

Neutrosophic Fuzzy Hyper h -Ideals in Hyper BCK-Algebras

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Abstract. In this paper, we introduce and investigate neutrosophic fuzzy hyper h -ideals in hyper BCK-algebras. The notions of neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper h -ideals are also defined and studied. We show that every neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper h -ideal is a corresponding neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper BCK-ideal, respectively. Moreover, we examine the relationships among these classes of ideals and provide characterizations in terms of level subsets. Finally, we study hyper homomorphic preimages and product structures of neutrosophic fuzzy hyper h -ideals. In particular, we prove that the product of two neutrosophic fuzzy hyper h -ideals is again a neutrosophic fuzzy hyper h -ideal.

1. INTRODUCTION

Set operations such as union, intersection, and set difference play important roles in algebra and are closely related to several algebraic structures, including Boolean algebras, distributive lattices, and semirings. While union and intersection have been widely studied in many algebraic systems,

Received: Jul. 1, 2026.

2020 *Mathematics Subject Classification.* 08A72.

Key words and phrases. Neutrosophic fuzzy hyper h -ideal, neutrosophic fuzzy weak hyper h -ideal, neutrosophic fuzzy strong hyper h -ideal, neutrosophic fuzzy reflexive hyper h -ideal.

set difference also gives rise to important algebraic structures. In 1966, Imai and Iséki [9] introduced BCK-algebras as algebraic systems related to BCK logic and the behavior of set difference. The corresponding implicational axiom schemes of BCK logic may be written as follows:

- Axiom B: $(C \supset G) \supset ((L \supset C) \supset (L \supset G))$;
- Axiom C: $(C \supset (G \supset L)) \supset (G \supset (C \supset L))$;
- Axiom K: $C \supset (G \supset C)$.

Here C , G , and L denote propositional formulae. Later, BCI-algebras were introduced as a generalization of BCK-algebras. These algebras have close connections with their corresponding logical systems and have been studied from both algebraic and logical points of view.

Hyperstructure theory was introduced by Marty [16] in 1934. In a hyperstructure, the product of two elements is a nonempty subset rather than a single element. This idea provides a natural extension of ordinary algebraic operations and has been applied to several areas of pure and applied mathematics. Imai and Iséki [9] introduced BCK-algebras, and Jun et al. [10, 11] later applied hyperstructures to BCK-algebras and introduced hyper BCK-algebras. Borzooei et al. [7] applied related ideas to K -algebras. Jun et al. [12] studied fuzzy strong $\mathcal{HBCKI}$ s. Fuzzy set theory, introduced by Zadeh [22], is a useful tool for describing uncertainty. Jun and Xin [13] applied fuzzy sets to hyper BCK-algebras. Touqeer and Çağman [19] introduced fuzzy hyper h -ideals in hyper BCK-algebras. Khademan et al. [15] introduced fuzzy soft positive implicative $\mathcal{HBCKI}$ s. Smarandache [18] introduced neutrosophic sets as a generalization of fuzzy-type structures. Anjaneyulu Naik et al. [3, 4] studied intuitionistic fuzzy hyper h -ideals and p -ideals in hyper BCK-algebras. Anjum et al. [21] studied ordered h -regular semirings. Bhattacharya and Mukherjee [5] introduced fuzzy relations and fuzzy groups.

Several variations of neutrosophic structures have been introduced to describe uncertainty and indeterminacy in algebraic systems. Borzooei et al. [8] studied a new concept of polarity of generalized neutrosophic subalgebras in BCK/BCI-algebras. Alsubie and Masarwah [1] introduced MBI-neutrosophic $\mathcal{HBCKI}$ s in hyper BCK-algebras by using interval-valued membership functions. Kaviyarasu et al. [14] recently studied fermatean neutrosophic soft structures and their ideals in hyper BCK-algebras. Satyanarayana and Anjaneyulu Naik et al. [2, 6, 17, 20] explored related classes of neutrosophic fuzzy H -ideals in BCK-algebras. In general, a neutrosophic fuzzy set is described by truth-membership, indeterminacy-membership, and falsity-membership functions. This framework is suitable for studying algebraic structures in which uncertainty and indeterminacy are considered simultaneously.

Motivated by the above studies, this paper introduces the concept of neutrosophic fuzzy hyper h -ideals in hyper BCK-algebras. We also define neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper h -ideals and investigate their basic properties. We show that every neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper h -ideal is a corresponding neutrosophic fuzzy

weak, ordinary, strong, and reflexive hyper BCK-ideal, respectively. Moreover, we examine the relationships among these classes of ideals, study their characterizations by level subsets, and discuss product structures, strongest neutrosophic fuzzy relations, and hyper homomorphic preimages.

ACRONYMS

- Hyper BCK-algebra: $\mathcal{HBCKA}$;
- Hyper BCK-ideal: $\mathcal{HBCKI}$;
- Neutrosophic fuzzy set: $\mathcal{NFS}$;
- Hyper h -ideal: $\mathcal{HHI}$;
- Neutrosophic fuzzy hyper h -ideal: $\mathcal{NFHHI}$;
- Neutrosophic fuzzy weak hyper h -ideal: $\mathcal{NFWHHI}$;
- Neutrosophic fuzzy strong hyper h -ideal: $\mathcal{NFSHHI}$;
- Neutrosophic fuzzy reflexive hyper h -ideal: $\mathcal{NFRHHI}$;
- Neutrosophic fuzzy weak, strong, and reflexive hyper BCK-ideal: $\mathcal{NF}(W, S, R)\mathcal{HBCKI}$.

2. PRELIMINARIES

In this section, we recall some basic notions and results that are needed in this paper.

Let $\mathcal{K}$ be a nonempty set endowed with a hyperoperation $\star$, that is, $\star$ is a mapping from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{P}^*(\mathcal{K})$, where $\mathcal{P}^*(\mathcal{K}) = \mathcal{P}(\mathcal{K}) \setminus \{\emptyset\}$. For any nonempty subsets $\mathfrak{C}$ and $\mathfrak{G}$ of $\mathcal{K}$, we define

$$\mathfrak{C} \star \mathfrak{G} = \bigcup_{m \in \mathfrak{C}, n \in \mathfrak{G}} (m \star n).$$

For simplicity, we write $\check{f} \star \check{g}$ instead of $\{\check{f}\} \star \{\check{g}\}$, $\check{f} \star \{\check{g}\}$, or $\{\check{f}\} \star \check{g}$ whenever no confusion can occur.

The relation $\ll$ on $\mathcal{K}$ is defined by

$$\check{f} \ll \check{g} \quad \text{if and only if} \quad 0 \in \check{f} \star \check{g}.$$

For nonempty subsets $\mathfrak{C}$ and $\mathfrak{G}$ of $\mathcal{K}$, we write $\mathfrak{C} \ll \mathfrak{G}$ if, for every $m \in \mathfrak{C}$, there exists $n \in \mathfrak{G}$ such that $m \ll n$. The relation $\ll$ is called the hyperorder on $\mathcal{K}$.

Definition 2.1. Following [11], a hyper BCK-algebra, briefly an $\mathcal{HBCKA}$, is a triple $(\mathcal{K}, \star, 0)$, where $\mathcal{K}$ is a nonempty set, $\star$ is a hyperoperation on $\mathcal{K}$, and 0 is a constant of $\mathcal{K}$ satisfying the following axioms:

- ($\mathcal{HBCKA1}$) $\check{f} \star (\check{g} \star \check{h}) \ll \check{f} \star \check{g}$;
 - ($\mathcal{HBCKA2}$) $(\check{f} \star \check{g}) \star \check{h} = (\check{f} \star \check{h}) \star \check{g}$;
 - ($\mathcal{HBCKA3}$) $\check{f} \star \mathcal{K} \ll \{\check{f}\}$;
 - ($\mathcal{HBCKA4}$) $\check{f} \ll \check{g}$ and $\check{g} \ll \check{f}$ imply $\check{f} = \check{g}$,
- for all $\check{f}, \check{g}, \check{h} \in \mathcal{K}$.

Clearly, every BCK-algebra $(\mathcal{K}, \circ, 0)$ becomes an $\mathcal{HBCKA}$ under the hyperoperation defined by

$$\check{f} \star \check{g} = \{\check{f} \circ \check{g}\}$$

for all $\check{f}, \check{g} \in \mathcal{K}$.

