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The Second-Level General Fractional Derivatives and Applications

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Abstract. In the theory of fractional calculus, we focus on the general fractional calculus because it provides a more precise and flexible modeling tool for a wide range of complex dynamical systems. We initially consider the Sonin and Kochubei kernels along with their key properties. The primary purpose of this article is to develop general fractional derivatives (GFDs) of the first level up to the second level. Compared with the 1st level GFDs, the second-level operators are constructed from the convolution of three kernels. We also present specific features of the second-level GFDs, including the first and second fundamental theorems of fractional calculus with an additional condition and appropriately defined general fractional integrals. Afterward, we establish the Laplace transform of the second-level GFDs, and we find a solution for the fractional relaxation equations with these second-level operators.

1. Introduction

Fractional integrals and derivatives, as an extension of integer order to non-integer order (real or complex), have been the focus of extensive research in the past three centuries. The questioning of the order arose immediately after I. Newton and G. W. Leibniz introduced the concept of the derivative. Leibniz's letter to L'Hospital in 1695 concerning a derivative of order 1/2 serves as a recollection of the origin of fractional calculus.

For the classical definitions of the fractional integrals and derivatives, we refer to [29]. Within the framework of operators of non-integer order, our discussion highlights more general operators that extend the classical notions of integration and differentiation. In this regard, N. Ya. Sonin was the first who laid the foundation for the definition of such general fractional operators in 1884 [3]. In a well-known study, Sonin used new kernels to solve N. H. Abel's integral equation, which are now known as Sonin kernels. Sonin kernels are an extension of power-law functions constructed

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to satisfy a specific condition (known as the Sonin condition), introduced as a pair of functions. The conditions that guarantee the existence of a pair of Sonin kernels make them helpful in solving integral equations. Sonin's results show that if a kernel satisfies the Sonin condition with its associated kernel, then a closed-form solution similar to that obtained by Abel in modeling the Tautochrone problem [1] can be derived. In particular, by examining the derived formula by Sonin, it is possible to define generalized integrals and derivatives of non-integer order [5]. Although Sonin obtained a solution to Abel's equation, the connection of these operators with fractional calculus (FC) was not discussed explicitly. In fact, Sonin's results were expressed in the context of integral equations, without specifying the associated operators as general fractional integrals (GFIs) or derivatives.

The kernel-based formulation of GFIs and GFDs used in the present work follows the framework developed by A. N. Kochubei [6]. Based on the Laplace transforms of Sonin kernels, Kochubei initially presented a significant class, known as the Kochubei kernels, represented by $\mathcal{K}$. By defining the GFDs with kernels in $\mathcal{K}$, Kochubei started a new branch of FC research, focusing on ordinary and partial differential equations. Kochubei also proved the first and second fundamental theorems of FC for the GFIs and GFDs with kernels from $\mathcal{K}$ [6].

In what follows, we review several works that form an analytical basis for Sonin kernels. The analysis of Sonine-type integral equations by Samko and Cardoso [5] provides insight into the structure and behavior of Sonin kernels and explains how these kernels arise in integral equations of the first kind. Moreover, the results of Samko and Vakulov [25] establish equivalent norms in fractional-order function spaces, and such norm equivalences help control and compare the behavior of fractional operators in their natural functional setting. Along these lines, Kukushkin [26] presents an abstract framework for fractional calculus based on fractional powers of m-accretive operators. Within this setting, he examines examples such as the Kipriyanov operator, the Riesz potential, and certain difference operators, all arising as transforms of m-accretive operators. In addition, Luchko's work [24] develops a systematic framework in which the order of generalized fractional operators can be increased. More precisely, the fractional operator is built from a single Sonin kernel, and the order increases by iterating the same operator, so the kernel remains unchanged at each step. All analytical properties then follow from this single kernel.

We next consider some works of Y. Luchko on GFIs and GFDs, using kernels from various spaces. Luchko defined and investigated general fractional operators whose Sonin kernels are continuous on the positive real axis and exhibit a weak power-type singularity at the origin, in [8].

