

Topological Approximations Using Maximal Neighborhoods and Primal Structures with Advanced Decision-Making Strategies for Heart Failure Diagnosis

M. A. El-Gayar¹, O. A. Embaby², M. K. El-Bably^{2,*}, R. A. Hosny³

¹*Department of Mathematics, Faculty of Science, Capital University (formerly Helwan), Helwan 11795, Egypt*

²*Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt*

³*Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt*

**Corresponding author: mkamel_bably@yahoo.com*

Abstract. This paper develops a unified framework for modeling uncertainty in structured and evolving data environments by integrating primal structures with generalized approximation methods based on κ -neighborhood systems. Classical rough set theory is primarily founded on equivalence relations, which may limit its ability to effectively represent imprecise, dynamic, and sensitive information arising in complex real-world applications. To overcome these limitations, we introduce novel lower and upper approximation operators defined through κ -neighborhood spaces and primal sets, with particular emphasis on maximal neighborhood structures. The proposed approximation operators provide enhanced flexibility in the construction of rough approximations, particularly within topological and supra-topological settings. They enable approximation sets to capture more effectively the uncertainty associated with structural asymmetry, deformation, and changes in the underlying data. The fundamental properties of the proposed operators are systematically established and illustrated through suitable examples. Furthermore, a comprehensive comparative analysis with classical and recent rough set models demonstrates the effectiveness of the proposed framework in improving data representation and analysis. The practical applicability of the proposed approach is demonstrated through applications in digital image processing, including pattern recognition, noise removal, and feature extraction. In addition, a decision-making case study illustrates the effectiveness of the proposed classification methods and indicates improved performance compared with conventional rough set approaches. Overall, the proposed framework establishes a versatile algebraic-topological approach to uncertainty modeling, offering a rigorous foundation for further theoretical developments and a broad range of practical applications in data analysis, image processing, and decision support.

Received: Jun. 29, 2026.

2020 *Mathematics Subject Classification.* 03E99; 54A05; 54A10; 54E99.

Key words and phrases. supra topology; maximal neighbourhood systems; primal; approximation spaces.

1. INTRODUCTION

Real-world information arising in fields such as engineering, healthcare, social sciences, artificial intelligence, and data mining is frequently characterized by incompleteness, ambiguity, inconsistency, and uncertainty. These characteristics pose substantial challenges to conventional mathematical and computational approaches for knowledge representation and decision support. Over the past several decades, a variety of mathematical frameworks have therefore been developed to model and analyze uncertain information. Among these, rough set theory has emerged as an effective and mathematically rigorous approach for describing imprecise knowledge and extracting meaningful information from complex datasets.

Classical rough set theory, introduced by Pawlak [1], provides a systematic mechanism for representing imprecise information through lower and upper approximation operators. In the classical Pawlak model, an equivalence relation partitions the universe into equivalence classes that represent the indiscernibility of objects. Although this formulation has played a fundamental role in the development of rough set theory, the requirement of an equivalence relation may be restrictive when dealing with incomplete, dynamic, or structurally complex information. This limitation has motivated the development of numerous generalized rough set models based on more flexible relational and neighborhood structures [2,3].

In particular, relational and neighborhood-based approaches have significantly extended the expressive capability of rough set theory. Generalized rough sets based on arbitrary relations, reflexive and transitive relations, and neighborhood systems have been investigated extensively, providing alternative mechanisms for constructing lower and upper approximations [4,5]. More recent developments have incorporated adaptive neighborhood systems and optimization techniques to improve the performance of rough approximations in feature selection and data analysis [6]. These approaches demonstrate that relaxing the structural restrictions imposed by classical equivalence-based rough sets can substantially enhance their applicability to complex information systems. Applications in variety of areas, including data clustering [7], multi-knowledge neighborhood structures [8,9], feature selection in machine learning [10–12], the development of novel mathematical structures [13–15], and medical diagnosis [16,17]. These developments demonstrate the importance of generalizing the underlying structures of rough set theory in order to enhance its expressive power and applicability to increasingly complex information systems.

In 2014, Abd El-Monsef et al. [18] introduced a topological framework that extends Pawlak's rough set theory through neighborhood systems generated by general binary relations. Their approach provides a flexible mechanism for constructing topological structures and approximation models from a wide variety of neighborhood systems, thereby allowing the underlying relational structure to be adapted to the characteristics of the data under consideration.

A central component of this framework is the concept of κ -neighborhood spaces, in which the elements of the universe are associated through neighborhoods induced by a binary relation. This approach is closely related to Yao's notions of left and right neighborhoods [19] and to the

neighborhood concepts previously introduced by Allam et al. [20, 21]. Furthermore, Abd El-Monsef et al. [18] introduced additional neighborhood types, including intersection and union neighborhoods, thereby providing a broader family of neighborhood systems characterized by distinct algebraic properties.

An important contribution of Abd El-Monsef et al. [18] was the development of a general mechanism for generating topological structures from neighborhood systems without imposing restrictive conditions on the form of the neighborhoods. In particular, this construction differs from traditional topological generation methods in that it does not require the specification of a basis or subbasis.

Relatedly, Abo-Khadra [22] developed a general approach for constructing topologies directly from binary relations without imposing specific restrictions on the underlying relation or requiring a basis or subbasis. In particular, two dual topologies were introduced, one generated by predecessor sets and the other by successor sets. This perspective represents an early attempt to establish a close connection between topological structures and rough set theory and was subsequently further developed in El-Bably's Master's thesis [23].

Within the neighborhood-based framework, Abd El-Monsef et al. considered the family

$$\mathfrak{T} = \{\mathbb{E} \subseteq \mathfrak{Q} : \forall s \in \mathfrak{Q}, \mathfrak{U}(s) \subseteq \mathbb{E}\},$$

thereby establishing a topology on $\mathfrak{Q}$. Here, $\mathfrak{Q}$ denotes the universal set and $\mathfrak{U}(s)$ represents a neighborhood associated with the element s . By varying the underlying neighborhood systems, different topological structures can be generated on the same universe. Numerous studies, including [24–27], have employed this general framework to construct and investigate topological structures derived from the eight neighborhood classes introduced by Abd El-Monsef et al. Consequently, this line of research has contributed significantly to the development of topological approaches to rough set theory [28–31].

Since its introduction, the neighborhood-based framework of Abd El-Monsef et al. has been extended and adapted to accommodate a broad range of neighborhood systems. Many of its fundamental properties and theoretical results have subsequently been employed and generalized while preserving the central idea of generating topological structures from neighborhood systems. The generality of this construction, which does not depend on a particular form of neighborhood, has consequently provided a useful foundation for the development of various rough approximation models.

For instance, Al-Shami [32] employed this general framework to further investigate the concept of maximal neighborhoods, originally introduced by Dai et al. [33]. Subsequently, additional studies examined maximal neighborhood systems and their associated approximation properties, including the investigations reported in [34, 35]. However, some developments based on these approaches have required further examination, particularly with regard to the validity of certain theoretical results and their implications for rough approximation models.

Building on the methodology of Abd El-Monsef et al. and the concept of ideals [36], Al-Shami and M. Hosny proposed a generalized rough set model in [37]. Subsequently, several issues concerning the formulation of some results, proofs, examples, and comparisons were identified. These issues were later examined by R. Hosny et al. [38], who provided counterexamples, corrected certain formulations, and offered further theoretical clarifications.

Within this line of research, Nawar et al. [39] introduced the concept of $\theta\beta$ -rough approximations and proposed a novel definition of $\theta\beta$ -open sets. By incorporating ideals into the approximation process, they developed two distinct extensions of Pawlak's rough set model. These constructions were illustrated through an application from chemistry. The first approach was based on topological structures generated from κ -neighborhood systems following the framework of Abd El-Monsef et al., whereas the second incorporated both κ -neighborhood systems and ideals, thereby extending the corresponding ideal-based approach [40]. The authors also introduced ideal $\theta\beta$ -open sets and $\theta\beta$ -rough sets within κ -neighborhood spaces, building on the neighborhood framework developed in [18].

Subsequently, El-Bably et al. [41] further developed this line of research by establishing additional theoretical results and proposing new algorithms, thereby extending the scope and applicability of the $\theta\beta$ -approximation framework. Their work builds directly upon and further develops the results introduced by Nawar et al.

The studies reported in [39,41] provide a systematic framework for defining $\theta\beta$ -related concepts and investigating their relationship with ideals. These results also establish a theoretical basis against which subsequent generalizations and alternative formulations may be evaluated.

The development of generalized approximation spaces has also benefited from the use of supra-topological structures. The concept of a supra-topology was introduced by Mashhour [42], who generalized the classical notion of topology by retaining closure under arbitrary unions while relaxing the requirement of closure under finite intersections. More precisely, a family $\mathcal{S}$ of subsets of a nonempty universe $\mathfrak{Q}$ is called a supra-topology if $\emptyset, \mathfrak{Q} \in \mathcal{S}$ and $\mathcal{S}$ is closed under arbitrary unions. This relaxation provides a broader structural setting for the construction of generalized approximation models and has subsequently been investigated in several applications [43].

Two closely related concepts in generalized topology are Kuratowski's ideals [36] and Choquet's grills [44]. Motivated by the duality between these structures, Acharjee et al. introduced the concept of a primal structure and investigated its role in the construction of primal topological spaces.

The systematic study of primal structures in [45] represents an important development in this area. Subsequently, R. A. Hosny et al. [46] extended this line of research by introducing approximate primal spaces. In particular, they proposed two approximation models: one based on a single primal structure and another based on a bidirectional construction. These models were further investigated in the context of decision-making and medical diagnosis. Moreover, R. A. Hosny and El-Bably [47] introduced an approach for constructing minimal topological structures

from κ -neighborhood systems and primal structures, thereby providing a further extension of the framework developed in [46]. In addition El-Bably et al. [48] study the concept of primal approximation spaces and their decisive role in advancing rough set theory. They introduce primal structures and present their basic properties. Furthermore, they introduce primal supra topology and primal approximation operators.

Motivated by these developments, the present study investigates the interaction between primal structures and different types of neighborhood systems, with particular emphasis on maximal neighborhoods. This perspective extends the role of primal structures beyond their conventional topological setting and explores their potential for constructing new rough approximation models. In the proposed framework, primal structures serve as algebraic tools for representing structural restrictions and uncertainty within information systems, while maximal neighborhoods provide a flexible mechanism for describing local relationships among objects.

Research Aims and Objectives. The primary aim of this study is to develop a novel framework for rough approximations based on the integration of primal structures and maximal neighborhood systems. The proposed approach is intended to provide a flexible mathematical mechanism for generating approximation operators within supra-topological settings and for modeling uncertainty in complex information and decision-making systems.

More specifically, the objectives of this study are as follows:

- (1) To develop new lower and upper approximation operators by integrating maximal neighborhood systems with primal structures and to investigate the supra-topological structures induced by these operators.
- (2) To establish and analyze the fundamental mathematical properties of the proposed approximation models and to examine their relationships with existing rough set and neighborhood-based approaches.
- (3) To demonstrate the applicability and effectiveness of the proposed models through appropriate practical applications, particularly in situations where greater structural flexibility is required than that provided by conventional ideal-based approaches.

Motivation. This study is motivated by the need for more flexible mathematical frameworks capable of representing ambiguity, vagueness, and uncertainty in complex information systems. Although classical rough set theory and its numerous extensions provide powerful tools for uncertainty modeling, their applicability may be constrained by restrictive structural assumptions. In particular, the requirement of closure under finite intersections in classical topological settings may limit the flexibility of approximation models constructed within such frameworks.

Motivated by the ability of supra-topological structures to relax this requirement and by the recently developed theory of primal structures, the present work investigates a new approach to the construction of rough approximations. The proposed framework combines primal structures

with maximal neighborhood systems in order to provide a more flexible setting for modeling uncertainty and analyzing complex information.

Accordingly, the present study is motivated by the following objectives:

- (1) To relax certain structural restrictions encountered in existing rough set and topological approximation models.
- (2) To provide a flexible and adaptive mathematical framework for modeling and analyzing information characterized by ambiguity, uncertainty, and structural variability.
- (3) To investigate the potential of primal structures, in conjunction with maximal neighborhood systems, for enhancing the expressive and analytical capabilities of rough set theory.
- (4) To explore the resulting approximation models within supra-topological settings and to examine their potential applicability to practical decision-making problems.

Main Contributions. The main contributions of this study can be summarized as follows:

- (1) A novel rough approximation framework is developed by integrating primal structures with maximal neighborhood systems. This integration provides a flexible mechanism for constructing lower and upper approximation operators in supra-topological settings.
- (2) Several new approximation operators are introduced and their fundamental mathematical properties are systematically investigated. The proposed constructions provide alternative mechanisms for representing uncertainty through the interaction between primal structures and neighborhood-based information.
- (3) The relationships between the proposed primal-based models and existing rough approximation frameworks are examined through a comparative theoretical analysis. This comparison clarifies the distinctive characteristics, potential advantages, and limitations of the proposed approach.
- (4) The practical applicability of the proposed framework is demonstrated through representative case studies involving digital image processing and multi-criteria decision-making. These applications illustrate the ability of the proposed approximation models to support tasks such as recognition, noise removal, feature extraction, and decision classification.

The remainder of this paper is organized as follows. Section 2 presents the preliminary concepts and mathematical background required for the development of the proposed framework. Section 3 introduces the proposed primal-based approximation models and investigates their fundamental properties. Section 4 provides a comparative analysis with related rough set models. Section 5 demonstrates the practical applicability of the proposed framework through representative applications in decision-making. Finally, Section 6 concludes the paper and outlines possible directions for future research.