A standard example of an $\mathcal{HBCKA}$ is the set of nonnegative real numbers $\mathcal{K} = [0, \infty)$ with the hyperoperation $\star$ defined by

$$\check{f} \star \check{g} := \begin{cases} [0, \check{f}], & \text{if } \check{f} < \check{g}, \\ [0, \check{g}], & \text{if } \check{f} \geq \check{g} \neq 0, \\ \{\check{f}\}, & \text{if } \check{g} = 0. \end{cases}$$

Proposition 2.1 ([11]). *In every $\mathcal{HBCKA}$ $\mathcal{K}$, the following properties hold:*

- (P₁) $\check{f} \star \check{g} \ll \{\check{f}\}$;
- (P₂) $\check{f} \star 0 \ll \{\check{f}\}$;
- (P₃) $0 \star 0 = \{0\}$;
- (P₄) $0 \ll \check{f}$;
- (P₅) $\mathfrak{C} \ll \mathfrak{G}$;
- (P₆) $\mathfrak{C} \subseteq \mathfrak{G}$ implies $\mathfrak{C} \ll \mathfrak{G}$;
- (P₇) $0 \star \check{f} = \{0\}$;
- (P₈) $0 \star \mathfrak{C} = \{0\}$;
- (P₉) $\mathfrak{C} \star \{0\} = \{0\}$ implies $\mathfrak{C} = \{0\}$;
- (P₁₀) $\check{g} \ll \check{h}$ implies $\check{f} \star \check{h} \ll \check{f} \star \check{g}$,

for all $\check{f}, \check{g}, \check{h} \in \mathcal{K}$ and for any nonempty subsets $\mathfrak{C}$ and $\mathfrak{G}$ of $\mathcal{K}$.

For the basic theory of hyper BCK-subalgebras and weak, strong, and reflexive $\mathcal{HBCKI}$ s, we refer to [11]. From now on, $\mathcal{K}$ will denote an $\mathcal{HBCKA}$ unless otherwise stated.

Lemma 2.1 ([11, 12]). *The following statements hold:*

- (i) Every $\mathcal{HBCKI}$ of $\mathcal{K}$ is a weak $\mathcal{HBCKI}$ of $\mathcal{K}$.
- (ii) Every reflexive $\mathcal{HBCKI}$ of $\mathcal{K}$ is a strong $\mathcal{HBCKI}$ of $\mathcal{K}$.
- (iii) Every strong $\mathcal{HBCKI}$ of $\mathcal{K}$ is an $\mathcal{HBCKI}$ of $\mathcal{K}$.

Lemma 2.2 ([12]). *Let $\mathfrak{S}$ be a reflexive $\mathcal{HBCKI}$ of $\mathcal{K}$. If $\check{f} \star \check{g} \cap \mathfrak{S} \neq \emptyset$, then $\check{f} \star \check{g} \ll \mathfrak{S}$ for all $\check{f}, \check{g} \in \mathcal{K}$.*

Proposition 2.2 ([10]). *Let $\mathfrak{C}$ be a subset of $\mathcal{K}$ and let $\mathfrak{S}$ be an $\mathcal{HBCKI}$ of $\mathcal{K}$. If $\mathfrak{C} \ll \mathfrak{S}$, then $\mathfrak{C} \subseteq \mathfrak{S}$.*

Definition 2.2. *Let $\mathfrak{S}$ be a nonempty subset of $\mathcal{K}$ containing 0. Following [19], $\mathfrak{S}$ is called:*

- (i) a weak $\mathcal{HHI}$ of $\mathcal{K}$ if, for all $\check{f}, \check{g}, \check{h} \in \mathcal{K}$,

$$\check{f} \star (\check{g} \star \check{h}) \subseteq \mathfrak{S} \text{ and } \check{g} \in \mathfrak{S} \text{ imply } \check{f} \star \check{h} \subseteq \mathfrak{S};$$

- (ii) an $\mathcal{HHI}$ of $\mathcal{K}$ if, for all $\check{f}, \check{g}, \check{h} \in \mathcal{K}$,

$$\check{f} \star (\check{g} \star \check{h}) \ll \mathfrak{S} \text{ and } \check{g} \in \mathfrak{S} \text{ imply } \check{f} \star \check{h} \subseteq \mathfrak{S};$$

- (iii) a strong $\mathcal{HHI}$ of $\mathcal{K}$ if, for all $\check{f}, \check{g}, \check{h} \in \mathcal{K}$,

$$\check{f} \star (\check{g} \star \check{h}) \cap \mathfrak{S} \neq \emptyset \text{ and } \check{g} \in \mathfrak{S} \text{ imply } \check{f} \star \check{h} \subseteq \mathfrak{S};$$

(iv) a reflexive $\mathcal{HHI}$ of $\mathcal{K}$ if $\mathfrak{S}$ is a strong $\mathcal{HHI}$ of $\mathcal{K}$ and

$$\check{f} \star \check{f} \subseteq \mathfrak{S}$$

for all $\check{f} \in \mathcal{K}$.

Theorem 2.1 ([19]). Every weak, strong, and reflexive $\mathcal{HHI}$ of $\mathcal{K}$ is a corresponding weak, strong, and reflexive $\mathcal{HBCKI}$ of $\mathcal{K}$, respectively.

Theorem 2.2 ([19]). For any $\mathcal{HBCKA}$ $\mathcal{K}$, the following statements hold:

- (i) Every $\mathcal{HHI}$ of $\mathcal{K}$ is a weak $\mathcal{HHI}$ of $\mathcal{K}$.
- (ii) Every strong $\mathcal{HHI}$ of $\mathcal{K}$ is an $\mathcal{HHI}$ of $\mathcal{K}$.
- (iii) Every reflexive $\mathcal{HHI}$ of $\mathcal{K}$ is a strong $\mathcal{HHI}$ of $\mathcal{K}$.

Remark. In general, the converses of the implications in Theorem 2.2 do not hold; see [19].

For a detailed study of fuzzy weak, strong, and reflexive $\mathcal{HBCKI}$ s, we refer to [13].

Theorem 2.3 ([13]). For any $\mathcal{HBCKA}$ $\mathcal{K}$, the following statements hold:

- (i) Every fuzzy $\mathcal{HBCKI}$ of $\mathcal{K}$ is a fuzzy weak $\mathcal{HBCKI}$ of $\mathcal{K}$.
- (ii) Every fuzzy strong $\mathcal{HBCKI}$ of $\mathcal{K}$ is a fuzzy $\mathcal{HBCKI}$ of $\mathcal{K}$.
- (iii) Every fuzzy reflexive $\mathcal{HBCKI}$ of $\mathcal{K}$ is a fuzzy strong $\mathcal{HBCKI}$ of $\mathcal{K}$.

Definition 2.3. Let $(\mathcal{K}_1, \star_1, 0_1)$ and $(\mathcal{K}_2, \star_2, 0_2)$ be $\mathcal{HBCKAs}$ and let $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$. For subsets $F \subseteq \mathcal{K}_1$ and $G \subseteq \mathcal{K}_2$, put

$$(F, G) = \{(\check{f}, \check{g}) \mid \check{f} \in F, \check{g} \in G\}.$$

Define a hyperoperation $\star$ on $\mathcal{K}$ by

$$(\check{f}, \check{g}) \star (\check{u}, \check{v}) = (\check{f} \star_1 \check{u}, \check{g} \star_2 \check{v})$$

for all $(\check{f}, \check{g}), (\check{u}, \check{v}) \in \mathcal{K}$, where the right-hand side is understood in the above sense for subsets. Let $0 = (0_1, 0_2)$, and define a hyperorder $\ll$ on $\mathcal{K}$ by

$$(\check{f}, \check{g}) \ll (\check{u}, \check{v}) \quad \text{if and only if} \quad \check{f} \ll \check{u} \text{ and } \check{g} \ll \check{v}.$$

Then $(\mathcal{K}, \star, 0)$ is an $\mathcal{HBCKA}$.

Let ξ and ω be fuzzy sets on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. The product $\xi \times \omega$ is the fuzzy set on $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$ defined by

$$(\xi \times \omega)(\check{f}, \check{g}) = \min\{\xi(\check{f}), \omega(\check{g})\}.$$

In the sequel, $\mathcal{K}_1$ and $\mathcal{K}_2$ will denote $\mathcal{HBCKAs}$ and $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$ [7].

3. BASIC PROPERTIES AND CHARACTERIZATIONS OF NEUTROSOPHIC FUZZY HYPER h -IDEALS

In this section, we introduce neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper h -ideals in hyper BCK-algebras and investigate their basic properties.

Definition 3.1. Let $\mathcal{K}$ be an $\mathcal{HBCKA}$. A neutrosophic fuzzy set, briefly an $\mathcal{NFS}$, in $\mathcal{K}$ is a triple $\mathcal{M} = (\xi, \zeta, \omega)$, where $\xi, \zeta, \omega : \mathcal{K} \rightarrow [0, 1]$.