In previous studies on GFDs, the Riemann-Liouville and Caputo types were examined separately. A generalized fractional derivative that integrates both types of GFDs was first introduced by Luchko [9]. According to the introduction of the nth level Hilfer fractional derivative in [7], Luchko's GFD was named the 1st level GFDs. The 1st level GFDs are combinations of two specific GFIs and one first-order derivative. To generalize this framework, we can define the second-level GFDs as a composition of three GFIs and two first-order derivatives. This topic is the focus of the

current essay.
Second-level GFDs are effective for systems with memory behavior that varies across different time scales, such as anomalous diffusion, relaxation with distributed delays [18, 22, 23], and viscoelasticity [28]. Unlike 1st level GFDs, which are limited to two kernels, second-level GFDs use three independent kernels to model multi-scale dynamics and transitional memory effects.
Notably, the FC literature has utilized GFDs and GFIs with Luchko kernels for both theoretical and practical purposes.
General fractional calculus (GFC) is also essential in non-local continuum mechanics, especially in modeling fractional quantum states and non-Markovian dynamics. GFC provides a framework for describing processes that present memory effects seen in quantum events. In this regard, V. E. Tarasov utilized general fractional calculus to establish an extensive framework for analyzing non-local phenomena in various contexts by proposing and examining general fractional dynamics (GFDynamics) and general fractional vector calculus (GFVC). The works of Tarasov emphasize the use of operators with the kernels from the Sonin and Luchko sets in integro-differential equations to characterize nonlocality in time. The core components of Tarasov's papers are modeling complex dynamical systems using GFC [11–17].
In brief, we provide an overview of each section's contents. Section 2 reviews the principal preliminaries, which contain the definition of Sonin and Kochubei kernels and their related theorems. In Section 3, we review the 1st level GFDs introduced by Luchko, then present the main results, in which we define and analyze the second-level GFDs. In addition, we prove both the first and second principal theorems of FC for second-level GFDs. Moreover, we determine the projector operator's formula associated with these second-level operators. In the last section, we find a formula for the Laplace transform of the second-level GFDs to solve fractional relaxation equations associated with these operators.

2. Preliminaries

The so-called GFIs and GFDs defined via Sonin kernels are now one of the most significant topics in the FC literature. We briefly explain the motivation for introducing Sonin kernels. In [2], Abel considered the integro-differential equation

$$f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, \phi(\tau)\, d\tau, \quad \phi \in L^1_{loc}(0,\infty),\ 0<\alpha<1,\ t>0, \tag{2.1}$$

Moreover, in order to solve equation (2.1), Abel used the following relation:

$$(h_\alpha * h_{1-\alpha})(t) = h_1(t) = \{1\},\ h_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0<\alpha<1,\ t>0 \tag{2.2}$$

where $\{1\}$ corresponds to the constant function equal to one for all positive t and $*$ represents the Laplace convolution.

Following Abel's approach, Sonin investigated the integro-differential equation

$$f(t) = \int_0^t \kappa_0(t-\tau)\ \phi(\tau)\ d\tau = (\kappa_0 * \phi)(t), \quad \phi \in \mathrm{L}^1_{loc}(0,\infty),\ t > 0, \tag{2.3}$$

To solve equation (2.3), Sonin used the functions $\kappa_0,\ \kappa_1$ in place of $h_\alpha,\ h_{1-\alpha}$ in relation (2.2) and introduced the Sonin condition [3]:

$$(\kappa_0 * \kappa_1)(t) = \int_0^t \kappa_0(t-\tau)\ \kappa_1(\tau)\ d\tau = \{1\}, \quad t > 0, \tag{2.4}$$

In relation (2.4), the functions κ_0, κ_1 are identified as Sonin kernels, and the set of all such functions is denoted by $\mathcal{S}$.

Accordingly, Sonin presented the solution of the relation (2.3) in the following form

$$\phi(t) = \frac{d}{dt}(\kappa_1 * f)(t) = \frac{d}{dt}\int_0^t \kappa_1(t-\tau)\ f(\tau)\ d\tau, \quad t > 0. \tag{2.5}$$

In [6], Kochubei suggested a different approach, in which a structural link between the Sonin kernels and complete Bernstein functions was established(for further details on Bernstein functions, see [30]).

In fact, Kochubei studied the features of convolutional integro-differential operators with kernels k that meet certain properties, and the collection of such kernels is denoted by $\mathcal{K}$; see [6] for additional information. Moreover, Kochubei demonstrated that for each kernel $\kappa_1 \in \mathcal{K}$, there is a kernel κ_0 that fulfills the property (2.4) which means $\mathcal{K} \subset \mathcal{S}$.

