

Fixed Point Theorems in Fuzzy Metric Spaces via Generalized Graphic Rational Θ -Contractions

Sidite Duraj¹, R. Pichailakshmi², Kastriot Zoto^{1,3}, E. Madhavi⁴, K. Dinesh^{5,*}

¹*Department of Mathematics, Faculty of Natural Sciences, University of Shkodra “Luigj Gurakuqi” 4001, Shkoder, Albania*

²*Department of Mathematics, SRM TRP Engineering College, Irungalur Tiruchirapalli -621105, Tamilnadu, India*

³*Department of Mathematics, Informatics and Physics, Faculty of Natural Sciences, University of Gjirokastra 6001 Gjirokastra, Albania*

⁴*Department of English, Gokaraju Rangaraju Institute of Engineering and Technology, Hyderabad, Telangana, India*

⁵*Department of Mathematics, K. Ramakrishnan College of Engineering (Autonomous), Trichy – 621 112, Tamilnadu, India*

*Corresponding author: dinesh.skksv93@gmail.com

Abstract. The aim of this paper is to investigate fixed point results for a new class of graphic rational Θ -contractions in complete fuzzy metric spaces endowed with a directed graph. By combining the graph-theoretic framework with rational contractive conditions involving Θ -functions, several existence and uniqueness theorems are established. The obtained results generalize and unify various well-known contraction principles in the literature, including Banach-type, Hardy–Rogers-type, Ćirić-type and integral-type contractions. The graph structure enables the treatment of mappings satisfying contractive conditions only on admissible pairs of points connected through graph edges. Sufficient conditions ensuring the convergence of Picard iterative sequences to unique fixed points are derived. A numerical example and convergence analysis are presented to illustrate the applicability of the theoretical findings. The established results extend several recent developments in fuzzy metric spaces and metric spaces endowed with graphs.

1. INTRODUCTION

The interaction between graph theory and fixed point (FP) theory has become an important area of contemporary research in nonlinear analysis. Graph structures provide additional information

Received: Jun. 30, 2026.

2020 *Mathematics Subject Classification.* 47H10, 54H25, 54E50.

Key words and phrases. Fuzzy metric space; Directed graph; Fixed point theorem; Rational contraction; Θ -contraction; Picard iteration.

on the relationships among elements of a space and permit the investigation of mappings whose contractive behaviour is restricted to admissible pairs of vertices. Such an approach is particularly useful in situations where global contractive conditions are difficult to verify. The pioneering work of Jachymski [3] established a connection between graph theory and FP theory, leading to numerous extensions and applications. Subsequent contributions by Onsod et al. [6] and Faraji and Radenovic [8] further demonstrated the effectiveness of graph-based contractive conditions.

In many practical and theoretical problems, uncertainty is naturally present and cannot be adequately described by classical metric structures. To overcome this limitation, CCs have been widely employed as a suitable framework for measuring the degree of nearness between elements. The development of FP theory in fuzzy settings has attracted considerable attention because of its ability to model vagueness while retaining many analytical properties of metric spaces. Recent studies have shown that fuzzy metric spaces (FMS) constitute an appropriate environment for investigating nonlinear problems under generalized contractive assumptions [7, 10, 12].

Another significant trend in FP theory concerns the use of auxiliary control functions to formulate generalized contraction conditions. In this direction, Wardowski [1] introduced the notion of a Θ -contraction, providing a powerful generalization of the classical contraction principle. Later, Jleli and Samet [4] and Ahmad et al. [5] obtained several refinements and extensions that broadened the scope of the theory. These developments have shown that Θ -type contractive conditions can be effectively combined with various generalized metric structures and graph frameworks.

Rational contractions constitute another important class of mappings in FP theory. Unlike linear contractive inequalities, rational expressions incorporate additional geometric information about the iterative process and frequently yield stronger convergence properties. Such contractive conditions have been successfully employed in metric, generalized metric and fuzzy environments [2, 11]. The flexibility offered by rational terms motivates the search for more general contraction principles that simultaneously incorporate graph structures and fuzzy distances.

Motivated by the above observations, we introduce in this paper a new class of graphic rational Θ -contractions defined on fuzzy metric spaces endowed with a directed graph. The proposed contraction combines three essential components: graph admissibility, fuzzy metric structure, and rational Θ -contractive behaviour. This unified framework enables us to establish existence and uniqueness results for FPs under assumptions that are substantially weaker than those required in classical settings.

The main objectives of this work are as follows:

- (1) To introduce the concept of graphic rational Θ -contractions in FMS endowed with a graph;
- (2) To establish existence and uniqueness theorems for FPs of such mappings in complete FMS;
- (3) To derive Hardy–Rogers-type, integral-type and Ćirić-type rational contractive conditions as consequences of the general theory;
- (4) To investigate the convergence behaviour of the corresponding Picard iterative sequences;
- (5) To provide an illustrative numerical example supporting the theoretical findings.

The obtained results extend several known FP theorems associated with graph contractions, rational contractions and Θ -contractions. Furthermore, the proposed framework offers a useful basis for future investigations in fuzzy analysis and related nonlinear problems.

The remainder of the paper is organized as follows. Section 2 contains the necessary preliminaries and auxiliary results. In Section 3, the notion of graphic rational Θ -contraction is introduced and the main FP theorems are established and presents illustrative examples and convergence analysis. Finally, concluding remarks are given in the last section.

2. PRELIMINARIES

This section collects several notions and auxiliary results that will be used throughout the paper. The concepts presented below are adapted from the theory of FMS, graph-based FP theory, and Θ -contractions [1, 3, 5, 13, 14].