(i) The $\mathcal{NFS} \mathcal{M} = (\xi, \zeta, \omega)$ is called a neutrosophic fuzzy weak hyper h -ideal, briefly an $\mathcal{NFWHHI}$, of $\mathcal{K}$ if

$$\xi(0) \geq \xi(\check{f}), \quad \zeta(0) \geq \zeta(\check{f}), \quad \omega(0) \leq \omega(\check{f})$$

for all $\check{f} \in \mathcal{K}$, and, for all $\check{f}, \check{g}, \check{h} \in \mathcal{K}$ and all $\check{u} \in \check{f} \star \check{h}$,

$$\xi(\check{u}) \geq \min\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}), \xi(\check{g})\right\},$$

$$\zeta(\check{u}) \geq \min\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \zeta(\mathfrak{m}), \zeta(\check{g})\right\},$$

$$\omega(\check{u}) \leq \max\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega(\mathfrak{m}), \omega(\check{g})\right\}.$$

(ii) The $\mathcal{NFS} \mathcal{M} = (\xi, \zeta, \omega)$ is called a neutrosophic fuzzy hyper h -ideal, briefly an $\mathcal{NFHHI}$, of $\mathcal{K}$ if

$$\check{f} \ll \check{g} \text{ implies } \xi(\check{f}) \geq \xi(\check{g}), \quad \zeta(\check{f}) \geq \zeta(\check{g}), \quad \omega(\check{f}) \leq \omega(\check{g})$$

for all $\check{f}, \check{g} \in \mathcal{K}$, and, for all $\check{f}, \check{g}, \check{h} \in \mathcal{K}$ and all $\check{u} \in \check{f} \star \check{h}$,

$$\xi(\check{u}) \geq \min\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}), \xi(\check{g})\right\},$$

$$\zeta(\check{u}) \geq \min\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \zeta(\mathfrak{m}), \zeta(\check{g})\right\},$$

$$\omega(\check{u}) \leq \max\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega(\mathfrak{m}), \omega(\check{g})\right\}.$$

(iii) The $\mathcal{NFS} \mathcal{M} = (\xi, \zeta, \omega)$ is called a neutrosophic fuzzy strong hyper h -ideal, briefly an $\mathcal{NFSHHI}$, of $\mathcal{K}$ if

$$\inf_{\mathfrak{m} \in \check{f} \star \check{f}} \xi(\mathfrak{m}) \geq \xi(\check{f}), \quad \inf_{\mathfrak{m} \in \check{f} \star \check{f}} \zeta(\mathfrak{m}) \geq \zeta(\check{f}), \quad \sup_{\mathfrak{m} \in \check{f} \star \check{f}} \omega(\mathfrak{m}) \leq \omega(\check{f})$$

for all $\check{f} \in \mathcal{K}$, and, for all $\check{f}, \check{g}, \check{h} \in \mathcal{K}$ and all $\check{u} \in \check{f} \star \check{h}$,

$$\xi(\check{u}) \geq \min\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}), \xi(\check{g})\right\},$$

$$\zeta(\check{u}) \geq \min\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \zeta(\mathfrak{m}), \zeta(\check{g})\right\},$$

$$\omega(\check{u}) \leq \max\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega(\mathfrak{m}), \omega(\check{g})\right\}.$$

(iv) The $\mathcal{NFS} \mathcal{M} = (\xi, \zeta, \omega)$ is called a neutrosophic fuzzy reflexive hyper h -ideal, briefly an $\mathcal{NFRHHI}$, of $\mathcal{K}$ if

$$\inf_{\mathfrak{m} \in \check{f} \star \check{f}} \xi(\mathfrak{m}) \geq \xi(\check{g}), \quad \inf_{\mathfrak{m} \in \check{f} \star \check{f}} \zeta(\mathfrak{m}) \geq \zeta(\check{g}), \quad \sup_{\mathfrak{m} \in \check{f} \star \check{f}} \omega(\mathfrak{m}) \leq \omega(\check{g})$$

for all $\check{f}, \check{g} \in \mathcal{K}$, and, for all $\check{f}, \check{g}, \check{h} \in \mathcal{K}$ and all $\check{u} \in \check{f} \star \check{h}$,

$$\begin{aligned}\xi(\check{u}) &\geq \min\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}), \xi(\check{g})\right\}, \\ \zeta(\check{u}) &\geq \min\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \zeta(\mathfrak{m}), \zeta(\check{g})\right\}, \\ \omega(\check{u}) &\leq \max\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega(\mathfrak{m}), \omega(\check{g})\right\}.\end{aligned}$$

Theorem 3.1. Every neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper h -ideal of $\mathcal{K}$ is a corresponding neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper BCK-ideal of $\mathcal{K}$, respectively.

Proof. This follows from Theorem 2.1. Indeed, every weak, ordinary, strong, and reflexive hyper h -ideal of $\mathcal{K}$ is a corresponding weak, ordinary, strong, and reflexive hyper BCK-ideal of $\mathcal{K}$, respectively. Therefore, the defining conditions of a neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper h -ideal imply the corresponding defining conditions of a neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper BCK-ideal. $\square$

Theorem 3.2. For any $\mathcal{HBCKA}$ $\mathcal{K}$, the following statements hold.

- (i) Every $\mathcal{NFHHI}$ of $\mathcal{K}$ is an $\mathcal{NFWHHI}$ of $\mathcal{K}$.
- (ii) Every $\mathcal{NFSHHI}$ of $\mathcal{K}$ is an $\mathcal{NFHHI}$ of $\mathcal{K}$.
- (iii) Every $\mathcal{NFRHHI}$ of $\mathcal{K}$ is an $\mathcal{NFSHHI}$ of $\mathcal{K}$.

Proof. (i) Let $\mathcal{M} = (\xi, \zeta, \omega)$ be an $\mathcal{NFHHI}$ of $\mathcal{K}$. By Proposition 2.1(P_4), we have $0 \ll \check{f}$ for all $\check{f} \in \mathcal{K}$. Hence, by Definition 3.1(ii),

$$\xi(0) \geq \xi(\check{f}), \quad \zeta(0) \geq \zeta(\check{f}), \quad \omega(0) \leq \omega(\check{f})$$

for all $\check{f} \in \mathcal{K}$. The remaining inequalities required in Definition 3.1(i) are exactly the same as those in Definition 3.1(ii). Therefore, $\mathcal{M}$ is an $\mathcal{NFWHHI}$ of $\mathcal{K}$.

(ii) Let $\mathcal{M} = (\xi, \zeta, \omega)$ be an $\mathcal{NFSHHI}$ of $\mathcal{K}$. By Theorem 3.1, $\mathcal{M}$ is a neutrosophic fuzzy strong hyper BCK-ideal of $\mathcal{K}$. Hence, by Theorem 2.3(ii), $\mathcal{M}$ is a neutrosophic fuzzy hyper BCK-ideal of $\mathcal{K}$. Therefore, the order condition in Definition 3.1(ii) is satisfied.

Let $\check{f}, \check{g}, \check{h} \in \mathcal{K}$ and let $\check{u} \in \check{f} \star \check{h}$. Since $\mathcal{M}$ is an $\mathcal{NFSHHI}$, we have

$$\xi(\check{u}) \geq \min\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}), \xi(\check{g})\right\}.$$

Since

$$\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}) \leq \sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}),$$

we get

$$\xi(\check{u}) \geq \min\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}), \xi(\check{g})\right\}.$$

Similarly,

$$\zeta(\check{u}) \geq \min\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \zeta(\mathfrak{m}), \zeta(\check{g})\right\}.$$

Moreover,

$$\max\{\inf_{m \in \check{f} \star (\check{g} \star \check{h})} \omega(m), \omega(\check{g})\} \leq \max\{\sup_{m \in \check{f} \star (\check{g} \star \check{h})} \omega(m), \omega(\check{g})\}.$$

Thus

$$\omega(\check{u}) \leq \max\{\sup_{m \in \check{f} \star (\check{g} \star \check{h})} \omega(m), \omega(\check{g})\}.$$

Therefore, $\mathcal{M}$ is an $\mathcal{NFHHI}$ of $\mathcal{K}$.

(iii) Let $\mathcal{M} = (\xi, \zeta, \omega)$ be an $\mathcal{NFRHHI}$ of $\mathcal{K}$. By Definition 3.1(iv), putting $\check{g} = \check{f}$ gives

$$\inf_{m \in \check{f} \star \check{f}} \xi(m) \geq \xi(\check{f}), \quad \inf_{m \in \check{f} \star \check{f}} \zeta(m) \geq \zeta(\check{f}), \quad \sup_{m \in \check{f} \star \check{f}} \omega(m) \leq \omega(\check{f})$$

for all $\check{f} \in \mathcal{K}$. The remaining inequalities in Definition 3.1(iii) are the same as those in Definition 3.1(iv). Hence $\mathcal{M}$ is an $\mathcal{NFSHHI}$ of $\mathcal{K}$. $\square$

Remark. In general, none of the converses of the implications in Theorem 3.2 holds, as shown in the following examples.

