

A Three-Term Conjugate Gradient Framework for Matrix-Scaled Consensus in Hybrid Multi-Agent Systems

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Abstract. This paper studies matrix-scaled consensus in hybrid multi-agent systems with both continuous-time and discrete-time dynamics. By introducing a coordinate transformation, the problem is reformulated as an agreement problem in a transformed space, leading to a regularized linear system. An optimization-based framework is developed, and a three-term conjugate-gradient (TTCG) algorithm is proposed to compute the consensus state efficiently. It is shown that the associated operator is symmetric positive definite, ensuring existence and uniqueness of the solution, while the TTCG iteration generates a strict descent sequence and converges to the projected agreement state. The framework is further extended to directed networks via a symmetrized formulation. Numerical results validate the theoretical findings and reveal that, although agreement is achieved in the transformed coordinates, the original states exhibit a clustered structure determined by the scaling matrices. The proposed approach provides a unified and computationally efficient solution for matrix-scaled consensus.

1. INTRODUCTION

Consensus problems in multi-agent systems (MASs) constitute a fundamental topic in distributed coordination, networked control, cooperative robotics, sensor networks, and cyber-physical systems. The central objective is to design local interaction rules under which a group of agents reaches an agreement by using only neighboring information. Classical consensus models typically drive all agent states toward a common value. However, many practical systems are heterogeneous: agents may have different physical units, local priorities, communication weights, sensing capabilities, or performance requirements. These features motivate scaled consensus, in which agreement is not imposed directly on the original states but on their scaled representations.

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Hybrid multi-agent systems (HMASs) further enrich the consensus problem by allowing continuous-time and discrete-time agents to coexist in the same network. Such models are more realistic for networked systems involving digital controllers, sampled-data communication, and continuous physical dynamics. A foundational hybrid consensus framework was developed in [2], where continuous-time and discrete-time agents were treated in a unified setting. More recently, asymptotic consensus problems for HMASs under network attacks were investigated in [10], showing that robustness and hybrid dynamics remain important issues in modern MAS research.

Finite-time consensus has also attracted increasing attention because asymptotic convergence may be insufficient in time-critical applications. Several recent studies have addressed finite-time or fast consensus by using sliding-mode, event-triggered, or hybrid control techniques. For instance, improved finite-time sliding-mode consensus under fuzzy topologies was studied in [1], while finite-time consensus for second-order MASs with input saturation was considered in [14]. Fast finite-time consensus protocols for high-order nonlinear MASs using event-triggered communication were developed in [13]. These works significantly advance finite-time consensus theory, but they primarily focus on direct state agreement and do not explicitly address matrix-scaled agreement in hybrid networks.

Scaled consensus provides a broader framework for heterogeneous coordination. In the hybrid setting, finite-time scaled consensus via conjugate gradient methods was studied in [3], where the consensus condition was transformed into a linear algebraic problem and solved using CGM-based ideas. Related edge-based extensions have also been developed. Hybrid edge consensus with scaled dynamics and protocol design was examined in [4], pulse-modulated scaled consensus for edge dynamics in HMASs was studied in [11], and multi-layered consensus protocols for modular edge-dynamic networks were proposed in [12]. These studies demonstrate the growing importance of scaled and structured consensus models. Nevertheless, most existing scaled consensus results rely on scalar scaling factors, edge-space formulations, or classical conjugate-gradient mechanisms. A systematic matrix-scaled consensus framework for HMASs, equipped with a three-term conjugate-gradient algorithm and a regularized optimization interpretation, remains insufficiently developed.

In parallel, conjugate-gradient-type algorithms have become powerful tools in optimization and distributed computation. Classical CG methods are well suited for symmetric positive definite linear systems and have been used to accelerate finite-time consensus. Recent developments in three-term conjugate-gradient methods further show that additional memory and correction terms can improve descent properties, robustness, and numerical performance. A new three-term conjugate gradient method for multiobjective optimization was proposed in [5], while a three-term conjugate-gradient-type method with sufficient descent property for vector optimization was developed in [6]. These works suggest that three-term structures can provide stronger descent behavior than classical two-term schemes. However, their application to matrix-scaled consensus in hybrid multi-agent systems has not yet been systematically explored.

Distributed optimization literature also provides useful insight into the relationship between consensus and iterative computation. The gradient-consensus method in [7] combines gradient descent with a finite-time terminated consensus mechanism over directed networks. Recent works on finite-time consensus in gradient tracking, such as [8] and [9], further indicate that consensus mechanisms can be integrated with optimization algorithms to improve convergence and coordination. These studies are conceptually related to the present work in that both consensus and optimization are combined. However, the objective here is different: rather than solving a distributed optimization problem with consensus as a subroutine, this paper uses an optimization-based formulation to compute the matrix-scaled consensus state itself.

