

Singularly Perturbed Nonlinear Volterra Integro-Differential Equations with Fading Memory: Analysis and Numerical Approximation

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Abstract. This paper investigates a class of singularly perturbed nonlinear Volterra integro-differential equations with fading memory effects. The model involves nonlinear reaction terms together with Volterra-type integral operators, leading to multiscale behavior characterized by the presence of an initial boundary layer. Under suitable assumptions, the existence, uniqueness, and positivity of solutions are established. An asymptotic analysis is carried out through the associated reduced problem and boundary layer correction, revealing the two-scale structure of the solution. Motivated by this analysis, a layer-adapted numerical method based on a Shishkin-type mesh is constructed. The differential term is discretized using a backward difference approximation, while the Volterra integral term is approximated by a composite trapezoidal rule. The resulting nonlinear discrete system is solved by Newton's iterative method. The proposed scheme is shown to be consistent and stable, and a convergence estimate is established in the asymptotic regime corresponding to sufficiently fine mesh discretizations. Numerical experiments confirm the theoretical results and demonstrate the effectiveness of the method in accurately resolving the boundary layer behavior.

1. INTRODUCTION

Mathematical models involving memory effects play a crucial role in describing systems where the present state depends on the past evolution. Volterra integro-differential equations (VIDEs) provide a natural framework for such phenomena by combining differential operators with integral terms representing hereditary effects. These models arise in a wide range of applications, including viscoelasticity, biological systems, and processes with fading memory. Linear VIDEs have been extensively studied due to their analytical tractability, providing explicit representations and qualitative insight that serve as a basis for more general formulations. Extensions

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to variable coefficients, higher-order structures, and generalized kernels have been investigated in the literature. In particular, Inpoonjai (2025 [6]) developed analytical and numerical results for extended linear VIDEs, while Inpoonjai (2026 [7]) further established existence and qualitative properties for generalized linear cases, highlighting the flexibility of the Volterra framework. From a computational perspective, the coupled structure of differential and integral terms presents significant challenges. Classical approaches include collocation methods (Zhao, 2014 [14]; Adesanya et al., 2020 [1]), higher-order time-stepping schemes (Bhalekar and Patade, 2016 [3]), Runge-Kutta and general linear methods with quadrature integration (Rabiei et al., 2019a,b [10, 11]), as well as spectral methods for weakly singular kernels (Liu et al., 2024 [8]). Hybrid function and block-pulse techniques further extend the available numerical tools (Hosry et al., 2020 [5]). For nonlinear VIDEs, iterative linearization techniques such as quasilinearization provide an effective framework for constructing convergent approximations by generating a sequence of linear subproblems (Rasoulizadeh, 2012 [12]). Foundational results on such techniques can be found in Ascher, Mattheij, and Russell (1995 [2]). A number of numerical approaches for nonlinear Volterra integro-differential equations have also been developed in the literature (Cakir et al., 2021 [4]), while in contrast singularly perturbed problems introduce additional analytical and numerical difficulties due to the presence of boundary layers and multiple scales. These features require specialized techniques for accurate approximation and stability analysis; see, for example, Roos et al. (2008 [13]) and Miller et al. (1996 [9]).

Despite these developments, most existing studies focus on regular or weakly perturbed nonlinear Volterra systems. Singularly perturbed nonlinear models with transformation structures, especially those incorporating fading memory effects, remain comparatively less explored. In many practical applications, the interaction between the singular perturbation parameter and the hereditary memory term may generate multiscale temporal behavior together with sharp initial transitions, making both the analytical treatment and numerical approximation substantially more difficult than in standard Volterra models. Moreover, many existing numerical approaches are primarily developed for linear or mildly nonlinear problems and may fail to maintain stability and accuracy uniformly with respect to the perturbation parameter in the presence of boundary layers. This gap is particularly significant in applications where small perturbation parameters strongly influence the qualitative behavior of solutions.

Motivated by these considerations, this paper studies singularly perturbed nonlinear Volterra integro-differential equations with fading memory effects. The principal contributions of the present work are as follows:

1. A theoretical framework is established for the continuous problem, including the existence, uniqueness, and positivity of solutions under suitable assumptions.
2. An asymptotic analysis is carried out through the associated reduced problem and boundary layer equation, revealing the multiscale structure generated by the singular perturbation parameter.

3. A layer-adapted numerical method based on a Shishkin-type mesh is constructed to accurately capture the sharp initial layer behavior.
4. The differential term is discretized using a backward difference approximation, while the Volterra integral operator is approximated by a composite trapezoidal quadrature rule, leading to a fully discrete nonlinear scheme.
5. The resulting nonlinear discrete system is solved using Newton's iterative method.
6. The proposed numerical method is shown to be consistent and stable, and an almost first-order convergence rate is established using discrete stability arguments and Grönwall-type inequalities.

2. CONTINUOUS NONLINEAR VOLTERRA EQUATIONS

We consider the nonlinear singularly perturbed Volterra integro-differential initial-value problem

$$\begin{aligned} \epsilon u'(t) + u^n(t)f(t, u^{1-n}(t)) + u^n(t) \int_0^t K(t, s, u^{1-n}(s))ds &= g(t), \quad 0 \leq t \leq T, \\ u(0) &= u_0 > 0, \end{aligned} \quad (2.1)$$

where $n \geq 0$ is a given constant and $0 < \epsilon \ll 1$. The equation consists of a singular perturbation term $\epsilon u'(t)$, a nonlinear local term, and a nonlinear Volterra-type memory integral. The nonlinear factor $u^n(t)$ acts as a coupling weight for both local and nonlocal contributions, while the transformation $u^{1-n}(t)$ introduces a nonlinear change of state variable inside both the function f and the kernel K . Consequently, the model describes a class of nonlinear hereditary systems with memory effects, where the present state depends simultaneously on instantaneous nonlinear interactions and accumulated past history.

We work in the Banach space $C([0, T])$ endowed with the norm $\|u\|_\infty = \max_{t \in [0, T]} |u(t)|$, and seek solutions in $C^1([0, T])$. The following assumptions are imposed throughout the paper.

(A1) Regularity, positivity, growth, and monotonicity of f . The function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and Lipschitz continuous with respect to its second argument; that is, there exists a constant $L_f > 0$ such that

$$|f(t, x) - f(t, y)| \leq L_f |x - y|, \quad \forall t \in [0, T], \quad x, y \in \mathbb{R}.$$

Moreover, f satisfies the linear growth condition

$$|f(t, x)| \leq C_f(1 + |x|), \quad \forall t \in [0, T], \quad x \in \mathbb{R}.$$

In addition, f is uniformly positive; that is, there exists a constant $\alpha > 0$ such that

$$f(t, x) \geq \alpha, \quad \forall t \in [0, T], \quad x \in \mathbb{R}.$$

Furthermore, for each fixed $t \in [0, T]$, the mapping $U \mapsto U^n f(t, U^{1-n})$ is continuously differentiable and strictly increasing on $(0, \infty)$.

(A2) Properties of the kernel K . The kernel $K : \{(t, s) \in [0, T]^2 : 0 \leq s \leq t\} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and Lipschitz continuous in its third argument (the transformed state variable), i.e., there exists $L_K > 0$ such that

$$|K(t, s, x) - K(t, s, y)| \leq L_K |x - y|, \quad \forall 0 \leq s \leq t \leq T, \quad x, y \in \mathbb{R}.$$

In addition, K satisfies a fading-memory growth condition: there exists a nonnegative function $M \in L^1(0, T)$ such that

$$|K(t, s, x)| \leq M(t - s)(1 + |x|), \quad \forall 0 \leq s \leq t \leq T, \quad x \in \mathbb{R}.$$

(A3) Forcing term. The function $g \in C([0, T])$ is uniformly positive; that is, there exists a constant $g_0 > 0$ such that $g(t) \geq g_0$, $\forall t \in [0, T]$.

(A4) Positivity of solutions. The initial value satisfies $u(0) = u_0 > 0$. Moreover, we assume that the solution remains strictly positive on $[0, T]$; that is, there exists a constant $m > 0$ such that

$$u(t) \geq m > 0, \quad \forall t \in [0, T],$$

which ensures that the nonlinear transformation $u^{1-n}(t)$ is well defined.

Under assumptions (A1)-(A4), the problem is well posed and admits a unique positive solution in $C^1([0, T])$. Moreover, these conditions provide a framework for deriving a priori estimates via Grönwall-type inequalities, which are essential for both the analytical and numerical analysis developed in the subsequent sections.

To establish the well-posedness of problem (2.1), we first reformulate the equation into an equivalent integral form and introduce an associated nonlinear operator. The following auxiliary results are used to verify the existence and uniqueness of solutions. Integrating equation (2.1) from 0 to t , we obtain the equivalent integral equation

$$u(t) = u_0 + \frac{1}{\epsilon} \int_0^t \left[g(\tau) - u^n(\tau) f(\tau, u^{1-n}(\tau)) - u^n(\tau) \int_0^\tau K(\tau, s, u^{1-n}(s)) ds \right] d\tau. \quad (2.2)$$

Define the operator $\mathcal{T} : C([0, T]) \rightarrow C([0, T])$ by

$$(\mathcal{T}u)(t) = u_0 + \frac{1}{\epsilon} \int_0^t \left[g(\tau) - u^n(\tau) f(\tau, u^{1-n}(\tau)) - u^n(\tau) \int_0^\tau K(\tau, s, u^{1-n}(s)) ds \right] d\tau. \quad (2.3)$$

We begin by verifying that the operator $\mathcal{T}$ is well defined on a suitable closed subset of $C([0, T])$, which is a fundamental step in establishing the existence and uniqueness of solutions via the Banach fixed-point theorem.

Lemma 2.1. *Under assumptions (A1)-(A4), the operator $\mathcal{T}$ is well defined on the set*

$$\mathcal{B}_{m,R} = \{u \in C([0, T]) : m \leq u(t) \leq R, \quad \forall t \in [0, T]\},$$

and maps bounded subsets of $\mathcal{B}_{m,R}$ into bounded subsets of $C([0, T])$.

