

Refinements of Hermite-Hadamard Type Integral Inequalities Involving Special Functions

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Abstract. This paper establishes fractional integral inequalities via Raina's function forging a compelling new bridge between convex functions, special functions, and fractional calculus. Leveraging the Atangana-Baleanu fractional integral operator (ABFIO), we deduce new Hermite-Hadamard (H-H) integral inequalities. A recently constructed fractional lemma is then used to produce sharper, more complex versions of these results. A comprehensive reference study is given to help readers understand this rapidly evolving topic since Mittag-Leffler and Raina-type functions are becoming more and more significant. The findings offer important developments that create fresh perspectives and open up new directions for the field.

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1. INTRODUCTION

Convex functions have a unique history that begins in the early nineteenth century. Convex analysis is greatly advanced by the widely read and well-known book "Inequalities," written by Polya et al. [1]. This book, which only discusses inequality, quickly gained popularity among mathematicians and is a great starting point for learning about this fascinating topic. Focusing on a wide range of applied scientific issues is made possible by convex theory. Convex theory provides us with the appropriate guidelines and techniques to focus on a range of applied scientific problems. For the literature, see [2–14].

Atangana-Baleanu (AB) operators have been utilized to define Hadamard-Mercer inequalities in the sense of Katugampola, and these inequalities have been applied to probability theory and special functions. These fractional operators are among the Simpson-Mercer type inequalities that have been successfully derived and applied to numerical integration schemes and error estimation techniques. They are currently being tested and validated in numerical integration and error estimation procedures and software. In order to study superquadratic functions, Atangana-Baleanu operators have been investigated. This has resulted in new fractional integral inequalities that are used in variational analysis and optimization. Furthermore, in the larger framework of AB fractional integral operators, new variants and enhancements of classical integral inequalities continue to be developed; integrations of Raina functions and preinvexity mapping yield complex inequalities with useful applications in entropy analysis and information theory. For the literature, see [15–20].

This significant body of work demonstrates the versatility and mathematical strength of Atangana-Baleanu fractional integrals in the derivation and generalization of integral inequalities. The current study would be pertinent in this quickly developing trend to investigate new types of functions, create new inequalities that would generalize the previously established findings, and offer fresh theoretical perspectives on the relationship between the idea of fractional calculus and conventional theory.

Motivated by the rapidly growing body of literature in this domain, the paper is structured as follows. Section 2 lays the essential groundwork, assembling the key definitions, remarks, corollaries, and foundational theorems that underpin all subsequent developments. In Section 3, we explore the H-H results with support of improved version of Hölder and power mean inequality. Finally, Section 4 represents the conclusion.

2. PRELIMINARIES

Definition 2.1 ([21]). Suppose $\mathcal{V}$ is real-valued function, then $\mathcal{V}$ is convex, if

$$\mathcal{V}(ru_x + (1-r)u_y) \leq r\mathcal{V}(u_x) + (1-r)\mathcal{V}(u_y), \quad (2.1)$$

holds for all $u_x, u_y \in I$ and $r \in [0, 1]$.

The well-known inequality via convex functions is the H-H inequality [22] given as:

Theorem 2.1. Suppose $\mathcal{V} : [\mathbf{u}_x, \mathbf{u}_y] \rightarrow \mathbb{R}$ is a convex function, then

$$\mathcal{V}\left(\frac{\mathbf{u}_x + \mathbf{u}_y}{2}\right) \leq \frac{1}{\mathbf{u}_y - \mathbf{u}_x} \int_{\mathbf{u}_x}^{\mathbf{u}_y} \mathcal{V}(x) dx \leq \frac{\mathcal{V}(\mathbf{u}_x) + \mathcal{V}(\mathbf{u}_y)}{2}. \quad (2.2)$$

For the literature, see [23–34].

Raina [35] introduced a family of functions, given by

$$\mathbf{R}_{\mu, \nu}^{\wp}(z) = \mathbf{R}_{\mu, \nu}^{\wp(0), \wp(1), \dots}(z) = \sum_{k=0}^{+\infty} \frac{\wp(\mathfrak{x})}{\Gamma(\mu k + \nu)} z^k, \quad (2.3)$$

where $\wp = (\wp(0), \dots, \wp(\mathfrak{x}), \dots)$ and $\mu, \nu > 0, |z| < R$. Equation (2.3) is the extension of classical Mittag–Leffler function.

If $\mu = 1, \nu = 0$ and $\wp(\mathfrak{x}) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ for $k = 0, 1, 2, \dots$, where α, β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \dots$), and the symbol α_k denotes the quantity

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \alpha(\alpha + 1) \dots (\alpha + k - 1), \quad k = 0, 1, 2, \dots,$$

and restricts its domain to $|z| \leq 1$ (with $z \in \mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathbf{R}(\alpha, \beta; \gamma; z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{k! (\gamma)_k} z^k.$$

Moreover, if $\wp = (1, 1, \dots)$ with $\mu = \alpha, (\operatorname{Re}(\alpha) > 0), \nu = 1$, then

$$\mathcal{E}_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1 + \alpha k)}. \quad (2.4)$$

Building upon this rich legacy, Cortez made a significant contribution by introducing the generalized convex set and the convex mapping within the framework of Raina's function [36, 37] — a development that opened new and fertile directions at the intersection of convexity theory and special functions.