Motivated by these observations, this paper develops a three-term conjugate-gradient framework for matrix-scaled consensus in hybrid multi-agent systems. Each agent is associated with a symmetric positive definite scaling matrix, and the matrix-scaled consensus condition is converted into ordinary agreement in a transformed coordinate. This transformation enables the consensus problem to be reformulated as a regularized linear system. The resulting operator is shown to be symmetric positive definite under an undirected connected graph, and the solution coincides exactly with the projected agreement component of the transformed initial condition. A three-term conjugate-gradient algorithm is then constructed to solve this system efficiently. The main contributions of this paper are summarized as follows.

- A matrix-scaled consensus framework is formulated for hybrid multi-agent systems with both continuous-time and discrete-time dynamics. Unlike scalar scaled consensus, the proposed formulation allows each agent to possess a local positive definite scaling matrix.
- The matrix-scaled consensus condition is transformed into an agreement problem in a scaled coordinate. This leads to a regularized linear system whose unique solution is exactly the projected agreement state.
- A three-term conjugate-gradient algorithm is developed for the regularized consensus system. The proposed direction incorporates residual, displacement, and correction terms, yielding a strict descent relation for the associated quadratic functional.
- Theoretical guarantees are established, including the characterization of the consensus manifold, positive definiteness of the regularized operator, uniqueness of the consensus solution, strict descent of the TTCG iteration, and finite-step convergence under additional conjugacy conditions.
- An extension to directed networks is discussed through a symmetrized regularization, preserving the optimization-based structure while clarifying the limitations of nonsymmetric formulations.
- Numerical simulations demonstrate that the transformed states reach agreement, whereas the original states form distinct clusters determined by the scaling matrices, thereby validating the matrix-scaled consensus interpretation.

The remainder of this paper is organized as follows. Section 2 introduces graph-theoretic preliminaries, the hybrid multi-agent model, matrix-scaled consensus, and the standing assumptions. Section 3 presents the scaled reformulation, the regularized quadratic model, and the TTCG algorithm. Section 4 establishes the main theoretical results. Section 5 provides numerical validation. Finally, Section 6 concludes the paper and discusses future research directions.

2. PRELIMINARIES AND PROBLEM FORMULATION

This paper investigates matrix-scaled consensus in hybrid multi-agent systems and develops a three-term conjugate-gradient framework in a unified analytical setting. In contrast to conventional approaches that rely on case-by-case protocol constructions, an operator-oriented perspective is adopted. Specifically, the distributed interaction mechanism is used to induce a structured sampled evolution in appropriately transformed coordinates, which allows the consensus problem to be reformulated as a regularized quadratic optimization problem.

Let $\mathbb{R}$ denote the set of real numbers. For $n \in \mathbb{N}$, let $\mathbf{1}_n$ and I_n denote the all-ones vector and the identity matrix of dimension n , respectively. For any matrix A , A^T denotes its transpose, and $\|\cdot\|$ denotes the Euclidean norm.

Graph-theoretic preliminaries. Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with node set $\mathcal{V} = \{1, \dots, N\}$ and adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$. The associated Laplacian matrix is defined as $L = D - \mathcal{A}$, where $D = \text{diag}(d_1, \dots, d_N)$ with $d_i = \sum_{j=1}^N a_{ij}$. Since $\mathcal{G}$ is undirected, L is symmetric positive semidefinite. Moreover, if $\mathcal{G}$ is connected, then $\ker(L) = \text{span}\{\mathbf{1}_N\}$, which characterizes the agreement structure of the network.

Hybrid multi-agent model. Each agent $i \in \mathcal{V}$ is associated with a state $x_i(t) \in \mathbb{R}^d$ and a symmetric positive definite matrix $S_i \in \mathbb{R}^{d \times d}$ that characterizes local scaling. Define the block-diagonal matrix $S = \text{blkdiag}(S_1, \dots, S_N) \in \mathbb{R}^{Nd \times Nd}$. Let $X(t) = \text{col}(x_1(t), \dots, x_N(t)) \in \mathbb{R}^{Nd}$ be the stacked state vector, and define the transformed variable $Z(t) = SX(t)$. This transformation plays a fundamental role, as it converts the matrix-scaled consensus condition in the original coordinates into a standard agreement problem in the transformed coordinates. We consider a hybrid multi-agent system in which the index set $\mathcal{V}$ is partitioned as $\mathcal{V} = \mathcal{V}_c \cup \mathcal{V}_d$, $\mathcal{V}_c \cap \mathcal{V}_d = \emptyset$, where $\mathcal{V}_c$ and $\mathcal{V}_d$ denote the sets of continuous-time and discrete-time agents, respectively. The hybrid nature arises from the coexistence of continuous-time evolution over sampling intervals and discrete-time updates at sampling instants. The system dynamics are described by

$$\begin{cases} S_i \dot{x}_i(t) = u_i(t), & i \in \mathcal{V}_c, \\ S_i x_i(t_{k+1}) = S_i x_i(t_k) + u_i(t_k), & i \in \mathcal{V}_d, \end{cases} \quad (2.1)$$

where $\{t_k\}$ is a sequence of sampling instants defined by $t_k = kh$ with $h > 0$.