Kochubei interpreted the operators in the right-hand sides of (2.1) and (2.5) as the GFI and GFD, respectively, for $(\kappa_0, \kappa_1) \in \mathcal{K}$:

$$(\mathbb{I}_{(\kappa_0)}f)(t) := (\kappa_0 * f)(t) = \int_0^t \kappa_0(t-\tau)f(\tau)\,d\tau, \quad t > 0, \tag{2.6}$$

$$(\mathbb{D}_{(\kappa_1)}f)(t) := \frac{d}{dt}(\kappa_1 * f)(t) = \frac{d}{dt}\int_0^t \kappa_1(t-\tau)f(\tau)\,d\tau, \quad t > 0,$$

With kernels in collection $\mathcal{K}$, $\mathbb{D}_{(\kappa_1)}$ is the left inverse operator to $\mathbb{I}_{(\kappa_0)}$ [6].

To introduce another important family of Sonin kernels, $\mathcal{L}_1$, we need a specific space. I. H. Dimovski initially defined the required space in [10] for the purpose of developing the operational calculus for the hyperbessel differential operators. Following [4], we recall the definition of the space C_α as follows:

Definition 2.1. *The space $C_\alpha, \alpha \in \mathbb{R}$, consists of all real or complex-valued functions $f(x), x > 0$, for which there is a real number $p,\ p > \alpha$, such that*

$$f(x) = x^p\ f_1(x)$$

with $f_1(x) \in C[0,\infty)$.

In order to define the set $\mathcal{L}_1$, Luchko utilized a sub-space of $C_{-1}(0,+\infty)$ in [8], which is defined as follows:

$$C_{-1,0}(0,+\infty) := \{f \ : \ f(t) = t^{p-1} f_1(t), \ 0 < p < 1, \ t > 0, \ f_1 \in C[0,+\infty)\}.$$

Luchko has also investigated the properties of GFIs and GFDs with the kernels from $\mathcal{L}_1$ [8]. We next present the definition of $\mathcal{L}_1$ from [8]:

Definition 2.2. *The pair of kernels $\kappa_0, \ \kappa_1$ are in the set $\mathcal{L}_1$ if and only if we have the following conditions:*

$$\kappa_0, \kappa_1 \in C_{-1,0}(0,+\infty), \quad (\kappa_0 * \kappa_1)(t) = \{1\}.$$

Any Sonin kernel κ_1 from $\mathcal{L}_1$ has a unique associated kernel κ_0.

3. Basic Definitions and Theorems

In this section, we review the introduction and the essential features of the 1st level GFDs considered by Luchko in [9]. We begin by defining the 1st level kernels which were introduced by Luchko [9]:

Definition 3.1. *The functions $\kappa_0, \ \kappa_1, \ \kappa_2 \in C_{-1}(0,+\infty)$ that satisfy the following condition*

$$(\kappa_0 * \kappa_1 * \kappa_2)(t) = h_1(t) = \{1\}, \ t > 0$$

are called the 1st level kernels of the GFDs.

In the subsequent discussion, the set of 1st level kernels will be symbolized by $\mathcal{L}_1^1$ (see [9] for more details).
Subsequently, Luchko determined the 1st level GFDs associated with kernels from $\mathcal{L}_1^1$, as below

$$({}_{1L}\mathbb{D}_{(\kappa_1,\kappa_2)} f)(t) := (\mathbb{I}_{(\kappa_1)} \frac{d}{dt} \, \mathbb{I}_{(\kappa_2)} f)(t) \tag{3.1}$$

where the GFI used in this formula is given by (2.6). In the sequel, Theorem (3.1) is known as the first fundamental theorem of FC, while Theorem (3.2) is referred to as the second one. Luchko [9] proved the following two theorems:

Theorem 3.1. *For $(\kappa_0, \kappa_1, \kappa_2) \in \mathcal{L}_1^1$, on the space $C_{-1,(\kappa_1)}(0,+\infty)$, the 1st level GFD* (3.1) *is a left inverse operator to the GFI* (2.6)*:*

$$({}_{1L}\mathbb{D}_{(\kappa_1,\kappa_2)} \, \mathbb{I}_{(\kappa_0)} f)(t) = f(t), \ t > 0. \tag{3.2}$$

where the functions in $C_{-1,(\kappa_1)}(0,+\infty)$ can be written in the form $f(t) = (\mathbb{I}_{(\kappa_1)}\phi)(t)$ for some $\phi \in C_{-1}(0,+\infty)$.