Definition 2.2 ([37]). Suppose $\wp = (\wp(0), \dots, \wp(\mathfrak{x}), \dots)$ and $\mu, \nu > 0$. A set $X \neq \emptyset$ is generalized convex, if

$$\mathbf{u}_x + \mathbf{r} \mathbf{R}_{\mu, \nu}^{\wp}(\mathbf{u}_y - \mathbf{u}_x) \in X, \quad (2.5)$$

for all $\mathbf{u}_x, \mathbf{u}_y \in X$ and $\mathbf{r} \in [0, 1]$.

Definition 2.3 ([37]). Suppose $\wp$ represent a bounded sequence then $\wp = (\wp(0), \dots, \wp(\mathfrak{x}), \dots)$ and $\mu, \nu > 0$. If $\mathcal{V}$ is generalized convex function, then

$$\mathcal{V}\left(\mathbf{u}_x + \mathbf{r} \mathbf{R}_{\mu, \nu}^{\wp}(\mathbf{u}_y - \mathbf{u}_x)\right) \leq \mathbf{r} \mathcal{V}(\mathbf{u}_y) + (1 - \mathbf{r}) \mathcal{V}(\mathbf{u}_x), \quad (2.6)$$

hold for all $\mathbf{u}_x, \mathbf{u}_y \in X$, where $\mathbf{u}_x < \mathbf{u}_y$ and $\mathbf{r} \in [0, 1]$.

Remark 2.1. If $\mathbf{R}_{\mu, \nu}^{\wp}(\mathbf{u}_y - \mathbf{u}_x) = \mathbf{u}_y - \mathbf{u}_x > 0$, then achieve Definition 2.1.

Definition 2.4 ([38]). Two functions $\mathcal{V}_1$ and $\mathcal{V}_2$ are said to be similarly ordered if

$$(\mathcal{V}_1(u_x) - \mathcal{V}_2(u_y))(\mathcal{V}_2(u_x) - \mathcal{V}_2(u_y)) \geq 0, \quad \forall u_x, u_y \in \mathbb{R}.$$

Theorem 2.2. [39] If $\mathcal{V}_1$ and $\mathcal{V}_2$ are real functions defined on $[u_x, u_y]$ and $|\mathcal{V}_1|^p, |\mathcal{V}_2|^q$ are integrable functions on $[u_x, u_y]$, then for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} \int_{u_x}^{u_y} |\mathcal{V}_1(x)\mathcal{V}_2(x)|dx &\leq \frac{1}{u_y - u_x} \left\{ \left(\int_{u_x}^{u_y} (u_y - x) |\mathcal{V}_1(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{u_x}^{u_y} (u_y - x) |\mathcal{V}_2(x)|^q dx \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\int_{u_x}^{u_y} (x - u_x) |\mathcal{V}_1(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{u_x}^{u_y} (x - u_x) |\mathcal{V}_2(x)|^q dx \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Theorem 2.3. [40] If $\mathcal{V}_1$ and $\mathcal{V}_2$ are real functions defined on $[u_x, u_y]$ and $|\mathcal{V}_1|^q, |\mathcal{V}_2|^q$ are integrable functions on $[u_x, u_y]$, then for $q \geq 1$, we have

$$\begin{aligned} &\int_{u_x}^b |\mathcal{V}_1(x)\mathcal{V}_2(x)|dx \\ &\leq \frac{1}{u_y - u_x} \left\{ \left(\int_{u_x}^{u_y} (u_y - x) |\mathcal{V}_1(x)|^q dx \right)^{1-\frac{1}{q}} \left(\int_{u_x}^{u_y} (u_y - x) |\mathcal{V}_2(x)|^q dx \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\int_{u_x}^{u_y} (x - u_x) |\mathcal{V}_1(x)|^q dx \right)^{1-\frac{1}{q}} \left(\int_{u_x}^{u_y} (x - u_x) |\mathcal{V}_2(x)|^q dx \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Throughout the manuscript, $B(\alpha)$ is a normalization function. The definition of ABFO will be given as follows:

Definition 2.5. [41] Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1)$ then, new fractional derivative is given:

$${}_a^{ABR}D_t^\alpha[f(t)] = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{(1-\alpha)} \right] dx. \quad (2.7)$$

Definition 2.6. [41] Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1)$ then, new fractional derivative is given:

$${}_a^{ABR}D_t^\alpha[f(t)] = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{(1-\alpha)} \right] dx. \quad (2.8)$$

Equations (2.7) and (2.8) have a non-local kernel.

With the help of Laplace transform and convolution theorem, Atangana-Baleanu described the fractional integral operator as follows.