Example 3.1. Let $\mathcal{K} = \{0, f, g\}$ be an $\mathcal{HBCKA}$ with the hyperoperation $\star$ given by the following Cayley table:

$\star$	0	f	g
0	{0}	{0}	{0}
f	{f}	{0}	{0}
g	{g}	{f, g}	{0, f, g}

Define an $\mathcal{NFS}$ $\mathcal{M} = (\xi, \zeta, \omega)$ in $\mathcal{K}$ by

$$\xi(0) = \xi(g) = 0.6, \quad \xi(f) = 0.03,$$

$$\zeta(0) = \zeta(g) = 0.25, \quad \zeta(f) = 0.01,$$

$$\omega(0) = \omega(g) = 0.2, \quad \omega(f) = 0.4.$$

It is straightforward to check that $\mathcal{M}$ is an $\mathcal{NFWHHI}$ of $\mathcal{K}$. However, $\mathcal{M}$ is not an $\mathcal{NFHHI}$ of $\mathcal{K}$. Indeed, since $f \star g = \{0\}$, we have $f \ll g$. If $\mathcal{M}$ were an $\mathcal{NFHHI}$ of $\mathcal{K}$, then the order condition in Definition 3.1(ii) would imply

$$\xi(f) \geq \xi(g), \quad \zeta(f) \geq \zeta(g), \quad \omega(f) \leq \omega(g).$$

But

$$\xi(f) = 0.03 < 0.6 = \xi(g),$$

$$\zeta(f) = 0.01 < 0.25 = \zeta(g),$$

$$\omega(f) = 0.4 > 0.2 = \omega(g).$$

This contradiction shows that $\mathcal{M}$ is not an $\mathcal{NFHHI}$ of $\mathcal{K}$.

Example 3.2. Let $\mathcal{K} = \{0, f, g\}$ be an $\mathcal{HBCKA}$ with the hyperoperation $\star$ given by the following Cayley table:

$\star$	0	f	g
0	$\{0\}$	$\{0\}$	$\{0\}$
f	$\{f\}$	$\{0, f\}$	$\{0, f\}$
g	$\{g\}$	$\{f, g\}$	$\{0, f, g\}$

Define an $\mathcal{NFS}$ $\mathcal{M} = (\xi, \zeta, \omega)$ in $\mathcal{K}$ by

$$\xi(0) = \xi(f) = 0.4, \quad \xi(g) = 0.04,$$

$$\zeta(0) = \zeta(f) = 0.3, \quad \zeta(g) = 0.11,$$

$$\omega(0) = \omega(f) = 0.2, \quad \omega(g) = 0.5.$$

It is straightforward to check that $\mathcal{M}$ is an $\mathcal{NFHHI}$ of $\mathcal{K}$. However, $\mathcal{M}$ is not an $\mathcal{NFSHHI}$ of $\mathcal{K}$. Indeed, take $\check{f} = g$, $\check{g} = f$, and $\check{h} = g$. Since $g \in g \star g$ and $f \star g = \{0, f\}$, we obtain

$$g \star (f \star g) = g \star \{0, f\} = (g \star 0) \cup (g \star f) = \{g\} \cup \{f, g\} = \{f, g\}.$$

Thus

$$\sup_{m \in g \star (f \star g)} \xi(m) = 0.4$$

and hence

$$\min\left\{\sup_{m \in g \star (f \star g)} \xi(m), \xi(f)\right\} = 0.4.$$

But

$$\xi(g) = 0.04 < 0.4.$$

Similarly,

$$\min\left\{\sup_{m \in g \star (f \star g)} \zeta(m), \zeta(f)\right\} = 0.3$$

and

$$\zeta(g) = 0.11 < 0.3.$$

Moreover,

$$\inf_{m \in g \star (f \star g)} \omega(m) = 0.2$$

and so

$$\max\left\{\inf_{m \in g \star (f \star g)} \omega(m), \omega(f)\right\} = 0.2.$$

But

$$\omega(g) = 0.5 > 0.2.$$

Therefore, the defining condition of an $\mathcal{NFSHHI}$ is not satisfied. Hence $\mathcal{M}$ is not an $\mathcal{NFSHHI}$ of $\mathcal{K}$.

Example 3.3. Let $\mathcal{K} = \{0, f, g\}$ be the $\mathcal{HBCKA}$ defined in Example 3.2. Define an $\mathcal{NFSM} = (\xi, \zeta, \omega)$ in $\mathcal{K}$ by

$$\begin{aligned}\xi(0) &= 0.5, & \xi(f) &= \xi(g) = 0.03, \\ \zeta(0) &= 0.2, & \zeta(f) &= \zeta(g) = 0.04, \\ \omega(0) &= 0, & \omega(f) &= \omega(g) = 0.6.\end{aligned}$$

It is straightforward to check that $\mathcal{M}$ is an $\mathcal{NFSHHI}$ of $\mathcal{K}$. However, $\mathcal{M}$ is not an $\mathcal{NFRHHI}$ of $\mathcal{K}$. Indeed, since

$$g \star g = \{0, f, g\},$$

we have

$$\begin{aligned}\inf_{m \in g \star g} \xi(m) &= 0.03 < 0.5 = \xi(0), \\ \inf_{m \in g \star g} \zeta(m) &= 0.04 < 0.2 = \zeta(0), \\ \sup_{m \in g \star g} \omega(m) &= 0.6 > 0 = \omega(0).\end{aligned}$$

Thus, the reflexive condition in Definition 3.1(iv) is not satisfied. Therefore, $\mathcal{M}$ is not an $\mathcal{NFRHHI}$ of $\mathcal{K}$.

For $s, t, v \in [0, 1]$, define the level subsets of an $\mathcal{NFSM} = (\xi, \zeta, \omega)$ in $\mathcal{K}$ by

$$\begin{aligned}\xi_s &= \{f \in \mathcal{K} \mid \xi(f) \geq s\}, \\ \zeta_t &= \{f \in \mathcal{K} \mid \zeta(f) \geq t\}, \\ \omega_v &= \{f \in \mathcal{K} \mid \omega(f) \leq v\}.\end{aligned}$$

Theorem 3.3. Let $\mathcal{M} = (\xi, \zeta, \omega)$ be an $\mathcal{NFS}$ in $\mathcal{K}$. Then $\mathcal{M}$ is an $\mathcal{NFHHI}$ of $\mathcal{K}$ if and only if, for every $s, t, v \in [0, 1]$, the nonempty level subsets ξ_s , ζ_t , and ω_v are $\mathcal{HHI}$ s of $\mathcal{K}$.

Proof. Assume that $\mathcal{M} = (\xi, \zeta, \omega)$ is an $\mathcal{NFHHI}$ of $\mathcal{K}$.

First, let $\xi_s \neq \emptyset$ and choose $\check{a} \in \xi_s$. Since $0 \ll \check{a}$, Definition 3.1(ii) gives

$$\xi(0) \geq \xi(\check{a}) \geq s.$$

Hence $0 \in \xi_s$. Now suppose that

$$\check{f} \star (\check{g} \star \check{h}) \ll \xi_s \quad \text{and} \quad \check{g} \in \xi_s.$$

For each $m \in \check{f} \star (\check{g} \star \check{h})$, there exists $n \in \xi_s$ such that $m \ll n$. By Definition 3.1(ii), we get

$$\xi(m) \geq \xi(n) \geq s.$$

Thus

$$\inf_{m \in \check{f} \star (\check{g} \star \check{h})} \xi(m) \geq s.$$

Since $\check{g} \in \xi_s$, we also have $\xi(\check{g}) \geq s$. Hence, for all $\check{u} \in \check{f} \star \check{h}$,

$$\xi(\check{u}) \geq \min\left\{\inf_{m \in \check{f} \star (\check{g} \star \check{h})} \xi(m), \xi(\check{g})\right\} \geq s.$$

Therefore, $\check{f} \star \check{h} \subseteq \xi_s$, and so ξ_s is an $\mathcal{HHI}$ of $\mathcal{K}$.

By the same argument, every nonempty level subset ζ_t is an $\mathcal{HHI}$ of $\mathcal{K}$.

Next, let $\omega_v \neq \emptyset$ and choose $\check{a} \in \omega_v$. Since $0 \ll \check{a}$, Definition 3.1(ii) gives

$$\omega(0) \leq \omega(\check{a}) \leq v.$$

Hence $0 \in \omega_v$. Suppose that

$$\check{f} \star (\check{g} \star \check{h}) \ll \omega_v \quad \text{and} \quad \check{g} \in \omega_v.$$

For each $m \in \check{f} \star (\check{g} \star \check{h})$, there exists $n \in \omega_v$ such that $m \ll n$. By Definition 3.1(ii), we get

$$\omega(m) \leq \omega(n) \leq v.$$

Thus

$$\sup_{m \in \check{f} \star (\check{g} \star \check{h})} \omega(m) \leq v.$$

Since $\check{g} \in \omega_v$, we also have $\omega(\check{g}) \leq v$. Hence, for all $\check{u} \in \check{f} \star \check{h}$,

$$\omega(\check{u}) \leq \max\left\{\sup_{m \in \check{f} \star (\check{g} \star \check{h})} \omega(m), \omega(\check{g})\right\} \leq v.$$

Therefore, $\check{f} \star \check{h} \subseteq \omega_v$, and so ω_v is an $\mathcal{HHI}$ of $\mathcal{K}$.