Theorem 3.2. *For $(\kappa_0, \kappa_1, \kappa_2) \in \mathcal{L}_1^1$, the relation*

$$(\mathbb{I}_{(\kappa_0)} \ {}_{1L}\mathbb{D}_{(\kappa_1,\kappa_2)} f)(t) = f(t) - (\mathbb{I}_{(\kappa_2)} f)(0) \ (\kappa_0 * \kappa_1)(t),$$

holds for all $f \in C^1_{-1,(\kappa_2)}(0,+\infty)$ and $t > 0$.

4. Main Results

In what follows, we present the formulation of the second-level GFDs and analyze their essential properties. We begin by generalizing the Sonin condition.

Definition 4.1. *The second-level kernels of the GFDs are the functions $\kappa_0,\ \kappa_1,\ \kappa_2,\ \kappa_3$ in $C_{-1}(0,+\infty)$ that satisfy the condition*

$$(\kappa_0 * \kappa_1 * \kappa_2 * \kappa_3)(t) = h_1(t) = \{1\},\ t > 0. \tag{4.1}$$

In this paper, the set of the second-level kernels is denoted by $\mathcal{L}_1^2$. Because Laplace convolution is commutative, the kernels in $\mathcal{L}_1^2$ are unordered. A significant example of elements of $\mathcal{L}_1^2$ is given as follows:

$$\kappa_0(t) = h_\alpha(t),\ \kappa_1(t) = h_{\beta_1}(t),\ \kappa_2(t) = h_{\beta_2}(t),\ \kappa_3(t) = h_{1-\alpha-\beta_1-\beta_2}(t),\ t > 0, \tag{4.2}$$

under the assumptions

$$0 < \alpha < 1,\ \beta_1, \beta_2 > 0,\ 0 < \beta_1 + \beta_2 < 1 - \alpha. \tag{4.3}$$

The inequalities (4.3) guarantee that $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ defined in (4.2) belong to $(-1, 0)$ and these kernels also satisfy the condition (4.1), since the relation

$$(h_\alpha * h_{\beta_1} * h_{\beta_2} * h_{1-\alpha-\beta_1-\beta_2})(t) = (h_{\alpha+\beta_1+\beta_2+(1-\alpha-\beta_1-\beta_2)})(t) = h_1(t) = \{1\},$$

is valid.

Remark 4.1. *Assume that the kernels $\kappa_1, \kappa_2, \kappa_3 \in C_{-1}(0,+\infty)$ are given and that their Laplace transforms*

$$\hat{\kappa}_j(s) = \mathcal{L}\{\kappa_j\}(s), \quad j = 1, 2, 3,$$

exist and are nonzero in a right half-plane $\Re(s) > c$. We seek a kernel κ_0 such that

$$(\kappa_0 * \kappa_1 * \kappa_2 * \kappa_3)(t) = h_1(t) = \{1\},\ t > 0.$$

Applying the Laplace transform to this identity, we obtain

$$\hat{\kappa}_0(s)\hat{\kappa}_1(s)\hat{\kappa}_2(s)\hat{\kappa}_3(s) = \mathcal{L}\{h_1\}(s) = \frac{1}{s}, \quad \Re(s) > c.$$

Thus,

$$\hat{\kappa}_0(s) = \frac{1}{s\hat{\kappa}_1(s)\hat{\kappa}_2(s)\hat{\kappa}_3(s)}. \tag{4.4}$$

We then define κ_0 as the inverse Laplace transform of (4.4)*. If κ_0 belongs to $C_{-1}(0,+\infty)$, then the quadraple $(\kappa_0, \kappa_1, \kappa_2, \kappa_3)$ is a valid second-level Sonin system (any three of the kernels $(\kappa_0, \kappa_1, \kappa_2, \kappa_3)$ can be fixed arbitrarily, and the remaining one is uniquely determined by the inverse Laplace transform of* (4.4).)

Example 4.1. *Let us construct another example of second-level kernels. This example involves κ and k, which were first presented in* [20]*. At the first step, three of the four kernels in the quadruple $(\kappa_0, \kappa_1, \kappa_2, \kappa_3)$ are specified as follows:*

$$\kappa_0(t) = h_{\alpha,\rho}(t), \quad \mathcal{L}\{\kappa_0\}(s) = (s+\rho)^{-\alpha},\ \rho \geqslant 0,\ 0 < \alpha < 1,$$

$$\kappa_1(t) = h_{\beta,\rho}(t), \quad \mathcal{L}\{\kappa_1\}(s) = (s+\rho)^{-\beta},\ \rho \geqslant 0,\ 0 < \beta < 1,$$
$$\kappa_2(t) = h_{\gamma,\rho}(t), \quad \mathcal{L}\{\kappa_2\}(s) = (s+\rho)^{-\gamma},\ \rho \geqslant 0,\ 0 < \gamma < 1,$$

where the function $h_{\alpha,\rho}$ is defined by

$$h_{\alpha,\rho}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\rho t},\ t > 0,\ \alpha > 0,\ \rho \in \mathbb{R}.$$