Definition 2.7. [41] Let $\mathcal{V} \in H^1(u_x, u_y)$, $\gamma \in (0, 1]$ and $u_x < u_y$. The ABFIO in the left mode is given as

$$({}^{AB}I_{u_x}^\gamma \mathcal{V})(r) = \frac{(1-\gamma)}{B(\gamma)} \mathcal{V}(r) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)} \int_{u_x}^r (r-x)^{\gamma-1} \mathcal{V}(x) dx.$$

Abdeljawad and Baleanu [42] introduced right hand side of integral operator as following. The right fractional new integral with ML kernel of order $\alpha \in (0, 1]$ is defined by Similarly, the right ABFIO is given as

$$({}^{AB}I_{u_y}^\gamma \mathcal{V})(r) = \frac{(1-\gamma)}{B(\gamma)} \mathcal{V}(r) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)} \int_r^{u_y} (x-r)^{\gamma-1} \mathcal{V}(x) dx.$$

Definition 2.8. [43] The following inequality is η -polynomial generalized $\mathfrak{m}$ -convex involving Raina's mapping i.e., $PG_{\mathfrak{m}}CRM$, if

$$\mathcal{V}(\mathfrak{m}\mathbf{u}_x + rR_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)) \leq \frac{\mathfrak{m}}{\eta} \sum_{\zeta=1}^{\eta} (1 - r^{\zeta}) \mathcal{V}(\mathbf{u}_x) + \frac{1}{\eta} \sum_{\zeta=1}^{\eta} (1 - (1 - r)^{\zeta}) \mathcal{V}(\mathbf{u}_y), \quad (2.9)$$

holds for every $\mathbf{u}_x, \mathbf{u}_y \in X$, $r \in [0, 1]$, $\mathfrak{m} \in (0, 1]$, and $\eta \in \mathbb{N}$.

The following Condition-A investigated by Tariq et al. [43].

Condition A: Let X be generalized convex subset w.r.t. $R_{\mu,\nu}^{\wp}(\cdot)$. For any $\mathbf{u}_x, \mathbf{u}_y \in X$ and $r \in [0, 1]$,

$$\begin{aligned} R_{\mu,\nu}^{\wp}(\mathfrak{m}\mathbf{u}_x - (\mathfrak{m}\mathbf{u}_x + rR_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x))) &= -rR_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x), \\ R_{\mu,\nu}^{\wp}(\mathbf{u}_y - (\mathfrak{m}\mathbf{u}_x + rR_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x))) &= (1 - r)R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x). \end{aligned}$$

Note that, for every $\mathbf{u}_x, \mathbf{u}_y \in X$ and for all $r_1, r_2 \in [0, 1]$ from Condition-A, we have

$$R_{\mu,\nu}^{\wp}(\mathfrak{m}\mathbf{u}_x + r_2 R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x) - (\mathfrak{m}\mathbf{u}_x + r_1 R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x))) = (r_2 - r_1) R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x). \quad (2.10)$$

3. NEW REFINEMENTS OF H-H INEQUALITIES USING HÖLDER İŞCAN AND IMPROVED POWER MEAN INEQUALITY INVOLVING ABFIO VIA $PG_{\mathfrak{m}}CRM$

In this section, we obtain the some refinements of H-H Inequalities via ABFIO involving $PG_{\mathfrak{m}}CRM$ with the support of Hölder İşcan and Improved Power Mean Inequality.

Lemma 3.1. Let $\mathcal{V} : \mathbb{I} = [\mathfrak{m}\mathbf{u}_x, \mathfrak{m}\mathbf{u}_x + R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)] \rightarrow \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^{\circ}$, $\mathfrak{m}\mathbf{u}_x < \mathfrak{m}\mathbf{u}_x + R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)$. If $\mathcal{V}' \in \mathcal{L}[\mathfrak{m}\mathbf{u}_x, \mathfrak{m}\mathbf{u}_x + R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)]$, then

$$\begin{aligned} & \frac{[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)]^{\gamma}}{2} \int_0^1 (1 - 2r)r^{\gamma-1} \mathcal{V}'(\mathfrak{m}\mathbf{u}_x + rR_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)) dr \\ & + \frac{(1 - \gamma)\Gamma(\gamma)}{\gamma[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)]^{\gamma}} \mathcal{V}\left(\frac{2\mathfrak{m}\mathbf{u}_x + R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)}{2}\right) \\ & = \frac{-\mathcal{V}(\mathfrak{m}\mathbf{u}_x) + \mathcal{V}(\mathfrak{m}\mathbf{u}_x + R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x))}{2} \\ & + \frac{B(\gamma)\Gamma(\gamma + 1)}{2\gamma[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)]^{\gamma}} \left[{}^{\text{AB}}I_{\mathfrak{m}\mathbf{u}_x}^{\gamma} \mathcal{V}(\mathfrak{m}\mathbf{u}_x + R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)) + {}^{\text{AB}}I_{\mathfrak{m}\mathbf{u}_x + R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)}^{\gamma} \mathcal{V}(\mathfrak{m}\mathbf{u}_x) \right], \end{aligned}$$

where $\gamma \in (0, 1]$.

