{f_S(r) \cup f_S(y)\} \\ &\supseteq \bigcap_{a=hrsy} \{f_S(hrsy)\} \\ &= f_S(xy) \\ &= f_S(a) \end{aligned}$$

Thus, we have $\tilde{O} *_{f_S} \tilde{O} *_{f_S} \cong f_S$. Moreover, in the case where $a = xy$ and $x \neq hn$ for $a \in S$, since $(\tilde{O} *_{f_S} \tilde{O})(x) = U$, $\tilde{O} *_{f_S} \tilde{O} *_{f_S} \cong f_S$ is satisfied.

Conversely, assume that $\tilde{O} *_{f_S} \tilde{O} *_{f_S} \cong f_S$. Let $a = xyz$ for $a, x, y, z \in S$. Then, we have

$$\begin{aligned} f_S(xyz) &= f_S(a) \\ &\subseteq (\tilde{O} *_{f_S} \tilde{O} *_{f_S})(a) \\ &= [(\tilde{O} *_{f_S} \tilde{O}) *_{f_S}](a) \\ &= \bigcap_{a=hn} \{(\tilde{O} *_{f_S} \tilde{O})(h) \cup f_S(n)\} \\ &\subseteq (\tilde{O} *_{f_S} \tilde{O})(xyz) \cup f_S(z) \\ &= \bigcap_{xyz=bc} \{(\tilde{O} *_{f_S})(b) \cup \tilde{O}(c)\} \cup f_S(z) \end{aligned}$$

$$\begin{aligned}
&\subseteq [(\tilde{\mathcal{O}} *_{\mathcal{F}_S})(xy) \cup \tilde{\mathcal{O}}(z)] \cup_{\mathcal{F}_S}(t) \\
&= \left[\bigcap_{xy=pq} \{\tilde{\mathcal{O}}(p) \cup_{\mathcal{F}_S}(q)\} \cup \tilde{\mathcal{O}}(z) \right] \cup_{\mathcal{F}_S}(t) \\
&\subseteq [\{\tilde{\mathcal{O}}(x) \cup_{\mathcal{F}_S}(y)\} \cup \tilde{\mathcal{O}}(z)] \cup_{\mathcal{F}_S}(t) \\
&= [\{\emptyset \cup_{\mathcal{F}_S}(y)\} \cup \emptyset] \cup_{\mathcal{F}_S}(t) \\
&= \mathcal{F}_S(y) \cup_{\mathcal{F}_S}(t)
\end{aligned}$$

Hence, $\mathcal{F}_S(xyzt) \subseteq \mathcal{F}_S(y) \cup_{\mathcal{F}_S}(t)$ implying that $\mathcal{F}_S$ is an S-uni L-QI ideal.

Proposition 3.4. Let $\mathcal{F}_S \in \mathcal{S}_S(U)$. Then, $\mathcal{F}_S$ is an S-uni R-QI ideal iff $\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} \cong \mathcal{F}_S$.

Proof: Assume that $\mathcal{F}_S$ is an S-uni R-QI ideal and $a \in S$. If $(\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})(a) = U$, then $\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} \cong \mathcal{F}_S$. Otherwise, there exist elements $x, y, h, n, r, s \in S$ such that $a = xy$, $y = hn$ and $n = rs$, for $a \in S$. Since $\mathcal{F}_S$ is an S-uni R-QI ideal of S , $\mathcal{F}_S(a) = \mathcal{F}_S(xy) = \mathcal{F}_S(x(hn)) = \mathcal{F}_S(xh(rs)) \subseteq \mathcal{F}_S(x) \cup_{\mathcal{F}_S}(r)$. Thus,

$$\begin{aligned}
&(\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})(a) = [\mathcal{F}_S * (\tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})](a) \\
&= \bigcap_{a=xy} \{\mathcal{F}_S(x) \cup (\tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})(y)\} \\
&= \bigcap_{a=xy} \left\{ \mathcal{F}_S(x) \cup \bigcap_{y=hn} \{\tilde{\mathcal{O}}(h) \cup (\mathcal{F}_S * \tilde{\mathcal{O}})(n)\} \right\} \\
&= \bigcap_{a=xy} \left\{ \mathcal{F}_S(x) \cup \bigcap_{y=hn} \left\{ \tilde{\mathcal{O}}(h) \cup \bigcap_{n=rs} \{\mathcal{F}_S(r) \cup \tilde{\mathcal{O}}(s)\} \right\} \right\} \\
&= \bigcap_{a=xhrs} \{\mathcal{F}_S(x) \cup_{\mathcal{F}_S}(r)\} \\
&\supseteq \bigcap_{a=xhrs} \{\mathcal{F}_S(xhrs)\} \\
&= \mathcal{F}_S(xy) \\
&= \mathcal{F}_S(a)
\end{aligned}$$

Hence, we have $\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} \cong \mathcal{F}_S$. Moreover, in the case where $a = xy$ and $y \neq hn$ for $a \in S$, since $(\tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})(y) = U$, $\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} \cong \mathcal{F}_S$ is satisfied.

Conversely, let $\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} \cong \mathcal{F}_S$. Let $a = xyz\ddagger$ for $a, x, y, z, \ddagger \in S$. Then, we have

$$\begin{aligned}
&\mathcal{F}_S(xyz\ddagger) = \mathcal{F}_S(a) \\
&\subseteq (\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})(a) \\
&= [\mathcal{F}_S * (\tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})](a) \\
&= \bigcap_{a=h\ddagger} \{\mathcal{F}_S(h) \cup (\tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})(n)\}
\end{aligned}$$

$$\begin{aligned}
&\subseteq f_S(x) \cup (\tilde{O} * f_S * \tilde{O})(yzt) \\
&= f_S(x) \cup \bigcap_{yzt=de} \{(\tilde{O} * f_S)(d) \cup \tilde{O}(e)\} \\
&\subseteq f_S(x) \cup [(\tilde{O} * f_S)(yz) \cup \tilde{O}(t)] \\
&= f_S(x) \cup \left[\bigcap_{yz=pq} \{\tilde{O}(p) \cup f_S(q)\} \cup \tilde{O}(t) \right] \\
&\subseteq f_S(x) \cup [(\tilde{O}(y) \cup f_S(z)) \cup \tilde{O}(t)] \\
&= f_S(x) \cup [\emptyset \cup f_S(z) \cup \emptyset] \\
&= f_S(x) \cup f_S(z)
\end{aligned}$$

Therefore, $f_S(xyzt) \subseteq f_S(x) \cup f_S(z)$, implying that f_S is an S-uni R-QI ideal.

Theorem 3.5. Let $f_S \in S_S(U)$. Then, f_S is an S-uni QI ideal iff $\tilde{O} * f_S * \tilde{O} * f_S \cong f_S$ and $f_S * \tilde{O} * f_S * \tilde{O} \cong f_S$.

Proof: It follows by Proposition 3.3 and Proposition 3.4.

Corollary 3.6. $\tilde{O}$ is an S-uni QI ideals.