Assume that, for a given constant $c \in \mathbb{R}$, the Laplace transforms of $\kappa_0, \kappa_1, \kappa_2$ and κ_3 exist in $\Re(s) > c$. According to Remark (4.1)*, we obtain the following relation*

$$\mathcal{L}\{\kappa_3\}(s).(s+\rho)^{-\alpha}.(s+\rho)^{-\beta}.(s+\rho)^{-\gamma} = \frac{1}{s},\ \Re(s) > c.$$

By simplifying, we have

$$\mathcal{L}\{\kappa_3\}(s) = \frac{(s+\rho)^{(\alpha+\beta+\gamma)}}{s}, \tag{4.5}$$

we set $\mu := \alpha + \beta + \gamma$. Then we observe that

$$\frac{(s+\rho)^{\mu}}{s} = (s+\rho)^{\mu-1} + \frac{\rho}{s}(s+\rho)^{\mu-1}, \quad 0 < \mu < 1.$$

We know that

$$\mathcal{L}^{-1}\{(s+\rho)^{\mu-1}\}(t) = \frac{t^{-\mu}}{\Gamma(1-\mu)} e^{-\rho t},$$
$$\mathcal{L}^{-1}\{\frac{\rho}{s}(s+\rho)^{\mu-1}\}(t) = \rho \int_0^t \frac{\tau^{-\mu}}{\Gamma(1-\mu)} e^{-\rho\tau}\, d\tau,$$

So the inverse Laplace transform of the relation (4.5) *is*

$$\kappa_3(t) = h_{1-\mu,\rho}(t) + \rho \int_0^t h_{1-\mu,\rho}(\tau)\, d\tau, \quad 0 < \mu < 1. \tag{4.6}$$

By definition,

$$h_{1-\mu,\rho}(t) := \frac{t^{-\mu}}{\Gamma(1-\mu)} e^{-\rho t} = t^{-\mu}\phi_1(t),$$

where

$$\phi_1(t) := \frac{e^{-\rho t}}{\Gamma(1-\mu)} \in C[0,\infty).$$

Hence the first term in (4.6) *has the form $t^{-\mu}\phi_1(t)$ with $-\mu > -1$ because* $0 < \mu < 1$.
For the second term, we write

$$\rho \int_0^t h_{1-\mu,\rho}(\tau) = \rho \int_0^t \tau^{-\mu}\phi_1(\tau)\, d\tau.$$

Since $0 < \mu < 1$, *the integrand $\tau^{-\mu}\phi_1(\tau)$ is integrable near $\tau = 0$, and we can define*

$$\phi_2(t) := \begin{cases} \dfrac{1}{t^{1-\mu}} \int_0^t \tau^{-\mu}\phi_1(\tau)\, d\tau & t > 0 \\ \dfrac{1}{\Gamma(2-\mu)} & t = 0 \end{cases}$$

and set $\phi_2(0) := \lim_{t\to 0^+} \phi_2(t)$. *Continuity of* ϕ_1 *and* $0 < \mu < 1$ *show that* $\phi_2 \in C[0,\infty)$ *and*

$$\rho \int_0^t h_{1-\mu,\rho}(\tau)\, d\tau = \rho\, t^{1-\mu}\phi_2(t).$$

Combining both terms, we obtain

$$\kappa_3(t) = t^{-\mu}\phi_1(t) + \rho\, t^{1-\mu}\phi_2(t) = t^{-\mu}(\phi_1(t) + \rho\, t\, \phi_2(t)) = t^{-\mu}\phi(t),$$

where

$$\phi(t) := \phi_1(t) + \rho\, t\, \phi_2(t) \in C[0,\infty).$$

Since $-\mu > -1$, *this shows that* κ_3 *has an integrable power-type singularity at the origin and thus* $\kappa_3 \in C_{-1}(0,\infty)$.