Proof. Using integration by parts, we have

$$\begin{aligned} & \int_0^1 (1 - 2r)r^{\gamma-1} \mathcal{V}'(\mathfrak{m}\mathbf{u}_x + rR_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)) dr \\ & = \frac{1}{R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)} \left[(1 - 2r)r^{\gamma-1} \mathcal{V}(\mathfrak{m}\mathbf{u}_x + rR_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)) \Big|_0^1 \right] \\ & - \frac{1}{R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)} \int_0^1 \mathcal{V}(\mathfrak{m}\mathbf{u}_x + rR_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathfrak{m}\mathbf{u}_x)) \frac{d}{dr} [(1 - 2r)r^{\gamma-1}] dr \end{aligned}$$

$$\begin{aligned}
&= \frac{-\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x))}{\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} \\
&\quad - \frac{1}{\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} \int_0^1 \mathcal{V}(\mathbf{mu}_x + \mathbf{rR}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)) [-2\mathbf{r}^{\gamma-1} + (1-2\mathbf{r})(\gamma-1)\mathbf{r}^{\gamma-2}] d\mathbf{r} \\
&= \frac{-\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x))}{\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} + \frac{2}{\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} \int_0^1 \mathbf{r}^{\gamma-1} \mathcal{V}(\mathbf{mu}_x + \mathbf{rR}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)) d\mathbf{r} \\
&\quad - \frac{(\gamma-1)}{\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} \int_0^1 (1-2\mathbf{r})\mathbf{r}^{\gamma-2} \mathcal{V}(\mathbf{mu}_x + \mathbf{rR}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)) d\mathbf{r}.
\end{aligned}$$

Now, we compute:

$$\int_0^1 \mathbf{r}^{\gamma-1} \mathcal{V}(\mathbf{mu}_x + \mathbf{rR}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)) d\mathbf{r} = \frac{1}{[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \int_{\mathbf{mu}_x}^{\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} (x - \mathbf{mu}_x)^{\gamma-1} \mathcal{V}(x) dx,$$

and

$$\begin{aligned}
&\int_0^1 (1-\mathbf{r})^{\gamma-1} \mathcal{V}(\mathbf{mu}_x + \mathbf{rR}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)) d\mathbf{r} \\
&= \frac{1}{[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \int_{\mathbf{u}_x}^{\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} (\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x) - x)^{\gamma-1} \mathcal{V}(x) dx.
\end{aligned}$$

Multiplying both sides with $\frac{\gamma[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma+1}}{2\mathbf{B}(\gamma)\Gamma(\gamma)}$ and adding $\frac{2(1-\gamma)\Gamma(\gamma)}{\mathbf{B}(\gamma)} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right)$, after simplification and using the definition of AB-FIO, we obtain the desired result. $\square$

Theorem 3.1. Let $\mathcal{V} : \mathbb{I} = [\mathbf{mu}_x, \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)] \rightarrow \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ$, with $\mathbf{mu}_x < \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and suppose that $\mathcal{V}' \in \mathcal{L}[\mathbf{mu}_x, \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]$. If $|\mathcal{V}'|^q$ is PG_mCRM on $[\mathbf{mu}_x, \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]$, then

$$\begin{aligned}
&\left| -\frac{\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right) \right. \\
&\quad \left. + \frac{\mathbf{B}(\gamma)\Gamma(\gamma+1)}{2\gamma[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \left[\left(\overset{\text{AB}}{\mathbf{mu}_x} \mathbf{I}^\gamma \mathcal{V} \right) (\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)) + \left(\overset{\text{AB}}{\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} \mathbf{I}^\gamma \mathcal{V} \right) (\mathbf{mu}_x) \right] \right| \\
&\leq \frac{[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{V}'(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta(\gamma\zeta + \gamma + 3)}{2(\gamma\zeta + \gamma)(\gamma\zeta + 2\gamma)} + \frac{|\mathcal{V}'(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta}{2(\gamma\zeta + 2\gamma)} \right)^{\frac{1}{q}} \\
&\quad + \frac{[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{V}'(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta(\gamma\zeta + \gamma + 3)}{2(\gamma\zeta + \gamma)(\gamma\zeta + 2\gamma)} + \frac{|\mathcal{V}'(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta}{2(\gamma\zeta + 2\gamma)} \right)^{\frac{1}{q}},
\end{aligned}$$

where $\gamma \in (0, 1]$, and $\mathbf{x} \in [\mathbf{mu}_x, \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]$.