2. PRELIMINARIES

This section outlines the fundamental concepts of primals, κ -neighborhoods, maximal κ -neighborhoods, and supra topological spaces.

As a dual counterpart of the notion of a *grill*, Acharjee et al. [45] introduced the concept of a *primal*, defined as follows:

Definition 2.1 ([45]). Let $\mathfrak{Q} \neq \emptyset$. A family $\mathfrak{P} \subseteq 2^{\mathfrak{Q}}$ is said to be a *primal* on $\mathfrak{Q}$ provided that the following conditions are satisfied:

- (1) $\mathfrak{Q} \notin \mathfrak{P}$;
- (2) whenever $\mathcal{H} \notin \mathfrak{P}$ and $\mathcal{H} \subseteq \mathcal{X}$, then $\mathcal{X} \notin \mathfrak{P}$;
- (3) for any $\mathcal{X}, \mathcal{H} \subseteq \mathfrak{Q}$, one has

$$\mathcal{X} \notin \mathfrak{P}, \quad \mathcal{H} \notin \mathfrak{P} \iff \mathcal{X} \cap \mathcal{H} \notin \mathfrak{P}.$$

Remark 2.1 ([46,47]). (1) The set $\mathfrak{P} = \{\emptyset\}$ cannot serve as a primal over any universe. To illustrate, take $\mathfrak{Q} = \{v, \rho\}$. In this case, the intersection of $\{v\}$ and $\{\rho\}$ is the empty set, which belongs to $\mathfrak{P}$, although neither $\{v\}$ nor $\{\rho\}$ belongs to $\mathfrak{P}$.

(2) If $\mathfrak{Q} = \{v\}$ consists of a single element, an ideal can be constructed, but a primal cannot be formed, as the definition of a primal necessitates a minimum of two elements.

(3) It should be noted that the intersection or union of two primals does not, in general, yield another primal. This is illustrated by Example 2.1.

Example 2.1. Let

$$\mathfrak{P} = \{\emptyset, \{v\}, \{\rho\}, \{\omega\}, \{v, \rho\}, \{\rho, \omega\}, \{v, \omega\}, \{v, \rho, \omega\}\},$$

and

$$\widetilde{\mathfrak{P}} = \{\emptyset, \{v\}, \{\rho\}, \{\varsigma\}, \{v, \rho\}, \{\rho, \varsigma\}, \{v, \varsigma\}, \{v, \rho, \varsigma\}\}$$

be two primals defined on the universe $\mathfrak{Q} = \{v, \rho, \varsigma, \omega\}$. Then the following observations hold:

- The union

$$\mathfrak{P} \cup \widetilde{\mathfrak{P}} = 2^{\mathfrak{Q}} \setminus \{\mathfrak{Q}, \{v, \varsigma, \omega\}, \{\rho, \varsigma, \omega\}, \{\varsigma, \omega\}\}$$

constitutes a primal on $\mathfrak{Q}$.

- Consider the intersection family

$$\Gamma = \{\mathcal{X} \cap \mathcal{X}' : \mathcal{X} \in \mathfrak{P}, \mathcal{X}' \in \widetilde{\mathfrak{P}}\} = \{\emptyset, \{v\}, \{\rho\}, \{v, \rho\}\}.$$

Then Γ does not constitute a primal. Indeed, although $\{v, \rho\} \in \Gamma$, neither $\{v, \rho, \varsigma\}$ nor $\{v, \rho, \omega\}$ belongs to Γ . Hence, condition (iii) in the definition of a primal is violated.

- The union family

$$\Sigma = \{\mathcal{X} \cup \mathcal{X}' : \mathcal{X} \in \mathfrak{P}, \mathcal{X}' \in \widetilde{\mathfrak{P}}\} = 2^{\mathfrak{Q}}$$

does not constitute a primal, since $\mathfrak{Q} \in \Sigma$, whereas a primal must exclude the entire universe.

Remark 2.2. It should be noted that the definition of a primal given by Acharjee et al. [45] is intended as the dual counterpart of the notion of a grill. However, the definition, as originally stated, does not explicitly require the family $\mathfrak{P}$ to be nonempty. This omission is mathematically significant. Indeed, if $\mathfrak{P} = \emptyset$, then all the stated conditions are satisfied vacuously: $\mathfrak{Q} \notin \mathfrak{P}$ holds trivially, while the hereditary condition and the intersection condition have no instances to verify. Nevertheless, in this case, $\emptyset \notin \mathfrak{P}$.

On the other hand, if $\mathfrak{P} \neq \emptyset$, then $\emptyset \in \mathfrak{P}$ follows immediately from the hereditary condition. In fact, for any $\mathcal{H} \in \mathfrak{P}$, since $\emptyset \subseteq \mathcal{H}$, we obtain $\emptyset \in \mathfrak{P}$. Consequently,

$$\mathfrak{P} \neq \emptyset \implies \emptyset \in \mathfrak{P}.$$

Thus, the original formulation admits the degenerate case $\mathfrak{P} = \emptyset$, which does not exhibit the intended dual relationship with a grill. To exclude this degenerate case and to ensure the proper axiomatic structure of a primal, the nonemptiness of $\mathfrak{P}$ should be explicitly incorporated into the definition.

Accordingly, in the present paper, we adopt the following corrected formulation of a primal, in which the essential condition $\mathfrak{P} \neq \emptyset$ is explicitly imposed.

Definition 2.2. Let $\mathfrak{Q} \neq \emptyset$. A nonempty family $\mathfrak{P} \subseteq 2^{\mathfrak{Q}}$ is said to be a **primal** on $\mathfrak{Q}$ provided that the following conditions are satisfied:

- (1) $\mathfrak{Q} \notin \mathfrak{P}$;
- (2) whenever $\mathcal{H} \in \mathfrak{P}$ and $\mathcal{O} \subseteq \mathcal{H}$, then $\mathcal{O} \in \mathfrak{P}$;
- (3) for any $\mathcal{H}, \mathcal{O} \subseteq \mathfrak{Q}$, if $\mathcal{H} \cap \mathcal{O} \in \mathfrak{P}$, then $\mathcal{H} \in \mathfrak{P}$ or $\mathcal{O} \in \mathfrak{P}$.

Lemma 2.1 ([45–47]). The following are representative examples of primals defined on a universe $\mathfrak{Q}$:

- (1) Trivial primal: $\mathfrak{P} = 2^{\mathfrak{Q}} \setminus \{\mathfrak{Q}\}$.
- (2) Point-omission primal: $\mathfrak{P}_z = \{\mathcal{H} \subseteq \mathfrak{Q} : z \notin \mathcal{H}\}$, where $z \in \mathfrak{Q}$.
- (3) Modified excluded set primal: $\mathfrak{P}_{\mathcal{F}} = \{\mathcal{H} \subseteq \mathfrak{Q} : \mathcal{H} \cup \mathcal{F} \neq \mathfrak{Q}\}$, where $\mathcal{F} \subseteq \mathfrak{Q}$.

Definition 2.3 ([42]). Let $(\mathfrak{Q}, \mathcal{S})$ be a supra-topological space. A set $\mathcal{H}$ of $\mathfrak{Q}$ is named:

- **Supra-open** if $\mathcal{H} \in \mathcal{S}$;
- **Supra-closed** if $\mathcal{H}^c \in \mathcal{S}$.

The **supra-interior** of a subset $\mathcal{H}$, represented by $\text{Sint}(\mathcal{H})$, is defined as the union of every supra-open set that is a subset of $\mathcal{H}$, which constitutes the most extensive supra-open set included in $\mathcal{H}$.

Correspondingly, the **supra-closure** of $\mathcal{H}$, denoted $\text{Scl}(\mathcal{H})$, is characterized as the intersection of all supra-closed sets that contain $\mathcal{H}$, representing the most restricted supra-closed superset of $\mathcal{H}$.

Different kinds of neighborhoods within a universe $\mathfrak{Q}$ can be constructed with respect to a two-element (binary) relation $\mathcal{R}$. These neighborhoods are distinguished by the specific modes of interaction among the elements of $\mathfrak{Q}$ induced by $\mathcal{R}$. An outline of these neighborhood categories is presented below:

Definition 2.4. [18, 19, 21] Let $\mathcal{R}$ be a relation defined between pairs of elements in a universe $\mathfrak{Q}$. For a given point $\omega \in \mathfrak{Q}$, we specify its κ -neighborhood, symbolized by $N_{\kappa}(\omega)$ for $\kappa \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$, as the coming cases:

- (1) **Right Neighborhood (r-neighborhood):**

$$N_r(\omega) = \{z \in \mathfrak{Q} : (\omega, z) \in \mathcal{R}\}.$$

(2) **Left Neighborhood (*l*-neighborhood):**

$$N_l(\omega) = \{z \in \mathfrak{Q} : (z, \omega) \in \mathcal{R}\}.$$

(3) **Intersection Neighborhood (*i*-neighborhood):**

$$N_i(\omega) = N_r(\omega) \cap N_l(\omega).$$

(4) **Union Neighborhood (*u*-neighborhood):**

$$N_u(\omega) = N_r(\omega) \cup N_l(\omega).$$

(5) **Minimal Right Neighborhood (*r*-neighborhood):**

$$N_{\langle r \rangle}(\omega) = \bigcap \{N_r(z) : \omega \in N_r(z)\},$$

with the convention that $N_{\langle r \rangle}(\omega) = \emptyset$ if the collection $\{N_r(z) : \omega \in N_r(z)\}$ is empty.

(6) **Minimal Left Neighborhood (*l*-neighborhood):**

$$N_{\langle l \rangle}(\omega) = \bigcap \{N_l(z) : \omega \in N_l(z)\},$$

with the convention that $N_{\langle l \rangle}(\omega) = \emptyset$ if the collection $\{N_l(z) : \omega \in N_l(z)\}$ is empty.

(7) **Minimal Intersection Neighborhood (*i*-neighborhood):**

$$N_{\langle i \rangle}(\omega) = N_{\langle r \rangle}(\omega) \cap N_{\langle l \rangle}(\omega).$$

(8) **Minimal Union Neighborhood (*u*-neighborhood):**

$$N_{\langle u \rangle}(\omega) = N_{\langle r \rangle}(\omega) \cup N_{\langle l \rangle}(\omega).$$

From this point onward, except indicated otherwise, κ is assumed to lie in the set $\{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$.

Definition 2.5 ([18]). If $\mathcal{R}$ represents a binary relation on $\mathfrak{Q}$, and $\zeta_\kappa : \mathfrak{Q} \rightarrow 2^\mathfrak{Q}$ a mapping that associates each element $\omega \in \mathfrak{Q}$ with its corresponding κ -neighborhood. The triple $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$ is then referred to as a κ -neighborhood space (κ -NS).

Proposition 2.1 ([16, 24, 49]). Let $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$ be a κ -neighborhood space and let ω be an element of $\mathfrak{Q}$. The following properties hold:

- (1) If the relation $\mathcal{R}$ is reflexive, then for every type κ , the neighborhood is non-empty if and only if it contains its generator:

$$N_\kappa(\omega) \neq \emptyset \quad \Leftrightarrow \quad \omega \in N_\kappa(\omega).$$

- (2) For a transitive relation $\mathcal{R}$, the standard neighborhoods are contained within their corresponding minimal counterparts:

$$N_\kappa(\omega) \subseteq N_{\langle j \rangle}(\omega) \quad \text{for } \kappa \in \{r, l, i, u\}.$$

- (3) In the case of a symmetric relation $\mathcal{R}$, the standard neighborhoods coincide, and similarly for the minimal neighborhoods:

$$N_r(\omega) = N_l(\omega) = N_i(\omega) = N_u(\omega), \quad \text{and} \\ N_{\langle r \rangle}(\omega) = N_{\langle l \rangle}(\omega) = N_{\langle i \rangle}(\omega) = N_{\langle u \rangle}(\omega).$$

- (4) If $\mathcal{R}$ is a preorder (i.e., both reflexive and transitive), then for every $\kappa \in \{r, l, i, u\}$, the minimal neighborhood is equal to the standard neighborhood:

$$N_{\langle j \rangle}(\omega) = N_\kappa(\omega).$$

Definition 2.6 ([19]). (*The Yao Rough Set Model*) Let a binary relation $\mathcal{R}$ be defined on a universe of discourse $\mathfrak{Q}$. For any subset $X \subseteq \mathfrak{Q}$, the fundamental constructs of the $\mathcal{Y}$ -rough set model are formally given by:

$\mathcal{Y}$ -Lower Approximation: The set of all elements ω in $\mathfrak{Q}$ whose $\mathcal{R}$ -neighborhood $N_r(\omega)$ is entirely contained within the target set X :

$$\underline{\mathcal{Y}}_r(X) = \{\omega \in \mathfrak{Q} \mid N_r(\omega) \subseteq X\}.$$

$\mathcal{Y}$ -Upper Approximation: The set of all elements ω in $\mathfrak{Q}$ for which the $\mathcal{R}$ -neighborhood $N_r(\omega)$ has a non-empty intersection with the set X :

$$\overline{\mathcal{Y}}_r(X) = \{\omega \in \mathfrak{Q} \mid N_r(\omega) \cap X \neq \emptyset\}.$$

$\mathcal{Y}$ -Boundary Region: The set difference between the upper and lower approximations, representing the region of uncertainty:

$$\mathcal{B}_{\mathcal{Y}}(X) = \overline{\mathcal{Y}}_r(X) \setminus \underline{\mathcal{Y}}_r(X).$$