Matrix-scaled consensus. Define the matrix-scaled consensus manifold $C_S = \{X \in \mathbb{R}^{Nd} : S_i x_i = S_j x_j, \forall i, j \in \mathcal{V}\}$ and the corresponding agreement subspace $C_Z = \{Z \in \mathbb{R}^{Nd} : Z = \mathbf{1}_N \otimes z^*, z^* \in \mathbb{R}^d\}$. It follows that $X \in C_S$ iff $Z = SX \in C_Z$.

Definition 2.1 (Definition 2.1). *The hybrid multi-agent system (2.1) is said to achieve matrix-scaled consensus if, for every initial condition,*

$$\begin{cases} \lim_{k \rightarrow \infty} \|S_i x_i(t_k) - S_j x_j(t_k)\| = 0, & \forall i, j, \\ \lim_{t \rightarrow \infty} \|S_i x_i(t) - S_j x_j(t)\| = 0, & \forall i, j \in \mathcal{V}_c. \end{cases} \quad (2.2)$$

When $S_i = I_d$ for all i , this definition reduces to classical consensus. In the general case, agreement is achieved in the transformed coordinates, while the original states may converge to different limiting values.

Assumptions.

- (A1) The graph $\mathcal{G}$ is connected and undirected.
- (A2) Each matrix S_i is symmetric positive definite.
- (A3) The sampling period $h > 0$ is chosen such that the operator $I_{Nd} - h(L \otimes I_d)$ is stable.
- (A4) Each control input depends only on locally available neighbor information.

Under Assumptions (A1)–(A4), the matrix-scaled consensus problem can be equivalently formulated in the transformed coordinates $Z = SX$, where the target set becomes the standard agreement subspace C_Z . This equivalence forms the foundation for the operator-based reformulation developed in the next section.

3. SCALED REFORMULATION AND THREE-TERM CONJUGATE GRADIENT FRAMEWORK

This section develops the computational framework underlying the proposed matrix-scaled consensus problem. The key idea is to reformulate the hybrid interaction dynamics into an algebraic problem in the transformed coordinates and to solve it via a three-term conjugate-gradient method.

Scaled dynamics and sampled reformulation. Consider the distributed protocol

$$u_i(t) = \sum_{j \in N_i} a_{ij}(S_j x_j(t_k) - S_i x_i(t_k)), \quad i \in \mathcal{V}_c, \quad (3.1)$$

and

$$u_i(t_k) = h \sum_{j \in N_i} a_{ij}(S_j x_j(t_k) - S_i x_i(t_k)), \quad i \in \mathcal{V}_d. \quad (3.2)$$

By integrating the continuous-time dynamics over each sampling interval and combining them with the discrete-time updates, one obtains the sampled recursion

$$Z_{k+1} = (I_{Nd} - h(L \otimes I_d))Z_k. \quad (3.3)$$

Matrix-scaled consensus is therefore equivalent to driving Z_k into $C_Z = \{\mathbf{1}_N \otimes z : z \in \mathbb{R}^d\}$.

Regularized quadratic formulation. Define

$$P_c = \left(\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right) \otimes I_d, \quad C = \rho I_{Nd} + h(L \otimes I_d), \quad b = \rho P_c Z_0.$$

Consider

$$CZ = b, \tag{3.4}$$

which corresponds to minimizing $\Phi(Z) = \frac{1}{2}Z^T CZ - b^T Z$. Since C is symmetric positive definite, Φ is strictly convex and admits a unique minimizer $Z^\star = P_c Z_0$.

Three-term conjugate gradient algorithm. To efficiently solve the transformed linear system associated with the matrix-scaled consensus problem, we adopt a three-term conjugate gradient (TTCG) framework, which extends classical two-term schemes by incorporating additional curvature information to enhance convergence and stability. Let the initial iterate be defined as $Z_0 = SX_0$, and initialize the residual and search direction by $R_0 = b - CZ_0$, $P_0 = R_0$. For each iteration $k \geq 0$, the iterate is updated along the search direction P_k as

$$Z_{k+1} = Z_k + \alpha_k P_k, \quad \alpha_k = \frac{R_k^T P_k}{P_k^T C P_k},$$

which corresponds to an exact line search for the quadratic functional associated with the system $CZ = b$. The residual is then updated via $R_{k+1} = R_k - \alpha_k C P_k$, $S_k = Z_{k+1} - Z_k$, where S_k represents the displacement vector encoding the most recent step information. To construct a robust three-term search direction, we introduce the conjugate parameters

$$\beta_k = \max \left\{ 0, \frac{R_{k+1}^T (R_{k+1} - R_k)}{\|R_k\|^2} \right\}, \quad \theta_k = -\frac{R_{k+1}^T S_k}{R_{k+1}^T R_k}.$$