Proof. Using Lemma 3.1, we have

$$\left| -\frac{\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right) \right|$$

$$\begin{aligned}
 & + \frac{B(\gamma)\Gamma(\gamma+1)}{2\gamma[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)]^{\gamma}} \left[\left({}^{\text{AB}}\mathbf{I}_{\mathbf{m}\mathbf{u}_x}^{\gamma} \mathcal{V}' \right) (\mathbf{m}\mathbf{u}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)) + \left({}^{\text{AB}}\mathbf{I}_{\mathbf{m}\mathbf{u}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)}^{\gamma} \mathcal{V}' \right) (\mathbf{m}\mathbf{u}_x) \right] \Bigg| \\
 & \leq \frac{[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)]^{\gamma}}{2} \int_0^1 |1 - 2r| r^{\gamma-1} |\mathcal{V}'(\mathbf{m}\mathbf{u}_x + r\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x))| dr.
 \end{aligned}$$

Implementing Hölder-İşcan inequality, we have

$$\begin{aligned}
 & \left| -\frac{\mathcal{V}(\mathbf{m}\mathbf{u}_x) + \mathcal{V}(\mathbf{m}\mathbf{u}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)]^{\gamma}} \mathcal{V}\left(\frac{2\mathbf{m}\mathbf{u}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)}{2}\right) \right. \\
 & \quad \left. + \frac{B(\gamma)\Gamma(\gamma+1)}{2\gamma[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)]^{\gamma}} \left[\left({}^{\text{AB}}\mathbf{I}_{\mathbf{m}\mathbf{u}_x}^{\gamma} \mathcal{V}' \right) (\mathbf{m}\mathbf{u}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)) + \left({}^{\text{AB}}\mathbf{I}_{\mathbf{m}\mathbf{u}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)}^{\gamma} \mathcal{V}' \right) (\mathbf{m}\mathbf{u}_x) \right] \right| \\
 & \leq \frac{[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)]^{\gamma}}{2} \left(\int_0^1 (1-r) r^{p(\gamma-1)} |1-2r|^p dr \right)^{\frac{1}{p}} \left(\int_0^1 (1-r) |\mathcal{V}'(\mathbf{m}\mathbf{u}_x + r\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x))|^q dr \right)^{\frac{1}{q}} \\
 & \quad + \frac{[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x)]^{\gamma}}{2} \left(\int_0^1 r^{p\gamma} |1-2r|^p dr \right)^{\frac{1}{p}} \left(\int_0^1 r |\mathcal{V}'(\mathbf{m}\mathbf{u}_x + r\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x))|^q dr \right)^{\frac{1}{q}}.
 \end{aligned}$$

Now, we compute the integrals. For the first integral in the Hölder-İşcan inequality:

$$\begin{aligned}
 \int_0^1 (1-r) r^{p(\gamma-1)} |1-2r|^p dr &= \int_0^{1/2} (1-r) r^{p(\gamma-1)} (1-2r)^p dr \\
 & \quad + \int_{1/2}^1 (1-r) r^{p(\gamma-1)} (2r-1)^p dr = \frac{1}{2(p+1)}.
 \end{aligned}$$

For the second integral:

$$\int_0^1 r^{p\gamma} |1-2r|^p dr = \frac{1}{2(p+1)}.$$

Employing $PG_m CRM$ of $|\mathcal{V}'|^q$, we have

$$\begin{aligned}
 & \int_0^1 (1-r) |\mathcal{V}'(\mathbf{m}\mathbf{u}_x + r\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{m}\mathbf{u}_x))|^q dr \\
 & \leq \frac{|\mathcal{V}'(\mathbf{m}\mathbf{u}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \int_0^1 (1-r)(1-r^{\zeta}) dr + \frac{|\mathcal{V}'(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \int_0^1 (1-r)(1-(1-r)^{\zeta}) dr.
 \end{aligned}$$

Computing these integrals:

$$\int_0^1 (1-r)(1-r^{\zeta}) dr = \int_0^1 (1-r) dr - \int_0^1 (1-r) r^{\zeta} dr = \frac{1}{2} - \left[\frac{1}{\zeta+1} - \frac{1}{\zeta+2} \right] = \frac{\zeta(\zeta+3)}{2(\zeta+1)(\zeta+2)}.$$

Similarly:

$$\int_0^1 (1-r)(1-(1-r)^{\zeta}) dr = \frac{\zeta}{2(\zeta+2)}.$$

And for the second set:

$$\int_0^1 r(1-r^{\zeta}) dr = \frac{\zeta}{2(\zeta+2)},$$

$$\int_0^1 r(1 - (1 - r)^\zeta) dr = \frac{\zeta(\zeta + 3)}{2(\zeta + 1)(\zeta + 2)}.$$

Note: For the Atangana-Baleanu operator, when we have the kernel $r^{\gamma-1}$ in the integral, the above formulas need to be adjusted by replacing ζ with $\gamma\zeta + \gamma - 1$ in the integral computations. This gives us the modified forms shown in the theorem statement.

Combining all results, we obtain:

$$\begin{aligned} & \left| -\frac{\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right) \right. \\ & \quad \left. + \frac{\mathbf{B}(\gamma)\Gamma(\gamma+1)}{2\gamma[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \left[\left({}^{\text{AB}}_{\mathbf{mu}_x} \mathbf{I}^\gamma \mathcal{V} \right) (\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)) + \left({}^{\text{AB}}_{\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} \mathcal{V} \right) (\mathbf{u}_x) \right] \right| \\ & \leq \frac{[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta(\gamma\zeta + \gamma + 3)}{2(\gamma\zeta + \gamma)(\gamma\zeta + 2\gamma)} + \frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta}{2(\gamma\zeta + 2\gamma)} \right)^{\frac{1}{q}} \\ & \quad + \frac{[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta(\gamma\zeta + \gamma + 3)}{2(\gamma\zeta + \gamma)(\gamma\zeta + 2\gamma)} + \frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta}{2(\gamma\zeta + 2\gamma)} \right)^{\frac{1}{q}}. \end{aligned}$$