$\mathcal{Y}$ -Accuracy Measure: A ratio quantifying the precision of the rough set representation, defined provided the upper approximation is non-empty:

$$\sigma_{\mathcal{Y}_r}(X) = \frac{|\underline{\mathcal{Y}}_r(X)|}{|\overline{\mathcal{Y}}_r(X)|}, \quad \text{where } |\overline{\mathcal{Y}}_r(X)| > 0.$$

Definition 2.7 ([20, 21]). (*Allam's Framework*) Let $\mathcal{R}$ be a binary relation on $\mathfrak{Q}$. For any $X \subseteq \mathfrak{Q}$, the $\mathcal{A}$ -lower approximation, $\mathcal{A}$ -upper approximation, $\mathcal{A}$ -boundary region, and $\mathcal{A}$ -accuracy degree are defined as:

- **$\mathcal{A}$ -Lower Approximation:** $\underline{\mathcal{A}}_{\langle r \rangle}(X) = \{\omega \in \mathfrak{Q} : N_{\langle r \rangle}(\omega) \subseteq X\};$
- **$\mathcal{A}$ -Upper Approximation:** $\overline{\mathcal{A}}_{\langle r \rangle}(X) = \{\omega \in \mathfrak{Q} : N_{\langle r \rangle}(\omega) \cap X \neq \emptyset\};$
- **$\mathcal{A}$ -Boundary Region:** $\mathcal{B}_{\mathcal{A}_{\langle r \rangle}}(X) = \overline{\mathcal{A}}_{\langle r \rangle}(X) \setminus \underline{\mathcal{A}}_{\langle r \rangle}(X);$
- **$\mathcal{A}$ -Accuracy Degree:** $\sigma_{\mathcal{A}_{\langle r \rangle}}(X) = \frac{|\underline{\mathcal{A}}_{\langle r \rangle}(X)|}{|\overline{\mathcal{A}}_{\langle r \rangle}(X)|},$ provided that $|\overline{\mathcal{A}}_{\langle r \rangle}(X)| \neq 0.$

Theorem 2.1 ([18]). Let $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$ be a κ -NS, and let $X \subseteq \mathfrak{Q}$. For each $\kappa \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$, the family

$$\mathcal{T}_\kappa = \{X \subseteq \mathfrak{Q} : \forall \omega \in X, N_\kappa(\omega) \subseteq X\}$$

constitutes a topology on $\mathfrak{Q}$. The structure $(\mathfrak{Q}, \mathcal{R}, \tau_\kappa)$ is designated as a κ -topological space (κ -TS).

Definition 2.8 ([18]). Consider a κ -neighborhood space (κ -NS), denoted by the triple $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$. A subset $\mathcal{X} \subseteq \mathfrak{Q}$ is classified as τ_κ -open if it is a member of the family τ_κ . Conversely, a subset $\mathcal{E} \subseteq \mathfrak{Q}$ is called τ_κ -closed if its complement, $\mathcal{E}^c$ is τ_κ -open. The family of all such τ_κ -closed subsets of $\mathfrak{Q}$ is consequently defined as:

$$\Upsilon_\kappa = \{\mathcal{E} \subseteq \mathfrak{Q} : \mathcal{E}^c \in \tau_\kappa\}.$$

Definition 2.9 ([18]). Consider a topology τ_κ generated from a κ -neighborhood structure (κ -NS) on a universe $\mathfrak{Q}$. For an arbitrary subset $\mathcal{H} \subseteq \mathfrak{Q}$, the following notions are specified:

- (1) **κ -Lower Approximation:** The interior of $\mathcal{H}$ in relation to the topology τ_κ :

$$\tau_\kappa L(\mathcal{H}) = \text{int}_{\tau_\kappa}(\mathcal{H}).$$

- (2) **κ -Upper Approximation:** The topological closure of $\mathcal{H}$ within τ_κ :

$$\tau_\kappa U(\mathcal{H}) = \text{cl}_{\tau_\kappa}(\mathcal{H}).$$

- (3) **κ -Boundary Region:** The set-theoretic difference between the upper and lower approximations:

$$\tau_\kappa \mathcal{B}(\mathcal{H}) = \tau_\kappa U(\mathcal{H}) \setminus \tau_\kappa L(\mathcal{H}).$$

- (4) **κ -Accuracy Measure:** The ratio of the cardinality of the lower approximation to that of the upper approximation, defined as:

$$\tau_\kappa \sigma(\mathcal{H}) = \frac{|\tau_\kappa L(\mathcal{H})|}{|\tau_\kappa U(\mathcal{H})|}, \quad \text{provided that } |\tau_\kappa U(\mathcal{H})| > 0.$$

Theorem 2.2 ([46]). Consider $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$ is a κ -neighborhood space and $\mathfrak{P}$ a primal structure on the universe $\mathfrak{Q}$. Thus, for any given type κ , the family of subsets defined by

$$\tau_\kappa^\mathfrak{P} = \{\mathcal{H} \subseteq \mathfrak{Q} : \forall \omega \in \mathcal{H}, N_\kappa(\omega) \setminus \mathcal{H} \in \mathfrak{P}\}$$

forms a supra-topological structure on $\mathfrak{Q}$.

Theorem 2.3 ([46]). Let $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$ be a κ -neighborhood space and $\mathfrak{P}$ a primal on $\mathfrak{Q}$. Then, for every κ , the original topology is contained in the primal supra-topology:

$$\tau_\kappa \subseteq \tau_\kappa^\mathfrak{P}.$$

Definition 2.10 ([46]). The structure $(\mathfrak{Q}, \tau_\kappa^\mathfrak{P})$, where $\tau_\kappa^\mathfrak{P}$ is a supra-topology on $\mathfrak{Q}$ as defined in Theorem 2.2, is called a κ -primal supra-topological space. A subset $\mathcal{H}$ of $\mathfrak{Q}$ is said to be:

- $\tau_\kappa^\mathfrak{P}$ -open if $\mathcal{H} \in \tau_\kappa^\mathfrak{P}$,
- $\tau_\kappa^\mathfrak{P}$ -closed if its complement $\mathfrak{Q} \setminus \mathcal{H}$ belongs to $\tau_\kappa^\mathfrak{P}$.

The family of all $\tau_\kappa^\mathfrak{P}$ -closed sets is symbolized as

$$\Upsilon_\kappa^\mathfrak{P} = \{\mathcal{F} \subseteq \mathfrak{Q} : \mathfrak{Q} \setminus \mathcal{F} \in \tau_\kappa^\mathfrak{P}\}.$$

Theorem 2.4 ([46]). Let $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$ be a κ -neighborhood space equipped with a primal $\mathfrak{P}$. Under the following properties of the relation $\mathcal{R}$, the associated primal supra-topologies satisfy these inclusion and equality relations:

- (1) **Reflexivity:** If $\mathcal{R}$ is reflexive, then for every $\kappa \in \{r, l, i, u\}$, the primal supra-topology is contained in its minimal counterpart:

$$\tau_\kappa^\mathfrak{P} \subseteq \tau_{\langle \kappa \rangle}^\mathfrak{P}.$$

- (2) **Symmetry:** If $\mathcal{R}$ is symmetric, then the four standard primal supra-topologies coincide, and similarly for the four minimal ones:

$$\begin{aligned} \tau_r^\mathfrak{P} &= \tau_l^\mathfrak{P} = \tau_i^\mathfrak{P} = \tau_u^\mathfrak{P}, \quad \text{and} \\ \tau_{\langle r \rangle}^\mathfrak{P} &= \tau_{\langle l \rangle}^\mathfrak{P} = \tau_{\langle i \rangle}^\mathfrak{P} = \tau_{\langle u \rangle}^\mathfrak{P}. \end{aligned}$$

- (3) **Preorder:** If $\mathcal{R}$ is a preorder (reflexive and transitive), then for every $\kappa \in \{r, l, i, u\}$, the standard and minimal primal supra-topologies are identical:

$$\tau_\kappa^\mathfrak{P} = \tau_{\langle \kappa \rangle}^\mathfrak{P}.$$

Proposition 2.2 ([46]). Let $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$ be a κ -neighborhood space and $\mathfrak{P}$ a primal on $\mathfrak{Q}$. The subsequent chain of inclusions holds among the primal supra-topologies:

- (1) For the standard topologies:

$$\tau_u^\mathfrak{P} \subseteq \tau_r^\mathfrak{P} \cap \tau_l^\mathfrak{P} \subseteq \tau_r^\mathfrak{P} \cup \tau_l^\mathfrak{P} \subseteq \tau_i^\mathfrak{P}.$$

- (2) For the minimal topologies:

$$\tau_{\langle u \rangle}^\mathfrak{P} \subseteq \tau_{\langle r \rangle}^\mathfrak{P} \cap \tau_{\langle l \rangle}^\mathfrak{P} \subseteq \tau_{\langle r \rangle}^\mathfrak{P} \cup \tau_{\langle l \rangle}^\mathfrak{P} \subseteq \tau_{\langle i \rangle}^\mathfrak{P}.$$

Definition 2.11 ([46]). Consider a κ -primal supra-topological space $(\mathfrak{Q}, \tau_\kappa^\mathfrak{P})$. For any subset $\mathcal{H} \subseteq \mathfrak{Q}$, we define:

- The $\tau_\kappa^\mathfrak{P}$ -**interior** of $\mathcal{H}$ as the union of all $\tau_\kappa^\mathfrak{P}$ -open sets contained within $\mathcal{H}$:

$$\text{int}_{\tau_\kappa^\mathfrak{P}}(\mathcal{H}) = \bigcup \{ W \in \tau_\kappa^\mathfrak{P} : W \subseteq \mathcal{H} \}.$$

- The $\tau_\kappa^\mathfrak{P}$ -**closure** of $\mathcal{H}$ as the intersection of all $\tau_\kappa^\mathfrak{P}$ -closed sets that contain $\mathcal{H}$:

$$\text{cl}_{\tau_\kappa^\mathfrak{P}}(\mathcal{H}) = \bigcap \{ \mathcal{F} \in \tau_\kappa^\mathfrak{P} : \mathcal{H} \subseteq \mathcal{F} \}.$$

Definition 2.12 ([46]). Let $\tau_\kappa^\mathfrak{P}$ denote the κ -supra topology induced by a κ -NS and a primal $\mathfrak{P}$. For any subset $\mathcal{H}$ of $\mathfrak{Q}$, define:

- (1) $\mathfrak{P}_\kappa$ -**Lower Approximation:** $\tau_\kappa^\mathfrak{P} L(\mathcal{H}) = \tau_\kappa^\mathfrak{P} \text{int}(\mathcal{H})$;
- (2) $\mathfrak{P}_\kappa$ -**Upper Approximation:** $\tau_\kappa^\mathfrak{P} U(\mathcal{H}) = \tau_\kappa^\mathfrak{P} \text{cl}(\mathcal{H})$;
- (3) $\mathfrak{P}_\kappa$ -**Boundary Region:** $\tau_\kappa^\mathfrak{P} \mathcal{B}(\mathcal{H}) = \tau_\kappa^\mathfrak{P} U(\mathcal{H}) \setminus \tau_\kappa^\mathfrak{P} L(\mathcal{H})$;
- (4) $\mathfrak{P}_\kappa$ -**Accuracy Degree:** $\tau_\kappa^\mathfrak{P} \sigma(\mathcal{H}) = \frac{|\tau_\kappa^\mathfrak{P} L(\mathcal{H})|}{|\tau_\kappa^\mathfrak{P} U(\mathcal{H})|}$, where $|\tau_\kappa^\mathfrak{P} U(\mathcal{H})| \neq 0$.

It is evident that the accuracy measure satisfies the inequality $0 \leq \sigma_{\top_{\kappa}^{\mathfrak{P}}}(\mathcal{H}) \leq 1$. A value of $\sigma_{\top_{\kappa}^{\mathfrak{P}}}(\mathcal{H})$ approaching 1 signifies that the lower and upper approximations of $\mathcal{H}$ are nearly identical, implying a high level of definability for the set. When $\sigma_{\top_{\kappa}^{\mathfrak{P}}}(\mathcal{H}) = 1$, the set $\mathcal{H}$ is termed $\top_{\kappa}^{\mathfrak{P}}$ -exact. Conversely, if $\sigma_{\top_{\kappa}^{\mathfrak{P}}}(\mathcal{H}) < 1$, then $\mathcal{H}$ is considered a $\top_{\kappa}^{\mathfrak{P}}$ -rough set.