The next search direction is defined as

$$P_{k+1} = R_{k+1} + \beta_k S_k + \beta_k \theta_k R_k.$$

This three-term structure is inspired by modern TTCG schemes, where additional correction terms are introduced to preserve sufficient descent properties and improve robustness compared to classical PRP-type methods. In particular, the use of the nonnegative truncation in β_k follows the PRP^+ strategy, which is well-known to guarantee global convergence and avoid jamming phenomena. A key property of the proposed direction is the orthogonality-like condition

$$R_{k+1}^T P_{k+1} = \|R_{k+1}\|^2,$$

which ensures that P_{k+1} remains a descent direction for the quadratic objective. Consequently, the associated energy functional $\Phi(Z_k)$ satisfies a strict descent condition along the generated sequence.

Moreover, under standard assumptions (e.g., C symmetric positive definite), the method inherits finite-step convergence in the quadratic case and global convergence in the general setting. The inclusion of the additional S_k and R_k terms enhances the conjugacy structure, leading to improved numerical performance and stability, as observed in recent TTCG developments.

4. MAIN RESULTS

This section establishes the main theoretical properties of the proposed matrix-scaled consensus framework. The analysis is carried out in two stages. The first stage concerns the undirected case, in which the regularized operator is symmetric positive definite and the matrix-scaled consensus problem admits a complete quadratic characterization. In this setting, the agreement subspace, the regularized operator, the exact solution of the associated linear system, and the descent mechanism of the TTCG iteration can all be analyzed in a unified manner. The second stage addresses a directed extension, where the original Laplacian matrix is generally nonsymmetric and the quadratic structure is recovered through a suitable symmetrization.

4.1. Fundamental properties under undirected graphs. The analysis begins with the transformed and regularized formulation developed in Section 3. In the scaled coordinates, the matrix-scaled consensus objective is equivalent to driving the transformed state toward the agreement subspace $C_Z = \{\mathbf{1}_N \otimes z : z \in \mathbb{R}^d\}$. At the same time, the regularized formulation leads to the quadratic functional

$$\Phi(Z) = \frac{1}{2}Z^T CZ - b^T Z,$$

where $C = \rho I_{Nd} + h(L \otimes I_d)$, $b = \rho P_c Z_0$. For the TTCG iteration introduced in Section 3, the residual associated with the iterate Z_k is $R_k = b - CZ_k$.

The purpose of this subsection is to show that the solution of the regularized system coincides exactly with the projected agreement component of the transformed initial condition and that the TTCG iteration provides a descent-based computational mechanism for reaching this state. The first result identifies the agreement subspace through the null space of the Laplacian operator.

Lemma 4.1. *Under Assumption (A1), $\ker(L \otimes I_d) = C_Z$.*

Proof. Since the communication graph is connected and undirected, the Laplacian matrix satisfies $\ker(L) = \text{span}\{\mathbf{1}_N\}$. Using the standard property of the Kronecker product, one has $\ker(L \otimes I_d) = \ker(L) \otimes \mathbb{R}^d$. Substituting the expression for $\ker(L)$ yields $\ker(L \otimes I_d) = \text{span}\{\mathbf{1}_N\} \otimes \mathbb{R}^d = \{\mathbf{1}_N \otimes z : z \in \mathbb{R}^d\}$. By definition, the right-hand side is exactly C_Z . Hence, $\ker(L \otimes I_d) = C_Z$. $\square$

Lemma 4.1 shows that agreement in the transformed coordinates is completely characterized by the null space of the Laplacian operator. This fact is fundamental because it links the graph-theoretic structure of the network directly to the target set of the regularized optimization problem. The next result establishes the basic analytical property of the regularized operator.

Lemma 4.2. *The matrix $C = \rho I_{Nd} + h(L \otimes I_d)$ is symmetric positive definite.*

Proof. Since the graph is undirected, the Laplacian matrix L is symmetric positive semidefinite. Therefore, the Kronecker product $L \otimes I_d$ is also symmetric positive semidefinite. It follows immediately that C is symmetric. To prove positive definiteness, let $Z \in \mathbb{R}^{Nd}$ be any nonzero vector. Then $Z^T CZ = \rho \|Z\|^2 + h Z^T (L \otimes I_d) Z$. Because $L \otimes I_d$ is positive semi-definite, $Z^T (L \otimes I_d) Z \geq 0$. Hence,

$Z^T CZ \geq \rho \|Z\|^2$. Since $\rho > 0$ and $Z \neq 0$, one obtains $Z^T CZ > 0$. Therefore, C is symmetric positive definite. $\square$

Lemma 4.2 implies that the quadratic functional Φ is strictly convex and therefore admits a unique minimizer. In particular, the regularized problem is well posed and the exact line-search parameter used in the TTCG iteration is well defined at every nontrivial iteration. The next proposition identifies the minimizer explicitly.