Definition 2.1 ([9]). Let $*$ be a continuous t -norm on $[0, 1]$. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t -norm if the following conditions hold:

- (1) $a * b = b * a$ for all $a, b \in [0, 1]$;
- (2) $(a * b) * c = a * (b * c)$ for all $a, b, c \in [0, 1]$;
- (3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
- (4) $a * 1 = a$ for every $a \in [0, 1]$;
- (5) $*$ is continuous.

Definition 2.2 ([13]). Let $\mathfrak{J}$ be a nonempty set and let $*$ be a continuous t -norm. A mapping $M : \mathfrak{J} \times \mathfrak{J} \times (0, \infty) \rightarrow (0, 1]$ is called a fuzzy metric on $\mathfrak{J}$ if, for all $\mathfrak{v}, \mathfrak{f}, \mathfrak{z} \in \mathfrak{J}$ and $t, s > 0$, the following conditions are satisfied:

- (1) $M(\mathfrak{v}, \mathfrak{f}, t) > 0$;
- (2) $M(\mathfrak{v}, \mathfrak{f}, t) = 1$ if and only if $\mathfrak{v} = \mathfrak{f}$;
- (3) $M(\mathfrak{v}, \mathfrak{f}, t) = M(\mathfrak{f}, \mathfrak{v}, t)$;
- (4) $M(\mathfrak{v}, \mathfrak{z}, t + s) \geq M(\mathfrak{v}, \mathfrak{f}, t) * M(\mathfrak{f}, \mathfrak{z}, s)$;
- (5) for every $\mathfrak{v}, \mathfrak{f} \in \mathfrak{J}$, the function $t \mapsto M(\mathfrak{v}, \mathfrak{f}, t)$ is continuous.

Then $(\mathfrak{J}, M, *)$ is called a FMS [13, 14].

Definition 2.3. A sequence $\{\mathfrak{v}_n\}$ in a FMS $(\mathfrak{J}, M, *)$ is said to converge to $\mathfrak{v} \in \mathfrak{J}$ if $\lim_{n \rightarrow \infty} M(\mathfrak{v}_n, \mathfrak{v}, t) = 1$ for every $t > 0$.

Definition 2.4. A sequence $\{\mathfrak{v}_n\}$ in $(\mathfrak{J}, M, *)$ is called Cauchy if $\lim_{n, m \rightarrow \infty} M(\mathfrak{v}_n, \mathfrak{v}_m, t) = 1$ for every $t > 0$. A FMS is said to be complete whenever every Cauchy sequence converges to a point of the space.

Definition 2.5. Let $G = (V(G), E(G))$ be a directed graph with $V(G) = \mathfrak{J}$ and containing all loops. A mapping $\mathfrak{J} : \mathfrak{J} \rightarrow \mathfrak{J}$ is said to preserve the graph structure if $(\mathfrak{v}, \mathfrak{f}) \in E(G)$ implies $(\mathfrak{J}\mathfrak{v}, \mathfrak{J}\mathfrak{f}) \in E(G)$ for all $\mathfrak{v}, \mathfrak{f} \in \mathfrak{J}$. [3]

Definition 2.6. A graph G is said to satisfy Property (P) if whenever $\{\eta_n\}$ is a sequence such that $(\eta_n, \eta_{n+1}) \in E(G)$ for every n and $\eta_n \rightarrow \eta$, there exists a subsequence $\{\eta_{n_k}\}$ satisfying $(\eta_{n_k}, \eta) \in E(G)$ for all k .

Definition 2.7. A function $\Theta : (0, \infty) \rightarrow (1, \infty)$ is called a Θ -function if:

- (1) Θ is nondecreasing;
- (2) for every sequence $\{r_n\} \subset (0, \infty)$,

$$\Theta(r_n) \rightarrow 1 \iff r_n \rightarrow 0;$$
- (3) there exist $l \in (0, 1)$ and $c \in (0, \infty]$ such that

$$\lim_{r \rightarrow 0^+} \frac{\Theta(r)-1}{r^l} = c.$$

The above class of functions was introduced in the study of Θ -contractions by Jleli Samet [4] and further investigated in [5].

Throughout the paper, $(\mathfrak{J}, M, *)$ denotes a complete fuzzy metric space endowed with a directed graph G , and $\mathfrak{I} : \mathfrak{J} \rightarrow \mathfrak{J}$ denotes a self-mapping.

3. MAIN RESULTS

Definition 3.1. Let $(\mathfrak{J}, M, *)$ be a FMS endowed with a directed graph $G = (V(G), E(G))$, where $V(G) = \mathfrak{J}$. Assume that $\Theta : (0, \infty) \rightarrow (1, \infty)$ belongs to the class of Θ -functions and let $\lambda \in (0, 1)$. A self-mapping $\mathfrak{I} : \mathfrak{J} \rightarrow \mathfrak{J}$ is said to satisfy a graphic rational Θ -contraction whenever, for every admissible pair $(\eta, \mathfrak{k}) \in E(G)$ with $\mathfrak{I}\eta \neq \mathfrak{I}\mathfrak{k}$, the inequality

$$\Theta\left(\frac{1}{M(\mathfrak{I}\eta, \mathfrak{I}\mathfrak{k}, t)} - 1\right) \leq \left[\Theta(\mathcal{R}(\eta, \mathfrak{k}))\right]^\lambda$$

holds, where

$$\mathcal{R}(\eta, \mathfrak{k}) = \frac{\Delta(\eta, \mathfrak{k}) + \Delta(\eta, \mathfrak{I}\eta) + \Delta(\mathfrak{k}, \mathfrak{I}\mathfrak{k})}{1 + \Delta(\eta, \mathfrak{k})},$$

and

$$\Delta(u, v) = \frac{1}{M(u, v, t)} - 1$$

for all $u, v \in \mathfrak{J}$.