Proof. Let $u \in \mathcal{B}_{m,R}$ be given. Since f and K are continuous and satisfy the growth conditions in (A1)-(A2), and $g \in C([0, T])$, the integrand in (2.2) is continuous in τ . Hence, $(\mathcal{T}u)(t)$ is well defined and continuous on $[0, T]$, which implies that $\mathcal{T}u \in C([0, T])$. Next, we show that $\mathcal{T}$ maps bounded sets into bounded sets. Since $m \leq u(t) \leq R$, $\forall t \in [0, T]$, we have

$$|u^{1-n}(t)| \leq \max\{R^{1-n}, m^{1-n}\} =: C_1,$$

which shows that u^{1-n} is bounded on $[0, T]$. Using the growth condition on f , we obtain

$$|f(\tau, u^{1-n}(\tau))| \leq C_f(1 + |u^{1-n}(\tau)|) \leq C_f(1 + C_1).$$

Similarly, by the fading-memory condition on K , we have

$$\left| \int_0^\tau K(\tau, s, u^{1-n}(s)) ds \right| \leq \int_0^\tau M(\tau - s)(1 + |u^{1-n}(s)|) ds.$$

Using the bound on u^{1-n} , this yields

$$\left| \int_0^\tau K(\tau, s, u^{1-n}(s)) ds \right| \leq (1 + C_1) \int_0^\tau M(\tau - s) ds \leq (1 + C_1) \|M\|_{L^1(0, T)}.$$

Moreover, since $g \in C([0, T])$, there exists $C_g > 0$ such that

$$|g(t)| \leq C_g, \quad \forall t \in [0, T].$$

Combining the above estimates, we obtain

$$|(\mathcal{T}u)(t)| \leq |u_0| + \frac{1}{\epsilon} \int_0^t \left[|g(\tau)| + |u^n(\tau)| |f(\tau, u^{1-n}(\tau))| + |u^n(\tau)| \left| \int_0^\tau K(\tau, s, u^{1-n}(s)) ds \right| \right] d\tau.$$

Since $u \in \mathcal{B}_{m,R}$, we have $|u^n(\tau)| \leq R^n$. Hence

$$|(\mathcal{T}u)(t)| \leq |u_0| + \frac{1}{\epsilon} \int_0^t \left[C_g + R^n C_f(1 + C_1) + R^n(1 + C_1) \|M\|_{L^1(0, T)} \right] d\tau.$$

Therefore, there exists a constant $C_0 > 0$, depending on $R, m, C_f, \|M\|_{L^1}, C_g$, such that

$$|(\mathcal{T}u)(t)| \leq |u_0| + \frac{C_0}{\epsilon} t \leq |u_0| + \frac{C_0 T}{\epsilon}.$$

Taking the supremum over $t \in [0, T]$, we conclude that

$$\|\mathcal{T}u\|_\infty \leq |u_0| + \frac{C_0 T}{\epsilon},$$

which shows that $\mathcal{T}$ maps bounded subsets of $\mathcal{B}_{m,R}$ into bounded subsets of $C([0, T])$. $\square$

Lemma 2.2. Under assumptions (A1)-(A4), the operator $\mathcal{T}$ is a contraction on the closed subset

$$\mathcal{B}_{m,R} = \{u \in C([0, T]) : m \leq u(t) \leq R, \quad \forall t \in [0, T]\},$$

for sufficiently small $T > 0$.

Proof. Let $u, v \in \mathcal{B}_{m,R}$. From the definition of $\mathcal{T}$, we have

$$|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)| \leq \frac{1}{\epsilon} \int_0^t |\Phi(u)(\tau) - \Phi(v)(\tau)| d\tau,$$

where

$$\Phi(u)(\tau) = u^n(\tau)f(\tau, u^{1-n}(\tau)) + u^n(\tau) \int_0^\tau K(\tau, s, u^{1-n}(s)) ds.$$

We estimate the difference term by term. First, for the local nonlinear term,

$$\begin{aligned} & |u^n(\tau)f(\tau, u^{1-n}(\tau)) - v^n(\tau)f(\tau, v^{1-n}(\tau))| \\ & \leq |u^n(\tau) - v^n(\tau)| \cdot |f(\tau, u^{1-n}(\tau))| + |v^n(\tau)| \cdot |f(\tau, u^{1-n}(\tau)) - f(\tau, v^{1-n}(\tau))|. \end{aligned}$$

Since $u, v \in \mathcal{B}_{m,R}$, we have

$$u(t), v(t) \in [m, R], \quad \forall t \in [0, T].$$

Hence, the functions $x \mapsto x^n$ and $x \mapsto x^{1-n}$ are Lipschitz continuous on $[m, R]$. Therefore,

$$|u^n(\tau) - v^n(\tau)| \leq C_1|u(\tau) - v(\tau)| \text{ and } |u^{1-n}(\tau) - v^{1-n}(\tau)| \leq C_2|u(\tau) - v(\tau)|.$$

Using the growth condition on f and its Lipschitz continuity, we obtain

$$|f(\tau, u^{1-n}(\tau))| \leq C_f(1 + C_3) \text{ and } |f(\tau, u^{1-n}(\tau)) - f(\tau, v^{1-n}(\tau))| \leq L_f C_2 |u(\tau) - v(\tau)|.$$

Combining these estimates yields

$$|u^n(\tau)f(\tau, u^{1-n}(\tau)) - v^n(\tau)f(\tau, v^{1-n}(\tau))| \leq C_4|u(\tau) - v(\tau)|.$$

Next, for the integral term,

$$\begin{aligned} & \left| u^n(\tau) \int_0^\tau K(\tau, s, u^{1-n}(s)) ds - v^n(\tau) \int_0^\tau K(\tau, s, v^{1-n}(s)) ds \right| \\ & \leq |u^n(\tau) - v^n(\tau)| \cdot \left| \int_0^\tau K(\tau, s, u^{1-n}(s)) ds \right| + |v^n(\tau)| \cdot \left| \int_0^\tau K(\tau, s, u^{1-n}(s)) - K(\tau, s, v^{1-n}(s)) ds \right|. \end{aligned}$$

Since $u, v \in \mathcal{B}_{m,R}$, the functions u^{1-n} and v^{1-n} are bounded on $[0, T]$. Consequently, by the growth condition on the kernel,

$$\left| \int_0^\tau K(\tau, s, u^{1-n}(s)) ds \right| \leq C_5.$$

Moreover, by the Lipschitz continuity of K ,

$$|K(\tau, s, u^{1-n}(s)) - K(\tau, s, v^{1-n}(s))| \leq L_K C_2 |u(s) - v(s)|.$$

Thus,

$$\begin{aligned} & \left| u^n(\tau) \int_0^\tau K(\tau, s, u^{1-n}(s)) ds - v^n(\tau) \int_0^\tau K(\tau, s, v^{1-n}(s)) ds \right| \\ & \leq C_6 |u(\tau) - v(\tau)| + C_7 \int_0^\tau |u(s) - v(s)| ds. \end{aligned}$$

Combining all estimates, we obtain

$$|\Phi(u)(\tau) - \Phi(v)(\tau)| \leq C_8 \|u - v\|_\infty,$$

for some constant $C_8 > 0$. Therefore,

$$\|\mathcal{T}u - \mathcal{T}v\|_\infty \leq \frac{C_8}{\epsilon} T \|u - v\|_\infty.$$

Choosing $T > 0$ sufficiently small such that $\frac{C_8}{\epsilon} T < 1$, we conclude that $\mathcal{T}$ is a contraction on $\mathcal{B}_{m,R}$. $\square$

We now establish the local well-posedness of problem (2.1). By the reformulation (2.2), solutions of the original problem correspond to fixed points of the operator $\mathcal{T}$ defined on $C([0, T])$. Using the properties established in Lemma 2.1 and Lemma 2.2, we obtain the following result.

Theorem 2.1. *Under assumptions (A1)-(A4), there exists a $T_0 \in (0, T]$ such that problem (2.1) admits a unique positive solution $u \in C^1([0, T_0])$.*

Proof. Let $0 < m < R$ be fixed and define

$$\mathcal{B}_{m,R} = \{u \in C([0, T_0]) : m \leq u(t) \leq R, \forall t \in [0, T_0]\}.$$

Clearly, $\mathcal{B}_{m,R}$ is a closed subset of $C([0, T_0])$, since it can be written as the intersection of the closed sets

$$\{u \in C([0, T_0]) : u(t) \geq m, \forall t \in [0, T_0]\} \text{ and } \{u \in C([0, T_0]) : u(t) \leq R, \forall t \in [0, T_0]\}.$$

We first show that $\mathcal{T}(\mathcal{B}_{m,R}) \subset \mathcal{B}_{m,R}$. Let $u \in \mathcal{B}_{m,R}$. From Lemma 2.1, there exists a constant $C_0 > 0$ such that

$$|(\mathcal{T}u)(t)| \leq |u_0| + \frac{C_0 T_0}{\epsilon}, \forall t \in [0, T_0].$$

Choosing $T_0 > 0$ sufficiently small such that $|u_0| + \frac{C_0 T_0}{\epsilon} \leq R$, we obtain

$$(\mathcal{T}u)(t) \leq R, \forall t \in [0, T_0].$$

Furthermore, using the integral formulation (2.3) together with the estimates in Lemma 2.1, there exists a constant $C_1 > 0$ such that

$$(\mathcal{T}u)(t) \geq u_0 - \frac{C_1 T_0}{\epsilon}, \forall t \in [0, T_0].$$

Choosing $T_0 > 0$ further such that $u_0 - \frac{C_1 T_0}{\epsilon} \geq m$, we obtain

$$(\mathcal{T}u)(t) \geq m, \forall t \in [0, T_0].$$

Hence, $\mathcal{T}(\mathcal{B}_{m,R}) \subset \mathcal{B}_{m,R}$. Next, by Lemma 2.2, there exists a constant $C_8 > 0$ such that

$$\|\mathcal{T}u - \mathcal{T}v\|_\infty \leq \frac{C_8}{\epsilon} T_0 \|u - v\|_\infty, \forall u, v \in \mathcal{B}_{m,R}.$$

Choosing $T_0 > 0$ sufficiently small such that $\frac{C_8}{\epsilon} T_0 < 1$, it follows that $\mathcal{T}$ is a contraction on $\mathcal{B}_{m,R}$. Since $\mathcal{B}_{m,R}$ is a closed subset of the Banach space $C([0, T_0])$, it is complete. Therefore, by the Banach fixed-point theorem, there exists a unique function $u \in \mathcal{B}_{m,R}$ such that $\mathcal{T}u = u$. Hence, u is a continuous solution of the integral equation (2.2), and therefore a solution of problem (2.1). Differentiating (2.2) with respect to t yields $u \in C^1([0, T_0])$. The positivity of the solution follows directly from the definition of $\mathcal{B}_{m,R}$. This completes the proof. $\square$

The above result establishes the local existence and uniqueness of a positive solution to problem (2.1) under assumptions (A1)-(A4). We now illustrate the applicability of the theoretical framework by considering a representative example.