Definition 2.13 ([16,32,33]). Suppose $\mathcal{R}$ is a binary relation on $\mathfrak{Q}$. The **maximal κ -neighborhoods** of $\omega \in \mathfrak{Q}$ are defined as:

$$\begin{aligned} (1) \quad N_{(r)}(\omega) &= \begin{cases} \bigcup_{\omega \in N_r(z)} N_r(z), & \text{if } \exists z \in \mathfrak{Q} \text{ such that } \omega \in N_r(z); \\ \emptyset, & \text{otherwise.} \end{cases} \\ (2) \quad N_{(l)}(\omega) &= \begin{cases} \bigcup_{\omega \in N_l(z)} N_l(z), & \text{if } \exists z \in \mathfrak{Q} \text{ such that } \omega \in N_l(z); \\ \emptyset, & \text{otherwise.} \end{cases} \\ (3) \quad N_{(i)}(\omega) &= N_{(r)}(\omega) \cap N_{(l)}(\omega); \\ (4) \quad N_{(u)}(\omega) &= N_{(r)}(\omega) \cup N_{(l)}(\omega). \end{aligned}$$

Proposition 2.3. Suppose that $\mathcal{R}$ represents a binary relation defined on a universe $\mathfrak{Q}$, and consider two elements $z, \omega \in \mathfrak{Q}$. Then, for every neighborhood type κ in the set $\{r, l, i, u\}$, the subsequent properties are valid:

- (1) $z \in N_{(\kappa)}(z)$ if $\mathcal{R}$ is inverse serial on $\mathfrak{Q}$ (i.e., for every $z \in \mathfrak{Q}$, there exists $\omega \in \mathfrak{Q}$ such that $\omega \mathcal{R} z$) [16,35,50].
- (2) $N_{(\kappa)}(\omega) \neq \emptyset$ if and only if $\omega \in N_{(\kappa)}(\omega)$ [16,32,35,50].
- (3) $z \in N_{(\kappa)}(\omega)$ if and only if $\omega \in N_{(\kappa)}(z)$ [16,32,35,50].
- (4) $N_{(\kappa)}(\omega) \subseteq N_{\kappa}(\omega) \subseteq N_{(\kappa)}(\omega)$, if $\mathcal{R}$ is reflexive [35,50].

Proposition 2.4. [16,32] Given a two-element relation $\mathcal{R}$ on $\mathfrak{Q}$ and an element $\omega \in \mathfrak{Q}$, the following properties Hold:

- (1) $N_{(i)}(\omega) \subseteq N_{(r)}(\omega) \subseteq N_{(u)}(\omega)$, and $N_{(i)}(\omega) \subseteq N_{(l)}(\omega) \subseteq N_{(u)}(\omega)$.
- (2) $N_{(i)}(\omega) = N_{(r)}(\omega) = N_{(l)}(\omega) = N_{(u)}(\omega)$, if $\mathcal{R}$ is symmetric.

Definition 2.14 ([33]). (**Dai's Approximation Framework**) Let $\mathcal{R}$ be a two-element relation on a universe $\mathfrak{Q}$. For a given subset $\mathcal{X} \subseteq \mathfrak{Q}$, the following operators, based on a specified neighborhood $N_{(r)}(\omega)$, are defined:

D-Lower Approximation: The set of all elements whose entire r -neighborhood is contained within $\mathcal{X}$:

$$\underline{\mathfrak{D}}_{(r)}(\mathcal{X}) = \{\omega \in \mathfrak{Q} : N_{(r)}(\omega) \subseteq \mathcal{X}\}.$$

D-Upper Approximation: The set of all elements whose r -neighborhood has a non-empty intersection with $\mathcal{X}$:

$$\overline{\mathfrak{D}}_{(r)}(\mathcal{X}) = \{\omega \in \mathfrak{Q} : N_{(r)}(\omega) \cap \mathcal{X} \neq \emptyset\}.$$

D-Boundary Region: The set of elements that are in the upper approximation but not in the lower approximation of $\mathcal{X}$:

$$\mathcal{B}_{\mathfrak{D}_{(r)}}(\mathcal{X}) = \overline{\mathfrak{D}_{(r)}}(\mathcal{X}) \setminus \underline{\mathfrak{D}_{(r)}}(\mathcal{X}).$$

D-Accuracy Measure: The ratio of the cardinality of the lower approximation to that of the upper approximation, defined as:

$$\sigma_{\mathfrak{D}_{(r)}}(\mathcal{X}) = \frac{|\underline{\mathfrak{D}_{(r)}}(\mathcal{X})|}{|\overline{\mathfrak{D}_{(r)}}(\mathcal{X})|}, \quad \text{provided that } |\overline{\mathfrak{D}_{(r)}}(\mathcal{X})| > 0.$$

Theorem 2.5. [35,51] Consider a κ -neighborhood structure $(\mathfrak{Q}, \mathcal{R}, \zeta_{\kappa})$. For any $\kappa \in \{r, l, i, u\}$, the family of sets defined by

$$\mathfrak{T}_{(\kappa)} = \{\mathcal{X} \subseteq \mathfrak{Q} : \forall \omega \in \mathcal{X}, N_{(\kappa)}(\omega) \subseteq \mathcal{X}\}$$

forms a topology on $\mathfrak{Q}$. The resulting topological space $(\mathfrak{Q}, \mathfrak{T}_{(\kappa)})$, induced by the relation $\mathcal{R}$, is referred to as a (κ) -topological space (abbreviated as (κ) -TS).

Definition 2.15. [35,51] Let $(\mathfrak{Q}, \mathcal{R}, \mathfrak{T}_{(\kappa)})$ be a (κ) -topological space, where $\kappa \in \{r, l, i, u\}$. A subset $\mathcal{X}$ of $\mathfrak{Q}$ is termed $\mathfrak{T}_{(\kappa)}$ -open if $\mathcal{X} \in \mathfrak{T}_{(\kappa)}$, and its complement is referred to as $\mathfrak{T}_{(\kappa)}$ -closed. The class of all $\mathfrak{T}_{(\kappa)}$ -closed sets is denoted by $\mathfrak{Y}_{(\kappa)} = \{\mathcal{E} \subseteq \mathfrak{Q} : \mathcal{E}^c \in \mathfrak{T}_{(\kappa)}\}$, where $\mathcal{E}^c$ represents the complement of $\mathcal{E}$ in $\mathfrak{Q}$.

Definition 2.16. [35,51] Consider a topology $\mathfrak{T}_{(\kappa)}$ on $\mathfrak{Q}$ that is generated by a (κ) -neighborhood framework. For any subset $\mathcal{H} \subseteq \mathfrak{Q}$, the corresponding rough approximations are described as follows:

(1) (κ) -Lower Approximation:

$$\mathfrak{T}_{(\kappa)}L(\mathcal{H}) = \mathfrak{T}_{(\kappa)}\text{int}(\mathcal{H}),$$

where $\mathfrak{T}_{(\kappa)}\text{int}(\mathcal{H})$ denotes the interior of $\mathcal{H}$ with respect to the topology $\mathfrak{T}_{(\kappa)}$.

(2) (κ) -Upper Approximation:

$$\mathfrak{T}_{(\kappa)}U(\mathcal{H}) = \mathfrak{T}_{(\kappa)}cl(\mathcal{H}),$$

where $\mathfrak{T}_{(\kappa)}cl(\mathcal{H})$ stands for the closure of $\mathcal{H}$ under $\mathfrak{T}_{(\kappa)}$.

(3) (κ) -Boundary Region:

$$\mathfrak{T}_{(\kappa)}\mathcal{B}(\mathcal{H}) = \mathfrak{T}_{(\kappa)}U(\mathcal{H}) \setminus \mathfrak{T}_{(\kappa)}L(\mathcal{H}).$$

(4) (κ) -Accuracy Measure:

$$\mathfrak{T}_{(\kappa)}\sigma(\mathcal{H}) = \frac{|\mathfrak{T}_{(\kappa)}L(\mathcal{H})|}{|\mathfrak{T}_{(\kappa)}U(\mathcal{H})|}, \quad \text{whenever } |\mathfrak{T}_{(\kappa)}U(\mathcal{H})| \neq 0.$$

Proposition 2.5. [35,51] Let $\mathfrak{T}_{(\kappa)}$ be the topology on $\mathfrak{Q}$ induced by a (κ) -neighborhood structure. Then, the following properties hold:

- (1) $\mathfrak{T}_{(u)} \subseteq \mathfrak{T}_{(r)} \subseteq \mathfrak{T}_{(i)}$ and $\mathfrak{T}_{(u)} \subseteq \mathfrak{T}_{(l)} \subseteq \mathfrak{T}_{(i)}$.
- (2) $\mathfrak{T}_{(u)} = \mathfrak{T}_{(r)} = \mathfrak{T}_{(l)} = \mathfrak{T}_{(i)}$, if $\mathcal{R}$ is symmetric.

Proposition 2.6. [51] Let $\mathcal{R}$ be a reflexive relation on $\mathfrak{Q}$, and let $\kappa \in \{r, l, i, u\}$. So,

$$\mathfrak{T}_{(\kappa)} \subseteq \mathfrak{T}_{\kappa}.$$

3. SUPRA TOPOLOGIES GENERATED BY VARIOUS CLASSES OF MAXIMAL NEIGHBORHOODS AND PRIMALS

This section presents innovative methods for generating various supra topologies using maximal neighborhoods in conjunction with primals. Furthermore, the fundamental relationships among these structures are systematically analyzed and discussed.

Theorem 3.1. Let $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$ be a κ -neighborhood system, with $\kappa \in \{r, l, i, u\}$, and let $\mathfrak{P}$ denote a primal on $\mathfrak{Q}$. Consider the collection

$$\tau_{(\kappa)}^{\mathfrak{P}} = \left\{ \mathcal{X} \subseteq \mathfrak{Q} \mid \forall z \in \mathcal{X}, N_{(\kappa)}(z) \setminus \mathcal{X} \in \mathfrak{P} \right\}.$$

Then, $\tau_{(\kappa)}^{\mathfrak{P}}$ constitutes a supra topology on $\mathfrak{Q}$.

Proof. We verify the axioms of a supra topology:

- (1) Clearly, both $\emptyset$ and $\mathfrak{Q}$ belong to $\tau_{(\kappa)}^{\mathfrak{P}}$.
- (2) Suppose $\{\mathcal{X}_\alpha\}_{\alpha \in \Omega} \subseteq \tau_{(\kappa)}^{\mathfrak{P}}$ and consider any $z \in \bigcup_{\alpha \in \Omega} \mathcal{X}_\alpha$. Then, there exists $\alpha_0 \in \Omega$ such that $z \in \mathcal{X}_{\alpha_0}$. By the definition of $\tau_{(\kappa)}^{\mathfrak{P}}$, we know that

$$N_{(\kappa)}(z) \setminus \mathcal{X}_{\alpha_0} \in \mathfrak{P}.$$

Since

$$N_{(\kappa)}(z) \setminus \bigcup_{\alpha \in \Omega} \mathcal{X}_\alpha \subseteq N_{(\kappa)}(z) \setminus \mathcal{X}_{\alpha_0},$$

and $\mathfrak{P}$ is a primal (hence closed under subsets), it follows that

$$N_{(\kappa)}(z) \setminus \bigcup_{\alpha \in \Omega} \mathcal{X}_\alpha \in \mathfrak{P}.$$

Consequently, $\bigcup_{\alpha \in \Omega} \mathcal{X}_\alpha \in \tau_{(\kappa)}^{\mathfrak{P}}$.

Thus, $\tau_{(\kappa)}^{\mathfrak{P}}$ satisfies the defining conditions of a supra topology on $\mathfrak{Q}$. □

It is shown by Example 3.1 that the structure $\tau_{(\kappa)}^{\mathfrak{P}}$ cannot be guaranteed to be a topology.

Example 3.1. Consider the set $\mathfrak{Q} = \{v, \rho, \varsigma\}$, the primal $\mathfrak{P} = \{\emptyset, \{v\}, \{\rho\}, \{\varsigma\}, \{v, \rho\}, \{v, \varsigma\}\}$ on $\mathfrak{Q}$, and the arbitrary relation $\mathcal{R} = \{(v, v), (v, \rho), (v, \varsigma), (\rho, \varsigma), (\varsigma, \rho)\}$ on $\mathfrak{Q}$. Then, the class

$$\tau_{(l)}^{\mathfrak{P}} = \{\emptyset, \mathfrak{Q}, \{\rho\}, \{\varsigma\}, \{v, \rho\}, \{v, \varsigma\}, \{\rho, \varsigma\}\}.$$

It is clear that the intersection $\{v, \rho\} \cap \{v, \varsigma\} = \{v\}$ does not belong to $\tau_{(l)}^{\mathfrak{P}}$, despite both $\{v, \rho\}$ and $\{v, \varsigma\}$ being elements of $\tau_{(l)}^{\mathfrak{P}}$.

Note that: $\tau_l^{\mathfrak{P}} = \tau_{\langle l \rangle}^{\mathfrak{P}} = 2^{\mathfrak{Q}}$, but $\tau_l^{\mathfrak{P}} \not\subseteq \tau_{(l)}^{\mathfrak{P}}$.

Theorem 3.2. Let $(\mathfrak{Q}, \mathcal{R}, \zeta_\kappa)$ be a κ -NS, and let $\mathfrak{P}$ be a primal on $\mathfrak{Q}$. Then, for every $\kappa \in \{r, l, i, u\}$, the subsequent inclusion is established:

$$\tau_{(\kappa)} \subseteq \tau_{(\kappa)}^{\mathfrak{P}}.$$

Proof. Let $X \in \tau_{(\kappa)}$. By definition, for every $z \in X$. It follows that $N_{(\kappa)}(z) \subseteq X$, which consequently implies that $N_{(\kappa)}(z) \setminus X = \emptyset \in \mathfrak{P}$. Therefore, $X \in \tau_{(\kappa)}^{\mathfrak{P}}$, and hence $\tau_{(\kappa)} \subseteq \tau_{(\kappa)}^{\mathfrak{P}}$. $\square$

The converse of Theorem 3.2 fails to hold, as evidenced by the forthcoming example.