The quantity $\mathcal{R}(\eta, \mathfrak{k})$ will be referred to as the associated rational control term of the mapping $\mathfrak{I}$.

Remark 3.1. The above definition combines the graph structure, the fuzzy metric, and a rational control mechanism through the function $\mathcal{R}(\eta, \mathfrak{k})$. When the graph G is complete, the admissibility restriction disappears and the definition reduces to a rational Θ -contractive condition on the underlying FMS.

Theorem 3.1. Let $(\mathfrak{J}, M, *)$ be a complete fuzzy metric space (CFMS) endowed with a weakly connected graph G . Suppose that $\mathfrak{I} : \mathfrak{J} \rightarrow \mathfrak{J}$ is a graphic rational Θ -contraction. Assume that

- (1) $\mathfrak{I}$ preserves edges, i.e., $(\eta, \mathfrak{k}) \in E(G) \implies (\mathfrak{I}\eta, \mathfrak{I}\mathfrak{k}) \in E(G)$;

- (2) there exists $\mathfrak{y}_0 \in \mathfrak{I}$ such that $(\mathfrak{y}_0, \mathfrak{I}\mathfrak{y}_0) \in E(G)$;
- (3) G satisfies Property (P);
- (4) $\mathfrak{I}$ is orbitally G -continuous.

Then $\mathfrak{I}$ possesses a unique fixed point (UFP) $\mathfrak{y}^* \in \mathfrak{I}$.

Moreover, $\lim_{n \rightarrow \infty} \mathfrak{I}^n \mathfrak{y} = \mathfrak{y}^*$ for every $\mathfrak{y} \in \mathfrak{I}$.

Proof. Let $\mathfrak{y}_0 \in \mathfrak{I}$ be such that $(\mathfrak{y}_0, \mathfrak{I}\mathfrak{y}_0) \in E(G)$. Define the Picard sequence $\mathfrak{y}_{n+1} = \mathfrak{I}\mathfrak{y}_n = \mathfrak{I}^{n+1}\mathfrak{y}_0$ for $n \geq 0$. Since $\mathfrak{I}$ preserves edges, we obtain inductively $(\mathfrak{y}_n, \mathfrak{y}_{n+1}) \in E(G)$, for $n \geq 0$. If there exists n_0 such that $\mathfrak{y}_{n_0} = \mathfrak{y}_{n_0+1}$, then $\mathfrak{y}_{n_0} = \mathfrak{I}\mathfrak{y}_{n_0}$, and hence $\mathfrak{y}_{n_0}$ is a FP of $\mathfrak{I}$. Therefore, we may assume that $\mathfrak{y}_n \neq \mathfrak{y}_{n+1}$ for every $n \in \mathbb{N}$.

Since $(\mathfrak{y}_n, \mathfrak{y}_{n+1}) \in E(G)$, the graphic rational Θ -contractive condition yields

$$\Theta\left(\frac{1}{M(\mathfrak{y}_{n+1}, \mathfrak{y}_{n+2}, t)} - 1\right) \leq [\Theta(R_n)]^\lambda,$$

where

$$R_n = \frac{\left(\frac{1}{M(\mathfrak{y}_n, \mathfrak{y}_{n+1}, t)} - 1\right) + \left(\frac{1}{M(\mathfrak{y}_n, \mathfrak{y}_{n+1}, t)} - 1\right)\left(\frac{1}{M(\mathfrak{y}_{n+1}, \mathfrak{y}_{n+2}, t)} - 1\right)}{1 + \left(\frac{1}{M(\mathfrak{y}_n, \mathfrak{y}_{n+1}, t)} - 1\right)}.$$

Since $\lambda \in (0, 1)$ and Θ is increasing, we obtain

$$\frac{1}{M(\mathfrak{y}_{n+1}, \mathfrak{y}_{n+2}, t)} - 1 < \frac{1}{M(\mathfrak{y}_n, \mathfrak{y}_{n+1}, t)} - 1.$$

Hence the sequence $a_n = \frac{1}{M(\mathfrak{y}_n, \mathfrak{y}_{n+1}, t)} - 1$ is decreasing and bounded below by 0. Therefore there exists $L \geq 0$ such that $a_n \rightarrow L$. Passing to the limit in the contractive inequality gives $\Theta(L) \leq [\Theta(L)]^\lambda$.