Example 2.1. Consider the nonlinear singularly perturbed Volterra integro-differential problem

$$\begin{aligned} \epsilon u'(t) + u^2(t)f(t, u^{-1}(t)) + u^2(t) \int_0^t K(t, s, u^{-1}(s))ds &= g(t), \quad 0 \leq t \leq T, \\ u(0) &= 1, \epsilon \in (0, 1]. \end{aligned} \quad (2.4)$$

Let $f(t, x) = x$, $K(t, s, x) = e^{-(t-s)}x$. We show that the function $u(t) = e^{-t}$, $t \in [0, T]$, is an exact solution of (2.4). Indeed, we compute $u'(t) = -e^{-t}$, $u^2(t) = e^{-2t}$, $u^{-1}(t) = e^t$. For the nonlinear term, we have $u^2(t)f(t, u^{-1}(t)) = e^{-2t} \cdot e^t = e^{-t}$. For the Volterra term,

$$\int_0^t e^{-(t-s)}u^{-1}(s)ds = \int_0^t e^{-(t-s)}e^sds = e^{-t} \int_0^t e^{2s}ds = e^{-t} \left(\frac{e^{2t} - 1}{2} \right).$$

Hence,

$$u^2(t) \int_0^t K(t, s, u^{-1}(s))ds = e^{-2t} \cdot e^{-t} \left(\frac{e^{2t} - 1}{2} \right) = \frac{e^{-t} - e^{-3t}}{2}.$$

Combining the above expressions, we obtain

$$\epsilon u'(t) + u^2(t)f(t, u^{-1}(t)) + u^2(t) \int_0^t K(t, s, u^{-1}(s))ds = -\epsilon e^{-t} + e^{-t} + \frac{e^{-t} - e^{-3t}}{2}.$$

Therefore, defining $g(t) = -\epsilon e^{-t} + \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}$, it follows that $u(t) = e^{-t}$ satisfies (2.4). Next, we verify assumptions (A1)-(A4). The function $f(t, x) = x$ is continuous and Lipschitz continuous in x with Lipschitz constant $L_f = 1$, and satisfies the linear growth condition

$$|f(t, x)| \leq C_f(1 + |x|), \quad \text{with } C_f = 1.$$

Similarly, the kernel $K(t, s, x) = e^{-(t-s)}x$ is continuous and Lipschitz continuous in x with Lipschitz constant $L_K = 1$, and satisfies

$$|K(t, s, x)| \leq M(t-s)(1 + |x|),$$

where $M(t-s) = e^{-(t-s)} \in L^1(0, T)$, ensuring the fading-memory property. The function $g(t)$ is continuous on $[0, T]$. Moreover, the exact solution $u(t) = e^{-t}$ is strictly positive on $[0, T]$, which guarantees that the term $u^{-1}(t)$ is well defined. Therefore, all assumptions of Theorem 2.1 are satisfied. It follows that the problem admits a unique positive solution in $C^1([0, T])$, which coincides with the exact solution $u(t) = e^{-t}$.

3. REDUCED PROBLEM AND BOUNDARY LAYER ANALYSIS

In this section, we study the asymptotic behavior of solutions to (2.1) as $\epsilon \rightarrow 0$. We derive the corresponding reduced problem governing the leading-order behavior and examine the boundary layer arising from the singular perturbation.

Reduced problem. Formally setting $\epsilon = 0$ in (2.1), we obtain the nonlinear Volterra integral equation

$$u_0^n(t)f(t, u_0^{1-n}(t)) + u_0^n(t) \int_0^t K(t, s, u_0^{1-n}(s))ds = g(t), \quad 0 \leq t \leq T. \quad (3.1)$$

The solution $u_0(t)$ of (3.1) is referred to as the *outer solution*, which describes the leading-order behavior of the full solution away from the boundary layer as $\epsilon \rightarrow 0$.

We first establish the well-posedness of the reduced problem, which is essential for ensuring that the limiting model is mathematically meaningful and suitable for further analysis.

Theorem 3.1. *Under assumptions (A1)-(A4), there exists a sufficiently small $T > 0$ such that the reduced problem (3.1) admits a unique positive solution $u_0 \in C([0, T])$.*

Proof. Define the operator $\mathcal{T}_0 : C([0, T]) \rightarrow C([0, T])$ by

$$(\mathcal{T}_0 u)(t) = \left[\frac{g(t)}{f(t, u^{1-n}(t)) + \int_0^t K(t, s, u^{1-n}(s))ds} \right]^{1/n}.$$

Let $0 < m < R$ be fixed and define

$$B_{m,R} = \{u \in C([0, T]) : m \leq u(t) \leq R, \quad \forall t \in [0, T]\}.$$

Step 1: Well-definedness and positivity. By assumptions (A1)-(A4), we have $g(t) > 0$, and f and K are continuous. Moreover, using the growth condition on K and boundedness of $u \in B_{m,R}$, we obtain

$$\left| \int_0^t K(t, s, u^{1-n}(s))ds \right| \leq C_T,$$

where $C_T > 0$ depends on T . Thus, for sufficiently small $T > 0$, the denominator satisfies

$$f(t, u^{1-n}(t)) + \int_0^t K(t, s, u^{1-n}(s))ds > 0,$$

which implies that $\mathcal{T}_0$ is well-defined and maps positive functions to positive functions.

Step 2: Invariance. Let $u \in B_{m,R}$, i.e., $m \leq u(t) \leq R$ for all $t \in [0, T]$. Then $u^{1-n}(t)$ remains in a bounded interval, and by the continuity of f , there exists a constant $M_f > 0$ such that

$$|f(t, u^{1-n}(t))| \leq M_f, \quad \forall t \in [0, T].$$

Moreover, by assumption (A1), there exists $\alpha > 0$ such that $f(t, u^{1-n}(t)) \geq \alpha$. Using the growth condition on K , we obtain

$$\left| \int_0^t K(t, s, u^{1-n}(s))ds \right| \leq CT,$$

for some constant $C > 0$. Hence, for sufficiently small $T > 0$, we have

$$f(t, u^{1-n}(t)) + \int_0^t K(t, s, u^{1-n}(s))ds \geq \alpha - CT > 0,$$

and

$$f(t, u^{1-n}(t)) + \int_0^t K(t, s, u^{1-n}(s))ds \leq M_f + CT.$$

Therefore,

$$\frac{g_{\min}}{M_f + CT} \leq \frac{g(t)}{f(t, u^{1-n}(t)) + \int_0^t K(t, s, u^{1-n}(s)) ds} \leq \frac{g_{\max}}{\alpha - CT},$$

where $g_{\min} = \min_{[0, T]} g(t)$ and $g_{\max} = \max_{[0, T]} g(t)$. Taking the n -th root, we obtain

$$\left(\frac{g_{\min}}{M_f + CT} \right)^{1/n} \leq (\mathcal{T}_0 u)(t) \leq \left(\frac{g_{\max}}{\alpha - CT} \right)^{1/n}.$$

Choosing m and R such that

$$0 < m \leq \left(\frac{g_{\min}}{M_f + CT} \right)^{1/n}, \left(\frac{g_{\max}}{\alpha - CT} \right)^{1/n} \leq R.$$

Then we conclude that $m \leq (\mathcal{T}_0 u)(t) \leq R, \forall t \in [0, T]$. Hence, $\mathcal{T}_0(B_{m, R}) \subset B_{m, R}$.