Proposition 4.1. *The unique solution of the regularized system $CZ = b$ is given by $Z^\star = P_c Z_0$.*

Proof. By construction, $b = \rho P_c Z_0$. Since $P_c Z_0 \in C_Z$, Lemma 4.1 implies that $(L \otimes I_d)P_c Z_0 = \mathbf{0}_{Nd}$. Therefore,

$$C(P_c Z_0) = (\rho I_{Nd} + h(L \otimes I_d))P_c Z_0 = \rho P_c Z_0 = b.$$

Thus, $P_c Z_0$ is a solution of $CZ = b$. Since C is symmetric positive definite by Lemma 4.2, it is invertible, and therefore the solution is unique. Hence, $Z^\star = P_c Z_0$. $\square$

Proposition 4.1 gives the precise interpretation of the regularized formulation: the minimizer of the quadratic problem is not an auxiliary algebraic quantity, but exactly the projected agreement component of the transformed initial state. The next lemma concerns the algebraic structure of the TTCG search direction.

Lemma 4.3. *For the TTCG iteration defined in Section 3, $R_{k+1}^T P_{k+1} = \|R_{k+1}\|^2$.*

Proof. By the TTCG update rule, $P_{k+1} = R_{k+1} + \beta_k S_k + \beta_k \theta_k R_k$. Taking the inner product with R_{k+1} , one obtains $R_{k+1}^T P_{k+1} = R_{k+1}^T R_{k+1} + \beta_k R_{k+1}^T S_k + \beta_k \theta_k R_{k+1}^T R_k$. If $R_{k+1}^T R_k \neq 0$, then by definition $\theta_k = -\frac{R_{k+1}^T S_k}{R_{k+1}^T R_k}$, and therefore $\beta_k \theta_k R_{k+1}^T R_k = -\beta_k R_{k+1}^T S_k$. Substituting this into the previous expression yields $R_{k+1}^T P_{k+1} = \|R_{k+1}\|^2$. If $R_{k+1}^T R_k = 0$, then the algorithm resets the direction to $P_{k+1} = R_{k+1}$, and the identity again holds trivially $R_{k+1}^T P_{k+1} = R_{k+1}^T R_{k+1} = \|R_{k+1}\|^2$. Hence the conclusion follows in both cases. $\square$

Lemma 4.3 is crucial because it shows that the TTCG direction remains strictly aligned with the residual. Since the residual is the negative gradient of the quadratic functional, this identity immediately yields a strict descent property for the proposed iteration. The main theorem for the undirected case is stated next.

Theorem 4.1. *Under Assumptions (A1)–(A4), the TTCG iteration generates a strictly decreasing sequence for the quadratic functional $\Phi(Z_k)$ and converges to $Z^\star = P_c Z_0$. If, in addition, the search directions are C -conjugate, then convergence occurs in at most Nd steps.*

Proof. Since $\nabla \Phi(Z) = CZ - b$, and the residual is defined by $R_k = b - CZ_k$, it follows that $\nabla \Phi(Z_k) = -R_k$. Applying Lemma 4.3 at iteration $k+1$ yields $\nabla \Phi(Z_{k+1})^T P_{k+1} = -R_{k+1}^T P_{k+1} = -\|R_{k+1}\|^2$. Whenever $R_{k+1} \neq 0$, one has $\nabla \Phi(Z_{k+1})^T P_{k+1} < 0$, so P_{k+1} is a strict descent direction for Φ . Consequently, the sequence $\{\Phi(Z_k)\}$ is strictly decreasing until a stationary point is reached. By

Lemma 4.2, the matrix C is symmetric positive definite. Hence Φ is strictly convex and admits a unique minimizer. Proposition 4.1 identifies this minimizer as $Z^* = P_c Z_0$. Therefore, the TTCG sequence converges to Z^* . If, in addition, the search directions are C -conjugate, then the TTCG iteration coincides with the exact conjugate-gradient mechanism associated with the symmetric positive definite system $CZ = b$. Standard conjugate-gradient theory then implies finite-step convergence in at most Nd steps. $\square$

Theorem 4.1 gives the principal conclusion of the undirected theory: the proposed TTCG scheme is not merely a computational routine appended to the consensus problem, but an intrinsic part of the matrix-scaled consensus mechanism itself. It provides both a descent-based interpretation and an efficient iterative realization of the agreement state. The corresponding consequence in the original coordinates follows immediately.