Proposition 3.7. $\emptyset \neq Z \subseteq S$ is an L- (R-) QI ideal iff the S-uni subsemigroup f_S defined by

$$f_S(\varsigma) = \begin{cases} \alpha, & \text{if } \varsigma \in S \setminus Z \\ \beta, & \text{if } \varsigma \in Z \end{cases}$$

is an S-uni L- (R-) QI ideal, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof: The proof is presented only for S-uni L-QI ideal, as the proof for S-uni R-QI ideal can be shown similarly. Suppose Z is an L-QI ideal and $x, a, y, b \in S$. If $a, b \in Z$, then $xyab \in Z$. Hence, $f_S(xyab) = f_S(a) = f_S(b) = \beta$ and so $f_S(xyab) \subseteq f_S(a) \cup f_S(b)$. If $a \notin Z$ and $b \notin Z$ then, $xyab \in Z$ or $xyab \notin Z$. In this case, if $xyab \in Z$, then $\beta = f_S(xyab) \subseteq f_S(a) \cup f_S(b) = \alpha$. If $xyab \notin Z$, then $\alpha = f_S(xyab) \subseteq f_S(a) \cup f_S(b) = \alpha$. If $a \in Z$ or $b \in Z$, then $xyab \in Z$ or $xyab \notin Z$. Here, firstly note that, if $a \in Z$ or $b \in Z$ then either $f_S(a) \cup f_S(b) = \beta$ (the case where $a \in Z$ and $b \in Z$) or $f_S(a) \cup f_S(b) = \alpha$ (the case where $a \in Z$ and $b \notin Z$ (or if $a \notin Z$ and $b \in Z$)). Thus, either $xyab \in Z$ or $xyab \notin Z$ in any case $f_S(xyab) \subseteq f_S(a) \cup f_S(b)$, since $\alpha \supseteq \beta$. Hence, f_S is an S-uni L-QI ideal. Conversely assume that S-uni subsemigroup f_S is an S-uni L-QI ideal. Let $a, b \in Z$ and $x, y \in S$. Then, $f_S(xyab) \subseteq f_S(a) = f_S(b) = \beta$. Since $\beta \subseteq \alpha$ and the function is two-valued function, $f_S(xyab) \neq \alpha$, implying that $f_S(xyab) = \beta$. Hence, $xyab \in Z$. By Theorem 2.20, Z is a subsemigroup. Thus, Z is an L-QI ideal.

Theorem 3.8. $\emptyset \neq Z \subseteq S$ is a QI ideal iff the S-uni subsemigroup f_S defined by

$$f_S(\varsigma) = \begin{cases} \alpha, & \text{if } \varsigma \in S \setminus Z \\ \beta, & \text{if } \varsigma \in Z \end{cases}$$

is an S-uni QI ideal, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proposition 3.9. Let β be a subsemigroup. Then, β is an L- (R-) QI ideal iff ζ_{β^c} is an S-uni L- (R-) QI ideal.

Proof: Since

$$\zeta_{\beta^c}(z) = \begin{cases} U, & \text{if } z \in S \setminus \beta \\ \emptyset, & \text{if } z \in \beta \end{cases}$$

and $U \supseteq \emptyset$, the remainder of the proof is completed based on proposition 3.7.

Theorem 3.10. Let β be a subsemigroup. Then, β is a QI ideal iff ζ_{β^c} is an S-uni QI ideal.

Example 3.11. We consider the semigroup in Example 3.2. One can show that $A = \{\alpha, \tilde{\alpha}\}$ is a QI ideal of S . By the definition of SCF, $\zeta_A = \{(\alpha, \emptyset), (\tilde{\alpha}, \emptyset), (\zeta, U)\}$. Then, ζ_{A^c} is an S-uni QI ideal. Conversely, by choosing the S-uni QI ideal as $f_S = \{(\alpha, \emptyset), (\tilde{\alpha}, U), (\zeta, U)\}$, which is the SCF of $K = \{\alpha\}$, one can show that K is a QI ideal of S .

Now, we continue with the relationships between S-uni QI ideals and other types of S-uni ideals of S .

Proposition 3.12. Every S-uni interior ideal is an S-uni L-QI ideal.

Proof: Let f_S be an S-uni interior ideal. Then, $\tilde{O} * f_S * \tilde{O} \cong f_S$. Since, $\tilde{O} * f_S * \tilde{O} * f_S \cong \tilde{O} * f_S * \tilde{O} * \tilde{O} \cong \tilde{O} * f_S * \tilde{O} \cong f_S$. Hence, f_S is an S-uni L-QI ideal of S .

We show with a counterexample that the converse of Proposition 3.12 is not true:

Example 3.13 Consider the semigroup $S = \{\varrho, \mathfrak{l}, r, \check{v}\}$ defined by the following table

.	ϱ	$\mathfrak{l}$	r	$\check{v}$
ϱ	ϱ	ϱ	ϱ	ϱ
$\mathfrak{l}$	ϱ	ϱ	ϱ	ϱ
r	ϱ	ϱ	ϱ	$\mathfrak{l}$
$\check{v}$	ϱ	ϱ	$\mathfrak{l}$	r

Let f_S be an SS over $U = \mathbb{Z}$ as follows:

$$f_S = \{(\varrho, \{-5\}), (\mathfrak{l}, \{-5, -4, -3\}), (r, \{-5, -4, -3, -1\}), (\check{v}, \{-5, -1\})\}$$

Here, f_S is an S-uni L-QI ideal. In fact,

$$\begin{aligned} (\tilde{O} * f_S * \tilde{O} * f_S)(\varrho) &= \{-5\} \supseteq f_S(\varrho) = \{-5\}, (\tilde{O} * f_S * \tilde{O} * f_S)(\mathfrak{l}) = U \supseteq f_S(\mathfrak{l}) = \{-5, -4, -3\} \\ (\tilde{O} * f_S * \tilde{O} * f_S)(r) &= U \supseteq f_S(r) = \{-5, -4, -3, -1\} \quad , \quad (\tilde{O} * f_S * \tilde{O} * f_S)(\check{v}) = U \supseteq f_S(\check{v}) = \{-5, -1\} \end{aligned}$$

thus, f_S is an S-uni L-QI ideal of S . However, since $f_S(\check{v}\check{v}\check{v}) = f_S(\mathfrak{l}) \not\subseteq f_S(\check{v})$, f_S is not an S-uni interior ideal.

Proposition 3.14 shows that the converse of Proposition 3.12 holds for L-simple semigroups.

Proposition 3.14. Let $f_S \in S_S(U)$ and S be an L-simple semigroup. Then, the following conditions are equivalent:

1. f_S is an S-uni interior ideal.
2. f_S is an S-uni L-QI ideal.

Proof: (1) implies (2) is obvious by Proposition 3.12. Assume that f_S is an S-uni L-QI ideal. Let $a, b, c, d \in S$. By assumption, for every $b, c \in S$, there exists $x \in S$ such that $c = xb$. Since S is an L-simple semigroup, $f_S(abc) = f_S(ab(xb)) \subseteq f_S(b) \cup f_S(b) = f_S(b)$. Hence, f_S is an S-uni interior ideal.

Proposition 3.15. Every S-uni interior ideal is an S-uni R-QI ideal.

Proof: Let f_S be an S-uni interior ideal. Then, $\tilde{O} * f_S * \tilde{O} \cong f_S$. Since, $f_S * \tilde{O} * f_S * \tilde{O} \cong \tilde{O} * \tilde{O} * f_S * \tilde{O} \cong \tilde{O} * f_S * \tilde{O} \cong f_S$. Hence, f_S is an S-uni R-QI ideal of S .

We show with a counterexample that the converse of Proposition 3.15 is not true:

Example 3.16 Consider the SS f_S in Example 3.13. The SS f_S in Example 3.13 is an S-uni R-QI ideal. In fact,

$$\begin{aligned} (f_S * \tilde{O} * f_S * \tilde{O})(\wp) &= \{-5\} \supseteq f_S(\wp) = \{-5\}, (f_S * \tilde{O} * f_S * \tilde{O})(\mathfrak{l}) = U \supseteq f_S(\mathfrak{l}) = \{-5, -4, -3\} \\ (f_S * \tilde{O} * f_S * \tilde{O})(r) &= U \supseteq f_S(r) = \{-5, -4, -3, -1\} \quad , \quad (f_S * \tilde{O} * f_S * \tilde{O})(\mathfrak{v}) = U \supseteq f_S(\mathfrak{v}) = \{-5, -1\} \end{aligned}$$

thus, f_S is an S-uni R-QI ideal of S . However, since $f_S(\mathfrak{v}\mathfrak{v}\mathfrak{v}) = f_S(\mathfrak{l}) \not\subseteq f_S(\mathfrak{v})$, f_S is not an S-uni interior ideal.

Proposition 3.17 shows that the converse of Proposition 3.15 holds for R-simple semigroups.

Proposition 3.17. Let $f_S \in S_S(U)$ and S be an R-simple semigroup. Then, the following conditions are equivalent:

1. f_S is an S-uni interior ideal.
2. f_S is an S-uni R-QI ideal.