Example 3.2. Building upon Example 3.1, we derive:

$$\tau_{(l)} = \{\emptyset, \mathfrak{Q}\}, \quad \tau_{(l)}^{\mathfrak{P}} = \{\emptyset, \mathfrak{Q}, \{\rho\}, \{\varsigma\}, \{v, \rho\}, \{v, \varsigma\}, \{\rho, \varsigma\}\}.$$

Clearly, $\tau_{(l)}^{\mathfrak{P}} \not\subseteq \tau_{(l)}$.

Definition 3.1. A structure of the form $(\mathfrak{Q}, \tau_{(\kappa)}^{\mathfrak{P}})$, where $\tau_{(\kappa)}^{\mathfrak{P}}$ is a supra-topology defined on the universe $\mathfrak{Q}$, is called a (κ) -**primal supra-topological space**.

Within this space, a subset $X \subseteq \mathfrak{Q}$ is said to be:

- $\tau_{(\kappa)}^{\mathfrak{P}}$ -**open** if X belongs to the family $\tau_{(\kappa)}^{\mathfrak{P}}$,
- $\tau_{(\kappa)}^{\mathfrak{P}}$ -**closed** if its complement, denoted $\mathfrak{Q} \setminus X$, is a member of $\tau_{(\kappa)}^{\mathfrak{P}}$.

Consequently, the family of all closed sets within this space is given by:

$$Y_{(\kappa)}^{\mathfrak{P}} = \{\mathcal{F} \subseteq \mathfrak{Q} : \mathfrak{Q} \setminus \mathcal{F} \in \tau_{(\kappa)}^{\mathfrak{P}}\}.$$

Proposition 3.1. Let $(\mathfrak{Q}, \mathcal{R}, \zeta_{\kappa})$ be a κ -neighborhood space equipped with a primal structure $\mathfrak{P}$. The following subset relations hold for the induced primal supra-topologies:

- (1) $\tau_{(u)}^{\mathfrak{P}} \subseteq \tau_{(r)}^{\mathfrak{P}} \subseteq \tau_{(i)}^{\mathfrak{P}}$,
- (2) $\tau_{(u)}^{\mathfrak{P}} \subseteq \tau_{(l)}^{\mathfrak{P}} \subseteq \tau_{(i)}^{\mathfrak{P}}$.

Proof. Let $\xi \in \{r, l\}$. Given that $N_{(i)} \subseteq N_{(\xi)} \subseteq N_{(u)}$, the desired inclusions follow immediately from the definition of $\tau_{(\kappa)}^{\mathfrak{P}}$. $\square$

The subsequent example illustrates that the converse of Proposition 3.1 is not valid.

Example 3.3. Let $\mathfrak{Q} = \{v, \rho, \varsigma, \omega, e\}$. Consider $\mathfrak{P}_e$ as a primal on $\mathfrak{Q}$, and let

$$\mathcal{R} = \{(v, v), (\rho, \varsigma), (\rho, \omega), (\varsigma, \omega), (\varsigma, e), (\omega, v), (\omega, e), (e, \rho), (e, e)\}$$

be a binary relation on $\mathfrak{Q}$. Then, the next supra topologies are defined:

- $\tau_r^{\mathfrak{P}} = \{\emptyset, \{v\}, \{\rho\}, \{e\}, \{v, \rho\}, \{v, e\}, \{\rho, e\}, \{\varsigma, e\}, \{\omega, e\}, \{v, \rho, e\}, \{v, \varsigma, e\}, \{v, \omega, e\}, \{\rho, \varsigma, e\}, \{\rho, \omega, e\}, \{\varsigma, \omega, e\}, \{v, \rho, \varsigma, e\}, \{v, \rho, \omega, e\}, \{v, \varsigma, \omega, e\}, \{\rho, \varsigma, \omega, e\}, \mathfrak{Q}\}$
- $\tau_{(r)}^{\mathfrak{P}} = \{\emptyset, \{v\}, \{\varsigma\}, \{\omega\}, \{e\}, \{v, \varsigma\}, \{v, \omega\}, \{v, e\}, \{\rho, e\}, \{\varsigma, \omega\}, \{\varsigma, e\}, \{\omega, e\}, \{v, \rho, e\}, \{v, \varsigma, \omega\}, \{v, \varsigma, e\}, \{v, \omega, e\}, \{\rho, \varsigma, e\}, \{\rho, \omega, e\}, \{\varsigma, \omega, e\}, \{v, \rho, \varsigma, e\}, \{v, \varsigma, \omega, e\}, \{v, \rho, \omega, e\}, \{\rho, \varsigma, \omega, e\}, \mathfrak{Q}\}$
- $\tau_{(u)}^{\mathfrak{P}} = \tau_{(r)}^{\mathfrak{P}} = \{\emptyset, \{\varsigma\}, \{e\}, \{v, e\}, \{\rho, e\}, \{\varsigma, e\}, \{\omega, e\}, \{v, \rho, e\}, \{v, \varsigma, e\}, \{v, \omega, e\}, \{\rho, \varsigma, e\}, \{\rho, \omega, e\}, \{\varsigma, \omega, e\}, \{v, \rho, \varsigma, e\}, \{v, \rho, \omega, e\}, \{v, \varsigma, \omega, e\}, \{\rho, \varsigma, \omega, e\}, \mathfrak{Q}\}$
- $\tau_{(i)}^{\mathfrak{P}} = \tau_{(l)}^{\mathfrak{P}} = \{\emptyset, \{v\}, \{\rho\}, \{\varsigma\}, \{e\}, \{v, \rho\}, \{v, \varsigma\}, \{v, e\}, \{\rho, \varsigma\}, \{\rho, e\}, \{\varsigma, e\}, \{\omega, e\}, \{v, \rho, \varsigma\}, \{v, \rho, e\}, \{v, \varsigma, e\}, \{v, \omega, e\}, \{\rho, \varsigma, e\}, \{\rho, \omega, e\}, \{\varsigma, \omega, e\}, \{v, \rho, \varsigma, e\}, \{v, \rho, \omega, e\}, \{v, \varsigma, \omega, e\}, \{\rho, \varsigma, \omega, e\}, \mathfrak{Q}\}.$

Thus, the following results hold:

- (1) $\tau_{\langle r \rangle}^{\mathfrak{P}} \neq \tau_{(r)}^{\mathfrak{P}}$, and $\tau_r^{\mathfrak{P}} \neq \tau_{(r)}^{\mathfrak{P}}$.
- (2) $\tau_{(i)}^{\mathfrak{P}} \not\subseteq \tau_{(r)}^{\mathfrak{P}}$, and $\tau_{(l)}^{\mathfrak{P}} \not\subseteq \tau_{(u)}^{\mathfrak{P}}$.

Remark 3.1. Example 3.3 demonstrates that $\tau_{(l)}^{\mathfrak{P}}$ is not the dual of $\tau_{(r)}^{\mathfrak{P}}$.

The next theorem establishes that the supra topologies generated by the proposed method are weaker than those introduced in previous studies, specifically when $\mathcal{R}$ is a reflexive relation on $\mathfrak{Q}$ (cf. [46]). However, for general binary relations, these supra topologies are mutually independent, as demonstrated in Example 3.3.

Theorem 3.3. Let $(\mathfrak{Q}, \mathcal{R}, \zeta_{\kappa})$ be a κ -neighborhood space equipped with a primal $\mathfrak{P}$. Under the following properties of the relation $\mathcal{R}$, the associated primal supra-topologies satisfy these inclusion and equality relations:

- (1) **Reflexivity:** If the relation $\mathcal{R}$ is reflexive, then for every neighborhood type $\kappa \in \{r, l, i, u\}$, the forthcoming sequence of inclusions is established:

$$\tau_{(\kappa)}^{\mathfrak{P}} \subseteq \tau_{\kappa}^{\mathfrak{P}} \subseteq \tau_{\langle \kappa \rangle}^{\mathfrak{P}}.$$

- (2) **Symmetry:** If the relation $\mathcal{R}$ is symmetric, then the four primal supra-topologies generated by the standard neighborhoods coincide:

$$\tau_{(r)}^{\mathfrak{P}} = \tau_{(l)}^{\mathfrak{P}} = \tau_{(i)}^{\mathfrak{P}} = \tau_{(u)}^{\mathfrak{P}}.$$

Proof. The proof is straightforwardly derived from Proposition 2.3. □

The converse of the assertions made in Theorem 3.3 fails to hold, as illustrated in the upcoming example:

Example 3.4. Let $\mathfrak{P} = \{\emptyset, \{v\}, \{\rho\}, \{\varsigma\}, \{v, \rho\}, \{v, \varsigma\}\}$ be a primal on $\mathfrak{Q} = \{v, \rho, \varsigma\}$. Consider the reflexive relation $\mathcal{R} = \{(v, v), (v, \rho), (v, \varsigma), (\rho, \rho), (\rho, \varsigma), (\varsigma, v), (\varsigma, \varsigma)\}$ on $\mathfrak{Q}$. The following supra topologies are then defined:

- $\tau_r^{\mathfrak{P}} = \{\emptyset, \mathfrak{Q}, \{\rho\}, \{\varsigma\}, \{v, \rho\}, \{v, \varsigma\}, \{\rho, \varsigma\}\}$
- $\tau_l^{\mathfrak{P}} = 2^{\mathfrak{Q}}$
- $\tau_{\langle l \rangle}^{\mathfrak{P}} = \tau_{\langle r \rangle}^{\mathfrak{P}} = 2^{\mathfrak{Q}}$
- $\tau_{(l)}^{\mathfrak{P}} = \tau_{(r)}^{\mathfrak{P}} = \{\emptyset, \mathfrak{Q}, \{\rho\}, \{\varsigma\}, \{v, \rho\}, \{v, \varsigma\}, \{\rho, \varsigma\}\}.$

Definition 3.2. Let $\tau_{(\kappa)}^{\mathfrak{P}}$ represent the (κ) -supra topology on $\mathfrak{Q}$ induced by the (κ) -neighborhood space and the primal $\mathfrak{P}$. For any subset $X \subseteq \mathfrak{Q}$, the $(\kappa, \mathfrak{P})$ -interior and $(\kappa, \mathfrak{P})$ -closure of X , denoted by $\text{int}_{(\kappa)}^{\mathfrak{P}}(X)$ and $\text{cl}_{(\kappa)}^{\mathfrak{P}}(X)$ respectively, are defined for each $\kappa \in \{r, l, i, u\}$ as follows:

- (1) $\text{int}_{(\kappa)}^{\mathfrak{P}}(X) = \bigcup \{ \mathcal{H} \in \tau_{(\kappa)}^{\mathfrak{P}} \mid \mathcal{H} \subseteq X \}.$
- (2) $\text{cl}_{(\kappa)}^{\mathfrak{P}}(X) = \bigcap \{ \mathcal{F} \in \mathcal{Y}_{(\kappa)}^{\mathfrak{P}} \mid X \subseteq \mathcal{F} \}.$

Lemma 3.1. Let $(\mathfrak{Q}, \tau_{(\kappa)}^{\mathfrak{P}})$ be a (κ) -supra topological space induced by the (κ) -NS and the primal $\mathfrak{P}$. If $X \subseteq \mathfrak{Q}$, hence the following statements hold:

- (1) $\text{int}_{(u)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{int}_{(r)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{int}_{(i)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{cl}_{(i)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{cl}_{(r)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{cl}_{(u)}^{\mathfrak{P}}(\mathcal{X})$.
- (2) $\text{int}_{(u)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{int}_{(l)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{int}_{(i)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{cl}_{(i)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{cl}_{(l)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \text{cl}_{(u)}^{\mathfrak{P}}(\mathcal{X})$.

Proof. Obvious. □

Remark 3.2. Example 3.3 demonstrates that the converse of Lemma 3.1 does not apply generally.

4. TOPOLOGICAL APPROXIMATION MODELS INDUCED BY MAXIMAL NEIGHBORHOODS AND PRIMAL STRUCTURES

This section introduces newly developed rough set approaches founded on the (κ) -supra topologies $\tau_{(\kappa)}^{\mathfrak{P}}$, for each $\kappa \in \{r, l, i, u\}$, constructed from maximal neighborhood systems (denoted (κ) -NS) in conjunction with a primal structure $\mathfrak{P}$. The fundamental properties of these models are highlighted and systematically analyzed.

Definition 4.1. Let $\tau_{(\kappa)}^{\mathfrak{P}}$ be a (κ) -supra topology on $\mathfrak{Q}$ constructed from a (κ) -neighborhood system and a primal structure $\mathfrak{P}$. For a subset $\mathcal{X} \subseteq \mathfrak{Q}$, the $\mathfrak{P}_{(\kappa)}$ -lower approximation, $\mathfrak{P}_{(\kappa)}$ -upper approximation, and $\mathfrak{P}_{(\kappa)}$ -accuracy degree are defined as follows for each $\kappa \in \{r, l, i, u\}$:

- (1) $\mathfrak{P}_{(\kappa)}$ -Lower Approximation:

$$L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = \text{int}_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$$

- (2) $\mathfrak{P}_{(\kappa)}$ -Upper Approximation:

$$U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = \text{cl}_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$$

- (3) $\mathfrak{P}_{(\kappa)}$ -Accuracy Degree:

$$\sigma_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = \frac{|L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})|}{|U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})|}, \quad \text{where } |U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})| \neq 0$$

Definition 4.2. Within a (κ) -primal supra-topological space $(\mathfrak{Q}, \tau_{(\kappa)}^{\mathfrak{P}})$, a subset $\mathcal{X} \subseteq \mathfrak{Q}$ is categorized based on its approximations as follows:

- (1) **Totally $\mathfrak{P}_{(\kappa)}$ -definable** if its lower and upper primal approximations coincide with the set itself:

$$L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = \mathcal{X} = U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}).$$

- (2) **Internally $\mathfrak{P}_{(\kappa)}$ -definable** if it equals its own lower primal approximation but is a proper subset of its upper primal approximation:

$$L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = \mathcal{X} \quad \text{and} \quad U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \supsetneq \mathcal{X}.$$

- (3) **Externally $\mathfrak{P}_{(\kappa)}$ -definable** if its lower primal approximation is a proper subset of itself, but it equals its own upper primal approximation:

$$L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \subsetneq \mathcal{X} \quad \text{and} \quad U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = \mathcal{X}.$$

(4) $\mathfrak{P}_{(\kappa)}$ -**Rough** if it differs from both its lower and upper primal approximations:

$$L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \neq \mathcal{X} \quad \text{and} \quad U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \neq \mathcal{X}.$$

Remark 4.1. Referring to Example 3.4, the subsets $\emptyset$, $\mathfrak{Q}$, $\{\rho\}$, $\{\varsigma\}$, $\{v, \rho\}$, and $\{v, \varsigma\}$ are all totally $\mathfrak{P}_{(r)}$ -definable. In contrast, the singleton $\{v\}$ is externally $\mathfrak{P}_{(r)}$ -definable, while the pair $\{\rho, \varsigma\}$ is classified as internally $\mathfrak{P}_{(r)}$ -definable.