Corollary 4.1. *If $S_i = I_d$ for all $i \in \mathcal{V}$, then the proposed framework reduces to ordinary consensus, and the limiting state is*

$$X^* = P_c X_0.$$

Proof. If $S_i = I_d$ for all $i \in \mathcal{V}$, then $S = I_{Nd}$, and hence the transformed and original coordinates coincide, that is, $Z = X$. Therefore, $Z^* = P_c Z_0$ leading to $X^* = P_c X_0$. This is exactly the classical consensus state. $\square$

4.2. Directed extension. The preceding analysis relies essentially on the symmetry of the Laplacian matrix. For directed graphs, however, the Laplacian is generally nonsymmetric, and the regularized operator constructed from it does not automatically preserve the quadratic structure required by the TTCG analysis. To extend the framework to directed networks, a symmetrized formulation is introduced. Define the symmetrized operator

$$\widehat{C}_d = \rho I_{Nd} + \frac{h}{2}((L_d + L_d^T) \otimes I_d).$$

If the directed graph is strongly connected and weight-balanced, then the symmetric part of the Laplacian preserves the agreement subspace, while the operator $\widehat{C}_d$ becomes symmetric positive definite under the same regularization mechanism used in the undirected case.

Theorem 4.2. *Suppose that the directed graph is strongly connected and weight-balanced, and that $\widehat{C}_d$ is symmetric positive definite. Then the TTCG iteration applied to the symmetrized system $\widehat{C}_d Z = b$ converges to $Z^* = P_c Z_0$, and the corresponding transformed state belongs to the agreement subspace CZ . Consequently, the associated original state satisfies the matrix-scaled consensus condition.*

Proof. Since $\widehat{C}_d$ is symmetric positive definite, the quadratic functional $\widehat{\Phi}(Z) = \frac{1}{2}Z^T \widehat{C}_d Z - b^T Z$ is strictly convex and admits a unique minimizer. The same descent mechanism established in the undirected case applies, because the TTCG direction depends only on the algebraic structure of the residual and search updates. Moreover, under strong connectivity and weight balance, the symmetric part $L_d + L_d^T$ has the same agreement subspace as the original directed Laplacian,

namely $\ker((L_d + L_d^T) \otimes I_d) = C_Z$. Hence, by the same argument used in Proposition 4.1, the unique minimizer of the symmetrized system is $Z^* = P_c Z_0$. Since $Z^* \in C_Z$, it follows from the equivalence between C_S and C_Z that the original state satisfies the matrix-scaled consensus condition. $\square$

Remark 4.1. *The directed extension preserves the optimization-based structure of the proposed framework, but does so through the symmetric part of the directed Laplacian rather than through the original nonsymmetric operator itself. As a consequence, strict descent and convergence remain available, while the direct algebraic interpretation is weaker than in the undirected setting. Nevertheless, this extension shows that the TTCG framework is sufficiently flexible to accommodate a broader class of interaction topologies beyond the symmetric case.*

The 16-agent interaction graph

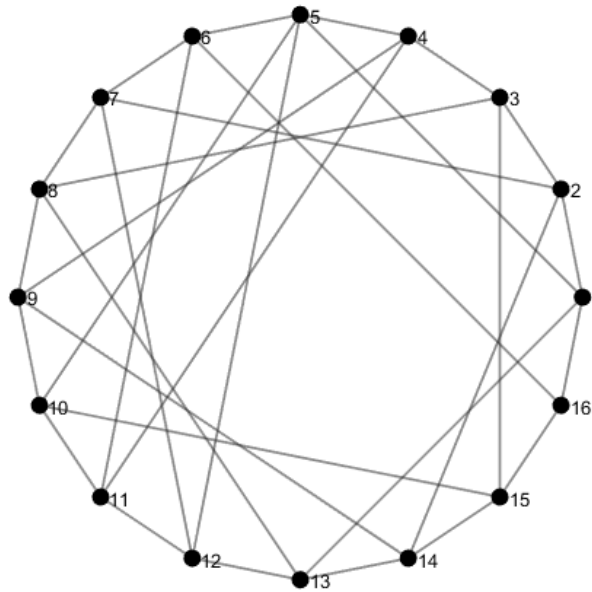


FIGURE 1. Interaction topology of the 16-agent network. The graph is undirected and connected, and its spectral properties ensure that the agreement subspace is one-dimensional. The ring-based structure with additional chords provides a balanced topology suitable for illustrating consensus behavior.

5. NUMERICAL EXAMPLE

This section presents a numerical example to illustrate the practical behavior of the proposed matrix-scaled consensus framework and to validate the theoretical results established in Section 4. In addition to qualitative visualization, quantitative performance indicators are reported to demonstrate consistency with the theoretical analysis.