Since $0 < \lambda < 1$, the above relation is possible only if $L = 0$. Consequently,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{M(\mathfrak{y}_n, \mathfrak{y}_{n+1}, t)} - 1\right) = 0,$$

which implies

$$\lim_{n \rightarrow \infty} M(\mathfrak{y}_n, \mathfrak{y}_{n+1}, t) = 1.$$

We now show that $\{\mathfrak{y}_n\}$ is a Cauchy sequence (CS). Assume to the contrary that $\{\mathfrak{y}_n\}$ is not a FCS. Then there exist $\varepsilon > 0$ and two increasing sequences $\{m_j\}$ and $\{n_j\}$ with

$$n_j > m_j, \quad M(\mathfrak{y}_{m_j}, \mathfrak{y}_{n_j}, t) \leq 1 - \varepsilon.$$

Choose n_j to be the smallest integer exceeding m_j satisfying the above inequality. Consequently,

$$M(\mathfrak{y}_{m_j}, \mathfrak{y}_{n_j-1}, t) > 1 - \varepsilon.$$

By the fuzzy triangle inequality,

$$M(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_j}, t) \geq M\left(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_{j-1}}, \frac{t}{2}\right) * M\left(\mathfrak{v}_{n_{j-1}}, \mathfrak{v}_{n_j}, \frac{t}{2}\right).$$

Since

$$M(\mathfrak{v}_n, \mathfrak{v}_{n+1}, s) \longrightarrow 1 \quad \text{for every } s > 0,$$

we have

$$M\left(\mathfrak{v}_{n_{j-1}}, \mathfrak{v}_{n_j}, \frac{t}{2}\right) \longrightarrow 1.$$

Moreover, by the minimality of n_j ,

$$M(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_{j-1}}, t) > 1 - \varepsilon.$$

Passing to the limit and using the continuity of the t -norm yields

$$\liminf_{j \rightarrow \infty} M(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_j}, t) \geq (1 - \varepsilon) * 1 = 1 - \varepsilon.$$

On the other hand,

$$M(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_j}, t) \leq 1 - \varepsilon$$

for every j , and therefore

$$\lim_{j \rightarrow \infty} M(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_j}, t) = 1 - \varepsilon.$$

Applying the contractive inequality to the pair $(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_j})$ and letting $j \rightarrow \infty$ forces

$$M(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_j}, t) \longrightarrow 1,$$

which contradicts

$$M(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_j}, t) \leq 1 - \varepsilon.$$

Hence $\{\mathfrak{v}_n\}$ is a FCS.

Since $(\mathfrak{F}, M, *)$ is complete, there exists $\mathfrak{v}^* \in \mathfrak{F}$ such that $\lim_{n \rightarrow \infty} \mathfrak{v}_n = \mathfrak{v}^*$.

Because $\mathfrak{F}$ is orbitally G -continuous and $(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \in E(G)$, we have $\mathfrak{F}\mathfrak{v}_n = \mathfrak{v}_{n+1} \longrightarrow \mathfrak{F}\mathfrak{v}^*$.

On the other hand, $\mathfrak{v}_{n+1} \longrightarrow \mathfrak{v}^*$. Therefore $\mathfrak{F}\mathfrak{v}^* = \mathfrak{v}^*$, and hence $\mathfrak{v}^*$ is a FP of $\mathfrak{F}$.

To prove uniqueness, let u and v be two fixed points of $\mathfrak{F}$. Since G is weakly connected, there exists a path joining u and v . Applying the contractive condition repeatedly along this path yields

$$\Theta\left(\frac{1}{M(u, v, t)} - 1\right) < \Theta\left(\frac{1}{M(u, v, t)} - 1\right),$$

which is impossible unless $\frac{1}{M(u, v, t)} - 1 = 0$.

Hence $M(u, v, t) = 1$, and therefore $u = v$. Thus $\mathfrak{F}$ has a UFP $\mathfrak{v}^*$ and $\lim_{n \rightarrow \infty} \mathfrak{F}^n \mathfrak{v} = \mathfrak{v}^*$ for every $\mathfrak{v} \in \mathfrak{F}$. $\square$

Example 3.1. Let $\mathfrak{I} = [0, 1]$ and define the fuzzy metric

$$M(\mathfrak{y}, \mathfrak{t}, t) = \frac{t}{t + |\mathfrak{y} - \mathfrak{t}|}, \quad t > 0.$$

Consider the mapping $\mathfrak{I}\mathfrak{y} = \frac{\mathfrak{y}}{2}$, $\mathfrak{y} \in \mathfrak{I}$. Taking the initial point $\mathfrak{y}_0 = 1$, the Picard sequence is $\mathfrak{y}_{n+1} = \mathfrak{I}\mathfrak{y}_n = \frac{\mathfrak{y}_n}{2}$. Thus 1, 0.5, 0.25, 0.125, 0.0625, ... and $\lim_{n \rightarrow \infty} \mathfrak{y}_n = 0$.

Since $\mathfrak{I}(0) = 0$, the point 0 is the UFP of $\mathfrak{I}$.

Hence all the conclusions of Theorem 3.1 are verified.

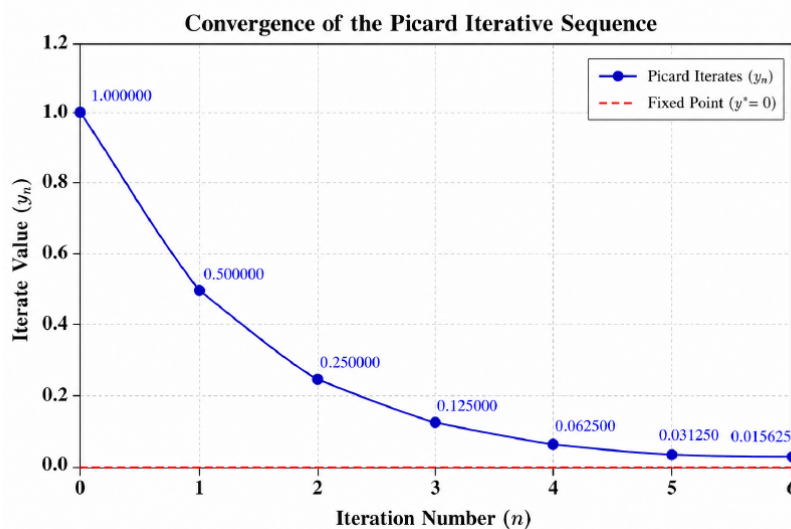


FIGURE 1. Convergence of the Picard iterative sequence generated by the mapping $\mathfrak{I}\mathfrak{y} = \frac{\mathfrak{y}}{2}$.