Step 3: Contraction property. Let $u, v \in B_{m, R}$. Denote

$$D_u(t) = f(t, u^{1-n}(t)) + \int_0^t K(t, s, u^{1-n}(s)) ds,$$

and define $D_v(t)$ analogously. Then

$$(\mathcal{T}_0 u)(t) = \left[\frac{g(t)}{D_u(t)} \right]^{1/n}.$$

Using the mean value theorem for the function

$$\phi(x) = g(t)^{1/n} x^{-1/n}, \quad x > 0,$$

together with the lower bound $D_u, D_v \geq c_1 > 0$ from Step 2 and the boundedness of g , we obtain

$$|(\mathcal{T}_0 u)(t) - (\mathcal{T}_0 v)(t)| \leq C_1 \left| \frac{1}{D_u(t)} - \frac{1}{D_v(t)} \right|.$$

Since

$$\left| \frac{1}{D_u} - \frac{1}{D_v} \right| = \frac{|D_v - D_u|}{D_u D_v},$$

and $D_u D_v \geq c_1^2$, it follows that

$$|(\mathcal{T}_0 u)(t) - (\mathcal{T}_0 v)(t)| \leq C_2 |D_u(t) - D_v(t)|.$$

Using the Lipschitz continuity of f and K , and the fact that $u, v \in B_{m, R}$ with $m > 0$, we obtain

$$|D_u(t) - D_v(t)| \leq C_3 |u(t) - v(t)| + C_4 \int_0^t |u(s) - v(s)| ds.$$

Hence,

$$|(\mathcal{T}_0 u)(t) - (\mathcal{T}_0 v)(t)| \leq C_5 |u(t) - v(t)| + C_6 \int_0^t |u(s) - v(s)| ds.$$

Taking the supremum over $t \in [0, T]$, we obtain

$$\|\mathcal{T}_0 u - \mathcal{T}_0 v\|_{\infty} \leq C_5 \|u - v\|_{\infty} + C_6 T \|u - v\|_{\infty}$$

and it follows that

$$\|\mathcal{T}_0 u - \mathcal{T}_0 v\|_{\infty} \leq (C_5 + C_6 T) \|u - v\|_{\infty}.$$

We now choose $T > 0$ sufficiently small such that $CT \leq \frac{\alpha}{2}$. Then $\alpha - CT \geq \frac{\alpha}{2}$. Consequently, the constant C_5 in the contraction estimate satisfies

$$C_5 \leq \frac{(n-1)L_f}{nm^n} (g_{\max})^{1/n} \left(\frac{\alpha}{2}\right)^{-1/n-1}.$$

Assume that $\frac{(n-1)L_f}{nm^n} (g_{\max})^{1/n} \left(\frac{\alpha}{2}\right)^{-1/n-1} < 1$. Hence, $C_5 < 1$. Choosing $T > 0$ sufficiently small such that $C_5 + C_6T < 1$, we conclude that $\mathcal{T}_0$ is a contraction on $B_{m,R}$.

Step 4: Conclusion. From Steps 1-3, the operator $\mathcal{T}_0$ is well-defined, maps $B_{m,R}$ into itself, and is a contraction on $B_{m,R}$ for sufficiently small $T > 0$. Since $(B_{m,R}, \|\cdot\|_\infty)$ is a closed subset of the Banach space $C([0, T])$, it is itself a complete metric space. Therefore, by the Banach fixed-point theorem, $\mathcal{T}_0$ admits a unique fixed point $u_0 \in B_{m,R}$. By construction, this fixed point satisfies $\mathcal{T}_0 u_0 = u_0$, and hence u_0 is the unique positive solution of the reduced problem (3.1) on $[0, T]$. $\square$

Boundary layer analysis. The reduced problem obtained in the singular limit generally does not satisfy the prescribed initial condition of the original problem. This mismatch indicates the presence of an initial boundary layer near $t = 0$, where the solution varies rapidly on a short time scale of order ϵ . To analyze this rapid transition, we introduce the stretched variable

$$\tau = \frac{t}{\epsilon}, \quad u(t) = U(\tau).$$

By the chain rule,

$$u'(t) = \frac{1}{\epsilon} U'(\tau).$$

Substituting $t = \epsilon\tau$ and $u(t) = U(\tau)$ into (2.1), we obtain

$$U'(\tau) = g(\epsilon\tau) - U^n(\tau)f(\epsilon\tau, U^{1-n}(\tau)) - U^n(\tau) \int_0^\tau \epsilon K(\epsilon\tau, \epsilon\sigma, U^{1-n}(\sigma)) d\sigma.$$

The transformed equation describes the evolution of the solution on the fast time scale associated with the initial layer.

The following result characterizes the leading-order asymptotic behavior of the solution in the singular limit as $\epsilon \rightarrow 0^+$, capturing the dominant dynamics of the initial layer.

Proposition 3.1. *As $\epsilon \rightarrow 0^+$, the leading-order inner solution satisfies the layer equation*

$$U'_0(\tau) = g(0) - U_0^n(\tau)f(0, U_0^{1-n}(\tau)).$$

Proof. Formally letting $\epsilon \rightarrow 0^+$ in the transformed equation yields $g(\epsilon\tau) \rightarrow g(0)$, while the integral term becomes negligible due to the factor ϵ . Consequently, the leading-order inner equation reduces to

$$U'_0(\tau) = g(0) - U_0^n(\tau)f(0, U_0^{1-n}(\tau)).$$

This completes the proof. $\square$

The limiting layer equation governs the rapid transient dynamics near the initial point. In order to study the long-time behavior of the layer solution, define

$$F(U) = g(0) - U^n f(0, U^{1-n}).$$

Then the layer equation can be rewritten in the autonomous form

$$U'_0(\tau) = F(U_0(\tau)).$$

The next result establishes the asymptotic connection between the initial layer dynamics and the reduced solution by showing that the layer solution converges to the corresponding equilibrium in the overlap region.

Theorem 3.2. *Assume that condition (A1) holds. Then the equilibrium state $U^* = u_0(0)$ of the layer equation is asymptotically stable. Moreover, the layer solution satisfies the matching condition*

$$U_0(\tau) \rightarrow u_0(0), \tau \rightarrow \infty.$$

Proof. The equilibrium state satisfies $F(U^*) = 0$, which is equivalent to

$$g(0) = (U^*)^n f(0, (U^*)^{1-n}).$$

On the other hand, evaluating the reduced problem (3.1) at $t = 0$ gives

$$g(0) = u_0^n(0) f(0, u_0^{1-n}(0)).$$

Hence, $U^* = u_0(0)$. Moreover, by assumption (A1),

$$F'(U) = -\frac{d}{dU} [U^n f(0, U^{1-n})] < 0.$$

Therefore, the equilibrium $U^* = u_0(0)$ is asymptotically stable according to the standard stability theory for autonomous ordinary differential equations. Consequently, every solution sufficiently close to U^* satisfies

$$U_0(\tau) \rightarrow U^* = u_0(0), \tau \rightarrow \infty.$$

This establishes the matching condition between the inner and outer solutions. $\square$

This matching condition confirms that the influence of the initial layer decays away from $t = 0$, allowing the solution to transition toward the slow dynamics governed by the reduced problem. Consequently, the solution exhibits a two-scale asymptotic structure consisting of:

- an outer solution describing the slow dynamics away from the initial layer, and
- an inner correction capturing the rapid transient behavior near $t = 0$.

This decomposition provides the analytical foundation for the fitted numerical method developed in the subsequent sections.

Composite structure. The preceding analysis indicates that the solution possesses a two-scale asymptotic structure consisting of a slowly varying outer component and a rapidly decaying inner correction near the initial point. To obtain an approximation that remains asymptotically valid throughout the interval $[0, T]$, we combine the outer and inner expansions using the method of

matched asymptotic expansions. In the overlap region, where $\epsilon \ll t \ll 1$, both the outer and inner scalings are simultaneously valid. Since the outer solution u_0 is continuous, we have

$$u_0(t) \rightarrow u_0(0), \quad t \rightarrow 0^+.$$

At the same time, $\tau = \frac{t}{\epsilon} \rightarrow \infty$. By the matching condition obtained in the boundary layer analysis,

$$U_0(\tau) \rightarrow u_0(0), \quad \tau \rightarrow \infty.$$

Hence, the outer and inner expansions share the common limiting value $u_0(0)$ in the overlap region. To avoid double counting of this common contribution, we subtract it when constructing the composite approximation. This leads to

$$u_{\text{comp}}(t) = u_0(t) + U_0\left(\frac{t}{\epsilon}\right) - u_0(0).$$

The resulting composite approximation combines the slow outer behavior with the rapidly varying boundary layer correction near $t = 0$, and therefore captures the leading-order behavior of the solution uniformly on $[0, T]$. This asymptotic decomposition also motivates the fitted numerical method developed in the next section.

4. NUMERICAL METHOD

The asymptotic analysis in Section 3 indicates that the solution of (2.1) possesses an initial boundary layer of width $O(\epsilon)$ near $t = 0$, where the solution varies rapidly, while remaining smooth in the outer region. Consequently, standard uniform meshes may fail to adequately resolve the sharp transition occurring in the layer region. To overcome this difficulty, we construct a layer-adapted numerical method based on a Shishkin-type mesh.

Layer-adapted mesh. Let N be an even positive integer. Motivated by the two-scale asymptotic structure obtained in Section 3, we employ a standard Shishkin-type mesh to efficiently resolve the initial boundary layer near $t = 0$. Following the classical Shishkin mesh construction, the transition parameter is chosen as

$$\sigma = \min\left\{\frac{T}{2}, \epsilon \ln N\right\}.$$

This choice of the transition parameter ensures that the exponential boundary layer terms are reduced to algebraic order in N , which is essential for obtaining convergence. The domain is then decomposed into the layer region and the outer region as

$$[0, T] = [0, \sigma] \cup [\sigma, T].$$

The mesh points $\{t_i\}_{i=0}^N$ are defined by

$$t_i = \begin{cases} \frac{2\sigma}{N}i, & 0 \leq i \leq \frac{N}{2}, \\ \sigma + \frac{2(T-\sigma)}{N}\left(i - \frac{N}{2}\right), & \frac{N}{2} < i \leq N. \end{cases}$$

This mesh construction is standard in the numerical treatment of singularly perturbed problems with boundary layers and is designed to provide a finer mesh distribution inside the layer region.

Consequently, the mesh is refined in the layer region $[0, \sigma]$, where rapid variation occurs, while a uniform mesh is used in the outer region $[\sigma, T]$.

Fully discrete scheme. To derive the fully discrete approximation, the differential equation (2.1) is considered on each mesh interval $[t_{i-1}, t_i]$. The differential term is discretized implicitly, while the nonlinear and Volterra integral terms are evaluated at the current mesh point to preserve stability in the layer region. Let $U_i \approx u(t_i)$ for $i = 0, 1, \dots, N$, and denote the local mesh size by $h_i = t_i - t_{i-1}$. The derivative term is approximated by the backward difference formula

$$u'(t_i) \approx \frac{U_i - U_{i-1}}{h_i}, \quad i = 1, 2, \dots, N.$$

The Volterra integral term is approximated using the composite trapezoidal rule:

$$\int_0^{t_i} K(t_i, s, u^{1-n}(s)) ds \approx \sum_{j=0}^{i-1} \frac{h_{j+1}}{2} \left[K(t_i, t_j, U_j^{1-n}) + K(t_i, t_{j+1}, U_{j+1}^{1-n}) \right].$$

Substituting these approximations into (2.1), we obtain the fully discrete scheme

$$\epsilon \frac{U_i - U_{i-1}}{h_i} + U_i^n f(t_i, U_i^{1-n}) + U_i^n \sum_{j=0}^{i-1} \frac{h_{j+1}}{2} \left[K(t_i, t_j, U_j^{1-n}) + K(t_i, t_{j+1}, U_{j+1}^{1-n}) \right] = g(t_i), \quad (4.1)$$

for $i = 1, 2, \dots, N$, together with the initial condition $U_0 = u(0)$.