Proof: (1) implies (2) is obvious by Proposition 3.15. Assume that f_S is an S-uni R-QI ideal and $a, b, c, d \in S$. By assumption, for every $a, b \in S$, there exists $y \in S$ such that $a = by$. Thus, $f_S(abc) = f_S((by)bc) \subseteq f_S(b) \cup f_S(b) = f_S(b)$. Hence, f_S is an S-uni interior ideal.

Theorem 3.18. Every S-uni interior ideal is an S-uni QI ideal.

Proof: It follows by Proposition 3.12 and Proposition 3.15.

Here note that the converse of Theorem 3.18 is not true follows from Example 3.13 and Example 3.16. Theorem 3.19 shows that the converse of Theorem 3.18 holds for groups as well.

Theorem 3.19. Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. f_S is an S-uni interior ideal.
2. f_S is an S-uni QI ideal.

Proof: (1) implies (2) is obvious by Theorem 3.18. Assume that f_S is an S-uni QI ideal of a group S . Then, by Theorem 2.1, S is both an L-simple and an R-simple semigroup. The rest of the proof follows from Proposition 3.14 and Proposition 3.17.

Proposition 3.20. Every S-uni L-ideal is an S-uni L-QI ideal.

Proof: Let f_S be an S-uni L-ideal of S . Then, $\tilde{O} * f_S \cong f_S$ ve $f_S * f_S \cong f_S$. Thus, $\tilde{O} * f_S * \tilde{O} * f_S \cong f_S * f_S \cong f_S$. Hence, f_S is an S-uni L-QI ideal of S .

We show with a counterexample that the converse of Proposition 3.20 is not true:

Example 3.21 Consider the semigroup $S = \{z, h, r, a\}$ defined by the following table:

*	z	h	r	a
z	z	z	z	z
h	z	z	z	z
r	z	z	h	z
a	z	z	h	h

Let f_S be an SS over $U = \mathbb{Z}_8^*$ as follows: $f_S = \{(z, \{5\}), (h, \mathbb{Z}_8^*), (r, \{\bar{3}, \bar{5}\}), (a, \{\bar{5}, \bar{7}\})\}$. Here, f_S is an S-uni L-QI ideal. In fact,

$$(\tilde{O} * f_S * \tilde{O} * f_S)(z) = \{\bar{5}\} \supseteq f_S(z) = \{\bar{5}\}, (\tilde{O} * f_S * \tilde{O} * f_S)(h) = U \supseteq f_S(h) = \mathbb{Z}_8^*$$

$$(\tilde{O} * f_S * \tilde{O} * f_S)(r) = U \supseteq f_S(r) = \{\bar{3}, \bar{5}\}, (\tilde{O} * f_S * \tilde{O} * f_S)(a) = U \supseteq f_S(a) = \{\bar{5}, \bar{7}\}$$

thus, f_S is an S-uni L-QI ideal of S . However, since $f_S(ar) = f_S(h) \not\subseteq f_S(r)$, f_S is not an S-uni L-ideal.

Proposition 3.22 shows that the converse of Proposition 3.20 holds for the idempotent SSs, and Proposition 3.23 shows that the converse of Proposition 3.20 holds for the regular semigroups as well.

Proposition 3.22. Let f_S be an idempotent SS over U . Then, the following conditions are equivalent:

1. f_S is an S-uni L-ideal.
2. f_S is an S-uni L-QI ideal.

Proof: (1) implies (2) is obvious by Proposition 3.20. Let f_S be an S-uni L-QI ideal of S . Since f_S is an idempotent S-uni L-QI ideal, $\tilde{O} * f_S = \tilde{O} * f_S * f_S = \tilde{O} * f_S * f_S * f_S \cong \tilde{O} * f_S * \tilde{O} * f_S \cong f_S$. Hence, f_S is an S-uni L-ideal.

Proposition 3.23. Let $f_S \in S_S(U)$ and S be a regular semigroup. Then, the following conditions are equivalent:

1. f_S is an S-uni L-ideal.
2. f_S is an S-uni L-QI ideal.

Proof: (1) implies (2) is obvious by Proposition 3.20. Assume that f_S is an S-uni L-QI ideal and $a, b \in S$. By assumption, for every $b \in S$, there exists $x \in S$ such that $b = bxb$. Thus, $f_S(ab) = f_S(a(bxb)) \subseteq f_S(b) \cup f_S(b) = f_S(b)$. Thus, f_S is an S-uni L-ideal.

Proposition 3.24. Every S-uni R-ideal is an S-uni R-QI ideal.

Proof: Let f_S be an S-uni R-ideal of S . Then, $f_S * \tilde{O} \cong f_S$ ve $f_S * f_S \cong f_S$. Thus, $f_S * \tilde{O} * f_S * \tilde{O} \cong f_S * f_S \cong f_S$. Hence, f_S is an S-uni R-QI ideal of S .

We show with a counterexample that the converse of Proposition 3.24 is not true:

Example 3.25 Consider the SS f_S in Example 3.21. The SS f_S in Example 3.21. is an S-uni R-QI ideal. In fact,

$$(f_S * \tilde{O} * f_S * \tilde{O})(z) = \{\bar{5}\} \supseteq f_S(z) = \{\bar{5}\}, (f_S * \tilde{O} * f_S * \tilde{O})(h) = U \supseteq f_S(h) = \mathbb{Z}_8^*$$

$$(f_S * \tilde{O} * f_S * \tilde{O})(x) = U \supseteq f_S(x) = \{\bar{3}, \bar{5}\}, (f_S * \tilde{O} * f_S * \tilde{O})(a) = U \supseteq f_S(a) = \{\bar{5}, \bar{7}\}$$

thus, f_S is an S-uni R-QI ideal of S . However, since $f_S(ax) = f_S(h) \not\subseteq f_S(a)$. f_S is not an S-uni R-ideal.

Proposition 3.26 shows that the converse of Proposition 3.24 holds for the idempotent SSs, and Proposition 3.27 shows that the converse of Proposition 3.24 holds for the regular semigroups as well.

Proposition 3.26. Let f_S be an idempotent SS over U . Then, the following conditions are equivalent:

1. f_S is an S-uni R-ideal.
2. f_S is an S-uni R-QI ideal.

Proof: (1) implies (2) is obvious by Proposition 3.24. Let f_S be an S-uni R-QI ideal of S . Since f_S is an idempotent S-uni R-QI ideal, $f_S * \tilde{O} = f_S * f_S * \tilde{O} = f_S * f_S * f_S * \tilde{O} \cong f_S * \tilde{O} * f_S * \tilde{O} \cong f_S$. Hence, f_S is an S-uni R-ideal.

Proposition 3.27. Let $f_S \in S_S(U)$ and S be a regular semigroup. Then, the following conditions are equivalent:

1. f_S is an S-uni R-ideal.
2. f_S is an S-uni R-QI ideal.

Proof: (1) implies (2) is obvious by Proposition 3.24. Assume that f_S is an S-uni R-QI ideal and $a, b \in S$. By assumption, for every $a \in S$, there exists $x \in S$ such that $a = axa$. Thus, $f_S(ab) = f_S((axa)b) \subseteq f_S(a) \cup f_S(a) = f_S(a)$. Thus, f_S is an S-uni R-ideal.

Theorem 3.28. Every S-uni ideal is an S-uni QI ideal.

Proof: It follows by Proposition 3.20 and Proposition 3.24.

Here note that the converse of Theorem 3.28 is not true follows from Example 3.21 and Example 3.25. Theorem 3.29 shows that the converse of Theorem 3.28 holds for the idempotent SSs, and Theorem 3.30 shows that the converse of Theorem 3.28 holds for the regular semigroups as well.

Theorem 3.29. Let f_S be an idempotent SS over U . Then, the following conditions are equivalent:

1. f_S is an S-uni ideal.
2. f_S is an S-uni QI ideal.

Proof: (1) implies (2) is obvious by Theorem 3.28 (2) implies (1) is obvious by Proposition 3.22 and Proposition 3.26.

Theorem 3.30. Let $f_S \in S_S(U)$ and S be a regular semigroup. Then, the following conditions are equivalent:

1. f_S is an S-uni ideal.
2. f_S is an S-uni QI ideal.