Figure 1 illustrates the convergence behavior of the Picard iterative sequence corresponding to the mapping $\mathfrak{I}\mathfrak{y} = \frac{\mathfrak{y}}{2}$. Starting from the initial point $\mathfrak{y}_0 = 1$, the successive iterates approach the FP $\mathfrak{y}^* = 0$. The graph confirms that the sequence converges rapidly, supporting the conclusion of Theorem 3.1 regarding the existence and uniqueness of the FP.

Theorem 3.2. Let $(\mathfrak{I}, M, *)$ be a CFMS endowed with a graph G . Suppose that for all $(\mathfrak{y}, \mathfrak{t}) \in E(G)$,

$$\Theta\left(\frac{1}{M(\mathfrak{I}\mathfrak{y}, \mathfrak{I}\mathfrak{t}, t)} - 1\right) \leq [\Theta(R(\mathfrak{y}, \mathfrak{t}))]^\lambda,$$

where

$$\begin{aligned} R(\mathfrak{y}, \mathfrak{t}) = & a\left(\frac{1}{M(\mathfrak{y}, \mathfrak{t}, t)} - 1\right) + b\left(\frac{1}{M(\mathfrak{y}, \mathfrak{I}\mathfrak{y}, t)} - 1\right) \\ & + c\left(\frac{1}{M(\mathfrak{t}, \mathfrak{I}\mathfrak{t}, t)} - 1\right) \\ & + d \frac{\left(\frac{1}{M(\mathfrak{y}, \mathfrak{I}\mathfrak{t}, t)} - 1\right) + \left(\frac{1}{M(\mathfrak{t}, \mathfrak{I}\mathfrak{y}, t)} - 1\right)}{1 + \left(\frac{1}{M(\mathfrak{y}, \mathfrak{t}, t)} - 1\right)}. \end{aligned}$$

If $a + b + c + 2d < 1$, then $\mathfrak{I}$ possesses a UFP in $\mathfrak{I}$.

Proof. Let $\mathfrak{v}_0 \in \mathfrak{I}$ be such that $(\mathfrak{v}_0, \mathfrak{I}\mathfrak{v}_0) \in E(G)$. Define the Picard sequence $\{\mathfrak{v}_n\}$ by

$$\mathfrak{v}_{n+1} = \mathfrak{I}\mathfrak{v}_n = \mathfrak{I}^{n+1}\mathfrak{v}_0, \quad n \geq 0.$$

Since $\mathfrak{I}$ preserves edges, we obtain

$$(\mathfrak{v}_n, \mathfrak{v}_{n+1}) \in E(G), \quad n \in \mathbb{N} \cup \{0\}.$$

If for some n_0 , $\mathfrak{v}_{n_0} = \mathfrak{v}_{n_0+1}$, then $\mathfrak{v}_{n_0} = \mathfrak{I}\mathfrak{v}_{n_0}$, and hence $\mathfrak{v}_{n_0}$ is a FP of $\mathfrak{I}$.

Thus, we may suppose that $\mathfrak{v}_n \neq \mathfrak{v}_{n+1}$ for all n .

Put $\Delta_n = \frac{1}{M(\mathfrak{v}_n, \mathfrak{v}_{n+1}, t)} - 1$. Applying the contractive condition with $\mathfrak{v} = \mathfrak{v}_n$ and $\mathfrak{f} = \mathfrak{v}_{n+1}$, we obtain

$$\Theta(\Delta_{n+1}) \leq [\Theta(R_n)]^\lambda,$$

where

$$R_n = a\Delta_n + b\Delta_n + c\Delta_{n+1} + d \frac{\left(\frac{1}{M(\mathfrak{v}_n, \mathfrak{v}_{n+2}, t)} - 1\right) + \left(\frac{1}{M(\mathfrak{v}_{n+1}, \mathfrak{v}_{n+1}, t)} - 1\right)}{1 + \Delta_n}.$$

Since $M(\mathfrak{v}_{n+1}, \mathfrak{v}_{n+1}, t) = 1$, we obtain $\frac{1}{M(\mathfrak{v}_{n+1}, \mathfrak{v}_{n+1}, t)} - 1 = 0$.

Hence

$$R_n = (a + b)\Delta_n + c\Delta_{n+1} + d \frac{\frac{1}{M(\mathfrak{v}_n, \mathfrak{v}_{n+2}, t)} - 1}{1 + \Delta_n}.$$

Using the monotonicity of the fuzzy metric and the triangular property, there exists a constant $L > 0$ such that $\frac{1}{M(\mathfrak{v}_n, \mathfrak{v}_{n+2}, t)} - 1 \leq \Delta_n + \Delta_{n+1}$. Therefore $R_n \leq (a + b + d)\Delta_n + (c + d)\Delta_{n+1}$.

Since $a + b + c + 2d < 1$, let $\eta = a + b + c + 2d < 1$. Then $R_n \leq \eta \max\{\Delta_n, \Delta_{n+1}\}$.