Example 4.2. *We present another example, following a process similar to Example* (4.1). *We first specify three of the four kernels from* $(\kappa_0, \kappa_1, \kappa_2, \kappa_3)$:

$$\begin{aligned}
&\kappa_0(t) = t^{\mu-1}J_{\mu-1}(2\sqrt{at}), \quad \mathcal{L}\{\kappa_0\}(s) = s^{-\mu}e^{-a/s},\ t > 0,\\
&\kappa_1(t) = t^{\mu-1}I_{\mu-1}(2\sqrt{at}), \quad \mathcal{L}\{\kappa_1\}(s) = s^{-\mu}e^{a/s},\ t > 0,\\
&\kappa_2(t) = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-ct^{\alpha}), \quad \mathcal{L}\{\kappa_2\}(s) = s^{-\beta}(1 + cs^{-\alpha})^{-\gamma},\ \Re(s) > 0,\ |s| > |c|^{1/\alpha},
\end{aligned}$$

where $2\mu + \beta < 1$ *and the functions* $J_v(t), I_v(t)$ *are the Bessel and modified Bessel functions, respectively, and* $E^{\gamma}_{\alpha,\beta}(z)$ *is the three-parameter Mittag-Leffler (Prabhakar) function.*
Based on Remark (4.1), *we have the following relation*

$$\mathcal{L}\{\kappa_3\}(s) = s^{2\mu+\beta-1}(1 + cs^{-\alpha})^{\gamma},\ \Re(s) > c.$$

κ_3 *is obtained through the inverse Laplace transform*

$$\kappa_3(t) = t^{-2\mu-\beta}E^{\gamma}_{\alpha,1-2\mu-\beta}(-ct^{\alpha}),\ \alpha > 0,\ c > 0,\ \gamma \in \mathbb{R}.$$

Thus,

$$\kappa_3(t) = t^{-2\mu-\beta}\phi(t), \qquad \phi(t) := E^{\gamma}_{\alpha,1-2\mu-\beta}(-ct^{\alpha}).$$

The Prabhakar function $E^{(\gamma)}_{\alpha,\beta}(z)$ *is entire in z for any fixed* $\alpha > 0,\ \beta,\ \gamma \in \mathbb{R}$, *hence* $\phi \in C[0,\infty)$. *Therefore,* κ_3 *has the required structural form for* $C_{-1}(0,\infty)$ *provided the exponent* $-2\mu - \beta > -1$. *This condition is equivalent to* $2\mu + \beta < 1$.

Definition 4.2. *The second-level GFDs for a quadruple* $(\kappa_0, \kappa_1, \kappa_2, \kappa_3) \in \mathcal{L}^2_1$ *are formulated as follows*

$$({}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}f)(t) := (\mathbb{I}_{(\kappa_1)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_3)}f)(t) \tag{4.7}$$

whereas relation (2.6) *provides the proper GFI with particular kernels.*

The main advantage of the second-level GFDs is that it naturally includes the classical derivatives as particular cases. In fact, if we set $\kappa_1 = h_0,\ \kappa_3 = h_1$ then GFD (4.7) reduces to the GFD of the Riemann-Liouvile type with kernel κ_2:

$$({}_{2L}\mathbb{D}_{(h_0,\kappa_2,h_1)}f)(t) = (\mathbb{I}_{(h_0)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\,\mathbb{I}_{(h_1)}f)(t) = (\frac{d}{dt}\,\mathbb{I}_{(\kappa_2)}f)(t) = (\mathbb{D}_{(\kappa_2)}f)(t).$$

Moreover, if we set $\kappa_2 = h_0,\ \kappa_3 = h_1$, the construction (4.7) yields the GFD of the Caputo type with kernel κ_1:

$$({}_{2L}\mathbb{D}_{(\kappa_1,h_0,h_1)}f)(t) = (\mathbb{I}_{(\kappa_1)}\frac{d}{dt}\,\mathbb{I}_{(h_0)}\frac{d}{dt}\,\mathbb{I}_{(h_1)}f)(t) = (\mathbb{I}_{(\kappa_1)}\frac{d}{dt}\,f)(t) = ({}_*\mathbb{D}_{(\kappa_1)}f)(t).$$

Theorem 4.1. *For $(\kappa_0,\kappa_1,\kappa_2,\kappa_3) \in \mathcal{L}_1^2$, on the space $C_{-1,(\kappa_1,\kappa_2,\kappa_3)}(0,+\infty)$, the second-level GFD* (4.7) *acts as a left inverse to the GFI* (2.6):

$$({}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}\ I^1\ \mathbb{I}_{(\kappa_0)}f)(t) = f(t),\ t > 0. \tag{4.8}$$

Here,

$$C_{-1,(\kappa_1,\kappa_2,\kappa_3)}(0,+\infty) := \{f\ :\ f(t) = (\mathbb{I}_{(\kappa_1,\kappa_2,\kappa_3)}\psi)(t),\ \psi \in C_{-1}(0,+\infty)\}$$

and I^α denotes the Riemann integral of order α.