Proof: (1) implies (2) is obvious by Theorem 3.28 (2) implies (1) is obvious by Proposition 3.23 and Proposition 3.27.

Proposition 3.31. Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. f_S is an S-uni bi-ideal.
2. f_S is an S-uni L-QI ideal.

Proof: Assume that f_S is an S-uni bi-ideal and $a, b, c, d \in S$. By assumption, for every $a, b, c \in S$, there exists $x, y \in S$ such that $a = bx$ and $c = by$. Thus, $f_S(abcd) = f_S((bx)b(by)d) \subseteq f_S(bxb) \cup f_S(byd) \subseteq (f_S(b) \cup f_S(b)) \cup (f_S(b) \cup f_S(d)) = f_S(b) \cup f_S(d)$. Thus, f_S is an S-uni L-QI ideal.

Conversely, assume that f_S is an S-uni L-QI ideal and $a, b \in S$. By assumption, for every $a, b \in S$, there exists $x \in S$ such that $a = xb$ and $b = xa$. Then, $f_S(ab) = f_S((xb)(xa)) = f_S(xbxa) \subseteq f_S(b) \cup f_S(a)$. Thus, f_S is an S-uni subsemigroup. Moreover, f_S is an S-uni L-QI ideal and $a, b, c \in S$. By assumption, for every $a, b \in S$, there exists $x \in S$ such that $b = ax$. Thus, $f_S(abc) = f_S(a(ax)c) = f_S(aaxc) \subseteq f_S(a) \cup f_S(c)$. Thus, f_S is an S-uni bi-ideal.

Proposition 3.32. Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. f_S is an S-uni bi-ideal.
2. f_S is an S-uni R-QI ideal.

Proof: Assume that f_S is an S-uni bi-ideal and $a, b, c, d \in S$. By assumption, for every $b, c, d \in S$, there exists $x, y \in S$ such that $b = xc$ and $d = yc$. Thus, $f_S(abcd) = f_S(a(xc)c(yc)) \subseteq f_S(axc) \cup f_S(cyc) \subseteq (f_S(a) \cup f_S(c)) \cup (f_S(c) \cup f_S(c)) = f_S(a) \cup f_S(c)$. Thus, f_S is an S-uni R-QI ideal.

Conversely, assume that f_S is an S-uni R-QI ideal and $a, b \in S$. By assumption, for every $a, b \in S$, there exists $x \in S$ such that $a = bx$ and $b = ax$. Then, $f_S(ab) = f_S((bx)(ax)) = f_S(bxax) \subseteq f_S(b) \cup f_S(a)$. Thus, f_S is an S-uni subsemigroup.

Moreover, f_S is an S-uni R-QI ideal and $a, b, c \in S$. By assumption, for every $b, c \in S$, there exists $x \in S$ such that $b = xc$. Thus, $f_S(abc) = f_S(a(xc)c) = f_S(axcc) \subseteq f_S(a) \cup f_S(c)$. Thus, f_S is an S-uni bi-ideal.

Theorem 3.33. Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. f_S is an S-uni bi-ideal.
2. f_S is an S-uni QI ideal.

Proof: It is implied from the proof of Proposition 3.31 and Proposition 3.32.

Proposition 3.34. Let f_S be an idempotent SS over U . Every S-uni L-QI ideal is an S-uni bi-ideal.

Proof: f_S is an idempotent S-uni L-QI ideal, $f_S * f_S = f_S \cong f_S$. Hence, f_S is an S-uni subsemigroup. Moreover, $f_S * \tilde{0} * f_S = f_S * f_S * \tilde{0} * f_S \cong \tilde{0} * f_S * \tilde{0} * f_S \cong f_S$. Hence, f_S is an S-uni bi-ideal.

Proposition 3.35. Let f_S be an idempotent SS over U . Every S-uni R-QI ideal is an S-uni bi-ideal.

Proof: f_S is an idempotent S-uni R-QI ideal, $f_S * f_S = f_S \cong f_S$. Hence, f_S is an S-uni subsemigroup. Moreover, $f_S * \tilde{0} * f_S = f_S * \tilde{0} * f_S * f_S \cong f_S * \tilde{0} * f_S * \tilde{0} \cong f_S$. Hence, f_S is an S-uni bi-ideal.

Theorem 3.36. Let f_S be an idempotent SS over U . Every S-uni QI ideal is an S-uni bi-ideal.

Proof: It is implied from the proof of Proposition 3.34 and Proposition 3.35.

Proposition 3.37. Let $f_S \in S_S(U)$ and S be a regular R-simple semigroup. Then, the following conditions are equivalent:

1. f_S is an S-uni quasi ideal.
2. f_S is an S-uni L-QI ideal.

Proof: Let f_S be an S-uni quasi ideal. Then, by Theorem 2.18 (2), f_S is an S-uni bi-ideal. Since, S is an R-simple semigroup, by Proposition 3.31, f_S is an S-uni L-QI ideal.

Conversely, let f_S be an S-uni L-QI ideal. Since, S is a regular semigroup, by Proposition 3.23, f_S is an S-uni L-ideal. Then, by Theorem 2.18 (1), f_S is an S-uni quasi ideal.

Proposition 3.38. Let $f_S \in S_S(U)$ and S be a regular L-simple semigroup. Then, the following conditions are equivalent:

1. f_S is an S-uni quasi ideal.
2. f_S is an S-uni R-QI ideal.

Proof: Let f_S be an S-uni quasi ideal. Then, by Theorem 2.18 (2), f_S is an S-uni bi-ideal. Since, S is an L-simple semigroup, by Proposition 3.32, f_S is an S-uni R-QI ideal.

Conversely, let f_S be an S-uni R-QI ideal. Since, S is a regular semigroup, by Proposition 3.27, f_S is an S-uni R-ideal. Then, by Theorem 2.18 (1), f_S is an S-uni quasi ideal.

Theorem 3.39. Let $f_S \in S_S(U)$ and S be a regular group. Then, the following conditions are equivalent:

1. f_S is an S-uni quasi ideal.
2. f_S is an S-uni QI ideal.

Proof: By Theorem 2.1, S is both an R-simple and an L-simple semigroup. The rest of the proof follows from Proposition 3.37 and Proposition 3.38.

Proposition 3.40. Let f_S be an idempotent SS over U . Every S-uni L-QI ideal is an S-uni L-bi-quasi ideal.

Proof: f_S is an idempotent S-uni L-QI ideal, $(\tilde{O} * f_S) \tilde{U} (f_S * \tilde{O} * f_S) \cong f_S * \tilde{O} * f_S = f_S * f_S * \tilde{O} * f_S \cong \tilde{O} * f_S * \tilde{O} * f_S \cong f_S$. Hence, f_S is an S-uni L-bi-quasi ideal.

Proposition 3.41. Let f_S be an idempotent SS over U . Every S-uni R-QI ideal is an S-uni R-bi-quasi ideal.

Proof: f_S is an idempotent S-uni R-QI ideal, $(f_S * \tilde{O}) \tilde{U} (f_S * \tilde{O} * f_S) \cong f_S * \tilde{O} * f_S = f_S * \tilde{O} * f_S * f_S \cong f_S * \tilde{O} * f_S * \tilde{O} \cong f_S$. Hence, f_S is an S-uni R-bi-quasi ideal.

Theorem 3.42. Let f_S be an idempotent SS over U . Every S-uni QI ideal is an S-uni bi-quasi ideal.

Proof: It is implied from the proof of Proposition 3.40 and Proposition 3.41.

Proposition 3.43. Let f_S be an idempotent SS over U . Every S-uni L-QI ideal is an S-uni bi-interior ideal.

Proof: f_S is an idempotent S-uni L-QI ideal, $(\tilde{O} * f_S * \tilde{O}) \tilde{U} (f_S * \tilde{O} * f_S) \cong f_S * \tilde{O} * f_S = f_S * f_S * \tilde{O} * f_S \cong \tilde{O} * f_S * \tilde{O} * f_S \cong f_S$. Hence, f_S is an S-uni bi-interior ideal.