Proposition 4.1. Consider $(\mathfrak{Q}, \tau_{(\kappa)}^{\mathfrak{P}})$ as a (κ) -primal supra-topological space, and let $\mathcal{X} \subseteq \mathfrak{Q}$ be any subset. For each neighborhood type $\kappa \in \{r, l, i, u\}$, the primal lower and upper approximations are related to their corresponding standard approximations through the following properties:

(1) The primal lower approximation contains the standard lower approximation:

$$L_{(\kappa)}(\mathcal{X}) \subseteq L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}).$$

(2) The primal upper approximation is contained within the standard upper approximation:

$$U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \subseteq U_{(\kappa)}(\mathcal{X}).$$

(3) The accuracy measure of the primal approximations is at least that of the standard approximations:

$$\sigma_{(\kappa)}(\mathcal{X}) \leq \sigma_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}).$$

Proof. These results are immediate from Theorem 3.2. □

Proposition 4.2. Let $\mathcal{X}, \mathcal{X}' \subseteq \mathfrak{Q}$. For each $\kappa \in \{r, l, i, u\}$, the following properties hold:

$(\mathfrak{P}L1) \quad L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \subseteq \mathcal{X}$	$(\mathfrak{P}U1) \quad \mathcal{X} \subseteq U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$
$(\mathfrak{P}L2) \quad L_{(\kappa)}^{\mathfrak{P}}(\emptyset) = \emptyset$	$(\mathfrak{P}U2) \quad U_{(\kappa)}^{\mathfrak{P}}(\emptyset) = \emptyset$
$(\mathfrak{P}L3) \quad L_{(\kappa)}^{\mathfrak{P}}(\mathfrak{Q}) = \mathfrak{Q}$	$(\mathfrak{P}U3) \quad U_{(\kappa)}^{\mathfrak{P}}(\mathfrak{Q}) = \mathfrak{Q}$
$(\mathfrak{P}L4) \quad \mathcal{X} \subseteq \mathcal{X}' \Rightarrow L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \subseteq L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}')$	$(\mathfrak{P}U4) \quad \mathcal{X} \subseteq \mathcal{X}' \Rightarrow U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \subseteq U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}')$
$(\mathfrak{P}L5) \quad L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X} \cap \mathcal{X}') = L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \cap L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}')$	$(\mathfrak{P}U5) \quad U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X} \cap \mathcal{X}') \subseteq U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \cap U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}')$
$(\mathfrak{P}L6) \quad L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X} \cup \mathcal{X}') \supseteq L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \cup L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}')$	$(\mathfrak{P}U6) \quad U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X} \cup \mathcal{X}') = U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) \cup U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}')$
$(\mathfrak{P}L7) \quad L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = \left[U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}^c) \right]^c$	$(\mathfrak{P}U7) \quad U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = \left[L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}^c) \right]^c$
$(\mathfrak{P}L8) \quad L_{(\kappa)}^{\mathfrak{P}}\left(L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})\right) = L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$	$(\mathfrak{P}U8) \quad U_{(\kappa)}^{\mathfrak{P}}\left(U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})\right) = U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$

Remark 4.2. The converses of the properties $\mathfrak{P}L1$, $\mathfrak{P}L4$, $\mathfrak{P}L6$, $\mathfrak{P}U1$, $\mathfrak{P}U4$, and $\mathfrak{P}U5$ are not generally applicable, as demonstrated in Example 4.1.

Example 4.1. Building upon Example 3.3, let $\kappa = r$. Then the (r) -supra topology is given by:

$$\tau_{(r)}^{\mathfrak{P}} = \{\emptyset, \{\varsigma\}, \{e\}, \{v, e\}, \{\rho, e\}, \{\varsigma, e\}, \{\omega, e\}, \{v, \rho, e\}, \{v, \varsigma, e\}, \dots, \mathfrak{Q}\}.$$

(1) If $\mathcal{X} = \{v, \varsigma\}$, then $L_{(r)}^{\mathfrak{P}}(\mathcal{X}) = \{\varsigma\}$, hence $\mathcal{X} \not\subseteq L_{(r)}^{\mathfrak{P}}(\mathcal{X})$.

- (2) If $X = \{v, \rho\}$ and $\dot{X} = \{\varsigma, e\}$, then $L_{(r)}^{\mathfrak{P}}(X) = \emptyset$, $L_{(r)}^{\mathfrak{P}}(\dot{X}) = \{\varsigma, e\}$, so $L_{(r)}^{\mathfrak{P}}(X) \subseteq L_{(r)}^{\mathfrak{P}}(\dot{X})$, but $X \not\subseteq \dot{X}$.
- (3) If $X = \{v\}$ and $\dot{X} = \{e\}$, then $L_{(r)}^{\mathfrak{P}}(X) = \emptyset$, $L_{(r)}^{\mathfrak{P}}(\dot{X}) = \{e\}$, while $L_{(r)}^{\mathfrak{P}}(X \cup \dot{X}) = \{v, e\}$. Consequently, $L_{(r)}^{\mathfrak{P}}(X \cup \dot{X}) \not\subseteq L_{(r)}^{\mathfrak{P}}(X) \cup L_{(r)}^{\mathfrak{P}}(\dot{X})$.

Proposition 4.3. Consider the (κ) -supra topology $\tau_{(\kappa)}^{\mathfrak{P}}$ produced by a (κ) -NS together with a primal structure $\mathfrak{P}$. Then, for any subset $X \subseteq \mathfrak{Q}$, the following relations are satisfied:

- (1) $L_{(u)}^{\mathfrak{P}}(X) \subseteq L_{(r)}^{\mathfrak{P}}(X) \subseteq L_{(i)}^{\mathfrak{P}}(X)$, and simultaneously $L_{(u)}^{\mathfrak{P}}(X) \subseteq L_{(l)}^{\mathfrak{P}}(X) \subseteq L_{(i)}^{\mathfrak{P}}(X)$.
- (2) $U_{(i)}^{\mathfrak{P}}(X) \subseteq U_{(r)}^{\mathfrak{P}}(X) \subseteq U_{(u)}^{\mathfrak{P}}(X)$, and likewise $U_{(i)}^{\mathfrak{P}}(X) \subseteq U_{(l)}^{\mathfrak{P}}(X) \subseteq U_{(u)}^{\mathfrak{P}}(X)$.
- (3) $\sigma_{(u)}^{\mathfrak{P}}(X) \leq \sigma_{(r)}^{\mathfrak{P}}(X) \leq \sigma_{(i)}^{\mathfrak{P}}(X)$, and in the same manner $\sigma_{(u)}^{\mathfrak{P}}(X) \leq \sigma_{(l)}^{\mathfrak{P}}(X) \leq \sigma_{(i)}^{\mathfrak{P}}(X)$.

Proposition 4.4. Consider two primal structures, $\mathfrak{P}$ and $\widetilde{\mathfrak{P}}$, defined on a universe $\mathfrak{Q}$, inducing the (κ) -supra topological spaces $(\mathfrak{Q}, \tau_{(\kappa)}^{\mathfrak{P}})$ and $(\mathfrak{Q}, \tau_{(\kappa)}^{\widetilde{\mathfrak{P}}})$, respectively. If $\mathfrak{P} \subseteq \widetilde{\mathfrak{P}}$, then for every $\kappa \in \{r, l, i, u\}$, the following inclusion holds:

$$\tau_{(\kappa)}^{\mathfrak{P}} \subseteq \tau_{(\kappa)}^{\widetilde{\mathfrak{P}}}.$$

Proof. The proof is straightforward. □

Corollary 4.1. Under the assumptions of Proposition 4.4, for each subset M of $\mathfrak{Q}$, the following inequalities hold:

- (1) $L_{(\kappa)}^{\mathfrak{P}}(X) \subseteq L_{(\kappa)}^{\widetilde{\mathfrak{P}}}(X)$,
- (2) $U_{(\kappa)}^{\mathfrak{P}}(X) \supseteq U_{(\kappa)}^{\widetilde{\mathfrak{P}}}(X)$,
- (3) $\sigma_{(\kappa)}^{\mathfrak{P}}(X) \leq \sigma_{(\kappa)}^{\widetilde{\mathfrak{P}}}(X)$.

5. APPLICATIONS: ADVANCED DECISION-MAKING STRATEGIES FOR HEART FAILURE DIAGNOSIS USING MAXIMAL APPROACHES AND PRIMAL STRUCTURES

This section stresses how important it is for medical sciences to use a strict and well-organized mathematical framework, especially when it comes to dealing with the inherent complexity of clinical decision-making processes. To illustrate the practical applicability of the proposed methodology, we implement it in a real clinical task—namely, the diagnosis of heart failure.

This research utilizes empirical data derived from twelve patients monitored in a clinical environment. Each record shows whether or not five clinically significant symptoms that are usually linked to heart failure are present. The Cardiology Department at Sayed Galal University Hospital, Al-Azhar University, Egypt, meticulously gathered the data [52].

The study includes twelve patients, each exhibiting distinct types of symptoms. Everyone got a complete diagnostic evaluation that included a full medical history review, a physical exam, lab tests, resting electrocardiography (ECG), and standard echocardiographic studies. Combining the results of these clinical tests helped us make the final decision about whether or not heart failure

was present. We selected the five symptoms for this study based on existing knowledge regarding heart disease and findings from prior research on heart disease symptoms.

Table 1 gives a short overview of the heart failure dataset. The columns show the condition attributes C , which are symptoms that are important for figuring out if someone has heart failure. Specifically, $\mathbb{H}_1$ refers to shortness of breath, $\mathbb{H}_2$ to orthopnea, $\mathbb{H}_3$ to paroxysmal nocturnal dyspnea, $\mathbb{H}_4$ to decreased exercise tolerance, and $\mathbb{H}_5$ to ankle swelling. Each symptom is assigned a code that indicates whether or not it is present. The decision attribute $\mathcal{D}$ shows the diagnostic result, which tells us if heart failure is confirmed or ruled out.

Note: The symbol “Yes” is abbreviated as $\hat{Y}$, whereas “No” is denoted by $\check{N}$.

In Table 1, the rows indexed by

$$\mathcal{Q} = \{S_1, S_2, \dots, S_{12}\}$$

represent individual patients, with each row specifying the set of symptoms observed for the corresponding patient. Thus, the table provides a systematic representation of symptom-based clinical information and illustrates how patient data can be incorporated into the proposed framework for structured medical decision-making.

The analysis begins by encoding the descriptive condition attributes

$$C = \{\mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_3, \mathbb{H}_4, \mathbb{H}_5\}$$

into qualitative categories, as illustrated in Table 1. This encoding makes it easier to compare patients based on the symptoms they have. The similarity measure quantifies the degree of similarity between two patients, S_i and S_j , as follows:

$$\mathfrak{O}(S_i, S_j) = \frac{\sum_{g=1}^n [a_g(S_i) = a_g(S_j)]}{n},$$

where:

- n denotes the total number of condition attributes,
- $i, j \in \{1, 2, \dots, 12\}$ index the patients,
- a_g represents the g -th attribute, with $a_g \in C$,
- $a_g(S_i)$ denotes the value of the g -th attribute for patient S_i .

Based on Table 1, the patient population is partitioned into two distinct classes: the class of patients diagnosed with heart failure, represented as

$$\mathbb{B} = \{S_1, S_3, S_7, S_8, S_9, S_{11}\};$$

and the class of patients without a heart failure diagnosis, represented as

$$\mathbb{W} = \{S_2, S_4, S_5, S_6, S_{10}, S_{12}\}.$$

Table 2 shows the similarity measures between the symptom profiles of the twelve patients.