Suppose that $\Delta_{n+1} \geq \Delta_n$. Then $R_n \leq \eta\Delta_{n+1}$. Consequently,

$$\Theta(\Delta_{n+1}) \leq [\Theta(\eta\Delta_{n+1})]^\lambda.$$

Since $\eta < 1$ and Θ is increasing,

$$\Theta(\eta\Delta_{n+1}) < \Theta(\Delta_{n+1}),$$

which yields

$$\Theta(\Delta_{n+1}) < [\Theta(\Delta_{n+1})]^\lambda.$$

This is impossible because $\lambda \in (0, 1)$.

Hence $\Delta_{n+1} < \Delta_n$ for all n . Thus $\{\Delta_n\}$ is a decreasing sequence of nonnegative numbers. Hence there exists $r \geq 0$ such that $\Delta_n \rightarrow r$. Passing to the limit in the contractive inequality gives $\Theta(r) \leq [\Theta(\eta r)]^\lambda$. Since $\eta < 1$, $\Theta(\eta r) < \Theta(r)$ whenever $r > 0$. Therefore $\Theta(r) < [\Theta(r)]^\lambda$, which is impossible. Hence $r = 0$. Therefore $\lim_{n \rightarrow \infty} \Delta_n = 0$, that is,

$$\lim_{n \rightarrow \infty} M(\eta_n, \eta_{n+1}, t) = 1.$$

Next, we prove that $\{\eta_n\}$ is a fuzzy CS. Suppose not. Then there exist $\varepsilon > 0$ and subsequences $\{\eta_{m_j}\}$, $\{\eta_{n_j}\}$ with $n_j > m_j$ such that $M(\eta_{m_j}, \eta_{n_j}, t) \leq 1 - \varepsilon$. Using the fuzzy triangle inequality repeatedly and the fact that $M(\eta_n, \eta_{n+1}, t) \rightarrow 1$, and in the same way as in the previous theorem one obtains a contradiction. Hence $\{\eta_n\}$ is fuzzy Cauchy. Since $(\mathfrak{J}, M, *)$ is complete, there exists $\eta^* \in \mathfrak{J}$ such that $\eta_n \rightarrow \eta^*$. Since $\mathfrak{J}$ is orbitally G -continuous, $\mathfrak{J}\eta_n = \eta_{n+1} \rightarrow \mathfrak{J}\eta^*$. But $\eta_{n+1} \rightarrow \eta^*$. Therefore $\mathfrak{J}\eta^* = \eta^*$. Hence η^* is a FP of $\mathfrak{J}$.

To prove uniqueness, let $u, v \in \text{Fix}(\mathfrak{J})$.

Applying the contractive condition, we get

$$\Theta\left(\frac{1}{M(u, v, t)} - 1\right) \leq \left[\Theta\left((a + d)\left(\frac{1}{M(u, v, t)} - 1\right)\right)\right]^\lambda.$$

Since $a + d < 1$,

$$\Theta\left((a + d)\left(\frac{1}{M(u, v, t)} - 1\right)\right) < \Theta\left(\frac{1}{M(u, v, t)} - 1\right).$$

Thus $\frac{1}{M(u, v, t)} - 1 = 0$. Consequently, $M(u, v, t) = 1$, which implies $u = v$. Therefore $\mathfrak{J}$ possesses a UFP in $\mathfrak{J}$. $\square$

Theorem 3.3. Let $(\mathfrak{J}, M, *)$ be a CFMS endowed with a graph G . Assume that there exists a continuous function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that

$$\int_0^{\Theta\left(\frac{1}{M(\mathfrak{J}\eta, \mathfrak{J}\mathfrak{k}, t)} - 1\right)} \varphi(s) ds \leq \lambda \int_0^{\Theta(R(\eta, \mathfrak{k}))} \varphi(s) ds$$

for all $(\eta, \mathfrak{k}) \in E(G)$, where

$$R(\eta, \mathfrak{k}) = \frac{\left(\frac{1}{M(\eta, \mathfrak{k}, t)} - 1\right) + \left(\frac{1}{M(\eta, \mathfrak{J}\eta, t)} - 1\right) + \left(\frac{1}{M(\mathfrak{k}, \mathfrak{J}\mathfrak{k}, t)} - 1\right)}{1 + \left(\frac{1}{M(\eta, \mathfrak{k}, t)} - 1\right)}.$$

Then $\mathfrak{J}$ admits a UFP in $\mathfrak{J}$.

Proof. Let $\eta_0 \in \mathfrak{J}$ be such that $(\eta_0, \mathfrak{J}\eta_0) \in E(G)$. Define the Picard sequence $\{\eta_n\}$ by $\eta_{n+1} = \mathfrak{J}\eta_n = \mathfrak{J}^{n+1}\eta_0, n \geq 0$. Since $\mathfrak{J}$ preserves the edges of G , it follows that $(\eta_n, \eta_{n+1}) \in E(G), n \geq 0$. If there

exists $n_0 \in \mathbb{N}$ such that $\mathfrak{v}_{n_0} = \mathfrak{v}_{n_0+1}$, then $\mathfrak{I}\mathfrak{v}_{n_0} = \mathfrak{v}_{n_0}$, and hence $\mathfrak{v}_{n_0}$ is a FP of $\mathfrak{I}$. Therefore, we may assume that $\mathfrak{v}_n \neq \mathfrak{v}_{n+1}$ for all $n \in \mathbb{N}$. For convenience, define

$$\Delta_n = \frac{1}{M(\mathfrak{v}_n, \mathfrak{v}_{n+1}, t)} - 1.$$

Applying the integral rational contraction condition with $\mathfrak{v} = \mathfrak{v}_n$, $\mathfrak{k} = \mathfrak{v}_{n+1}$, we obtain

$$\int_0^{\Theta(\Delta_{n+1})} \varphi(s) ds \leq \lambda \int_0^{\Theta(R_n)} \varphi(s) ds, \quad (3.1)$$

where

$$R_n = \frac{\Delta_n + \Delta_n + \Delta_{n+1}}{1 + \Delta_n} = \frac{2\Delta_n + \Delta_{n+1}}{1 + \Delta_n}.$$