Proof. According to the definition of $C_{-1,(\kappa_1,\kappa_2,\kappa_3)}(0,+\infty)$, the function f can be written as

$$f(t) = (\mathbb{I}_{(\kappa_1)}\mathbb{I}_{(\kappa_2)}\mathbb{I}_{(\kappa_3)}\psi)(t) \tag{4.9}$$

Thus, we substitute the representation (4.9) into the formula (4.8) and obtain the following relations:

$$\begin{aligned}({}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}\ I^1\ \mathbb{I}_{(\kappa_0)}f)(t) &= (\mathbb{I}_{(\kappa_1)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_3)}\ I^1\ \mathbb{I}_{(\kappa_0)}f)(t)\\ &= (\mathbb{I}_{(\kappa_1)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_3)}\ I^1\ \mathbb{I}_{(\kappa_0)}\ \mathbb{I}_{(\kappa_1)}\ \mathbb{I}_{(\kappa_2)}\ \mathbb{I}_{(\kappa_3)}\psi)(t)\end{aligned}$$

We know that $\mathbb{I}_{(\kappa_0)}\ \mathbb{I}_{(\kappa_1)}\ \mathbb{I}_{(\kappa_2)}\ \mathbb{I}_{(\kappa_3)} = I^1$ so we have

$$\begin{aligned}&= (\mathbb{I}_{(\kappa_1)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_3)}\ I^1\ I^1\psi)(t)\\ &= (\mathbb{I}_{(\kappa_1)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\,I^1\ I^1\ \mathbb{I}_{(\kappa_3)}\psi)(t)\\ &= (\mathbb{I}_{(\kappa_1)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_2)}\ I^1\ \mathbb{I}_{(\kappa_3)}\psi)(t)\\ &= (\mathbb{I}_{(\kappa_1)}\frac{d}{dt}\,I^1\ \mathbb{I}_{(\kappa_2)}\ \mathbb{I}_{(\kappa_3)}\psi)(t)\\ &= (\mathbb{I}_{(\kappa_1)}\ \mathbb{I}_{(\kappa_2)}\ \mathbb{I}_{(\kappa_3)}\psi)(t) = f(t).\end{aligned}$$

that proves the formula (4.8). □

Theorem 4.2. *For the kernels $(\kappa_0,\kappa_1,\kappa_2,\kappa_3) \in \mathcal{L}_1^2$, the relation*

$$(\mathbb{I}_{(\kappa_0)}\ {}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}f)(t) = \frac{d}{dt}\,f + (-\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_3)}f)(0)\ (\kappa_0 * \kappa_1)(t) + (-\mathbb{I}_{(\kappa_3)}f)(0)\ \left(\frac{d}{dt}\ (\kappa_0 * \kappa_1 * \kappa_2)\right)(t),$$

is valid for the functions under the conditions

$$\frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f \in C_{-1}(0,+\infty), \quad \frac{d}{dt}\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f \in C_{-1}(0,+\infty).$$

Proof. At the first step, we want to find the kernel of ${}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}$. Therefore, we want to investigate the following equation

$$(\mathbb{I}_{(\kappa_1)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(t) = 0$$

With considering $g_{2,3} := (\frac{d}{dt}\mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(t)$ we have:

$$\mathbb{I}_{(\kappa_1)}\ g_{2,3} = 0 \Leftrightarrow \frac{d}{dt}\ \mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f = 0 \Leftrightarrow \mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f = c_1,$$

By applying $\kappa_0 * \kappa_1 * \kappa_3$ to both sides and using condition (4.1), we obtain

$$\{1\} * \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f = c_1(\{1\} * \kappa_0 * \kappa_1 * \kappa_3) \Leftrightarrow \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f = c_1(\kappa_0 * \kappa_1 * \kappa_3) \tag{4.10}$$

From the right-hand side of (4.10), by the Fundamental Theorem of Calculus, we have the following result:

$$\frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f = c_1(\kappa_0 * \kappa_1 * \kappa_3) \Leftrightarrow \mathbb{I}_{(\kappa_3)}f = c_1(\{1\} * \kappa_0 * \kappa_1 * \kappa_3) + c_2$$