At this point, we build a minimally structured approximation space based on binary relations that show the decision problem’s inherent properties. Domain experts set the rules for these

TABLE 1. Overview of the Original Medical Information System

Patient S	Symptoms C					Decision $\mathcal{D}$
	H_1	H_2	H_3	H_4	H_5	
S_1	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$	$\check{N}$	$\hat{Y}$
S_2	$\check{N}$	$\check{N}$	$\check{N}$	$\hat{Y}$	$\hat{Y}$	$\check{N}$
S_3	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$
S_4	$\check{N}$	$\check{N}$	$\check{N}$	$\hat{Y}$	$\check{N}$	$\check{N}$
S_5	$\hat{Y}$	$\check{N}$	$\check{N}$	$\hat{Y}$	$\hat{Y}$	$\check{N}$
S_6	$\check{N}$	$\check{N}$	$\check{N}$	$\hat{Y}$	$\check{N}$	$\check{N}$
S_7	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$
S_8	$\hat{Y}$	$\hat{Y}$	$\check{N}$	$\hat{Y}$	$\hat{Y}$	$\hat{Y}$
S_9	$\hat{Y}$	$\check{N}$	$\hat{Y}$	$\hat{Y}$	$\check{N}$	$\hat{Y}$
S_{10}	$\check{N}$	$\check{N}$	$\check{N}$	$\hat{Y}$	$\hat{Y}$	$\check{N}$
S_{11}	$\hat{Y}$	$\check{N}$	$\hat{Y}$	$\hat{Y}$	$\check{N}$	$\hat{Y}$
S_{12}	$\hat{Y}$	$\check{N}$	$\check{N}$	$\hat{Y}$	$\hat{Y}$	$\check{N}$

TABLE 2. Assessment of Symptom Similarities in Twelve Patients

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
S_1	1	0.2	0.8	0.4	0.4	0.4	0.8	0.6	0.8	0.2	0.8	0.4
S_2	0.2	1	0.4	0.8	0.8	0.8	0.4	0.6	0.4	1	0.4	0.8
S_3	0.8	0.4	1	0.2	0.6	0.2	1	0.8	0.6	0.4	0.6	0.6
S_4	0.4	0.8	0.2	1	0.6	1	0.2	0.4	0.6	0.8	0.6	0.6
S_5	0.4	0.8	0.6	0.6	1	0.6	0.6	0.8	0.6	0.8	0.6	1
S_6	0.4	0.8	0.2	1	0.6	1	0.2	0.4	0.6	0.8	0.6	0.6
S_7	0.8	0.4	1	0.2	0.6	0.2	1	0.8	0.6	0.4	0.6	0.6
S_8	0.6	0.6	0.8	0.4	0.8	0.4	0.8	1	0.4	0.6	0.4	0.8
S_9	0.8	0.4	0.6	0.6	0.6	0.6	0.6	0.4	1	0.4	1	0.6
S_{10}	0.2	1	0.4	0.8	0.8	0.8	0.4	0.6	0.4	1	0.4	0.8
S_{11}	0.8	0.4	0.6	0.6	0.6	0.6	0.6	0.4	1	0.4	1	0.6
S_{12}	0.4	0.8	0.6	0.6	1	0.6	0.6	0.8	0.6	0.8	0.6	1

relationships, which makes sure that the resulting structure accurately reflects expert knowledge and real-world concerns.

To accomplish this, we establish a binary relation $\mathcal{R}$ on the patient set $\mathfrak{Q}$ as follows:

$$S_i \mathcal{R} S_j \iff \mathfrak{U}(S_i, S_j) \geq 0.8, \quad \forall i, j \in \{1, 2, \dots, 12\},$$

where $\mathfrak{U}(S_i, S_j)$ represents the proportion of shared symptoms between patients S_i and S_j relative to the total number of considered symptoms.

Remark. The construction above gives us a similarity relation that is based on experts. The threshold value of 0.8 serves as a similarity cutoff; higher values indicate a greater similarity between patients and are thus anticipated to yield more reliable diagnostic results. Medical professionals can change this threshold to fit their conceptual needs. It is important to note that the relation $\mathcal{R}$ is reflexive and symmetric, but it does not satisfy transitivity. Thus, the traditional Pawlak approximation space, which is based on equivalence relations, is not good enough to accurately model this situation.

To get around this problem, we give each patient two types of neighborhoods based on the relation $\mathcal{R}$ and the underlying attribute information: the *right neighborhood* (r -neighborhood) and the *maximal neighborhood* ((r) -neighborhood). These neighborhood systems furnish the structural foundation for the development of non-equivalence-based approximation spaces. Table 3 shows a summary of the neighborhoods that were created for all patients.

TABLE 3. Right and Maximal Right Neighborhoods Associated with Each Patient

Patient S	r -Neighborhood	(r) -Neighborhood
S ₁	{S ₁ , S ₃ , S ₇ , S ₉ , S ₁₁ }	{S ₁ , S ₃ , S ₇ , S ₈ , S ₉ , S ₁₁ }
S ₂	{S ₂ , S ₄ , S ₅ , S ₆ , S ₁₀ , S ₁₂ }	{S ₂ , S ₄ , S ₅ , S ₆ , S ₈ , S ₁₀ , S ₁₂ }
S ₃	{S ₁ , S ₃ , S ₇ , S ₈ }	{S ₁ , S ₃ , S ₅ , S ₇ , S ₈ , S ₉ , S ₁₁ , S ₁₂ }
S ₄	{S ₂ , S ₄ , S ₆ , S ₁₀ }	{S ₂ , S ₄ , S ₅ , S ₆ , S ₁₀ , S ₁₂ }
S ₅	{S ₂ , S ₅ , S ₈ , S ₁₀ , S ₁₂ }	$\mathcal{Q} \setminus \{S_1, S_9, S_{11}\}$
S ₆	{S ₂ , S ₄ , S ₆ , S ₁₀ }	{S ₂ , S ₄ , S ₅ , S ₆ , S ₁₀ , S ₁₂ }
S ₇	{S ₁ , S ₃ , S ₇ , S ₈ }	{S ₁ , S ₂ , S ₃ , S ₅ , S ₇ , S ₈ , S ₁₂ }
S ₈	{S ₃ , S ₅ , S ₇ , S ₈ , S ₁₂ }	$\mathcal{Q} \setminus \{S_4, S_6, S_9, S_{11}\}$
S ₉	{S ₁ , S ₉ , S ₁₁ }	{S ₁ , S ₃ , S ₇ , S ₉ , S ₁₁ }
S ₁₀	{S ₂ , S ₄ , S ₅ , S ₆ , S ₁₀ , S ₁₂ }	{S ₂ , S ₄ , S ₅ , S ₆ , S ₈ , S ₁₀ , S ₁₂ }
S ₁₁	{S ₁ , S ₉ , S ₁₁ }	{S ₁ , S ₃ , S ₇ , S ₉ , S ₁₁ }
S ₁₂	{S ₂ , S ₅ , S ₈ , S ₁₂ }	$\mathcal{Q} \setminus \{S_1, S_{11}\}$

If $\mathfrak{P}_{S_1}$ is a primal on $\mathcal{Q}$, then Table 4 presents a comparison between the proposed technique and Dai's method [33], considering the class of patients diagnosed with the disease, $\mathbb{B}$, and the patient set not diagnosed with heart failure, $\mathbb{W}$.

TABLE 4. Comparison of the Proposed Method with Dai's Approach

	Dai et al. Approach			Proposed Method		
	$\tau_{(r)}L(\cdot)$	$\tau_{(r)}U(\cdot)$	$\tau_{(r)}\sigma(\cdot)$	$L_{(r)}^{\mathfrak{P}}(\cdot)$	$U_{(r)}^{\mathfrak{P}}(\cdot)$	$\sigma_{(r)}^{\mathfrak{P}}(\cdot)$
$\mathbb{B}$	{S ₁ , S ₉ , S ₁₁ }	$\mathcal{Q} \setminus \{S_4, S_6\}$	$\frac{3}{10}$	$\mathcal{Q}$	$\mathcal{Q}$	1
$\mathbb{W}$	{S ₄ , S ₆ }	$\mathcal{Q} \setminus \{S_1, S_9, S_{11}\}$	$\frac{2}{9}$	$\mathcal{Q}$	$\mathcal{Q}$	1

Based on the definition of the primal $\mathfrak{P}_{S_1}$ and the data shown in Table 4, we conclude that our findings are consistent with those reported in Table 16 of [35], thereby confirming the reliability and validity of the proposed framework.

Algorithm: Accuracy Computation and Exactness Verification

Input:

- A finite universe $\mathfrak{Q}$.
- A target subset $\mathcal{X} \subseteq \mathfrak{Q}$.
- A κ -maximal neighborhood system (denoted as κ -NS).
- A primal neighborhood system $\mathfrak{P}$ over $\mathfrak{Q}$.
- Rules for computing the lower approximation $L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$, upper approximation $U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$, and accuracy $\sigma_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$.

Procedure:

(1) Neighborhood Construction:

For each $z \in \mathfrak{Q}$, compute the κ -maximal neighborhood $N_{(\kappa)}(z)$.

(2) Initialization:

- Set $L_{(\kappa)}^{\mathfrak{P}}(\emptyset)$.
- Set $U_{(\kappa)}^{\mathfrak{P}}(\emptyset)$.

(3) Lower Approximation Computation:

For each $z \in \mathcal{X}$, do:

- If $N_{(\kappa)}(z) \setminus \mathcal{X} \in \mathfrak{P}$, then add z to $L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$.

(4) Upper Approximation Computation:

For each $z \in \mathfrak{Q}$, do:

- If $N_{(\kappa)}(z) \cap \mathcal{X} \neq \emptyset$, then add z to $U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$.

(5) Accuracy Computation:

$$\sigma_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = \begin{cases} \frac{|L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})|}{|U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})|}, & \text{if } |U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})| \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

(6) Exactness Check:

- If $L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X}) = U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$, classify $\mathcal{X}$ as *exact*.
- Otherwise, classify $\mathcal{X}$ as *rough*.

Output:

- $L_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$: Lower approximation of $\mathcal{X}$.
- $U_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$: Upper approximation of $\mathcal{X}$.
- $\sigma_{(\kappa)}^{\mathfrak{P}}(\mathcal{X})$: Accuracy measure of $\mathcal{X}$.
- Classification of $\mathcal{X}$ as either *exact* or *rough*.

6. DISCUSSION

The models developed in this study demonstrate the utility of topological methods in enhancing rough set theory through their expressiveness and adaptability. We present adaptive approximation operators that can encapsulate intricate and nuanced relationships among elements within a specified universe by integrating κ -maximal neighborhoods with primal-based neighborhood systems.

Approximations derived from κ -maximal neighborhoods exhibit significant sensitivity to the structural characteristics of the underlying supra-topological space. This responsiveness makes it easier to move between areas that are open and closed, which helps to make the boundary areas more clearer. The primal-based framework, on the other hand, is modular and very flexible. You can change the approximation boundaries based on what you know about the domain, what experts say, or what the situation calls for.

These features make the proposed framework especially good for problems that aren't clear, aren't complete, or are only partially included, like classification problems, medical diagnosis, and decision-support systems. Adding graded similarity measures makes the model better at showing uncertainty in the real world, where relationships between objects are rarely binary.

The proposed approach is better at handling non-equivalence-based information granules than classical rough set and neighborhood-based models, like those made by Pawlak, Yao, and Dai et al., because it is more flexible. This advantage is most evident in contexts where transitivity cannot be presumed. The medical case study presented previously demonstrates that the suggested approximation models are both theoretically robust and practically applicable. In general, the results show that the system can be made bigger and used in places where decisions are made based on data and intelligence.

7. FINAL THOUGHTS AND FUTURE DIRECTIONS

This paper has created a new group of rough set approximation models based on topological ideas. It examines primal structures and κ -maximal neighborhood systems delineated in spaces that transcend topological dimensions. By relaxing classical constraints, especially the need for a finite intersection, the proposed framework can handle data structures that are more complex, varied, and changing than what traditional equivalence- or tolerance-based rough set models can do. We have done a full mathematical study of the new approximation operators, as well as a close look at their most important features. These show that they are very flexible, modular, and able to say a lot of different things. These traits make the suggested method especially good for modeling uncertainty in situations where traditional rough set methods don't work well. Decision-making applications, like a case study of a medical diagnosis, have shown that the framework is useful in the real world. When compared to models that have been around for a while, like those by Pawlak, Yao, Dai et al., and R. A. Hosny et al., it shows big improvements in how well it can adapt, how well it can classify, and how well it can handle different types of information. This paper shows

how primal structures can be used as a powerful and unified tool for creating supra-topological rough approximations. It also explains their benefits through careful theoretical analyses and real-world examples. The suggested framework naturally leads to more research on dynamic and time-evolving topological systems, applications that use a lot of data and are big, algebraic and categorical formulations that are more complicated, and intelligent decision-support systems that are more advanced. This opens up new areas of research at the intersection of topology, rough set theory, and data analytics that looks at uncertainty, for examples in the directions [53,54].

Availability of Data and Materials: The data supporting this study are publicly available in the previously published work cited as Reference [52]. No new datasets were generated during this study.