As $1 + \Delta_n > 1$, we have $R_n < 2\Delta_n + \Delta_{n+1}$. Since φ is nonnegative and continuous, the function $\Phi(u) = \int_0^u \varphi(s) ds$ is increasing on $[0, \infty)$.

Therefore, from (3.1),

$$\Phi(\Theta(\Delta_{n+1})) \leq \lambda \Phi(\Theta(R_n)). \quad (3.2)$$

Since $0 < \lambda < 1$, relation (3.2) yields

$$\Phi(\Theta(\Delta_{n+1})) < \Phi(\Theta(R_n)).$$

By the monotonicity of Φ and Θ , we obtain

$$\Delta_{n+1} < R_n. \quad (3.3)$$

Using the expression of R_n , $\Delta_{n+1} < \frac{2\Delta_n + \Delta_{n+1}}{1 + \Delta_n}$. After simplification, $\Delta_{n+1} < 2\Delta_n$. Consequently, the sequence $\{\Delta_n\}$ is bounded and decreasing. Hence there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \Delta_n = r$.

Passing to the limit in (3.1), we obtain $\Phi(\Theta(r)) \leq \lambda \Phi(\Theta(r))$. Since $0 < \lambda < 1$, it follows that $(1 - \lambda)\Phi(\Theta(r)) \leq 0$. Thus $\Phi(\Theta(r)) = 0$. Since $\varphi(s) > 0$ for $s > 0$, $\Theta(r) = 0$. Since Θ is positive and increasing, we conclude that $r = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{M(\mathfrak{v}_n, \mathfrak{v}_{n+1}, t)} - 1 \right) = 0,$$

which implies

$$\lim_{n \rightarrow \infty} M(\mathfrak{v}_n, \mathfrak{v}_{n+1}, t) = 1. \quad (3.4)$$

Next we show that $\{\mathfrak{v}_n\}$ is a fuzzy CS.

Assume to the contrary that it is not Cauchy. Then there exist $\varepsilon > 0$ and subsequences $\{\mathfrak{v}_{m_j}\}$ and $\{\mathfrak{v}_{n_j}\}$ with $n_j > m_j$ such that $M(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_j}, t) \leq 1 - \varepsilon$. As in the theorems above, using the fuzzy triangle inequality repeatedly together with (3.4), we obtain $M(\mathfrak{v}_{m_j}, \mathfrak{v}_{n_j}, t) \rightarrow 1$, which contradicts the above inequality. Hence $\{\mathfrak{v}_n\}$ is a fuzzy CS.

Since $(\mathfrak{I}, M, *)$ is complete, there exists $\mathfrak{v}^* \in \mathfrak{I}$ such that

$$\eta_n \longrightarrow \eta^*. \quad (3.5)$$

Assume that $\mathfrak{I}$ is orbitally G -continuous. Then from (3.5), $\mathfrak{I}\eta_n = \eta_{n+1} \longrightarrow \mathfrak{I}\eta^*$. But $\eta_{n+1} \longrightarrow \eta^*$. Hence $\mathfrak{I}\eta^* = \eta^*$. Thus η^* is a FP of $\mathfrak{I}$.

To prove uniqueness, let $u, v \in \text{Fix}(\mathfrak{I})$. Applying the integral contraction condition, we obtain

$$\int_0^{\Theta\left(\frac{1}{M(u,v,t)}-1\right)} \varphi(s) ds \leq \lambda \int_0^{\Theta\left(\frac{1}{M(u,v,t)}-1\right)} \varphi(s) ds.$$

Therefore,

$$(1 - \lambda) \int_0^{\Theta\left(\frac{1}{M(u,v,t)}-1\right)} \varphi(s) ds \leq 0.$$

Since $\lambda < 1$, we obtain

$$\int_0^{\Theta\left(\frac{1}{M(u,v,t)}-1\right)} \varphi(s) ds = 0.$$

Hence $\frac{1}{M(u,v,t)} - 1 = 0$. Thus $M(u,v,t) = 1$, which implies $u = v$. Therefore $\mathfrak{I}$ possesses a UFP in $\mathfrak{I}$. $\square$

Theorem 3.4. Let $(\mathfrak{I}, M, *)$ be a CFMS endowed with a graph G . Suppose that $\Theta\left(\frac{1}{M(\mathfrak{I}\eta, \mathfrak{I}\mathfrak{k}, t)} - 1\right) \leq \left[\Theta(Q(\eta, \mathfrak{k}))\right]^\lambda$, where

$$Q(\eta, \mathfrak{k}) = \max\left\{\frac{1}{M(\eta, \mathfrak{k}, t)} - 1, \frac{1}{M(\eta, \mathfrak{I}\eta, t)} - 1, \frac{1}{M(\mathfrak{k}, \mathfrak{I}\mathfrak{k}, t)} - 1, \frac{\left(\frac{1}{M(\eta, \mathfrak{I}\eta, t)} - 1\right)\left(\frac{1}{M(\mathfrak{k}, \mathfrak{I}\mathfrak{k}, t)} - 1\right)}{1 + \left(\frac{1}{M(\eta, \mathfrak{k}, t)} - 1\right)}\right\}.$$

Then $\mathfrak{I}$ possesses a UFP in $\mathfrak{I}$.