Conversely, assume that every nonempty level subset ξ_s , ζ_t , and ω_v is an $\mathcal{HHI}$ of $\mathcal{K}$. Let $\check{f} \ll \check{g}$. Put $s = \xi(\check{g})$. Then $\check{g} \in \xi_s$, and hence $\xi_s \neq \emptyset$. Since ξ_s is an $\mathcal{HHI}$ of $\mathcal{K}$, it is an $\mathcal{HBCKI}$ of $\mathcal{K}$ by Theorem 2.1. Since $\check{f} \ll \check{g}$ and $\check{g} \in \xi_s$, we have $\{\check{f}\} \ll \xi_s$. By Proposition 2.2, $\check{f} \in \xi_s$. Therefore,

$$\xi(\check{f}) \geq s = \xi(\check{g}).$$

Similarly, putting $t = \zeta(\check{g})$ and $v = \omega(\check{g})$, we obtain

$$\zeta(\check{f}) \geq \zeta(\check{g}) \quad \text{and} \quad \omega(\check{f}) \leq \omega(\check{g}).$$

It remains to prove the hyper h -ideal inequalities. Let $\check{f}, \check{g}, \check{h} \in \mathcal{K}$. Put

$$s_0 = \min\left\{\inf_{m \in \check{f} \star (\check{g} \star \check{h})} \xi(m), \xi(\check{g})\right\}.$$

Then $\check{g} \in \xi_{s_0}$ and

$$\check{f} \star (\check{g} \star \check{h}) \subseteq \xi_{s_0}.$$

By Proposition 2.1(P_6),

$$\check{f} \star (\check{g} \star \check{h}) \ll \xi_{s_0}.$$

Since ξ_{s_0} is an $\mathcal{HHI}$ of $\mathcal{K}$, we get

$$\check{f} \star \check{h} \subseteq \xi_{s_0}.$$

Thus, for all $\check{u} \in \check{f} \star \check{h}$,

$$\xi(\check{u}) \geq \min\left\{\inf_{m \in \check{f} \star (\check{g} \star \check{h})} \xi(m), \xi(\check{g})\right\}.$$

In the same way, put

$$t_0 = \min\left\{\inf_{m \in \check{f} \star (\check{g} \star \check{h})} \zeta(m), \zeta(\check{g})\right\}.$$

Then, for all $\check{u} \in \check{f} \star \check{h}$,

$$\zeta(\check{u}) \geq \min\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \zeta(\mathfrak{m}), \zeta(\check{g})\right\}.$$

Finally, put

$$v_0 = \max\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega(\mathfrak{m}), \omega(\check{g})\right\}.$$

Then $\check{g} \in \omega_{v_0}$ and

$$\check{f} \star (\check{g} \star \check{h}) \subseteq \omega_{v_0}.$$

By Proposition 2.1(P_6),

$$\check{f} \star (\check{g} \star \check{h}) \ll \omega_{v_0}.$$

Since ω_{v_0} is an $\mathcal{HHI}$ of $\mathcal{K}$, we get

$$\check{f} \star \check{h} \subseteq \omega_{v_0}.$$

Thus, for all $\check{u} \in \check{f} \star \check{h}$,

$$\omega(\check{u}) \leq \max\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega(\mathfrak{m}), \omega(\check{g})\right\}.$$

Therefore, $\mathcal{M}$ is an $\mathcal{NFHHI}$ of $\mathcal{K}$. □

Remark 3.1. The weak, strong, and reflexive versions of Theorem 3.3 are obtained by replacing $\mathcal{HHI}$ with weak, strong, and reflexive $\mathcal{HHIs}$, respectively, and by using the corresponding defining conditions in Definition 3.1.

Theorem 3.4. Let $\mathcal{M} = (\xi, \zeta, \omega)$ be an $\mathcal{NFHHI}$ of $\mathcal{K}$. Then

$$\mathcal{P} = \{\check{f} \in \mathcal{K} \mid \xi(\check{f}) = \xi(0), \zeta(\check{f}) = \zeta(0), \omega(\check{f}) = \omega(0)\}$$

is an $\mathcal{HHI}$ of $\mathcal{K}$.

Proof. Clearly, $0 \in \mathcal{P}$. Let $\check{f}, \check{g}, \check{h} \in \mathcal{K}$, and suppose that

$$\check{f} \star (\check{g} \star \check{h}) \ll \mathcal{P} \quad \text{and} \quad \check{g} \in \mathcal{P}.$$

Let $\check{u} \in \check{f} \star \check{h}$. Since $\check{g} \in \mathcal{P}$, we have

$$\xi(\check{g}) = \xi(0), \quad \zeta(\check{g}) = \zeta(0), \quad \omega(\check{g}) = \omega(0).$$

Now let $\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})$. Since

$$\check{f} \star (\check{g} \star \check{h}) \ll \mathcal{P},$$

there exists $\mathfrak{n} \in \mathcal{P}$ such that $\mathfrak{m} \ll \mathfrak{n}$. Since $\mathcal{M}$ is an $\mathcal{NFHHI}$, the order condition gives

$$\xi(\mathfrak{m}) \geq \xi(\mathfrak{n}) = \xi(0),$$

$$\zeta(\mathfrak{m}) \geq \zeta(\mathfrak{n}) = \zeta(0),$$

$$\omega(\mathfrak{m}) \leq \omega(\mathfrak{n}) = \omega(0).$$

On the other hand, by Theorem 3.2(i), $\mathcal{M}$ is an $\mathcal{NFWHHI}$ of $\mathcal{K}$. Hence

$$\xi(0) \geq \xi(\mathfrak{m}), \quad \zeta(0) \geq \zeta(\mathfrak{m}), \quad \omega(0) \leq \omega(\mathfrak{m}).$$

Therefore,

$$\xi(\mathfrak{m}) = \xi(0), \quad \zeta(\mathfrak{m}) = \zeta(0), \quad \omega(\mathfrak{m}) = \omega(0)$$

for all $\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})$. Thus

$$\begin{aligned} \inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}) &= \xi(0), \\ \inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \zeta(\mathfrak{m}) &= \zeta(0), \\ \sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega(\mathfrak{m}) &= \omega(0). \end{aligned}$$

Since $\mathcal{M}$ is an $\mathcal{NFHHI}$, we get

$$\xi(\check{u}) \geq \min\{\xi(0), \xi(0)\} = \xi(0),$$

$$\zeta(\check{u}) \geq \min\{\zeta(0), \zeta(0)\} = \zeta(0),$$

$$\omega(\check{u}) \leq \max\{\omega(0), \omega(0)\} = \omega(0).$$

Again, since $\mathcal{M}$ is an $\mathcal{NFWHHI}$, we have

$$\xi(0) \geq \xi(\check{u}), \quad \zeta(0) \geq \zeta(\check{u}), \quad \omega(0) \leq \omega(\check{u}).$$

Therefore,

$$\xi(\check{u}) = \xi(0), \quad \zeta(\check{u}) = \zeta(0), \quad \omega(\check{u}) = \omega(0).$$

Hence $\check{u} \in \mathcal{P}$. Since $\check{u} \in \check{f} \star \check{h}$ was arbitrary, we obtain

$$\check{f} \star \check{h} \subseteq \mathcal{P}.$$

Thus, $\mathcal{P}$ is an $\mathcal{HHI}$ of $\mathcal{K}$. □

Theorem 3.5. Let $\mathcal{P}$ be a nonempty subset of an $\mathcal{HBCKA} \mathcal{K}$, and let $\mathfrak{s}, \mathfrak{t}, \mathfrak{v} \in (0, 1]$. Define an $\mathcal{NFS}$ $\mathcal{M} = (\xi, \zeta, \omega)$ in $\mathcal{K}$ by

$$\xi(\check{f}) := \begin{cases} \mathfrak{s}, & \text{if } \check{f} \in \mathcal{P}, \\ 0, & \text{if } \check{f} \notin \mathcal{P}, \end{cases}$$

$$\zeta(\check{f}) := \begin{cases} \mathfrak{t}, & \text{if } \check{f} \in \mathcal{P}, \\ 0, & \text{if } \check{f} \notin \mathcal{P}, \end{cases}$$

$$\omega(\check{f}) := \begin{cases} 0, & \text{if } \check{f} \in \mathcal{P}, \\ \mathfrak{v}, & \text{if } \check{f} \notin \mathcal{P}. \end{cases}$$

Then $\mathcal{P}$ is an $\mathcal{HHI}$ of $\mathcal{K}$ if and only if $\mathcal{M}$ is an $\mathcal{NFHHI}$ of $\mathcal{K}$.

Proof. Assume that $\mathcal{P}$ is an $\mathcal{HHI}$ of $\mathcal{K}$. By Theorem 2.1, $\mathcal{P}$ is an $\mathcal{HBCKI}$ of $\mathcal{K}$.