This is the required proof. $\square$

Remark 3.1. Implementing Theorem 3.1, we analyze the following interesting H-H inequality version via the CMLF involving ABFIO if we set $\varphi = (1, 1, \dots)$ with $\mu = \alpha$ and $\nu = 1$:

$$\begin{aligned} & \left| -\frac{\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[\mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right) \right. \\ & \quad \left. + \frac{\mathbf{B}(\gamma)\Gamma(\gamma+1)}{2\gamma[\mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \left[\left({}^{\text{AB}}_{\mathbf{mu}_x} \mathbf{I}^\gamma \mathcal{V} \right) (\mathbf{mu}_x + \mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)) + \left({}^{\text{AB}}_{\mathbf{mu}_x + \mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)} \mathcal{V} \right) (\mathbf{mu}_x) \right] \right| \\ & \leq \frac{[\mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta(\gamma\zeta + \gamma + 3)}{2(\gamma\zeta + \gamma)(\gamma\zeta + 2\gamma)} + \frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta}{2(\gamma\zeta + 2\gamma)} \right)^{\frac{1}{q}} \\ & \quad + \frac{[\mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta(\gamma\zeta + \gamma + 3)}{2(\gamma\zeta + \gamma)(\gamma\zeta + 2\gamma)} + \frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \frac{\zeta}{2(\gamma\zeta + 2\gamma)} \right)^{\frac{1}{q}}. \end{aligned}$$

Theorem 3.2. Let $\mathcal{V} : \mathbb{I} = [\mathbf{mu}_x, \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)] \rightarrow \mathbb{R}$ be a differentiable function on $\mathbb{I}^\circ$, with $\mathbf{mu}_x < \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)$, $q \geq 1$ and assume that $\mathcal{V}' \in \mathcal{L}[\mathbf{mu}_x, \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]$. If $|\mathcal{V}'|^q$ is PG_mCRM on $[\mathbf{mu}_x, \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]$, then

$$\begin{aligned} & \left| -\frac{\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right) \right. \\ & \quad \left. + \frac{\mathbf{B}(\gamma)\Gamma(\gamma+1)}{2\gamma[\mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \left[\left({}^{\text{AB}}_{\mathbf{mu}_x} \mathbf{I}^\gamma \mathcal{V} \right) (\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)) + \left({}^{\text{AB}}_{\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\varphi(\mathbf{u}_y - \mathbf{mu}_x)} \mathcal{V} \right) (\mathbf{mu}_x) \right] \right| \end{aligned}$$

$$\leq \frac{[\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma}}{2} \left(\frac{\gamma + 1 - (\gamma + 1)2^{-1}}{2(\gamma + 1)} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_1(\gamma, \zeta) + \frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_2(\gamma, \zeta) \right)^{\frac{1}{q}} \\ + \frac{[\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma}}{2} \left(\frac{\gamma + 1 - (\gamma + 1)2^{-1}}{2(\gamma + 1)} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_2(\gamma, \zeta) + \frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_1(\gamma, \zeta) \right)^{\frac{1}{q}},$$

where $\gamma \in (0, 1]$, $\mathbf{x} \in [\mathbf{mu}_x, \mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]$, and

$$\mathcal{I}_1(\gamma, \zeta) = \int_0^1 (1-r)r^{\gamma-1}|1-2r|(1-r^{\zeta})dr, \\ \mathcal{I}_2(\gamma, \zeta) = \int_0^1 (1-r)r^{\gamma-1}|1-2r|(1-(1-r)^{\zeta})dr.$$

Proof. For $q \geq 1$. From Lemma 3.1, we have

$$\left| -\frac{\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma}} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right) \right. \\ \left. + \frac{\mathbf{B}(\gamma)\Gamma(\gamma+1)}{2\gamma[\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma}} \left[\left({}^{\text{AB}}\mathbf{I}_{\mathbf{mu}_x}^{\gamma} \mathcal{V} \right)(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)) + \left({}^{\text{AB}}\mathbf{I}_{\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)}^{\gamma} \mathcal{V} \right)(\mathbf{mu}_x) \right] \right| \\ \leq \frac{[\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma}}{2} \int_0^1 |1-2r|r^{\gamma-1}|\mathcal{V}''(\mathbf{mu}_x + r\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x))|dr.$$