A network consisting of $N = 16$ agents with state dimension $d = 2$ is considered. The communication topology is given by the undirected graph shown in Fig. 1. The graph is connected and undirected, and its spectral properties satisfy $\lambda_2(L) = 1.788737, \lambda_{\max}(L) = 7.854174$, which confirm that Assumption **(A1)** is satisfied and that the network possesses sufficient connectivity for information propagation. The agents are partitioned into three groups, each associated with a symmetric positive definite scaling matrix. Numerical verification confirms that all scaling matrices satisfy Assumption **(A2)**. The global scaling matrix is defined as $S = \text{blkdiag}(S_1, S_2, \dots, S_{16})$. The initial states are selected in $\mathbb{R}^2$ so that the three groups are initially separated in the state space. Let $X_0 = \text{col}(x_1(0), \dots, x_{16}(0))$ denote the stacked initial state, and define the transformed initial condition as $Z_0 = SX_0$. Following the regularized formulation introduced in Section 3, the agreement projector is defined by $P_c = \left(\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\right) \otimes I_2$, and the regularized operator is constructed as $C = \rho I_{Nd} + h(L \otimes I_2)$, with parameters $\rho = 0.05, h = 0.095491$. The spectral characteristics of C are given by $\lambda_{\min}(C) = 0.05, \lambda_{\max}(C) = 0.8, \text{cond}(C) = 16$, which confirm that C is symmetric positive definite in accordance with Lemma 4.2. Furthermore, the residual of the exact projected solution satisfies $\|CZ^* - b\| = 2.957742 \times 10^{-16}$, providing numerical validation of Proposition 4.1.

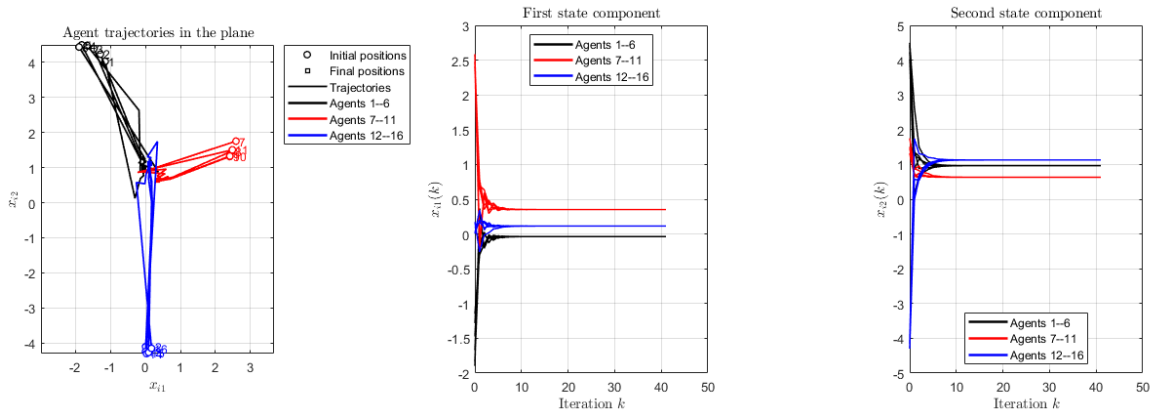


FIGURE 2. Evolution of the agent states in the original coordinates. Left: trajectories in the (x_{i1}, x_{i2}) -plane with initial and final positions. Middle and right: evolution of the first and second state components, respectively. The trajectories illustrate the emergence of three distinct clusters corresponding to the underlying scaling groups.

The TTCG iteration is applied with initial iterate Z_0 . The algorithm converges in 41 iterations, achieving a final residual norm of $\|R_k\| = 6.398517 \times 10^{-11}$, and an approximation error $\|Z_k - Z^*\| = 2.074000 \times 10^{-10}$. Moreover, the identity $R_{k+1}^T P_{k+1} = \|R_{k+1}\|^2$ is satisfied up to machine precision,

confirming the strict descent property established in Lemma 4.3. These results are consistent with the convergence behavior described in Theorem 4.1. The evolution of the agents in the original coordinates is depicted in Fig. 2. The trajectories do not converge to a common point; instead, they approach three distinct limiting groups corresponding to the three scaling matrices. Agents within the same group converge toward a common limiting value, while different groups remain separated. This phenomenon reflects the intrinsic clustering behavior induced by matrix scaling. The transformed-state trajectories are shown in Fig. 3. In contrast to the clustered behavior observed in the original coordinates, all transformed components converge to a common limiting value. Quantitatively, the maximum agreement error satisfies $\max_{i,j} \|z_i - z_j\| = 1.531006 \times 10^{-10}$, which confirms that Z_k approaches the agreement subspace C_Z as predicted by Lemma 4.1 and Theorem 4.1. Since $Z = SX$, this implies that $\max_{i,j} \|S_i x_i - S_j x_j\| = 1.531006 \times 10^{-10}$, verifying the matrix-scaled consensus condition.