Solution of the nonlinear discrete system. At each mesh point t_i , scheme (4.1) leads to a scalar nonlinear equation for U_i , since all previous values U_j for $j < i$ are already known. To compute U_i , we employ Newton's iterative method. Define the nonlinear function

$$F_i(z) = \epsilon \frac{z - U_{i-1}}{h_i} + z^n f(t_i, z^{1-n}) + z^n \sum_{j=0}^{i-1} \frac{h_{j+1}}{2} \left[K(t_i, t_j, U_j^{1-n}) + K(t_i, t_{j+1}, U_{j+1}^{1-n}) \right] - g(t_i).$$

Then U_i is obtained by solving the nonlinear equation $F_i(U_i) = 0$.

Starting from an initial guess $U_i^{[0]} = U_{i-1}$, Newton's iteration is defined by

$$U_i^{[k+1]} = U_i^{[k]} - \frac{F_i(U_i^{[k]})}{F_i'(U_i^{[k]})}, \quad k = 0, 1, 2, \dots$$

where $F_i'(z)$ denotes the derivative of F_i with respect to z . The iteration is terminated when

$$|U_i^{[k+1]} - U_i^{[k]}| < \text{tol},$$

where $\text{tol} > 0$ is a prescribed tolerance (in numerical experiments, we take $\text{tol} = 10^{-6}$). The converged value $U_i^{[k]}$ is taken as the numerical solution at node t_i .

Numerical algorithm: Fully discrete procedure.

- (1) Construct the Shishkin-type mesh $\{t_i\}_{i=0}^N$.
- (2) Set initial condition $U_0 = u(0)$.
- (3) For $i = 1, 2, \dots, N$ do:
 - Set initial guess $U_i^{[0]} = U_{i-1}$.

- For $k = 0, 1, 2, \dots$ compute

$$U_i^{[k+1]} = U_i^{[k]} - \frac{F_i(U_i^{[k]})}{F'_i(U_i^{[k]})},$$

until

$$|U_i^{[k+1]} - U_i^{[k]}| < \text{tol.}$$

- Set $U_i = U_i^{[k+1]}$.

(4) Output $\{U_i\}_{i=0}^N$.

5. ERROR ANALYSIS

In this section, we establish the stability and convergence of the proposed numerical method. The analysis is based on a discrete comparison principle together with a discrete Grönwall-type argument applied to the error equation. We assume that the exact solution exhibits boundary layer behavior near $t = 0$, i.e., there exist constants $C, \alpha > 0$, independent of ϵ , such that

$$|u^{(k)}(t)| \leq B \left(1 + \epsilon^{-k} e^{-\alpha t / \epsilon} \right), \quad k = 1, 2.$$

This regularity property is standard for singularly perturbed Volterra integro-differential equations and reflects the presence of an initial boundary layer. It will be used to derive uniform bounds for the truncation error and to ensure convergence on the Shishkin-type mesh.

The following lemma provides a discrete stability estimate for a first-order difference scheme, which will be used in the subsequent error analysis.

Lemma 5.1 (Cakir et al. (2021) [4]). *Let*

$$l_h g_i := \frac{g_i - g_{i-1}}{h_i} + a_i g_i = F_i, \quad i = 1, 2, \dots, N,$$

with $g_0 = A$. Assume that $a_i \geq \alpha > 0$ and $|F_i| \leq \mathcal{F}_i$, where $\{\mathcal{F}_i\}$ is a nondecreasing sequence. Then

$$|g_i| \leq |A| + \alpha^{-1} \mathcal{F}_i, \quad i = 1, 2, \dots, N.$$

To apply Lemma 5.1 in the subsequent analysis, we first derive the discrete error equation governing the evolution of the numerical approximation error.

Lemma 5.2 (Discrete error equation). *Let $e_i = u(t_i) - U_i$, $i = 0, 1, \dots, N$, denote the numerical error. Then*

$$\frac{e_i - e_{i-1}}{h_i} + a_i e_i = T_i - F_i, \quad i = 1, 2, \dots, N,$$

with $e_0 = 0$, where $a_i \geq \alpha > 0$, T_i collects the local truncation and quadrature errors, and F_i denotes the Volterra coupling term, which satisfies

$$|F_i| \leq \sum_{j=0}^{i-1} b_{ij} |e_j|, \quad b_{ij} \geq 0.$$

Proof. Subtracting the discrete scheme from the continuous problem evaluated at $t = t_i$, we compare the corresponding terms.

Differential term. For the derivative approximation,

$$\epsilon u'(t_i) - \epsilon \frac{U_i - U_{i-1}}{h_i},$$

we add and subtract the exact backward difference to obtain

$$\epsilon \left[u'(t_i) - \frac{u(t_i) - u(t_{i-1})}{h_i} \right] + \epsilon \frac{e_i - e_{i-1}}{h_i}.$$

The first term is the local truncation error.

Reaction term. Applying the mean value theorem to

$$\Phi(U) = U^n f(t_i, U^{1-n}),$$

gives

$$\Phi(u(t_i)) - \Phi(U_i) = a_i e_i,$$

where $a_i = \Phi'(\xi_i)$ for some ξ_i between $u(t_i)$ and U_i . Since Φ is strictly increasing by assumption, $a_i \geq \alpha > 0$.

Volterra term. Let

$$I[u](t_i) = \int_0^{t_i} K(t_i, s, u(s)) ds,$$

and let $I_h[\cdot]$ denote the discrete Volterra integral operator defined by the composite trapezoidal rule, i.e.,

$$I_h[v]_i := \sum_{j=0}^{i-1} w_{ij} K(t_i, t_j, v_j).$$

We split the Volterra discretization error as

$$I[u](t_i) - I_h[U]_i = (I[u](t_i) - I_h[u]_i) + (I_h[u]_i - I_h[U]_i).$$

The first term represents the quadrature error and is absorbed into the local truncation term. For the second term, define

$$F_i := I_h[u]_i - I_h[U]_i.$$

Then using the composite quadrature formula,

$$F_i = \sum_{j=0}^{i-1} w_{ij} \left[K(t_i, t_j, u(t_j)) - K(t_i, t_j, U_j) \right].$$

By the Lipschitz continuity of K ,

$$|K(t_i, t_j, u(t_j)) - K(t_i, t_j, U_j)| \leq L_K |e_j|.$$

Hence,

$$|F_i| \leq L_K \sum_{j=0}^{i-1} w_{ij} |e_j| = \sum_{j=0}^{i-1} b_{ij} |e_j|, \quad b_{ij} = L_K w_{ij} \geq 0.$$

Collecting all contributions into T_i , we obtain

$$\frac{e_i - e_{i-1}}{h_i} + a_i e_i = T_i - F_i,$$

where T_i contains all local truncation and quadrature errors, and F_i satisfies

$$|F_i| \leq \sum_{j=0}^{i-1} b_{ij} |e_j|.$$

This completes the proof. $\square$

The above representation separates the local discretization errors from the accumulated Volterra coupling terms. We next estimate the truncation component T_i .

Lemma 5.3 (Consistency estimate). *The truncation term satisfies*

$$|T_i| \leq Dh_i \left(\epsilon + \epsilon^{-1} e^{-\alpha t_i / \epsilon} \right) + Dh_i^2, \quad i = 1, 2, \dots, N,$$

where D is independent of ϵ and N .

Proof. From the definition of T_i in Lemma 5.2, T_i consists of the discretization error of the differential part and the quadrature approximation error of the Volterra integral term.

Differential part. Using Taylor expansion,

$$u(t_{i-1}) = u(t_i) - h_i u'(t_i) + \frac{h_i^2}{2} u''(\eta_i),$$

we obtain

$$u'(t_i) - \frac{u(t_i) - u(t_{i-1}))}{h_i} = \frac{h_i}{2} u''(\eta_i).$$

Hence the contribution of the differential term is

$$|T_i^{(1)}| \leq D_1 h_i \epsilon |u''(\eta_i)|.$$

Using the regularity estimate (boundary layer behavior),

$$|u''(t)| \leq B \left(1 + \epsilon^{-2} e^{-\alpha t / \epsilon} \right),$$

we obtain

$$|T_i^{(1)}| \leq D_2 h_i \left(\epsilon + \epsilon^{-1} e^{-\alpha t_i / \epsilon} \right).$$

Volterra quadrature part. From the composite trapezoidal rule and smoothness of the kernel,

$$|T_i^{(2)}| \leq D_3 h_i^2.$$

Combination. Collecting both contributions,

$$|T_i| \leq |T_i^{(1)}| + |T_i^{(2)}| \leq Dh_i \left(\epsilon + \epsilon^{-1} e^{-\alpha t_i / \epsilon} \right) + Dh_i^2,$$

which completes the proof. $\square$

We now derive a stability estimate for the numerical error, which will allow us to control the global error in terms of the local truncation contributions.