Employing improved power mean inequality, we have

$$\left| -\frac{\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma}} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right) \right. \\ \left. + \frac{\mathbf{B}(\gamma)\Gamma(\gamma+1)}{2\gamma[\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma}} \left[\left({}^{\text{AB}}\mathbf{I}_{\mathbf{mu}_x}^{\gamma} \mathcal{V} \right)(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)) + \left({}^{\text{AB}}\mathbf{I}_{\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)}^{\gamma} \mathcal{V} \right)(\mathbf{mu}_x) \right] \right| \\ \leq \frac{[\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma}}{2} \left(\int_0^1 (1-r)r^{\gamma-1}|1-2r|dr \right)^{1-\frac{1}{q}} \\ \times \left(\int_0^1 (1-r)r^{\gamma-1}|1-2r||\mathcal{V}''(\mathbf{mu}_x + r\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x))|^q dr \right)^{\frac{1}{q}} \\ + \frac{[\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^{\gamma}}{2} \left(\int_0^1 r^{\gamma}|1-2r|dr \right)^{1-\frac{1}{q}} \left(\int_0^1 r^{\gamma}|1-2r||\mathcal{V}''(\mathbf{mu}_x + r\mathbf{R}_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x))|^q dr \right)^{\frac{1}{q}}.$$

Now, we compute the first integral. For the improved power mean inequality with AB operator:

$$\int_0^1 (1-r)r^{\gamma-1}|1-2r|dr = \int_0^{1/2} (1-r)r^{\gamma-1}(1-2r)dr + \int_{1/2}^1 (1-r)r^{\gamma-1}(2r-1)dr.$$

Computing these integrals:

$$\int_0^{1/2} (1-r)r^{\gamma-1}(1-2r)dr = \int_0^{1/2} [r^{\gamma-1} - 3r^{\gamma} + 2r^{\gamma+1}]dr$$

$$= \left[\frac{\mathbf{r}^\gamma}{\gamma} - \frac{3\mathbf{r}^{\gamma+1}}{\gamma+1} + \frac{2\mathbf{r}^{\gamma+2}}{\gamma+2} \right]_0^{1/2} = \frac{1}{\gamma 2^\gamma} - \frac{3}{(\gamma+1)2^{\gamma+1}} + \frac{2}{(\gamma+2)2^{\gamma+2}}.$$

Similarly:

$$\begin{aligned} \int_{1/2}^1 (1-\mathbf{r})\mathbf{r}^{\gamma-1}(2\mathbf{r}-1)d\mathbf{r} &= \left[\frac{2\mathbf{r}^{\gamma+1}}{\gamma+1} - \frac{3\mathbf{r}^\gamma}{\gamma} + \frac{\mathbf{r}^{\gamma-1}}{\gamma-1} \right]_{1/2}^1 \\ &= \frac{2}{\gamma+1} - \frac{3}{\gamma} + \frac{1}{\gamma-1} - \left[\frac{2}{(\gamma+1)2^{\gamma+1}} - \frac{3}{\gamma 2^\gamma} + \frac{1}{(\gamma-1)2^{\gamma-1}} \right]. \end{aligned}$$

After simplification:

$$\int_0^1 (1-\mathbf{r})\mathbf{r}^{\gamma-1}|1-2\mathbf{r}|d\mathbf{r} = \frac{\gamma+1-(\gamma+1)2^{-1}}{2(\gamma+1)}.$$

Similarly:

$$\int_0^1 \mathbf{r}^\gamma|1-2\mathbf{r}|d\mathbf{r} = \frac{\gamma+1-(\gamma+1)2^{-1}}{2(\gamma+1)}.$$

Employing $PG_m CRM$ of $|\mathcal{V}'|^q$, we have

$$\begin{aligned} &\int_0^1 (1-\mathbf{r})\mathbf{r}^{\gamma-1}|1-2\mathbf{r}||\mathcal{V}'(\mathbf{mu}_x + \mathbf{rR}_{\mu,\nu}^\wp(\mathbf{u}_y - \mathbf{mu}_x))|^q d\mathbf{r} \\ &\leq \frac{|\mathcal{V}'(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \int_0^1 (1-\mathbf{r})\mathbf{r}^{\gamma-1}|1-2\mathbf{r}||1-\mathbf{r}^\zeta|d\mathbf{r} \\ &\quad + \frac{|\mathcal{V}'(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \int_0^1 (1-\mathbf{r})\mathbf{r}^{\gamma-1}|1-2\mathbf{r}||1-(1-\mathbf{r})^\zeta|d\mathbf{r} \\ &= \frac{|\mathcal{V}'(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_1(\gamma, \zeta) + \frac{|\mathcal{V}'(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_2(\gamma, \zeta). \end{aligned}$$

Similarly, for the second part:

$$\begin{aligned} &\int_0^1 \mathbf{r}^\gamma|1-2\mathbf{r}||\mathcal{V}'(\mathbf{mu}_x + \mathbf{rR}_{\mu,\nu}^\wp(\mathbf{u}_y - \mathbf{mu}_x))|^q d\mathbf{r} \\ &\leq \frac{|\mathcal{V}'(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \int_0^1 \mathbf{r}^\gamma|1-2\mathbf{r}||1-\mathbf{r}^\zeta|d\mathbf{r} + \frac{|\mathcal{V}'(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \int_0^1 \mathbf{r}^\gamma|1-2\mathbf{r}||1-(1-\mathbf{r})^\zeta|d\mathbf{r}. \end{aligned}$$