Proof. Let $\eta_0 \in \mathfrak{I}$ be such that $(\eta_0, \mathfrak{I}\eta_0) \in E(G)$. Define the Picard sequence $\eta_{n+1} = \mathfrak{I}\eta_n = \mathfrak{I}^{n+1}\eta_0, n \geq 0$. Since $\mathfrak{I}$ preserves edges, we have $(\eta_n, \eta_{n+1}) \in E(G), n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $\eta_{n_0} = \eta_{n_0+1}$, then $\mathfrak{I}\eta_{n_0} = \eta_{n_0}$, and hence η_{n_0} is a FP of $\mathfrak{I}$. Therefore, we assume that $\eta_n \neq \eta_{n+1}$ for all $n \geq 0$. Set $\Delta_n = \frac{1}{M(\eta_n, \eta_{n+1}, t)} - 1$. Applying the contractive condition with $\eta = \eta_n, \mathfrak{k} = \eta_{n+1}$, we obtain

$$\Theta(\Delta_{n+1}) \leq [\Theta(Q_n)]^\lambda, \quad (4.1)$$

where

$$Q_n = \max\left\{\Delta_n, \Delta_n, \Delta_{n+1}, \frac{\Delta_n \Delta_{n+1}}{1 + \Delta_n}\right\}.$$

Since $\frac{\Delta_n \Delta_{n+1}}{1 + \Delta_n} \leq \Delta_n$, it follows that $Q_n = \max\{\Delta_n, \Delta_{n+1}\}$. Suppose that $\Delta_{n+1} \geq \Delta_n$. Then $Q_n = \Delta_{n+1}$. Consequently, from (4.1), $\Theta(\Delta_{n+1}) \leq [\Theta(\Delta_{n+1})]^\lambda$. Since $0 < \lambda < 1$, the above inequality can hold only if $\Delta_{n+1} = 0$. This contradicts $\eta_{n+1} \neq \eta_{n+2}$. Hence $\Delta_{n+1} < \Delta_n$ for every n .

Thus $\{\Delta_n\}$ is a decreasing sequence of nonnegative real numbers. Therefore, there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \Delta_n = r$. Passing to the limit in (4.1), we obtain $\Theta(r) \leq [\Theta(r)]^\lambda$. Since $0 < \lambda < 1$ and Θ is increasing, the above inequality is possible only when $r = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{M(\eta_n, \eta_{n+1}, t)} - 1 \right) = 0,$$

which implies

$$\lim_{n \rightarrow \infty} M(\eta_n, \eta_{n+1}, t) = 1. \quad (4.2)$$

Next, we show that $\{\eta_n\}$ is a fuzzy CS.

Assume to the contrary that it is not. Then there exist $\varepsilon > 0$ and subsequences $\{\eta_{m_j}\}$ and $\{\eta_{n_j}\}$ with $n_j > m_j$ such that $M(\eta_{m_j}, \eta_{n_j}, t) \leq 1 - \varepsilon$. Using the fuzzy triangle inequality and (4.2), one obtains $M(\eta_{m_j}, \eta_{n_j}, t) \rightarrow 1$, which contradicts the above inequality. Hence $\{\eta_n\}$ is a fuzzy CS. Since $(\mathfrak{I}, M, *)$ is complete, there exists $\eta^* \in \mathfrak{I}$ such that

$$\eta_n \rightarrow \eta^*. \quad (4.3)$$

Since $\mathfrak{I}$ is orbitally G -continuous, $\mathfrak{I}\eta_n = \eta_{n+1} \rightarrow \mathfrak{I}\eta^*$.

On the other hand, $\eta_{n+1} \rightarrow \eta^*$. Therefore, $\mathfrak{I}\eta^* = \eta^*$, and hence η^* is a FP of $\mathfrak{I}$.

To prove uniqueness, let $u, v \in \text{Fix}(\mathfrak{I})$. Then $\mathfrak{I}u = u$, $\mathfrak{I}v = v$.

Applying the contractive condition, we get

$$\Theta\left(\frac{1}{M(u, v, t)} - 1\right) \leq \left[\Theta\left(\frac{1}{M(u, v, t)} - 1\right)\right]^\lambda.$$

This implies $\frac{1}{M(u, v, t)} - 1 = 0$. Hence $M(u, v, t) = 1$, which yields $u = v$. Therefore, $\mathfrak{I}$ possesses a UFP η^* in $\mathfrak{I}$. $\square$

4. CONCLUSION

A new class of graphic rational Θ -contractions in fuzzy metric spaces endowed with a directed graph has been introduced and studied. Several existence and uniqueness results were established in complete fuzzy metric spaces under suitable graph-theoretic assumptions. Moreover, Hardy–Rogers-type, integral-type, and Ćirić-type rational contractions were obtained as consequences of the proposed framework. The convergence of Picard iterative sequences to the unique FP was proved and illustrated through a numerical example.

The results presented herein extend a number of existing FP theorems involving graphs, Θ -contractions, and fuzzy metric spaces. Future work may include applications to nonlinear integral

equations, differential equations, multivalued mappings, and more general fuzzy metric-type structures.

Authors' Contributions: All authors contribute equally in this paper.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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