Lemma 5.4 (Stability estimate). *There exists a constant $E > 0$, independent of ϵ and N , such that*

$$\|e\|_{\infty} \leq E \max_{1 \leq i \leq N} |T_i|.$$

Proof. From Lemma 5.2, the discrete error equation is

$$\frac{e_i - e_{i-1}}{h_i} + a_i e_i = T_i - F_i,$$

where $|F_i| \leq \sum_{j=0}^{i-1} b_{ij}|e_j|$. Taking absolute values and using the triangle inequality, we obtain

$$\left| \frac{e_i - e_{i-1}}{h_i} + a_i e_i \right| \leq |T_i| + |F_i|.$$

Hence,

$$\left| \frac{e_i - e_{i-1}}{h_i} + a_i e_i \right| \leq |T_i| + \sum_{j=0}^{i-1} b_{ij}|e_j|.$$

By applying Lemma 5.1 (discrete stability inequality) with $g_i = e_i$ and using $a_i \geq \alpha > 0$, we obtain

$$|e_i| \leq E_1 \max_{1 \leq k \leq i} \left(|T_k| + \sum_{j=0}^{k-1} b_{kj}|e_j| \right).$$

Taking the maximum over i yields

$$\|e\|_{\infty} \leq E_1 \left(\max_{1 \leq i \leq N} |T_i| + \max_{1 \leq i \leq N} \sum_{j=0}^{i-1} b_{ij}|e_j| \right).$$

Since $b_{ij} \geq 0$ and $\sum_{j=0}^{i-1} b_{ij}$ is uniformly bounded under the mesh assumptions, a discrete Grönwall inequality gives

$$\|e\|_{\infty} \leq E \max_{1 \leq i \leq N} |T_i|.$$

This completes the proof. $\square$

The stability result obtained above, together with the truncation error estimate, enables us to establish the convergence of the proposed numerical method.

Theorem 5.1 (Convergence estimate). *Assume that the number of mesh intervals N is sufficiently large such that $\epsilon \ln N \geq \frac{T}{2}$, and hence the transition parameter satisfies $\sigma = T/2$. Let $u(t)$ be the exact solution of (2.1) and U_i be the numerical solution obtained on the Shishkin-type mesh. Then*

$$\max_{0 \leq i \leq N} |u(t_i) - U_i| \leq FN^{-1} \ln N,$$

where F is independent of ϵ and N .

Proof. By Lemma 5.4, $\|e\|_{\infty} \leq E \max_{1 \leq i \leq N} |T_i|$. Furthermore, Lemma 5.3 yields

$$|T_i| \leq Dh_i \left(\epsilon + \epsilon^{-1} e^{-\alpha t_i / \epsilon} \right) + Dh_i^2.$$

Since $\sigma = T/2$, the mesh widths satisfy $h_i \leq F_1 N^{-1}$. Moreover, the assumption $\epsilon \ln N \geq \frac{T}{2}$ implies

$$\epsilon^{-1} \leq \frac{2 \ln N}{T}.$$

Since $e^{-\alpha t_i/\epsilon} \leq 1$, it follows that

$$h_i \epsilon^{-1} e^{-\alpha t_i/\epsilon} \leq F_1 N^{-1} \epsilon^{-1} \leq \frac{2F_1}{T} N^{-1} \ln N.$$

In addition, $h_i \epsilon \leq F_1 N^{-1}$, $h_i^2 \leq F_1^2 N^{-2}$. Combining the foregoing estimates and absorbing lower-order terms into the generic constant F , we obtain

$$|T_i| \leq F N^{-1} \ln N, \quad i = 1, 2, \dots, N.$$

Therefore,

$$\max_{1 \leq i \leq N} |T_i| \leq F N^{-1} \ln N.$$

Substituting this estimate into Lemma 5.4, we obtain

$$\|e\|_\infty \leq F N^{-1} \ln N.$$

Hence,

$$\max_{0 \leq i \leq N} |u(t_i) - U_i| = \|e\|_\infty \leq F N^{-1} \ln N.$$

This completes the proof. $\square$

Remark 5.1. The convergence analysis presented above is restricted to the regime

$$\epsilon \ln N \geq \frac{T}{2},$$

for which the transition parameter satisfies $\sigma = T/2$. Numerical experiments indicate that the proposed method exhibits larger errors when

$$\epsilon \ln N < \frac{T}{2},$$

corresponding to the case $\sigma = \epsilon \ln N$. A rigorous convergence analysis for this regime remains an interesting topic for future research.

6. NUMERICAL RESULTS

In this section, we present numerical experiments to illustrate the efficiency of the proposed numerical method on the Shishkin-type mesh. The experiments are designed to verify the theoretical results established in Section 5, particularly the convergence of the scheme in the presence of boundary layers.

Example 6.1. Consider the nonlinear singularly perturbed Volterra integro-differential equation

$$\epsilon u'(t) + u^2(t)(1 + u^{-1}(t)) + u^2(t) \int_0^t e^{-(t-s)} ds = g(t), \quad 0 < t \leq 1, \quad (6.1)$$

subject to the initial condition $u(0) = 2$.

The forcing term is chosen such that the exact solution is given by $u(t) = 1 + e^{-t/\epsilon}$. Since

$$\int_0^t e^{-(t-s)} ds = 1 - e^{-t},$$

substitution of the exact solution into the differential equation yields

$$g(t) = -e^{-t/\epsilon} + (1 + e^{-t/\epsilon})(2 + e^{-t/\epsilon}) + (1 + e^{-t/\epsilon})^2(1 - e^{-t}).$$

The exact solution contains the exponential layer term $e^{-t/\epsilon}$, which produces a sharp initial boundary layer near $t = 0$ when ϵ is small. Therefore, this problem provides a suitable test example for examining the capability of the proposed layer-adapted method to resolve singular behavior accurately. It is straightforward to verify that Example 6.1 satisfies assumptions (A1)-(A4) introduced in Section 2. In particular, the nonlinear function $f(t, z) = 1 + z$ is continuous, positive, and Lipschitz continuous with respect to z , while the kernel $K(t, s, z) = e^{-(t-s)}$ is continuous, bounded, and independent of z . Furthermore, the forcing term $g(t)$ is continuous on $[0, 1]$, and the initial condition satisfies $u_0 > 0$.

Numerical solution profiles. The exact and numerical solutions are displayed in Figure 1 for several values of the perturbation parameter ϵ . A magnified view of the initial layer region near $t = 0$ is presented in Figure 2. Figures 1 and 2 demonstrate that the proposed numerical method accurately resolves the layer behavior for all tested values of the perturbation parameter. The numerical solutions closely match the exact solutions throughout the computational domain, including the sharp initial layer region near $t = 0$.

Maximum nodal errors and convergence rates. Since the exact solution is available, the maximum nodal error is computed by

$$E_\epsilon^N = \max_{0 \leq i \leq N} |u(t_i) - U_i|,$$

where U_i denotes the numerical solution obtained by the proposed numerical scheme. The experimental order of convergence is calculated using

$$p^N = \log_2 \left(\frac{E_\epsilon^N}{E_\epsilon^{2N}} \right).$$

The numerical results are computed for several values of the perturbation parameter ϵ and mesh size N . The maximum nodal errors and the corresponding convergence orders are reported in Table 1. The numerical results demonstrate that the proposed method remains stable as the perturbation parameter decreases. In all cases, the maximum nodal errors decrease monotonically as the mesh is refined, confirming the convergence of the method. It is further observed that the convergence rates deteriorate slightly for smaller values of ϵ , which is expected due to the increasingly sharp initial layer. Nevertheless, the proposed layer-adapted scheme continues to provide stable and accurate numerical approximations. For moderate values of the perturbation parameter, the observed convergence orders approach first-order accuracy, in agreement with the almost first-order convergence established in Theorem 5.1.

TABLE 1. Maximum nodal errors and observed convergence orders for Example 6.1.

ϵ	N	E_ϵ^N	Order
2^{-2}	16	0.01518171	—
	32	0.00808665	0.90871993
	64	0.00418055	0.95185025
2^{-3}	16	0.02108332	—
	32	0.01377005	0.61456820
	64	0.00829686	0.73089633
2^{-4}	16	0.02152785	—
	32	0.01403526	0.61714806
	64	0.00874023	0.68331240

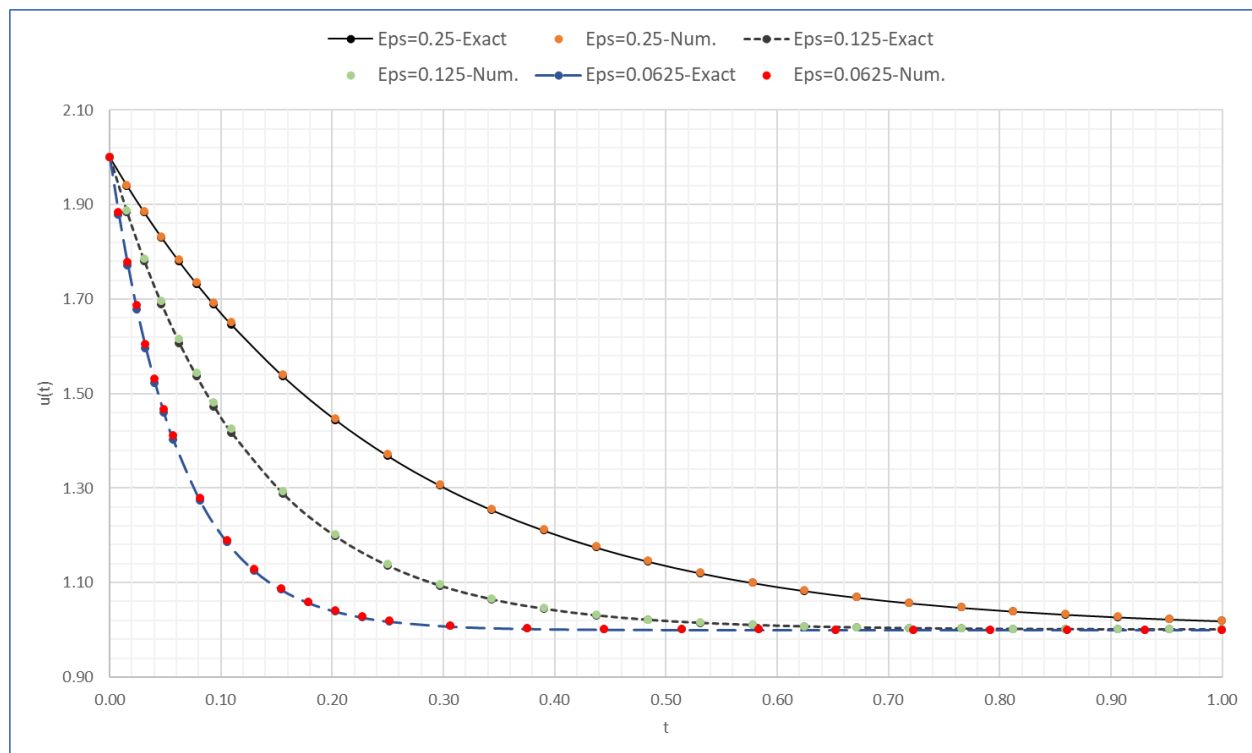


FIGURE 1. Exact and numerical solutions for Example 6.1 with $N = 64$ and different values of ϵ . The numerical solutions closely agree with the exact solutions over the entire interval.