First, we verify the order condition. Let $\check{f}, \check{g} \in \mathcal{K}$ and suppose that $\check{f} \ll \check{g}$. If $\check{g} \in \mathcal{P}$, then $\{\check{f}\} \ll \mathcal{P}$. Since $\mathcal{P}$ is an $\mathcal{HBCKI}$, Proposition 2.2 gives $\check{f} \in \mathcal{P}$. Hence

$$\xi(\check{f}) = \xi(\check{g}) = s, \quad \zeta(\check{f}) = \zeta(\check{g}) = t, \quad \omega(\check{f}) = \omega(\check{g}) = 0.$$

If $\check{g} \notin \mathcal{P}$, then $\xi(\check{g}) = 0$, $\zeta(\check{g}) = 0$, and $\omega(\check{g}) = v$. Since all values of ξ and ζ are nonnegative and all values of ω are at most v , we have

$$\xi(\check{f}) \geq \xi(\check{g}), \quad \zeta(\check{f}) \geq \zeta(\check{g}), \quad \omega(\check{f}) \leq \omega(\check{g}).$$

Thus the order condition in Definition 3.1(ii) is satisfied.

Next, let $\check{f}, \check{g}, \check{h} \in \mathcal{K}$ and let $\check{u} \in \check{f} \star \check{h}$. Put

$$A = \check{f} \star (\check{g} \star \check{h}).$$

For the ξ -condition, if $A \subseteq \mathcal{P}$ and $\check{g} \in \mathcal{P}$, then, by Proposition 2.1(P_6), $A \ll \mathcal{P}$. Since $\mathcal{P}$ is an $\mathcal{HHI}$, we get

$$\check{f} \star \check{h} \subseteq \mathcal{P}.$$

Thus $\check{u} \in \mathcal{P}$, and hence

$$\xi(\check{u}) = s.$$

Also,

$$\inf_{m \in A} \xi(m) = s \quad \text{and} \quad \xi(\check{g}) = s.$$

Therefore,

$$\xi(\check{u}) \geq \min\{\inf_{m \in A} \xi(m), \xi(\check{g})\}.$$

If $A \not\subseteq \mathcal{P}$ or $\check{g} \notin \mathcal{P}$, then

$$\min\{\inf_{m \in A} \xi(m), \xi(\check{g})\} = 0.$$

Hence the above inequality also holds.

The proof of the ζ -condition is the same. Thus

$$\zeta(\check{u}) \geq \min\{\inf_{m \in A} \zeta(m), \zeta(\check{g})\}.$$

For the ω -condition, if $A \subseteq \mathcal{P}$ and $\check{g} \in \mathcal{P}$, then $\check{u} \in \mathcal{P}$, and hence

$$\omega(\check{u}) = 0.$$

Also,

$$\sup_{m \in A} \omega(m) = 0 \quad \text{and} \quad \omega(\check{g}) = 0.$$

Therefore,

$$\omega(\check{u}) \leq \max\{\sup_{m \in A} \omega(m), \omega(\check{g})\}.$$

If $A \not\subseteq \mathcal{P}$ or $\check{g} \notin \mathcal{P}$, then

$$\max\{\sup_{m \in A} \omega(m), \omega(\check{g})\} = v.$$

Since $\omega(\check{u}) \leq v$, the required inequality holds. Hence $\mathcal{M}$ is an $\mathcal{NFHHI}$ of $\mathcal{K}$.

Conversely, assume that $\mathcal{M}$ is an $\mathcal{NFHHI}$ of $\mathcal{K}$. Since

$$\mathcal{P} = \xi_s = \zeta_t = \omega_0,$$

Theorem 3.3 implies that $\mathcal{P}$ is an $\mathcal{HHI}$ of $\mathcal{K}$. □

4. PRODUCT STRUCTURES AND HYPER HOMOMORPHIC PREIMAGES

In this section, we study product constructions of neutrosophic fuzzy hyper h -ideals in hyper BCK-algebras. We also investigate strongest neutrosophic fuzzy relations and hyper homomorphic preimages.

Definition 4.1. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be $\mathcal{HBCKA}$ s and let $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$. Let $\mathcal{M}_1 = (\xi_1, \zeta_1, \omega_1)$ and $\mathcal{M}_2 = (\xi_2, \zeta_2, \omega_2)$ be $\mathcal{NFS}$ s in $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. The product $\mathcal{M}_1 \times \mathcal{M}_2$ is the $\mathcal{NFSM} = (\xi, \zeta, \omega)$ in $\mathcal{K}$ defined by

$$\begin{aligned}\xi(\check{f}, \check{g}) &= \min\{\xi_1(\check{f}), \xi_2(\check{g})\}, \\ \zeta(\check{f}, \check{g}) &= \min\{\zeta_1(\check{f}), \zeta_2(\check{g})\}, \\ \omega(\check{f}, \check{g}) &= \max\{\omega_1(\check{f}), \omega_2(\check{g})\}\end{aligned}$$

for all $(\check{f}, \check{g}) \in \mathcal{K}_1 \times \mathcal{K}_2$.

Theorem 4.1. Let $\mathcal{M}_1 = (\xi_1, \zeta_1, \omega_1)$ and $\mathcal{M}_2 = (\xi_2, \zeta_2, \omega_2)$ be $\mathcal{NFHHI}$ s of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Then $\mathcal{M}_1 \times \mathcal{M}_2$ is an $\mathcal{NFHHI}$ of $\mathcal{K}_1 \times \mathcal{K}_2$.

Proof. Let $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 = (\xi, \zeta, \omega)$. First, let

$$(\check{f}_1, \check{f}_2) \ll (\check{g}_1, \check{g}_2)$$

in $\mathcal{K}_1 \times \mathcal{K}_2$. Then $\check{f}_1 \ll \check{g}_1$ in $\mathcal{K}_1$ and $\check{f}_2 \ll \check{g}_2$ in $\mathcal{K}_2$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{NFHHI}$ s, we have

$$\begin{aligned}\xi_1(\check{f}_1) &\geq \xi_1(\check{g}_1), & \xi_2(\check{f}_2) &\geq \xi_2(\check{g}_2), \\ \zeta_1(\check{f}_1) &\geq \zeta_1(\check{g}_1), & \zeta_2(\check{f}_2) &\geq \zeta_2(\check{g}_2), \\ \omega_1(\check{f}_1) &\leq \omega_1(\check{g}_1), & \omega_2(\check{f}_2) &\leq \omega_2(\check{g}_2).\end{aligned}$$

Therefore,

$$\begin{aligned}\xi(\check{f}_1, \check{f}_2) &\geq \xi(\check{g}_1, \check{g}_2), \\ \zeta(\check{f}_1, \check{f}_2) &\geq \zeta(\check{g}_1, \check{g}_2), \\ \omega(\check{f}_1, \check{f}_2) &\leq \omega(\check{g}_1, \check{g}_2).\end{aligned}$$

Now let

$$(\check{u}_1, \check{u}_2) \in (\check{f}_1, \check{f}_2) \star (\check{h}_1, \check{h}_2).$$

Then $\check{u}_1 \in \check{f}_1 \star_1 \check{h}_1$ and $\check{u}_2 \in \check{f}_2 \star_2 \check{h}_2$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{NFHHI}$ s, we have

$$\xi_1(\check{u}_1) \geq \min\left\{\inf_{m_1 \in \check{f}_1 \star_1 (\check{g}_1 \star_1 \check{h}_1)} \xi_1(m_1), \xi_1(\check{g}_1)\right\},$$

and

$$\xi_2(\check{u}_2) \geq \min\{\inf_{m_2 \in \check{f}_2 \star_2 (\check{g}_2 \star_2 \check{h}_2)} \xi_2(m_2), \xi_2(\check{g}_2)\}.$$

Hence

$$\begin{aligned} \xi(\check{u}_1, \check{u}_2) &= \min\{\xi_1(\check{u}_1), \xi_2(\check{u}_2)\} \\ &\geq \min\{\inf_{(m_1, m_2) \in (\check{f}_1, \check{f}_2) \star (\check{g}_1, \check{g}_2) \star (\check{h}_1, \check{h}_2)} \xi(m_1, m_2), \xi(\check{g}_1, \check{g}_2)\}. \end{aligned}$$

Similarly,

$$\zeta(\check{u}_1, \check{u}_2) \geq \min\{\inf_{(m_1, m_2) \in (\check{f}_1, \check{f}_2) \star (\check{g}_1, \check{g}_2) \star (\check{h}_1, \check{h}_2)} \zeta(m_1, m_2), \zeta(\check{g}_1, \check{g}_2)\}.$$

Moreover, since $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{NFHHI}$ s, we have

$$\omega_1(\check{u}_1) \leq \max\{\sup_{m_1 \in \check{f}_1 \star_1 (\check{g}_1 \star_1 \check{h}_1)} \omega_1(m_1), \omega_1(\check{g}_1)\},$$

and

$$\omega_2(\check{u}_2) \leq \max\{\sup_{m_2 \in \check{f}_2 \star_2 (\check{g}_2 \star_2 \check{h}_2)} \omega_2(m_2), \omega_2(\check{g}_2)\}.$$

Therefore,

$$\omega(\check{u}_1, \check{u}_2) \leq \max\{\sup_{(m_1, m_2) \in (\check{f}_1, \check{f}_2) \star (\check{g}_1, \check{g}_2) \star (\check{h}_1, \check{h}_2)} \omega(m_1, m_2), \omega(\check{g}_1, \check{g}_2)\}.$$

Thus, $\mathcal{M}_1 \times \mathcal{M}_2$ is an $\mathcal{NFHHI}$ of $\mathcal{K}_1 \times \mathcal{K}_2$. □

Remark 4.1. The weak, strong, and reflexive versions of Theorem 4.1 are obtained in the same way by using the corresponding defining conditions.