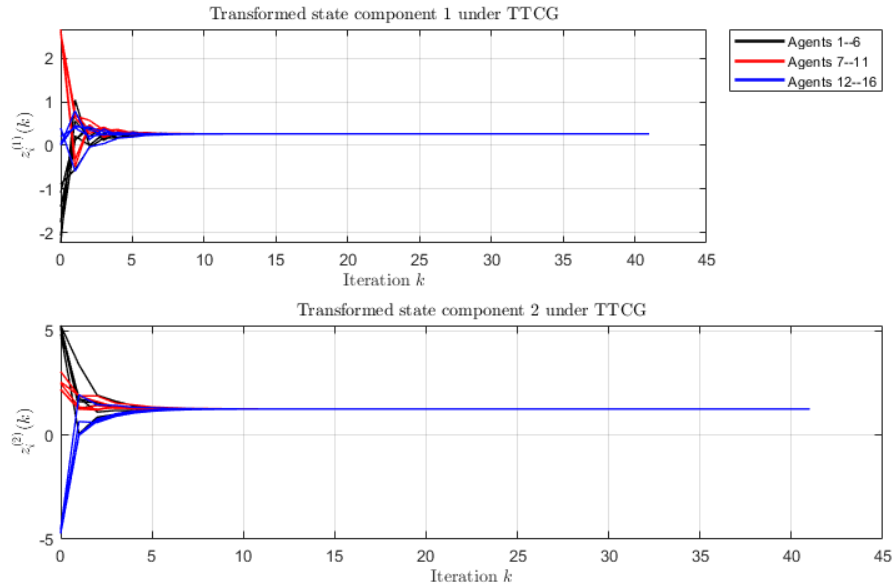


FIGURE 3. Evolution of the transformed-state components under the TTCG iteration. All components converge to a common limiting value, demonstrating convergence to the agreement subspace C_Z .

To further clarify the structure of the limiting configuration, Fig. 4 presents representative trajectories from each scaling group. These curves highlight that the steady-state values in the original coordinates depend explicitly on the associated scaling matrices. The approximate group centers are given by $(-0.0370, 0.9649)$, $(0.3516, 0.6330)$, $(0.1147, 1.1285)$, and the pairwise distances between these centers remain strictly positive, confirming that the clusters are well separated.

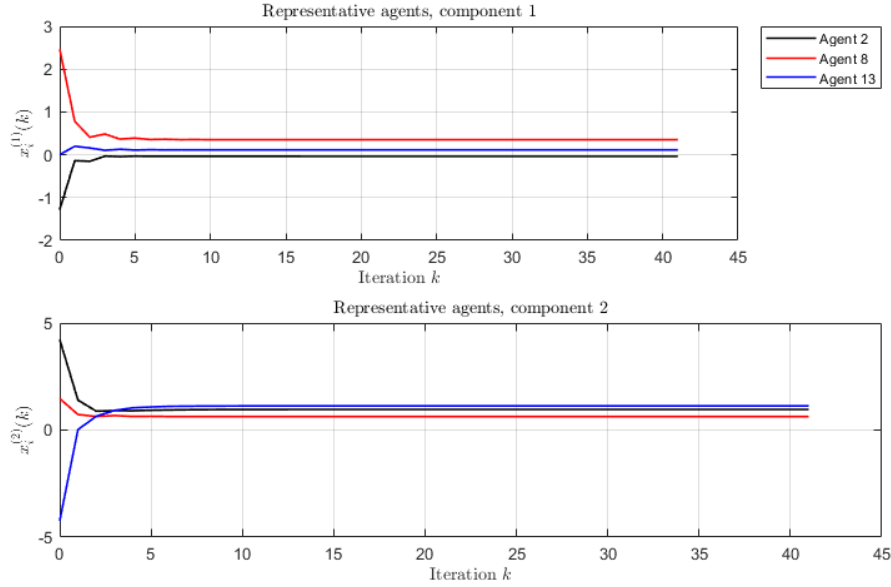


FIGURE 4. Representative trajectories of selected agents from each scaling group. The figure highlights the dependence of the steady-state values on the local scaling matrices and provides a clearer interpretation of the clustered structure.

6. CONCLUSION

This paper has developed an optimization-based framework for matrix-scaled consensus in hybrid multi-agent systems. By introducing a suitable coordinate transformation, the problem was reformulated as an agreement problem in the transformed space, leading to a regularized linear system whose solution coincides with the projected agreement component of the initial condition. A three-term conjugate-gradient (TTCG) algorithm was proposed to solve the resulting system efficiently. It was shown that the associated operator is symmetric positive definite, ensuring a unique solution, while the TTCG iteration generates a strict descent sequence and converges to the desired agreement state. An extension to directed networks was also discussed via a symmetrized formulation that preserves the underlying optimization structure. Numerical results validated the theoretical findings and demonstrated that, although the transformed states reach agreement, the original states exhibit a clustered structure determined by the local scaling matrices, which characterizes matrix-scaled consensus. Future work will focus on extensions to switching or time-varying networks, nonsymmetric formulations without symmetrization, and enhanced iterative schemes for improved efficiency in large-scale systems.

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