Example 6.2. Consider the nonlinear singularly perturbed Volterra integro-differential equation

$$\epsilon u'(t) + u^2(t)[t + u^{-1}(t)] + u^2(t) \int_0^t e^{-(t-s)} \sin(u^{-1}(s)) ds = g(t), \quad (6.2)$$

subject to the initial condition $u(0) = 1$, on the interval $0 < t \leq 1$.

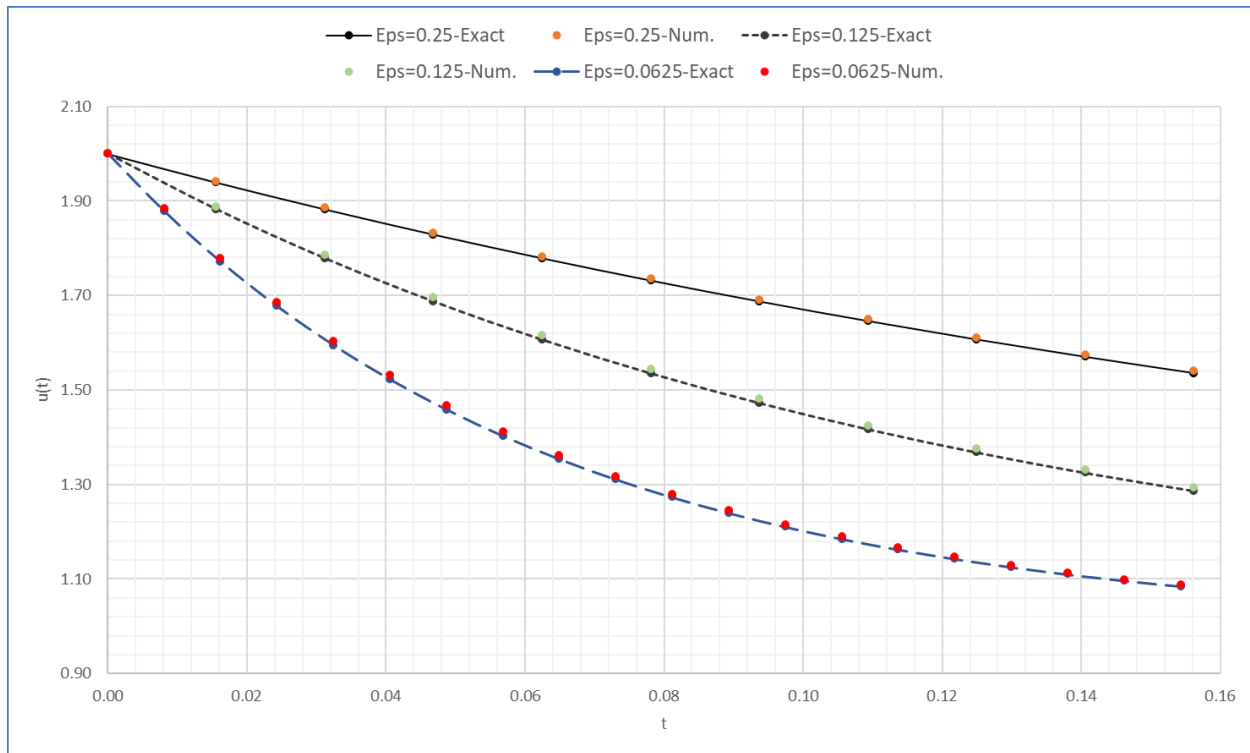


FIGURE 2. Magnified view of the initial layer near $t = 0$ for Example 6.1 with $N = 64$. The numerical method accurately resolves the sharp layer behavior for different values of ϵ .

The forcing term is chosen such that the exact solution is given by $u(t) = e^{-t}$. Since

$$\int_0^t e^{-(t-s)} \sin(u^{-1}(s)) ds = e^{-t} [\cos(1) - \cos(e^t)],$$

substitution of the exact solution into the differential equation yields

$$g(t) = -\epsilon e^{-t} + t e^{-2t} + e^{-t} + e^{-3t} [\cos(1) - \cos(e^t)].$$

In contrast to Example 6.1, the present problem does not exhibit boundary layer behavior. Instead, this example is intended to examine the performance of the proposed numerical method for nonlinear Volterra integro-differential equations involving fading-memory effects and nonlinear transformed terms. The nonlinear reaction term and the nonlinear memory term both depend explicitly on the transformed variable u^{-1} . Moreover, the exponentially decaying kernel $e^{-(t-s)}$ introduces a fading-memory effect, while the oscillatory nonlinear component $\sin(u^{-1}(s))$ further increases the nonlinearity of the problem. Consequently, this example provides an additional test for assessing the robustness and stability of the proposed numerical method in the presence of nonlinear memory interactions. It can be verified that assumptions (A1)-(A4) remain satisfied for this example.

Numerical solution profiles. The exact and numerical solutions corresponding to different values of the perturbation parameter ϵ are displayed in Figure 3. Unlike Example 6.1, where particular

emphasis was placed on resolving the sharp boundary layer near $t = 0$, the present example focuses on the capability of the proposed numerical method to accurately capture nonlinear fading-memory effects throughout the computational interval. The numerical results demonstrate that the computed solutions remain stable and closely follow the exact solution for all tested values of ϵ . In particular, accurate numerical approximations are obtained even on relatively coarse meshes. The numerical solutions with $N = 16$ already exhibit excellent agreement with the exact solution, indicating the efficiency of the proposed numerical method for problems without boundary layer behavior.

Maximum nodal errors and convergence rates. The maximum nodal errors and the corresponding convergence orders are reported in Table 2 for several values of the perturbation parameter ϵ and mesh sizes N . The numerical results further confirm the robustness and stability of the proposed numerical method for nonlinear singularly perturbed Volterra integro-differential equations with fading-memory effects. In all tested cases, the maximum nodal errors decrease monotonically as the mesh is refined, confirming the convergence of the method. Furthermore, the observed convergence orders are generally close to first-order accuracy and remain consistent with the theoretical convergence analysis established in Section 5. The computed results therefore demonstrate that the proposed numerical method effectively handles nonlinear transformed terms and nonlinear memory interactions while maintaining stable and accurate numerical approximations.

TABLE 2. Maximum nodal errors and observed convergence orders for Example 6.2.

ϵ	N	E_ϵ^N	Order
2^{-2}	16	0.00318546	—
	32	0.00166865	0.93281612
	64	0.00085472	0.96515847
2^{-3}	16	0.00153743	—
	32	0.00088828	0.79142923
	64	0.00052543	0.75751501
2^{-4}	16	0.00119577	—
	32	0.00057867	1.04712739
	64	0.00026809	1.11002429

7. APPLICATION TO A SINGULARLY PERTURBED POPULATION MODEL WITH FADING MEMORY

In this section, we demonstrate the applicability of the proposed numerical method to a nonlinear population dynamics model involving hereditary effects. The model belongs to the class of singularly perturbed Volterra integro-differential equations considered throughout this work and illustrates the capability of the proposed method for handling nonlinear memory-dependent systems arising in applied problems. Consider the nonlinear singularly perturbed population

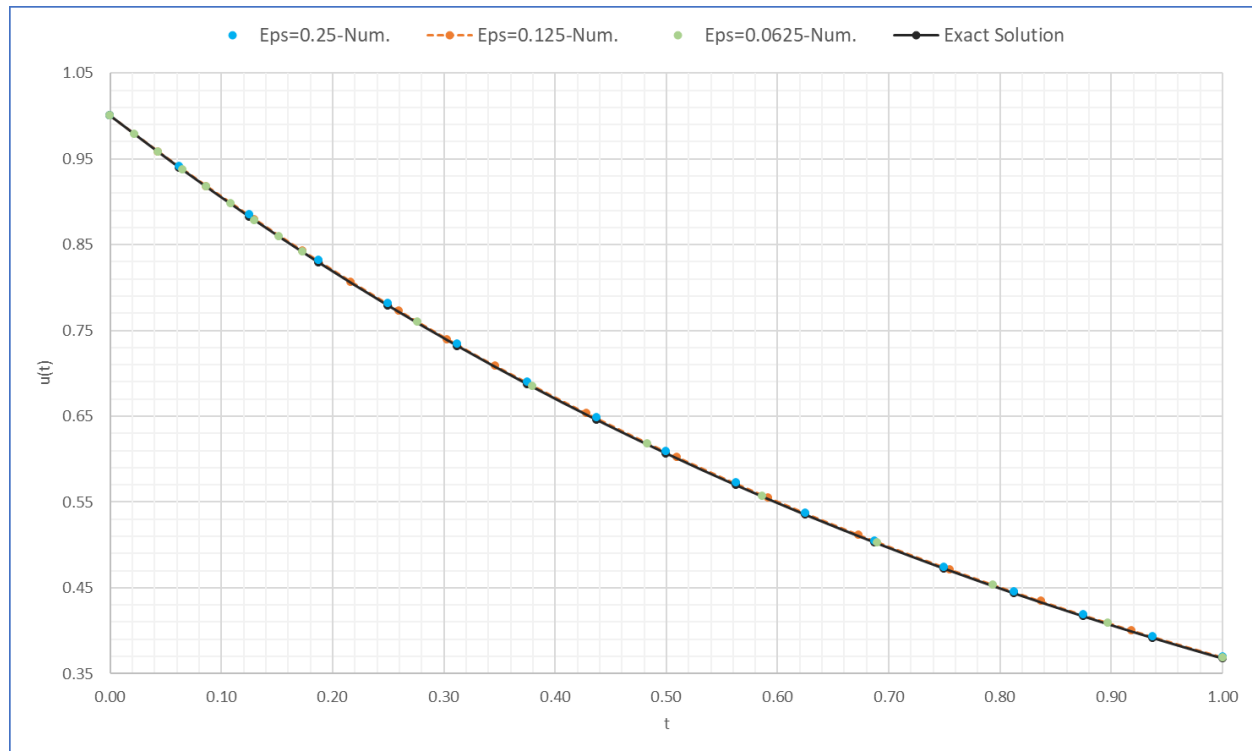


FIGURE 3. Exact and numerical solutions for Example 6.2 with $N = 16$ and different values of ϵ . The numerical solutions closely agree with the exact solutions without spurious oscillations.