Definition 4.2. Let $\mathcal{M} = (\xi, \zeta, \omega)$ be an $\mathcal{NFS}$ in an $\mathcal{HBCKA} \mathcal{K}$. The strongest neutrosophic fuzzy relation on $\mathcal{K}$ induced by $\mathcal{M}$ is the $\mathcal{NFS} \mathcal{Y}_{\mathcal{M}} = (\xi_Y, \zeta_Y, \omega_Y)$ in $\mathcal{K} \times \mathcal{K}$ defined by

$$\begin{aligned} \xi_Y(\check{f}, \check{g}) &= \min\{\xi(\check{f}), \xi(\check{g})\}, \\ \zeta_Y(\check{f}, \check{g}) &= \min\{\zeta(\check{f}), \zeta(\check{g})\}, \\ \omega_Y(\check{f}, \check{g}) &= \max\{\omega(\check{f}), \omega(\check{g})\} \end{aligned}$$

for all $(\check{f}, \check{g}) \in \mathcal{K} \times \mathcal{K}$.

Theorem 4.2. Let $\mathcal{M} = (\xi, \zeta, \omega)$ be an $\mathcal{NFS}$ in $\mathcal{K}$. Then $\mathcal{M}$ is an $\mathcal{NFSHHI}$ of $\mathcal{K}$ if and only if $\mathcal{Y}_{\mathcal{M}} = (\xi_Y, \zeta_Y, \omega_Y)$ is an $\mathcal{NFSHHI}$ of $\mathcal{K} \times \mathcal{K}$.

Proof. Assume first that $\mathcal{M}$ is an $\mathcal{NFSHHI}$ of $\mathcal{K}$. Since $\mathcal{Y}_{\mathcal{M}}$ is exactly the product $\mathcal{M} \times \mathcal{M}$, the strong version of Theorem 4.1 implies that $\mathcal{Y}_{\mathcal{M}}$ is an $\mathcal{NFSHHI}$ of $\mathcal{K} \times \mathcal{K}$.

Conversely, assume that $\mathcal{Y}_{\mathcal{M}}$ is an $\mathcal{NFSHHI}$ of $\mathcal{K} \times \mathcal{K}$. Let $\check{f} \in \mathcal{K}$. Applying the first strong condition of Definition 3.1(iii) to $(\check{f}, \check{f})$, we get

$$\inf_{(m, n) \in (\check{f}, \check{f}) \star (\check{f}, \check{f})} \xi_Y(m, n) \geq \xi_Y(\check{f}, \check{f}).$$

Since $(\check{f}, \check{f}) \star (\check{f}, \check{f}) = (\check{f} \star \check{f}) \times (\check{f} \star \check{f})$ and $\xi_Y(\check{f}, \check{f}) = \xi(\check{f})$, we obtain

$$\inf_{\mathfrak{m} \in \check{f} \star \check{f}} \xi(\mathfrak{m}) \geq \xi(\check{f}).$$

Similarly,

$$\inf_{\mathfrak{m} \in \check{f} \star \check{f}} \zeta(\mathfrak{m}) \geq \zeta(\check{f})$$

and

$$\sup_{\mathfrak{m} \in \check{f} \star \check{f}} \omega(\mathfrak{m}) \leq \omega(\check{f}).$$

Next, let $\check{f}, \check{g}, \check{h} \in \mathcal{K}$ and let $\check{u} \in \check{f} \star \check{h}$. Then

$$(\check{u}, \check{u}) \in (\check{f}, \check{f}) \star (\check{h}, \check{h}).$$

Since Y_M is an $\mathcal{NF}SHHI$ of $\mathcal{K} \times \mathcal{K}$, we have

$$\xi_Y(\check{u}, \check{u}) \geq \min\left\{\sup_{(\mathfrak{m}, \mathfrak{n}) \in (\check{f}, \check{f}) \star (\check{g}, \check{g}) \star (\check{h}, \check{h})} \xi_Y(\mathfrak{m}, \mathfrak{n}), \xi_Y(\check{g}, \check{g})\right\}.$$

Since

$$\xi_Y(\check{u}, \check{u}) = \xi(\check{u})$$

and

$$\sup_{(\mathfrak{m}, \mathfrak{n}) \in (\check{f}, \check{f}) \star (\check{g}, \check{g}) \star (\check{h}, \check{h})} \xi_Y(\mathfrak{m}, \mathfrak{n}) = \sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}),$$

it follows that

$$\xi(\check{u}) \geq \min\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}), \xi(\check{g})\right\}.$$

The same argument gives

$$\zeta(\check{u}) \geq \min\left\{\sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \zeta(\mathfrak{m}), \zeta(\check{g})\right\}.$$

Moreover, using the falsity component of Y_M , we obtain

$$\omega(\check{u}) \leq \max\left\{\inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega(\mathfrak{m}), \omega(\check{g})\right\}.$$

Hence M is an $\mathcal{NF}SHHI$ of $\mathcal{K}$. □

Theorem 4.3. Let $\psi : \mathcal{K} \rightarrow \mathcal{K}'$ be a hyper homomorphism from an $\mathcal{HBCKA} \mathcal{K}$ onto an $\mathcal{HBCKA} \mathcal{K}'$. Let $\mu = (\xi^\mu, \zeta^\mu, \omega^\mu)$ be an $\mathcal{NF}SHHI$ of $\mathcal{K}'$. Define the hyper homomorphic preimage $M = (\xi, \zeta, \omega)$ of μ under ψ by

$$\xi(\check{f}) = \xi^\mu(\psi(\check{f})),$$

$$\zeta(\check{f}) = \zeta^\mu(\psi(\check{f})),$$

$$\omega(\check{f}) = \omega^\mu(\psi(\check{f}))$$

for all $\check{f} \in \mathcal{K}$. Then M is an $\mathcal{NF}SHHI$ of $\mathcal{K}$.

Proof. Let $\check{f} \in \mathcal{K}$. Since ψ is a hyper homomorphism, we have

$$\psi(\check{f} \star \check{f}) \subseteq \psi(\check{f}) \star \psi(\check{f}).$$

Thus

$$\inf_{\mathfrak{m} \in \check{f} \star \check{f}} \xi(\mathfrak{m}) = \inf_{\mathfrak{m} \in \check{f} \star \check{f}} \xi^\mu(\psi(\mathfrak{m})) \geq \inf_{\mathfrak{a} \in \psi(\check{f}) \star \psi(\check{f})} \xi^\mu(\mathfrak{a}) \geq \xi^\mu(\psi(\check{f})) = \xi(\check{f}).$$

Similarly,

$$\inf_{\mathfrak{m} \in \check{f} \star \check{f}} \zeta(\mathfrak{m}) \geq \zeta(\check{f})$$

and

$$\sup_{\mathfrak{m} \in \check{f} \star \check{f}} \omega(\mathfrak{m}) \leq \omega(\check{f}).$$

Now let $\check{f}, \check{g}, \check{h} \in \mathcal{K}$ and let $\check{u} \in \check{f} \star \check{h}$. Since ψ is a hyper homomorphism, we have

$$\psi(\check{u}) \in \psi(\check{f}) \star \psi(\check{h})$$

and

$$\psi(\check{f} \star (\check{g} \star \check{h})) \subseteq \psi(\check{f}) \star (\psi(\check{g}) \star \psi(\check{h})).$$

Since μ is an $\mathcal{NF}SHHI$ of $\mathcal{K}'$, we obtain

$$\begin{aligned} \xi(\check{u}) &= \xi^\mu(\psi(\check{u})) \\ &\geq \min\left\{ \sup_{\mathfrak{a} \in \psi(\check{f}) \star (\psi(\check{g}) \star \psi(\check{h}))} \xi^\mu(\mathfrak{a}), \xi^\mu(\psi(\check{g})) \right\} \\ &\geq \min\left\{ \sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi^\mu(\psi(\mathfrak{m})), \xi^\mu(\psi(\check{g})) \right\} \\ &= \min\left\{ \sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \xi(\mathfrak{m}), \xi(\check{g}) \right\}. \end{aligned}$$