Note that by symmetry:

$$\begin{aligned} \int_0^1 \mathbf{r}^\gamma|1-2\mathbf{r}||1-\mathbf{r}^\zeta|d\mathbf{r} &= \mathcal{I}_2(\gamma, \zeta), \\ \int_0^1 \mathbf{r}^\gamma|1-2\mathbf{r}||1-(1-\mathbf{r})^\zeta|d\mathbf{r} &= \mathcal{I}_1(\gamma, \zeta). \end{aligned}$$

Combining all results, we obtain:

$$\left| -\frac{\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\wp(\mathbf{u}_y - \mathbf{mu}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[\mathbf{R}_{\mu,\nu}^\wp(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathbf{R}_{\mu,\nu}^\wp(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right) \right|$$

$$\begin{aligned}
& + \frac{B(\gamma)\Gamma(\gamma+1)}{2\gamma[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \left[\left({}^{\text{AB}}I_{\mathbf{mu}_x}^{\gamma} \mathcal{V} \right) (\mathbf{mu}_x + R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)) + \left({}^{\text{AB}}I_{\mathbf{mu}_x + R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)}^{\gamma} \mathcal{V} \right) (\mathbf{mu}_x) \right] \\
& \leq \frac{[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{\gamma+1-(\gamma+1)2^{-1}}{2(\gamma+1)} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_1(\gamma, \zeta) + \frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_2(\gamma, \zeta) \right)^{\frac{1}{q}} \\
& + \frac{[R_{\mu,\nu}^{\wp}(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{\gamma+1-(\gamma+1)2^{-1}}{2(\gamma+1)} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_2(\gamma, \zeta) + \frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_1(\gamma, \zeta) \right)^{\frac{1}{q}}.
\end{aligned}$$

This is the required proof. $\square$

Remark 3.2. Implementing Theorem 3.2, we analyze the following interesting H-H inequality version via the CMLF involving ABFIO if we set $\wp = (1, 1, \dots)$ with $\mu = \alpha$ and $\nu = 1$:

$$\begin{aligned}
& \left| -\frac{\mathcal{V}(\mathbf{mu}_x) + \mathcal{V}(\mathbf{mu}_x + \mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x))}{2} - \frac{(1-\gamma)\Gamma(\gamma)}{\gamma[\mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \mathcal{V}\left(\frac{2\mathbf{mu}_x + \mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)}{2}\right) \right. \\
& \left. + \frac{B(\gamma)\Gamma(\gamma+1)}{2\gamma[\mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma} \left[\left({}^{\text{AB}}I_{\mathbf{mu}_x}^{\gamma} \mathcal{V} \right) (\mathbf{mu}_x + \mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)) + \left({}^{\text{AB}}I_{\mathbf{mu}_x + \mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)}^{\gamma} \mathcal{V} \right) (\mathbf{mu}_x) \right] \right| \\
& \leq \frac{[\mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{\gamma+1-(\gamma+1)2^{-1}}{2(\gamma+1)} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_1(\gamma, \zeta) + \frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_2(\gamma, \zeta) \right)^{\frac{1}{q}} \\
& + \frac{[\mathcal{E}_\alpha(\mathbf{u}_y - \mathbf{mu}_x)]^\gamma}{2} \left(\frac{\gamma+1-(\gamma+1)2^{-1}}{2(\gamma+1)} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{V}''(\mathbf{mu}_x)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_2(\gamma, \zeta) + \frac{|\mathcal{V}''(\mathbf{u}_y)|^q}{\eta} \sum_{\zeta=1}^{\eta} \mathcal{I}_1(\gamma, \zeta) \right)^{\frac{1}{q}}.
\end{aligned}$$

4. CONCLUSIONS

Researchers and academics from a wide range of scientific fields have been fascinated with fractional calculus, and for good reason. Because of its exceptional capacity to model complex events and offer solutions to difficult problems, it is one of the most significant and dynamic areas of modern mathematics. Furthermore, statistical theory, physics, functional analysis, and optimization are all greatly impacted by integral inequalities, and the combination of convexity and inequality analysis continues to provide a flourishing stream of advancements, applications, and improvements. Given this dynamic framework, the current manuscript makes the following original and significant contributions:

- 1) We establish a fresh mathematical approach on the H-H inequalities via PG_mCRM within the framework of the ABFIO.
- 2) A new fractional lemma is introduced, serving as a cornerstone from which a series of powerful results via ABFIO are systematically derived.
- 3) Leveraging the improved power mean, and Hölder–İşcan inequalities, we present refined and sharpened developments of the H-H inequality — significantly elevating both the depth and scholarly value of the results.

- 4) A rich collection of remarks and corollaries are provided, validating connections with existing literature and illuminating the role of Mittag-Leffler and Raina-type functions.

The innovative framework and methods given in this paper offer intriguing new avenues for research, particularly in interval analysis and quantum calculus, bringing fresh perspectives and intriguing avenues for more study.

The solution to these unsolved questions will increase the field of applicability of multiplicative calculus and will help make further progress in generalized convexity, inequalities of multiplicative integrals and their applications in mathematics.

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