model

$$\epsilon u'(t) + u(t) \left[\frac{r}{K} u(t) - r \right] + \alpha u(t) \int_0^t e^{-\beta(t-s)} u(s) ds = g(t), \quad 0 < t \leq 1, \quad (7.1)$$

subject to the initial condition $u(0) = u_0 > 0$, where $\epsilon > 0$ is a small perturbation parameter. Here, $u(t)$ denotes the population density at time t , while $r > 0$ and $K > 0$ represent the intrinsic growth rate and environmental carrying capacity, respectively. The nonlinear term

$$u(t) \left(\frac{r}{K} u(t) - r \right)$$

describes logistic-type self-regulated population growth. The integral term

$$\int_0^t e^{-\beta(t-s)} u(s) ds$$

introduces fading-memory effects, where recent population states exert stronger influence on the present dynamics than distant past states.

Test configuration. To investigate the numerical performance of the proposed method, we consider the parameter values $r = 2, K = 1, \alpha = 1, \beta = 1, u_0 = 2$. The forcing term is chosen such that the

exact solution is given by $u(t) = 1 + e^{-t/\epsilon}$. Since

$$\int_0^t e^{-(t-s)}(1 + e^{-s/\epsilon})ds = 1 - e^{-t} + \frac{\epsilon}{1-\epsilon}(e^{-t} - e^{-t/\epsilon}),$$

substitution of the exact solution into equation (7.1) yields

$$g(t) = -e^{-t/\epsilon} + 2(1 + e^{-t/\epsilon})^2 - 2(1 + e^{-t/\epsilon}) + (1 + e^{-t/\epsilon})\left[1 - e^{-t} + \frac{\epsilon}{1-\epsilon}(e^{-t} - e^{-t/\epsilon})\right].$$

The exact solution contains the exponential layer term $e^{-t/\epsilon}$, which generates a sharp initial layer near $t = 0$ for small values of ϵ .

Numerical results and discussion. The exact and numerical solutions corresponding to different values of the perturbation parameter ϵ are displayed in Figure 4. The figure shows that the proposed numerical method successfully captures the rapid initial transition near $t = 0$ while preserving stable numerical behavior throughout the computational interval. In particular, the numerical solutions remain in close agreement with the exact solutions for all tested values of ϵ , indicating that the proposed Shishkin-type mesh effectively resolves the boundary layer structure. The maximum nodal errors and the corresponding convergence orders are presented in Table 3 for several values of the perturbation parameter ϵ and mesh sizes N . The computed errors decrease monotonically as the mesh is refined, demonstrating the convergence and robustness of the proposed numerical method for nonlinear singularly perturbed population models with fading-memory effects. Although the observed convergence orders slightly deteriorate for smaller values of ϵ , the numerical results still indicate satisfactory performance of the method in resolving multiscale temporal behavior and nonlinear hereditary interactions. Overall, the numerical experiments confirm that the proposed scheme provides stable and reliable approximations for nonlinear Volterra integro-differential population models involving memory effects and initial boundary layers.

TABLE 3. Maximum nodal errors and observed convergence orders for the singularly perturbed population model with fading-memory effects.

ϵ	N	E_ϵ^N	Order
2^{-2}	16	0.01450161	—
	32	0.00766676	0.91952380
	64	0.00394513	0.95854320
2^{-3}	16	0.02058888	—
	32	0.01330029	0.63040692
	64	0.00788132	0.75494830
2^{-4}	16	0.02102185	—
	32	0.01354580	0.63404379
	64	0.00830964	0.70498755

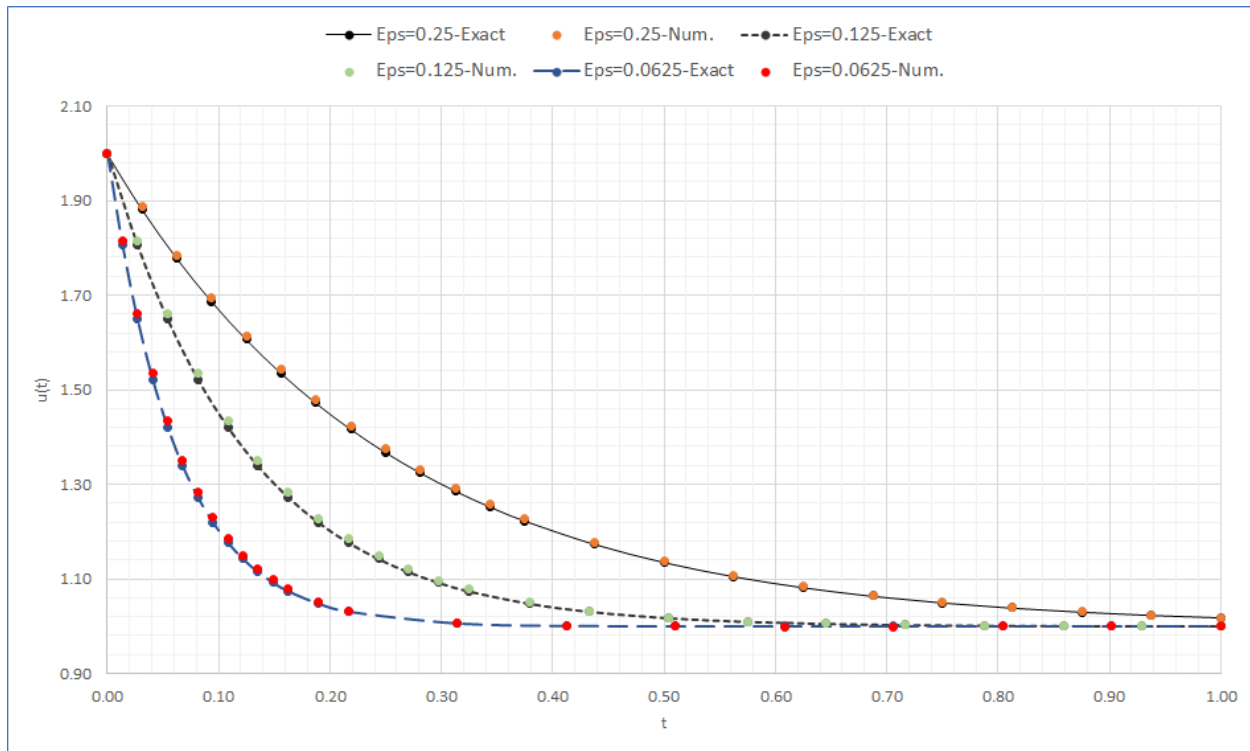


FIGURE 4. Exact and numerical solutions for the singularly perturbed population model with fading-memory effects for $N = 32$ and different values of the perturbation parameter ϵ .

From a broader perspective, the population model considered in this application is also relevant to the United Nations Sustainable Development Goals (SDGs), particularly SDG 15, which promotes the protection, restoration, and sustainable use of terrestrial ecosystems. The incorporation of environmental carrying capacity and fading-memory effects provides a mathematical framework for describing population dynamics under changing environmental conditions, where past population states may influence current behavior. Although the primary focus of this study is the theoretical analysis and numerical approximation of singularly perturbed nonlinear Volterra integro-differential equations, this application illustrates the potential of the proposed framework for supporting mathematical modeling of population and environmental processes relevant to sustainable ecosystem management.

8. CONCLUSION

In this paper, we have studied a class of nonlinear singularly perturbed Volterra integro-differential equations exhibiting initial boundary layer behavior due to the presence of a small perturbation parameter ϵ . The problem is characterized by multiscale dynamics, where rapid transient behavior occurs near the initial point and smooth behavior persists away from the layer region. A layer-adapted numerical method based on a Shishkin-type mesh was constructed to

efficiently resolve the boundary layer. The method combines a backward difference scheme for the differential term, a composite trapezoidal rule for the Volterra integral term, and Newton's method for the resulting nonlinear discrete system. A rigorous theoretical analysis was carried out. Stability of the discrete scheme was established in the maximum norm, and a convergence estimate was derived in the asymptotic regime where the transition parameter satisfies $\sigma = T/2$. In particular, the proposed method was shown to achieve an almost first-order rate of convergence. Furthermore, the application to a nonlinear population dynamics model with memory effects illustrated the applicability and robustness of the proposed framework in modeling real-world systems with hereditary interactions. The proposed methodology can be extended to higher-order schemes, adaptive mesh strategies, delay integro-differential systems, and multidimensional problems with memory effects.

An interesting direction for future research is the rigorous convergence analysis in the regime $\epsilon \ln N < T/2$, where the transition parameter is given by $\sigma = \epsilon \ln N$. Numerical results indicate that the boundary layer is less effectively resolved in this regime, leading to larger discretization errors. Extending the present analysis to this case would provide a more complete theoretical understanding of nonlinear singularly perturbed Volterra integro-differential equations on layer-adapted meshes.

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