Proposition 3.44. Let f_S be an idempotent SS over U . Every S-uni R-QI ideal is an S-uni bi-interior ideal.

Proof: f_S is an idempotent S-uni R-QI ideal, $(\tilde{O} * f_S * \tilde{O}) \tilde{U} (f_S * \tilde{O} * f_S) \cong f_S * \tilde{O} * f_S = f_S * \tilde{O} * f_S * f_S \cong f_S * \tilde{O} * f_S * \tilde{O} \cong f_S$. Hence, f_S is an S-uni bi-interior ideal.

Theorem 3.45. Let f_S be an idempotent SS over U . Every S-uni QI ideal is an S-uni bi-interior ideal.

Proof: It is implied from the proof of Proposition 3.43 and Proposition 3.44.

Proposition 3.46. Let f_S and f_T be S-uni L- (R-) QI ideals of S and T , respectively. Then, $f_S \vee f_T$ is an S-uni L- (R-) QI ideal of $S \times T$.

Proof: The proof is presented only for S-uni L-QI ideal, as the proof for S-uni R-QI ideal can be shown similarly. Let $(x_1, u_1), (x_2, u_2), (x_3, u_3), (x_4, u_4) \in S \times T$. Then,

$$\begin{aligned} f_{SVT}((x_1, u_1)(x_2, u_2)(x_3, u_3)(x_4, u_4)) &= f_{SVT}(x_1 x_2 x_3 x_4, u_1 u_2 u_3 u_4) = f_S(x_1 x_2 x_3 x_4) \cup \\ f_T(u_1 u_2 u_3 u_4) &\subseteq (f_S(x_2) \cup f_S(x_4)) \cup (f_T(u_2) \cup f_T(u_4)) = (f_S(x_2) \cup f_T(u_2)) \cup (f_S(x_4) \cup f_T(u_4)) = \\ f_{SVT}(x_2, u_2) \cup f_{SVT}(x_4, u_4) \end{aligned}$$

Thus, $f_S \vee f_T$ is an S-uni L-QI ideal of $S \times T$.

Theorem 3.47. Let f_S and f_T be S-uni QI ideals of S and T . Then, $f_S \vee f_T$ is an S-uni QI ideal of $S \times T$.

Proposition 3.48. Let f_S and g_S be S-uni L- (R-) QI ideals. Then, $f_S \tilde{\cup} g_S$ is an S-uni L- (R-) QI ideal.

Proof: The proof is presented only for S-uni L-QI ideal, as the proof for S-uni R-QI ideal can be shown similarly. Let f_S and g_S be S-uni L-QI ideals of S . Then, $\tilde{0} * f_S * \tilde{0} * f_S \cong f_S$ and $\tilde{0} * g_S * \tilde{0} * g_S \cong g_S$.

Thus, $\tilde{0} * (f_S \tilde{\cup} g_S) * \tilde{0} * (f_S \tilde{\cup} g_S) \cong \tilde{0} * f_S * \tilde{0} * f_S \cong f_S$ and $\tilde{0} * (f_S \tilde{\cup} g_S) * \tilde{0} * (f_S \tilde{\cup} g_S) \cong \tilde{0} * g_S * \tilde{0} * g_S \cong g_S$. Hence, $\tilde{0} * (f_S \tilde{\cup} g_S) * \tilde{0} * (f_S \tilde{\cup} g_S) \cong f_S \tilde{\cup} g_S$. Thus, $f_S \tilde{\cup} g_S$ is an S-uni L-QI ideal.

Theorem 3.49. Let f_S and g_S be S-uni QI ideals. Then, $f_S \tilde{\cup} g_S$ is an S-uni QI ideals.

Proposition 3.50. Let f_S and g_S be S-uni L- (R-) ideals. Then, $f_S \tilde{\cup} g_S$ is an S-uni L- (R-) QI ideal.

Proof: The proof is presented only for S-uni L-QI ideal, as the proof for S-uni R-QI ideal can be shown similarly. Let f_S and g_S be S-uni L-ideals of S . Then, by Proposition 3.20, f_S and g_S are S-uni QI ideals. The rest of the proof is clear from Proposition 3.48.

Theorem 3.51. Let f_S and g_S be S-uni ideals. Then, $f_S \tilde{\cup} g_S$ is an S-uni QI ideal.

Proposition 3.52. Let g_S be an S-uni L-QI ideal and f_S be an S-uni L-ideal. Then, $g_S \tilde{\cup} f_S$ is an S-uni L-QI ideal.

Proof: Let g_S be an S-uni L-QI ideal and f_S be an S-uni L-ideal. Then, by Proposition 3.20, f_S is an S-uni L-QI ideal. The rest of the proof is clear from Proposition 3.48.

Theorem 3.53. Let g_S be an S-uni QI ideal and f_S be an S-uni ideal. Then, $g_S \tilde{\cup} f_S$ is an S-uni QI ideal.

Proof: Let g_S be an S-uni QI ideal and f_S be an S-uni ideal. Then, by Theorem 3.28, f_S is an S-uni QI ideal. The rest of the proof is clear from Theorem 3.49.

Proposition 3.54. Let g_S be an SS over U and f_S be an S-uni L-QI ideal. Then, $f_S * g_S$ is an S-uni L-QI ideal.

Proof: Let g_S be an SS over U and f_S be an S-uni L-QI ideal. Then, $\tilde{0} * (f_S * g_S) * \tilde{0} * (f_S * g_S) \cong \tilde{0} * f_S * (\tilde{0} * \tilde{0}) * f_S * g_S \cong (\tilde{0} * f_S * \tilde{0} * f_S) * g_S \cong f_S * g_S$. Hence, $f_S * g_S$ is an S-uni L-QI ideal.

Proposition 3.55. Let g_S be an SS over U and f_S be an S-uni R-QI ideal. Then, $g_S * f_S$ is an S-uni R-QI ideal.

Proof: Let g_S be an SS over U and f_S be an S-uni R-QI ideal. Then, $(g_S * f_S) * \tilde{0} * (g_S * f_S) * \tilde{0} \cong g_S * f_S * (\tilde{0} * \tilde{0}) * f_S * \tilde{0} \cong g_S * (f_S * \tilde{0} * f_S * \tilde{0}) \cong g_S * f_S$. Hence, $g_S * f_S$ is an S-uni R-QI ideal.

Theorem 3.56. Let f_S and g_S be S-uni QI ideals. Then, $g_S * f_S$ is an S-uni QI ideal.

Proof: Let f_S and g_S be S-uni QI ideals. Then, $\tilde{O} * (g_S * f_S) * \tilde{O} * (g_S * f_S) \cong \tilde{O} * g_S * (\tilde{O} * \tilde{O}) * g_S * f_S \cong (\tilde{O} * g_S * \tilde{O} * g_S) * f_S \cong g_S * f_S$. Hence, $g_S * f_S$ is an S-uni L-QI ideal.

Similarly, $(g_S * f_S) * \tilde{O} * (g_S * f_S) * \tilde{O} \cong g_S * f_S * (\tilde{O} * \tilde{O}) * f_S * \tilde{O} \cong g_S * (f_S * \tilde{O} * f_S * \tilde{O}) \cong g_S * f_S$. Hence, $g_S * f_S$ is an S-uni R-QI ideal. Thus, $g_S * f_S$ is an S-uni QI ideal.

Proposition 3.57. Let g_S be an SS over U and f_S be an S-uni L-ideal. Then, $f_S * g_S$ is an S-uni L-QI ideal.

Proof: Let g_S be an SS over U and f_S be an S-uni L-ideal. Then, by Proposition 3.20, f_S is an S-uni L-QI ideal. The rest of the proof is clear from Proposition 3.54.

Proposition 3.58. Let g_S be an SS over U and f_S be an S-uni R-ideal. Then, $g_S * f_S$ is an S-uni R-QI ideal.

Proof: Let g_S be an SS over U and f_S be an S-uni R-ideal. Then, by Proposition 3.24, f_S is an S-uni R-QI ideal. The rest of the proof is clear from Proposition 3.55.