Similarly,

$$\zeta(\check{u}) \geq \min\left\{ \sup_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \zeta(\mathfrak{m}), \zeta(\check{g}) \right\}.$$

For the falsity component, we have

$$\begin{aligned} \omega(\check{u}) &= \omega^\mu(\psi(\check{u})) \\ &\leq \max\left\{ \inf_{\mathfrak{a} \in \psi(\check{f}) \star (\psi(\check{g}) \star \psi(\check{h}))} \omega^\mu(\mathfrak{a}), \omega^\mu(\psi(\check{g})) \right\} \\ &\leq \max\left\{ \inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega^\mu(\psi(\mathfrak{m})), \omega^\mu(\psi(\check{g})) \right\} \\ &= \max\left\{ \inf_{\mathfrak{m} \in \check{f} \star (\check{g} \star \check{h})} \omega(\mathfrak{m}), \omega(\check{g}) \right\}. \end{aligned}$$

Therefore, $\mathcal{M}$ is an $\mathcal{NF}SHHI$ of $\mathcal{K}$. □

5. CONCLUSION

In this paper, we introduced and investigated neutrosophic fuzzy hyper h -ideals in hyper BCK-algebras. We also defined the notions of neutrosophic fuzzy weak, strong, and reflexive hyper h -ideals and studied their fundamental properties. It was shown that every neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper h -ideal is a corresponding neutrosophic fuzzy weak, ordinary, strong, and reflexive hyper BCK-ideal, respectively. Moreover, the relationships among these classes were clarified by proving that every $NFRHHI$ is an $NFSHHI$, every $NFSHHI$ is an $NFHHI$, and every $NFHHI$ is an $NFWHHI$.

We further provided examples showing that the converses of these implications do not hold in general. Characterization results in terms of level subsets were established, and a characteristic-type construction was used to connect ordinary hyper h -ideals with neutrosophic fuzzy hyper h -ideals. Finally, product structures, strongest neutrosophic fuzzy relations, and hyper homomorphic preimages were studied. In particular, we proved that the product of two neutrosophic fuzzy hyper h -ideals is again a neutrosophic fuzzy hyper h -ideal, and that the hyper homomorphic preimage of a neutrosophic fuzzy strong hyper h -ideal preserves the corresponding structure.

These results enrich the theory of hyper BCK-algebras under neutrosophic fuzzy uncertainty and provide a basis for further investigations of generalized fuzzy ideal structures in hyperalgebraic systems. Future research may consider neutrosophic fuzzy hyper h -ideals in other hyperalgebraic structures, study additional closure properties, and develop applications of these structures in algebraic approaches to uncertainty, decision-making, and information systems.

Acknowledgments. This research project was supported by the University of Phayao and the Fundamental Fund (FF) 2027 (Grant No. 2285/2569), under the Thailand RISE Fund.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] A. Alsubie, A. Al-Masarwah, MBJ-Neutrosophic Hyper BCK-Ideals in Hyper BCK-Algebras, AIMS Math. 6 (2021), 6107–6121. <https://doi.org/10.3934/math.2021358>.
- [2] K. Anjaneyulu Naik, D. Bhuvaneswari, U. Bindu Madhavi, B. Tandava Krishna, B. Satyanarayana, On Neutrosophic Fuzzy (Implicative, Commutative, Positive Implicative) Hyper BCK-Ideals of Hyper BCK-Algebras, IAENG Int. J. Appl. Math. 56 (2026), 716–735.
- [3] K. Anjaneyulu Naik, D. Ramesh, A. Iampan, B. Satyanarayana, P. Rajani, et al., Intuitionistic Fuzzy Hyper h -Ideals in Hyper BCK-Algebras, Int. J. Anal. Appl. 24 (2026), 166. <https://doi.org/10.28924/2291-8639-24-2026-166>.
- [4] K. Anjaneyulu Naik, D. Ramesh, A. Iampan, B. Satyanarayana, R. Durga Prasad, Intuitionistic Fuzzy Hyper p -Ideals in Hyper BCK-Algebras, Asia Pac. J. Math. 13 (2026), 52. <https://doi.org/10.28924/APJM/13-52>.
- [5] P. Bhattacharya, N.P. Mukherjee, Fuzzy Relations and Fuzzy Groups, Inf. Sci. 36 (1985), 267–282. [https://doi.org/10.1016/0020-0255\(85\)90057-x](https://doi.org/10.1016/0020-0255(85)90057-x).
- [6] D. Bhuvaneswari, B. Satyanarayana, K. Anjaneyulu Naik, Explorations of Derivations in Neutrosophic Fuzzy Implicative Ideals of BCK-Algebras, in: S. Singh, S. Kaur, E. Güler, B. Bhushan, S. Malik, (eds) Analysis, Algebra,

- and Associated Topics, Trends in Mathematics, Springer Nature Switzerland, Cham, pp. 165–183, (2026). https://doi.org/10.1007/978-3-032-04418-1_14.
- [7] R.A. Borzooei, A. Hasankhani, M.M. Zahedi, Y.B. Jun, On Hyper K-Algebras, Math. Jpn. 52 (2000), 113–121.
 - [8] R.A. Borzooei, F. Smarandache, Y.B. Jun, Polarity of Generalized Neutrosophic Subalgebras in BCK/BCI-Algebras, Neutrosophic Sets Syst. 32 (2020), 123–145. <https://doi.org/10.5281/zenodo.3723613>.
 - [9] Y. Imai, K. Iséki, On Axiom Systems of Propositional Calculi, XIV, Proc. Japan Acad. 42 (1966), 19–22. <https://doi.org/10.3792/pja/1195522169>.
 - [10] Y.B. Jun, X.L. Xin, Scalar Elements and Hyperatoms of Hyper BCK-Algebras, Sci. Math. 2 (1999), 303–309.
 - [11] Y.B. Jun, M.M. Zahedi, X.L. Xin, R.A. Borzooei, On Hyper BCK-Algebras, Ital. J. Pure Appl. Math. 8 (2000), 127–136.
 - [12] Y.B. Jun, X.L. Xin, M.M. Zahedi, E.H. Roh, Strong Hyper BCK-Ideals of Hyper BCK-Algebras, Math. Jpn. 51 (2000), 493–498.
 - [13] Y.B. Jun, X.L. Xin, Fuzzy Hyper BCK-Ideals of Hyper BCK-Algebras, Sci. Math. Jpn. 53 (2001), 353–360.
 - [14] M. Kaviyarasu, M. Alqahtani, M. Rajeshwari, K. Alnefaie, Fermatean Neutrosophic Soft Structure and its Ideals in Hyper BCK-Algebras, Contemp. Math. 6 (2025), 1478–1500. <https://doi.org/10.37256/cm.6220255814>.
 - [15] S. Khademan, M.M. Zahedi, R.A. Borzooei, Y.B. Jun, Fuzzy Soft Positive Implicative Hyper BCK-Ideals of Several Types, Miskolc Math. Notes 22 (2021), 299–315. <https://doi.org/10.18514/MMN.2021.2855>.
 - [16] F. Marty, Sur une Généralisation de la Notion de Groupe, in: 8th Congress Math. Scandinaves, Stockholm, 1934, pp. 45–49.
 - [17] B. Satyanarayana, D. Bhuvaneswari, K. Anjaneyulu Naik, On Foldness of Neutrosophic Fuzzy H -Ideal in BCK-Algebra, Ind. Eng. J. 54 (2025), 42–49.
 - [18] F. Smarandache, Neutrosophic Set – a Generalization of the Intuitionistic Fuzzy Set, Int. J. Pure Appl. Math. 24 (2005), 287–297.
 - [19] M. Touqeer, N. Çağman, Fuzzy Hyper h -Ideals of Hyper BCK-Algebras, J. Intell. Fuzzy Syst. 31 (2016), 2051–2058. <https://doi.org/10.3233/ifs-162194>.
 - [20] K. Anjaneyulu Naik, B. Satyanarayana, D. Ramesh, B. Tandava Krishna, Interval-Valued Neutrosophic Fuzzy n -Fold Positive Implicative Ideals of BCK-Algebras: Properties and Characterizations, Neutrosophic Sets Syst. 95 (2026), 345–362. <https://doi.org/10.5281/zenodo.17088824>.
 - [21] R. Anjum, S. Ullah, Y.-M. Chu, M. Munir, N. Kausar, et al., Characterizations of Ordered h -Regular Semirings by Ordered h -Ideals, AIMS Math. 5 (2020), 5768–5790. <https://doi.org/10.3934/math.2020370>.
 - [22] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338–353. [https://doi.org/10.1016/S0019-9958\(65\)90241-X](https://doi.org/10.1016/S0019-9958(65)90241-X).