Acknowledgments: The authors gratefully acknowledge the Editor-in-Chief, the Section Editor, and the anonymous reviewers for their valuable comments and constructive suggestions, which substantially improved the manuscript.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] Z. Pawlak, Rough Sets, *Int. J. Comput. Inf. Sci.* 11 (1982), 341–356. <https://doi.org/10.1007/BF01001956>.
- [2] W. Zhu, Generalized Rough Sets Based on Relations, *Inf. Sci.* 177 (2007), 4997–5011. <https://doi.org/10.1016/j.ins.2007.05.037>.
- [3] K. Qin, J. Yang, Z. Pei, Generalized Rough Sets Based on Reflexive and Transitive Relations, *Inf. Sci.* 178 (2008), 4138–4141. <https://doi.org/10.1016/j.ins.2008.07.002>.
- [4] Y.-L. Zhang, J. Li, C. Li, Topological Structure of Relation-Based Generalized Rough Sets, *Fundam. Inform.* 147 (2016), 477–491. <https://doi.org/10.3233/FI-2016-1418>.
- [5] X. Yang, M. Zhang, H. Dou, J. Yang, Neighborhood Systems-Based Rough Sets in Incomplete Information System, *Knowl. Based Syst.* 24 (2011), 858–867. <https://doi.org/10.1016/j.knsys.2011.03.007>.
- [6] C. Liu, B. Lin, D. Miao, A Novel Adaptive Neighborhood Rough Sets Based on Sparrow Search Algorithm and Feature Selection, *Inf. Sci.* 679 (2024), 121099. <https://doi.org/10.1016/j.ins.2024.121099>.
- [7] T. Hachaj, J. Wąs, Rough Neighborhood Graph: A Method for Proximity Modeling and Data Clustering, *Appl. Soft Comput.* 171 (2025), 112789. <https://doi.org/10.1016/j.asoc.2025.112789>.
- [8] A.A. Abdien, R.A. Hosny, M. Abd Elaziz, Topological Methodologies for Generalized Multi-Granulation, *Bull. Fac. Sci. Zagazig Univ.* 2024 (2025), 136–146. <https://doi.org/10.21608/bfszu.2024.287533.1387>.
- [9] H.M. Abu-Donia, Multi Knowledge Based Rough Approximations and Applications, *Knowl. Based Syst.* 26 (2012), 20–29. <https://doi.org/10.1016/j.knsys.2011.06.010>.
- [10] R.A. Ibrahim, M. Abd Elaziz, M. Hazman, R.A. Hosny, Feature Selection Based on Multi-Knowledge Adhesion Neighborhoods and Modified Walrus Optimizer, *Comput. Appl. Math.* 45 (2026), 443. <https://doi.org/10.1007/s40314-026-03825-y>.
- [11] M. Abd Elaziz, H.M. Abu-Donia, R.A. Hosny, S.L. Hazae, R.A. Ibrahim, Improved Evolutionary-Based Feature Selection Technique Using Extension of Knowledge Based on the Rough Approximations, *Inf. Sci.* 594 (2022), 76–94. <https://doi.org/10.1016/j.ins.2022.01.026>.

- [12] R.A. Hosny, M. Abd Elaziz, R.A. Ibrahim, Enhanced Feature Selection Based on Integration Containment Neighborhoods Rough Set Approximations and Binary Honey Badger Optimization, *Comput. Intell. Neurosci.* 2022 (2022), 1–17. <https://doi.org/10.1155/2022/3991870>.
- [13] R.A. Hosny, R. Abu-Gdairi, M.K. El-Bably, Approximations by Ideal Minimal Structure with Chemical Application, *Intell. Autom. Soft Comput.* 36 (2023), 3073–3085. <https://doi.org/10.32604/iasc.2023.034234>.
- [14] M. Kondo, W.A. Dudek, Topological Structures of Rough Sets Induced by Equivalence Relations, *J. Adv. Comput. Intell. Intell. Inform.* 10 (2006), 621–624. <https://doi.org/10.20965/jaciii.2006.p0621>.
- [15] M. Kondo, On the Structure of Generalized Rough Sets, *Inf. Sci.* 176 (2006), 589–600. <https://doi.org/10.1016/j.ins.2005.01.001>.
- [16] M.K. El-Bably, R.A. Hosny, M.A. El-Gayar, Innovative Rough Set Approaches Using Novel Initial-Neighborhood Systems: Applications in Medical Diagnosis of COVID-19 Variants, *Inf. Sci.* 708 (2025), 122044. <https://doi.org/10.1016/j.ins.2025.122044>.
- [17] R.A. Hosny, R. Abu-Gdairi, M.K. El-Bably, Enhancing Dengue Fever Diagnosis with Generalized Rough Sets: Utilizing Initial-Neighborhoods and Ideals, *Alexandria Eng. J.* 94 (2024), 68–79. <https://doi.org/10.1016/j.aej.2024.03.028>.
- [18] M.E. Abd El-Monsef, O.A. Embaby, M.K. El-Bably, Comparison between Rough Set Approximations Based on Different Topologies, *Int. J. Granul. Comput. Rough Sets Intell. Syst.* 3 (2014), 292–305. <https://doi.org/10.1504/IJGCRSIS.2014.068032>.
- [19] Y.Y. Yao, Two Views of the Theory of Rough Sets in Finite Universes, *Int. J. Approx. Reason.* 15 (1996), 291–317. [https://doi.org/10.1016/S0888-613X\(96\)00071-0](https://doi.org/10.1016/S0888-613X(96)00071-0).
- [20] A.A. Allam, M.Y. Bakeir, E.A. Abo-Tabl, New Approach for Closure Spaces by Relations, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* 22 (2006), 285–304.
- [21] A.A. Allam, M.Y. Bakeir, E.A. Abo-Tabl, New Approach for Basic Rough Set Concepts, in: D. Ślęzak et al. (Eds.), *Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing*, Lecture Notes in Computer Science 3641, Springer, Berlin, 2005, pp. 64–73. https://doi.org/10.1007/11548669_7.
- [22] A.A. Abo Khadra, B.M. Taher, M.K. El-Bably, Generalization of Pawlak Approximation Space, in: *Proceedings of the International Conference on Mathematics: Trends and Developments*, Egyptian Mathematical Society, 2007, pp. 335–346.
- [23] M.K. El-Bably, Generalized Approximation Spaces, M.Sc. Thesis, Faculty of Science, Tanta University, 2008.
- [24] R. Abu-Gdairi, On Structures of Rough Topological Spaces Based on Neighborhood Systems, *Ital. J. Pure Appl. Math.* 50 (2023), 61–74.
- [25] R.A. Hosny, B.A. Asaad, A.A. Azzam, T.M. Al-Shami, Various Topologies Generated from E_j -Neighbourhoods via Ideals, *Complexity* 2021 (2021), 4149368. <https://doi.org/10.1155/2021/4149368>.
- [26] O.A. Embaby, N.A. Toumi, A Method for Improving Rough Set Approximation Accuracy in Terms of j -Neighborhood Spaces, *Delta J. Sci.* 38 (2017), 43–52. <https://doi.org/10.21608/djs.2017.139424>.
- [27] A.Ç. Güler, E.D. Yildirim, O.B. Özbakir, Some New Approaches to Neighborhoods via Graphs, *Soft Comput.* 27 (2023), 1303–1315. <https://doi.org/10.1007/s00500-022-07732-2>.
- [28] M.A. El-Gayar, S.H. Shalil, S.A. Kandil, M.K. El-Bably, Generalized Ordered Nano-Topological Rough Approximations for Decision-Making and COVID-19 Risk Assessment, *Int. J. Anal. Appl.* 24 (2026), 227. <https://doi.org/10.28924/2291-8639-24-2026-227>.
- [29] M.A. El-Gayar, S.A. Kandil, M.K. El-Bably, On Nearly Basic Rough Sets and Their Medical Applications, *Malays. J. Math. Sci.* 20 (2026), 597–623. <https://doi.org/10.47836/mjms.20.2.08>.
- [30] M.A. El-Gayar, N.M. Madbouly, A.S. Nawar, Enhancing Economic Decision-Making through the Application of Alpha Basic-Rough Sets, *Int. J. Fuzzy Log. Intell. Syst.* 26 (2026), 98–115. <https://doi.org/10.5391/IJFIS.2026.26.1.98>.

- [31] M.K. El-Bably, R. Abu-Gdairi, K.K. Fleifel, M.A. El-Gayar, Exploring β -Basic Rough Sets and Their Applications in Medicine, *Eur. J. Pure Appl. Math.* 17 (2024), 3743–3771. <https://doi.org/10.29020/nybg.ejpam.v17i4.5545>.
- [32] T.M. Al-Shami, Maximal Rough Neighborhoods with a Medical Application, *J. Ambient Intell. Humaniz. Comput.* 14 (2023), 16373–16384. <https://doi.org/10.1007/s12652-022-03858-1>.
- [33] J. Dai, S. Gao, G. Zheng, Generalized Rough Set Models Determined by Multiple Neighborhoods Generated from a Similarity Relation, *Soft Comput.* 22 (2018), 2081–2094. <https://doi.org/10.1007/s00500-017-2672-x>.
- [34] D.I. Taher, R. Abu-Gdairi, M.K. El-Bably, M.A. El-Gayar, Correction: Decision-Making in Diagnosing Heart Failure Problems Using Basic Rough Sets, *AIMS Math.* 9 (2024), 34270–34271. <https://doi.org/10.3934/math.20241632>.
- [35] D.I. Taher, R. Abu-Gdairi, M.K. El-Bably, M.A. El-Gayar, Decision-Making in Diagnosing Heart Failure Problems Using Basic Rough Sets, *AIMS Math.* 9 (2024), 21816–21847. <https://doi.org/10.3934/math.20241061>.
- [36] K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.
- [37] T.M. Al-Shami, M. Hosny, Generalized Approximation Spaces Generation from $\mathbb{I}_j$ -Neighborhoods and Ideals with Application to Chikungunya Disease, *AIMS Math.* 9 (2024), 10050–10077. <https://doi.org/10.3934/math.2024492>.
- [38] R.A. Hosny, N.M. Madbouly, M.K. El-Bably, Theoretical Advances and Improvements of “Generalized Approximation Spaces Generation from $\mathbb{I}_j$ -Neighborhoods and Ideals with Application to Chikungunya Disease”, *Eur. J. Pure Appl. Math.* 18 (2025), 6358. <https://doi.org/10.29020/nybg.ejpam.v18i4.6358>.
- [39] A.S. Nawar, M.A. El-Gayar, M.K. El-Bably, R.A. Hosny, $\theta\beta$ -Ideal Approximation Spaces and Their Applications, *AIMS Math.* 7 (2022), 2479–2497. <https://doi.org/10.3934/math.2022139>.
- [40] M. Hosny, Idealization of j -Approximation Spaces, *Filomat* 34 (2020), 287–301. <https://doi.org/10.2298/FIL2002287H>.
- [41] M.K. El-Bably, M.A. El-Gayar, A.S. Nawar, R.A. Hosny, Theoretical Refinements and Practical Implications of $\theta\beta$ -Ideal Rough Approximations, *Ann. Fuzzy Math. Inform.* 30 (2025), 113–131. <https://doi.org/10.30948/afmi.2025.30.2.113>.
- [42] A.S. Mashhour, A.A. Allam, F.S. Mahmoud, F.H. Khedr, On Supratopological Spaces, *Indian J. Pure Appl. Math.* 14 (1983), 502–510.
- [43] A.M. Kozae, M. Shokry, M. Zidan, Supra Topologies for Digital Plane, *AASCIT Commun.* 3(1) (2016), 1–10.
- [44] G. Choquet, Sur les notions de filtre et de grille, *C. R. Acad. Sci. Paris* 224 (1947), 171–173.
- [45] S. Acharjee, M. Ozkoc, F.Y. Issaka, Primal Topological Spaces, *Bol. Soc. Parana. Mat.* 43 (2025), 1–9. <https://doi.org/10.5269/bspm.66792>.
- [46] R.A. Hosny, M.K. El-Bably, M.A. El-Gayar, Primal Approximation Spaces by κ -Neighborhoods with Applications, *Eur. J. Pure Appl. Math.* 18 (2025), 5827. <https://doi.org/10.29020/nybg.ejpam.v18i1.5827>.
- [47] R.A. Hosny, M.K. El-Bably, Generating Minimal Topologies from κ -Neighborhoods and Primals, *Math. Syst. Sci.* 3 (2025), 3202. <https://doi.org/10.54517/mss3202>.
- [48] M.K. El-Bably, A.A. Nasef, R.A. Hosny, Primals Associated with Rough Sets, in: *SuperHyperTopologies and SuperHyperStructures with their Applications*, Biblio Publishing, pp. 222–232, (2025).
- [49] M.K. El-Bably, E.A. Abo-Tabl, A Topological Reduction for Predicting of a Lung Cancer Disease Based on Generalized Rough Sets, *J. Intell. Fuzzy Syst.* 41 (2021), 3045–3060. <https://doi.org/10.3233/JIFS-210167>.
- [50] M. El Sayed, M.A. El Safty, M.K. El-Bably, Topological Approach for Decision-Making of COVID-19 Infection via a Nano-Topology Model, *AIMS Math.* 6 (2021), 7872–7894. <https://doi.org/10.3934/math.2021457>.
- [51] M. Hosny, Rough Topological Structures by Various Types of Maximal Neighborhoods, *AIMS Math.* 9 (2024), 29662–29688. <https://doi.org/10.3934/math.20241437>.
- [52] A.A. Azzam, A.M. Khalil, S.-G. Li, Medical Applications via Minimal Topological Structure, *J. Intell. Fuzzy Syst.* 39 (2020), 4723–4730. <https://doi.org/10.3233/JIFS-200651>.
- [53] M.A. El-Gayar, R. Abu-Gdairi, Extension of Topological Structures Using Lattices and Rough Sets, *AIMS Math.* 9 (2024), 7552–7569. <https://doi.org/10.3934/math.2024366>.

- [54] R. Abu-Gdairi, A.A. Nasef, M.A. El-Gayar, M.K. El-Bably, On Fuzzy Point Applications of Fuzzy Topological Spaces, Int. J. Fuzzy Log. Intell. Syst. 23 (2023), 162–172. <https://doi.org/10.5391/IJFIS.2023.23.2.162>.