Proposition 3.59. Let g_S be an S-uni L-ideal and f_S be an S-uni R-ideal. Then, $g_S * f_S$ is an S-uni QI ideal.

Proof: Let g_S be an S-uni L-ideal and f_S be an S-uni R-ideal. Then, by Proposition 3.20, g_S is an S-uni L-QI ideal and by Proposition 3.54, $g_S * f_S$ is an S-uni L-QI ideal. Similarly, by Proposition 3.24, f_S is an S-uni R-QI ideal and by Proposition 3.55, $g_S * f_S$ is an S-uni R-QI ideal. Thus, $g_S * f_S$ is an S-uni QI ideal.

Conclusion 3.60. Let g_S and f_S be S-uni ideals. Then, $g_S * f_S$ are S-uni QI ideals.

Proposition 3.61. Let f_S be a nonempty SS over U . Then, every soft superset of f_S which is contained by $\tilde{O} * f_S * \tilde{O} * f_S$ ($f_S * \tilde{O} * f_S * \tilde{O}$) is an S-uni L- (R-) QI ideal.

Proof: The proof is presented only for S-uni L-QI ideal, as the proof for S-uni R-QI ideal can be shown similarly. Let f_S be an SS over U and g_S be a soft superset of f_S which is contained by $\tilde{O} * f_S * \tilde{O} * f_S$. Since, $\tilde{O} * g_S * \tilde{O} * g_S \cong \tilde{O} * f_S * \tilde{O} * f_S \cong g_S$. Hence, g_S is an S-uni L-QI ideal.

Theorem 3.62. Let f_S be a nonempty SS over U . Then, every soft superset of f_S which is contained by $\tilde{O} * f_S * \tilde{O} * f_S$ and $f_S * \tilde{O} * f_S * \tilde{O}$ is an S-uni QI ideal.

Proposition 3.63. Let f_S be an S-uni subsemigroup over U , g_S be an S-uni L-ideal and $f_S = g_S * g_S$. Then, f_S is an S-uni L-QI ideal.

Proof: Let f_S be an S-uni subsemigroup over U , g_S be an S-uni L-ideal and $f_S = g_S * g_S$. Then, $\tilde{O} * f_S * \tilde{O} * f_S = \tilde{O} * (g_S * g_S) * \tilde{O} * (g_S * g_S) = (\tilde{O} * g_S) * g_S * (\tilde{O} * g_S) * g_S \cong (g_S * g_S) * (g_S * g_S) = f_S * f_S \cong f_S$. Hence, f_S is an S-uni L-QI ideal.

Proposition 3.64. Let f_S be an S-uni subsemigroup over U , g_S be an S-uni R-ideal and $f_S = g_S * g_S$. Then, f_S is an S-uni R-QI ideal.

Proof: Let f_S be an S-uni subsemigroup over U , g_S be an S-uni R-ideal and $f_S = g_S * g_S$. Then, $f_S * \tilde{O} * f_S * \tilde{O} = (g_S * g_S) * \tilde{O} * (g_S * g_S) * \tilde{O} = g_S * (g_S * \tilde{O}) * g_S * (g_S * \tilde{O}) \cong (g_S * g_S) * (g_S * g_S) = f_S * f_S \cong f_S$. Hence, f_S is an S-uni R-QI ideal.

Theorem 3.65. Let f_S be an S-uni subsemigroup over U , g_S be an S-uni ideal and $f_S = g_S * g_S$. Then, f_S is an S-uni QI ideal.

Proposition 3.66. Let f_S be an SS over U and $\tilde{O} * \tilde{O} * \tilde{O} * f_S \cong f_S$. Then, f_S is an S-uni L-QI ideal.

Proof: Let f_S be an SS over U and $\tilde{O} * \tilde{O} * \tilde{O} * f_S \cong f_S$. Then, $\tilde{O} * f_S * \tilde{O} * f_S \cong \tilde{O} * \tilde{O} * \tilde{O} * f_S \cong f_S$. Hence, f_S is an S-uni L-QI ideal.

Proposition 3.67. Let f_S be an SS over U and $f_S * \tilde{O} * \tilde{O} * \tilde{O} \cong f_S$. Then, f_S is an S-uni R-QI ideal.

Proof: Let f_S be an SS over U and $f_S * \tilde{O} * \tilde{O} * \tilde{O} \cong f_S$. Then, $f_S * \tilde{O} * f_S * \tilde{O} \cong f_S * \tilde{O} * \tilde{O} * \tilde{O} \cong f_S$. Hence, f_S is an S-uni R-QI ideal.

Theorem 3.68. Let f_S be an SS over U , $\tilde{O} * \tilde{O} * \tilde{O} * f_S \cong f_S$ and $f_S * \tilde{O} * \tilde{O} * \tilde{O} \cong f_S$. Then, f_S is an S-uni QI ideal.

Proposition 3.69. Let f_S be an S-uni subsemigroup over U , α be a subset of U , $Im(f_S)$ be the image of f_S such that $\alpha \in Im(f_S)$. If f_S is an S-uni L- (R-) QI ideal of S , then $\mathcal{L}(f_S; \alpha)$ is an L- (R-) QI ideal of S .

Proof: The proof is presented only for S-uni L-QI ideal, as the proof for S-uni R-QI ideal can be shown similarly. Since, $f_S(x) = \alpha$ for some $x \in S$, $\emptyset \neq \mathcal{L}(f_S; \alpha) \subseteq S$. Let $k \in [S \cdot \mathcal{L}(f_S; \alpha) \cdot S \cdot \mathcal{L}(f_S; \alpha)]$. Then, there exist $x, y \in \mathcal{L}(f_S; \alpha)$ and $a, b \in S$ such that $k = axby$. Thus, $f_S(x) \subseteq \alpha$, $f_S(y) \subseteq \alpha$. Since f_S is an S-uni L-QI ideal, $f_S(k) = f_S(axby) \subseteq f_S(x) \cup f_S(y) \subseteq \alpha \cup \alpha = \alpha$.

Hence, $f_S(k) \subseteq \alpha$, implying that $k \in \mathcal{L}(f_S; \alpha)$. Therefore, $S \cdot \mathcal{L}(f_S; \alpha) \cdot S \cdot \mathcal{L}(f_S; \alpha) \subseteq \mathcal{L}(f_S; \alpha)$. Moreover, since f_S is an S-uni subsemigroup over U , by Proposition 2.19, $\mathcal{L}(f_S; \alpha)$ is a subsemigroup of S . Thus, $\mathcal{L}(f_S; \alpha)$ is an L-QI ideal.

Theorem 3.70. Let f_S be an S-uni subsemigroup over U , α be a subset of U , $Im(f_S)$ be the image of f_S such that $\alpha \in Im(f_S)$. If f_S is an S-uni QI ideal of S , then $\mathcal{L}(f_S; \alpha)$ is a QI ideal of S .

We illustrate Theorem 3.70 with Example 3.71.

Example 3.71. Consider the SS $\mathcal{F}_S$ in Example 3.2. It is clear that The SS $\mathcal{F}_S$ in Example 3.2 is an S-uni QI ideal. By considering the image set of $\mathcal{F}_S$, that is, $Im(\mathcal{F}_S) = \{\{1,5\}, \{1,5,7\}, \{1,5,7,11\}\}$ we obtain the following:

$$\mathcal{L}(\mathcal{F}_S; \alpha) = \begin{cases} \{\alpha\}, & \alpha = \{1,5\} \\ \{\alpha, \tilde{\omega}\}, & \alpha = \{1,5,7\} \\ \{\alpha, \tilde{\omega}, \zeta\}, & \alpha = \{1,5,7,11\} \end{cases}$$

Here, $\{\alpha\}$, $\{\alpha, \tilde{\omega}\}$ and $\{\alpha, \tilde{\omega}, \zeta\}$ are all subsemigroups of S . In fact, since

$$\{\alpha\} \cdot \{\alpha\} = \{\alpha\} \subseteq \{\alpha\} \quad , \quad \{\alpha, \tilde{\alpha}\} \cdot \{\alpha, \tilde{\alpha}\} = \{\alpha, \tilde{\alpha}\} \subseteq \{\alpha, \tilde{\alpha}\} \quad , \quad \{\alpha, \tilde{\alpha}, \zeta\} \cdot \{\alpha, \tilde{\alpha}, \zeta\} = \{\alpha, \tilde{\alpha}\} \subseteq \{\alpha, \tilde{\alpha}, \zeta\}$$

each $\mathcal{L}(\mathcal{F}_S; \alpha)$ is a subsemigroup of S . Here, $\{\alpha\}$, $\{\alpha, \tilde{\alpha}\}$ and $\{\alpha, \tilde{\alpha}, \zeta\}$ are all QI ideals of S . In fact, since

$$S \cdot \{\alpha\} \cdot S \cdot \{\alpha\} = \{\alpha\} \subseteq \{\alpha\} \quad , \quad S \cdot \{\alpha, \tilde{\alpha}\} \cdot S \cdot \{\alpha, \tilde{\alpha}\} = \{\alpha, \tilde{\alpha}\} \subseteq \{\alpha, \tilde{\alpha}\} \quad , \quad S \cdot \{\alpha, \tilde{\alpha}, \zeta\} \cdot S \cdot \{\alpha, \tilde{\alpha}, \zeta\} = \{\alpha, \tilde{\alpha}\} \subseteq \{\alpha, \tilde{\alpha}, \zeta\}$$

each $\mathcal{L}(\mathcal{F}_S; \alpha)$ is an L-QI ideal of S . Similarly, since

$$\{\alpha\} \cdot S \cdot \{\alpha\} \cdot S = \{\alpha\} \subseteq \{\alpha\} \quad , \quad \{\alpha, \tilde{\alpha}\} \cdot S \cdot \{\alpha, \tilde{\alpha}\} \cdot S = \{\alpha, \tilde{\alpha}\} \subseteq \{\alpha, \tilde{\alpha}\} \quad , \quad \{\alpha, \tilde{\alpha}, \zeta\} \cdot S \cdot \{\alpha, \tilde{\alpha}, \zeta\} \cdot S = \{\alpha, \tilde{\alpha}\} \subseteq \{\alpha, \tilde{\alpha}, \zeta\},$$

each $\mathcal{L}(\mathcal{F}_S; \alpha)$ is an R-QI ideal of S , and thus each of $\mathcal{L}(\mathcal{F}_S; \alpha)$ is a QI ideal of S .

Now, consider the SS g_S in Example 3.2. By taking into account, $Im(g_S) = \{\{6,12\}, \{2,4,6,10\}, \{13\}\}$ we obtain the following:

$$\mathcal{L}(g_S; \alpha) = \begin{cases} \{\zeta\}, & \alpha = \{13\} \\ \{\alpha\}, & \alpha = \{6,12\} \\ \{\tilde{\alpha}\}, & \alpha = \{2,4,6,10\} \end{cases}$$

Here, $\{\zeta\}$ is not a QI ideal of S . In fact, since $S \cdot \{\zeta\} \cdot S \cdot \{\zeta\} = \{\alpha, \tilde{\alpha}\} \not\subseteq \{\zeta\}$. One of the $\mathcal{L}(g_S; \alpha)$ is not a L quasi interior ideal of S , hence it is not a QI ideal of S . It is seen that each of $\mathcal{L}(g_S; \alpha)$ is not a QI ideal of S . On the other hand, in Example 3.2 it was shown that g_S is not an S-uni QI ideal of S .

Definition 3.72. Let f_S be an S-uni subsemigroup and S-uni L- (R-) QI ideal of S . Then, the L- (R-) QI ideals $\mathcal{L}(f_S; \alpha)$ are called lower α -L- (R-) QI ideals of f_S .

Proposition 3.73. Let f_S be an SS over U , $\mathcal{L}(f_S; \alpha)$ be the lower α -L- (R-) QI ideal of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an S-uni L- (R-) QI ideal.

Proof: The proof is presented only for S-uni L-QI ideal, as the proof for S-uni R-QI ideal can be shown similarly. Let $a, x, b, y \in S$ and $f_S(x) = \alpha_1$ and $f_S(y) = \alpha_2$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in \mathcal{L}(f_S; \alpha_1)$ and $y \in \mathcal{L}(f_S; \alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, $x, y \in \mathcal{L}(f_S; \alpha_2)$ and since $\mathcal{L}(f_S; \alpha)$ is an L-QI of S for all $\alpha \subseteq U$, it follows that $axby \in \mathcal{L}(f_S; \alpha_2)$. Hence, $f_S(axby) \subseteq \alpha_2 = \alpha_1 \cup \alpha_2 = f_S(x) \cup f_S(y)$. Thus, f_S is an S-uni L-QI ideal.

Theorem 3.74. Let f_S be an SS over U , $\mathcal{L}(f_S; \alpha)$ be lower α -QI ideal of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an S-uni QI ideal.

Proposition 3.75. Let f_S and f_T be SSs over U , and Ψ be a semigroup isomorphism from S to T . If f_S is an S-uni L- (R-) QI ideal of S , then $\Psi^*(f_S)$ is an S-uni L- (R-) QI ideal of T .

Proof: The proof is presented only for S-uni L-QI ideal, as the proof for S-uni R-QI ideal can be shown similarly. Let $t_1, t_2, t_3, t_4 \in T$. Since Ψ is surjective, there exist $s_1, s_2, s_3, s_4 \in S$ such that $\Psi(s_1) = t_1$, $\Psi(s_2) = t_2$, $\Psi(s_3) = t_3$ and $\Psi(s_4) = t_4$. Then,

$$(\Psi^*(f_S))(t_1 t_2 t_3 t_4) = \cap \{f_S(s) : s \in S, \Psi(s) = t_1 t_2 t_3 t_4\}$$

$$\begin{aligned}
&= \bigcap \{f_S(s) : s \in S, s = \Psi^{-1}(t_1 t_2 t_3 t_4)\} \\
&= \bigcap \{f_S(s) : s \in S, s = \Psi^{-1}(\Psi(s_1 s_2 s_3 s_4)) = s_1 s_2 s_3 s_4\} \\
&= \bigcap \{f_S(s_1 s_2 s_3 s_4) : s_i \in S, \Psi(s_i) = t_i, i = 1, 2, 3, 4\} \\
&\subseteq \bigcap \{f_S(s_2) \cup f_S(s_4) : s_2, s_4 \in S, \Psi(s_2) = t_2 \text{ and } \Psi(s_4) = t_4\} \\
&= (\Psi^*(f_S))(s_2) \cup (\Psi^*(f_S))(s_4)
\end{aligned}$$

Hence, $\Psi^*(f_S)$ is an S-uni L-QI ideal of T .

Theorem 3.76. Let f_S and f_T be SSs over U , and Ψ be a semigroup isomorphism from S to T . If f_S is an S-uni QI ideal of S , then $\Psi^*(f_S)$ is an S-uni QI ideal of T .

Proposition 3.77. Let f_S and f_T be SSs over U , and Ψ be a semigroup isomorphism from S to T . If f_T is an S-uni L- (R-) QI ideal of T , then $\Psi^{-1}(f_T)$ is an S-uni L- (R-) QI ideal of S .

Proof: The proof is presented only for S-uni L-QI ideal, as the proof for S-uni R-QI ideal can be shown similarly. Let $s_1, s_2, s_3, s_4 \in S$. Then,

$$\begin{aligned}
(\Psi^{-1}(f_T))(s_1 s_2 s_3 s_4) &= f_T(\Psi(s_1 s_2 s_3 s_4)) = f_T(\Psi(s_1) \Psi(s_2) \Psi(s_3) \Psi(s_4)) \subseteq f_T(\Psi(s_2)) \cup f_T(\Psi(s_4)) = \\
&= (\Psi^{-1}(f_T))(s_2) \cup (\Psi^{-1}(f_T))(s_4)
\end{aligned}$$

Thus, $\Psi^{-1}(f_T)$ is an S-uni L-QI ideal of S .

Theorem 3.78. Let f_S and f_T be SSs over U , and Ψ be a semigroup isomorphism from S to T . If f_T is an S-uni QI ideal of T , then $\Psi^{-1}(f_T)$ is an S-uni QI ideal of S .

Proposition 3.79. Let $f_S \in S_S(U)$. Then, the following conditions are equivalent:

1. S is regular.
2. $f_S = \tilde{O} * f_S * \tilde{O} * f_S$ for every S-uni L-QI ideal of S .

Proof: First assume that (1) holds. Let S be a regular semigroup, f_S be an S-uni L-QI ideal of S and $x \in S$. Then, $\tilde{O} * f_S * \tilde{O} * f_S \supseteq f_S$ and there exists an element $y \in S$ such that $x = xyx$. Thus,

$$\begin{aligned}
(\tilde{O} * f_S * \tilde{O} * f_S)(x) &= \bigcap_{x=ab} \{(\tilde{O} * f_S * \tilde{O})(a) \cup f_S(b)\} \\
&\subseteq (\tilde{O} * f_S * \tilde{O})(xy) \cup f_S(x) \\
&= \left[\bigcap_{xy=ab} \{(\tilde{O} * f_S)(a) \cup \tilde{O}(b)\} \right] \cup f_S(x) \\
&\subseteq \{(\tilde{O} * f_S)(x) \cup \tilde{O}(yxy)\} \cup f_S(x) \\
&= \left[\bigcap_{x=pq} \tilde{O}(p) \cup f_S(q) \right] \cup \emptyset \cup f_S(x) \\
&\subseteq \{\tilde{O}(xy) \cup f_S(x)\} \cup f_S(x)
\end{aligned}$$

$$\begin{aligned}
&= \emptyset \cup \mathcal{F}_S(x) \cup \mathcal{F}_S(x) \\
&= \mathcal{F}_S(x)
\end{aligned}$$

Therefore, $\tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S \subseteq \mathcal{F}_S$ implying that $\mathcal{F}_S = \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S$.

Conversely, let $\mathcal{F}_S = \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S$, where $\mathcal{F}_S$ is an S-uni L-QI ideal of S . In order to show that S is regular, we need to show that $A = SASA$ for every L-QI ideal A of S . It is obvious that $SASA \subseteq A$. Thus, it is enough to show that $A \subseteq SASA$. On the contrary, let there exist $a \in A$ such that $a \notin SASA$. By proposition 3.9, ζ_{Ac} is an S-uni L-QI ideal. Since $a \in A$, thus $\zeta_{Ac}(a) = \emptyset$. On the other hand, since $a \notin SASA$, this implies that there do not exist $u, t \in A$ and $x, z \in S$ such that $a = xutz$. Thus, $(\zeta_{Sc} * \zeta_{Ac} * \zeta_{Sc} * \zeta_{Ac})(a) = (\tilde{\mathcal{O}} * \zeta_{Ac} * \tilde{\mathcal{O}} * \zeta_{Ac})(a) = U$. However, this conflicts with our hypothesis. Thus, $A \subseteq SASA$ and so $A = SASA$. Therefore, S is a regular semigroup.

Proposition 3.80. Let $\mathcal{F}_S \in \mathcal{S}_S(U)$. Then, the following conditions are equivalent:

1. S is regular.
2. $\mathcal{F}_S = \mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}}$ for every S-uni R-QI ideal of S .

Proof: First assume that (1) holds. Let S be a regular semigroup, $\mathcal{F}_S$ be an S-uni R-QI ideal of S and $x \in S$. Then, $\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} \subseteq \mathcal{F}_S$ and there exists an element $y \in S$ such that $x = xyx$. Thus,

$$\begin{aligned}
(\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})(x) &= \bigcap_{x=ab} \{ \mathcal{F}_S(a) \cup (\tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})(b) \} \\
&\subseteq \mathcal{F}_S(x) \cup (\tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}})(yx) \\
&= \mathcal{F}_S(x) \cup \left[\bigcap_{yx=ab} \{ \tilde{\mathcal{O}}(a) \cup (\mathcal{F}_S * \tilde{\mathcal{O}})(b) \} \right] \\
&\subseteq \mathcal{F}_S(x) \cup \tilde{\mathcal{O}}(yxy) \cup (\mathcal{F}_S * \tilde{\mathcal{O}})(x) \\
&= \mathcal{F}_S(x) \cup \emptyset \cup \left[\bigcap_{x=pq} \mathcal{F}_S(p) \cup \tilde{\mathcal{O}}(q) \right] \\
&\subseteq \mathcal{F}_S(x) \cup \{ \mathcal{F}_S(x) \cup \tilde{\mathcal{O}}(yx) \} \\
&= \mathcal{F}_S(x) \cup \mathcal{F}_S(x) \cup \emptyset \\
&= \mathcal{F}_S(x)
\end{aligned}$$

Therefore, $\mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}} \subseteq \mathcal{F}_S$ implying that $\mathcal{F}_S = \mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}}$.

Conversely, let $\mathcal{F}_S = \mathcal{F}_S * \tilde{\mathcal{O}} * \mathcal{F}_S * \tilde{\mathcal{O}}$, where $\mathcal{F}_S$ is an S-uni R-QI ideal of S . In order to show that S is regular, we need to show that $A = ASAS$ for every R-QI ideal A of S . It is obvious that $ASAS \subseteq A$. Thus, it is enough to show that $A \subseteq ASAS$. On the contrary, let there exist $a \in A$ such that $a \notin ASAS$. By proposition 3.9, ζ_{Ac} is an S-uni R-QI ideal. Since $a \in A$, thus $\zeta_{Ac}(a) = \emptyset$. On the other hand, since $a \notin ASAS$, this implies that there do not exist $x, z \in A$ and $u, t \in S$ such that $a =$

Thus, $(\zeta_{A^c} * \zeta_{S^c} * \zeta_{A^c} * \zeta_{S^c})(a) = (\zeta_{A^c} * \tilde{\theta} * \zeta_{A^c} * \tilde{\theta})(a) = U$. However, this conflicts with our hypothesis. Thus, $A \subseteq ASAS$ and so $A = ASAS$. Therefore, S is a regular semigroup.

Theorem 3.81. Let $f_S \in S_S(U)$. Then, the following conditions are equivalent:

1. S is regular.
2. $f_S = \tilde{\theta} * f_S * \tilde{\theta} * f_S = f_S * \tilde{\theta} * f_S * \tilde{\theta}$ for every S -uni QI ideal of S .

4. Conclusion

As a generalization of quasi-ideals and interior ideals of semigroups, Rao [9] introduced the concept of QI ideals and explored their properties in semigroups. In this study, we introduced "S-uni QI ideals of semigroups" and applied this concept to both SS theory and semigroup theory. We derived the relationships between S-uni QI ideals and various types of S-uni ideals in a semigroup. We showed that an S-uni ideal and an S-uni interior ideal are both S-uni QI ideals. Furthermore, the S-uni bi-ideal of a group is an S-uni QI ideal, the S-uni quasi-ideal of a regular group is an S-uni QI ideal, the idempotent S-uni QI ideal is an S-uni bi-quasi-ideal, and an S-uni bi-interior ideal. However, the converses of these statements are not true, as demonstrated by counterexamples. For the converse statements, we show that the semigroup must be either a group or regular, and the S-uni QI ideal should be idempotent. Using our key theorem, we proved that if a subsemigroup of a semigroup is a QI ideal, then its SCF is an S-uni QI ideal, and vice versa. We illustrated how this concept connects to the existing algebraic structures in classical semigroup theory, establishing a clear relationship between semigroup theory and SS theory. Additionally, we provided conceptual characterizations and an analysis of the novel idea in the context of soft anti image, soft inverse image, and SS operations, supporting our claims with specific and insightful examples. Moreover, we demonstrated the connection between a regular semigroup and the structure of S-uni QI ideals. In future studies, different S-uni ideals of semigroups can be investigated, and the S-uni QI ideals introduced in this work can be characterized within intra-regular, zero semigroups, and semisimple semigroups.

The relation between several S-uni ideals and their generalized ideals is depicted in the following figure, where $A \rightarrow B$ denotes that A is B but B may not always be A .

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