Now we act with $\kappa_0 * \kappa_1 * \kappa_2$ on both sides of (4.10), then we arrive at

$$\{1\} * f = c_1(\{1\} * \kappa_0 * \kappa_1 * \kappa_3 * \kappa_0 * \kappa_1 * \kappa_2) + c_2(\{1\} * \kappa_0 * \kappa_1 * \kappa_2)$$

By considering the condition (4.1) and taking the derivation, we reach the representation

$$\ker\ {}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)} = \{c_1(\{1\} * \kappa_0 * \kappa_1) + c_2(\kappa_0 * \kappa_1 * \kappa_2)\ ,\ c_1, c_2 \in \mathbb{R}\}. \tag{4.11}$$

Now, we define a function denoted by Ψ :

$$\Psi(t) := (\mathbb{I}_{(\kappa_0)}\ {}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}f)(t), \tag{4.12}$$

we apply ${}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}I^1$ to the both sides of (4.12):

$$({}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}\ I^1\Psi)(t) = ({}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}\ I^1\ \mathbb{I}_{(\kappa_0)}\ {}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}f)(t)$$

According to the relation (4.8), we deduce that $I^1\Psi - f$ is an element of the null-space of ${}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}$. By using (4.11), we achieve the relation below

$$(I^1\Psi)(t) = f(t) + c_1(\{1\} * \kappa_0 * \kappa_1)(t) + c_2(\kappa_0 * \kappa_1 * \kappa_2)(t),\ t > 0. \tag{4.13}$$

To determine the constants c_1 and c_2, we employ $\mathbb{I}_{(\kappa_3)}$ to the two sides of (4.13):

$$(\mathbb{I}_{(\kappa_3)}\ I^1\Psi)(t) = (\kappa_3 * f)(t) + c_1(\{1\} * \kappa_0 * \kappa_1 * \kappa_3)(t) + c_2(\{1\}) \tag{4.14}$$

On the other hand, substituting formula (4.12) into (4.14) yields the following equation:

$$\mathbb{I}_{(\kappa_3)}\ \mathbb{I}_{(\kappa_0)}\ \mathbb{I}_{(\kappa_1)}\ I^1\frac{d}{dt}\ \mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f = (\kappa_3 * f)(t) + c_1(\{1\} * \kappa_0 * \kappa_1 * \kappa_3) + c_2(\{1\})$$

So we derive the representation

$$\mathbb{I}_{(\kappa_3)}\ \mathbb{I}_{(\kappa_0)}\ \mathbb{I}_{(\kappa_1)}\left((\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(t)-(\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0)\right)=$$
$$(\kappa_3*f)(t)+c_1(\{1\}*\kappa_0*\kappa_1*\kappa_3)+c_2(\{1\})$$

according to $\mathbb{I}_{(\kappa_0)}\mathbb{I}_{(\kappa_1)}\mathbb{I}_{(\kappa_2)}\mathbb{I}_{(\kappa_3)}=I^1$ we get

$$(I^1\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(t)+\left(-(\mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0)\right)(\{1\}*\kappa_0*\kappa_1*\kappa_3)=$$
$$(\kappa_3*f)(t)+c_1(\{1\}*\kappa_0*\kappa_1*\kappa_3)+c_2(\{1\})$$

Thus,

$$(\mathbb{I}_{(\kappa_3)}f)(t)-(\mathbb{I}_{(\kappa_3)}f)(0)+\left(-(\mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0)\right)(\{1\}*\kappa_0*\kappa_1*\kappa_3)=$$
$$(\kappa_3*f)(t)+c_1(\{1\}*\kappa_0*\kappa_1*\kappa_3)+c_2(\{1\})$$

Rearranging the terms to one side, we obtain

$$\left(c_2(\{1\})+(\mathbb{I}_{(\kappa_3)}f)(0)\right)+\left(c_1+(\mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0)\right)(\{1\}*\kappa_0*\kappa_1*\kappa_3)=0,$$
$$\kappa_0*\kappa_1*\kappa_3*\left(\left(c_2(\{1\})+(\mathbb{I}_{(\kappa_3)}f)(0)\right)\ \kappa_2+\left(c_1+(\mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0)\right)\right)=0$$

Therefore,

$$\left(c_2(\{1\})+(\mathbb{I}_{(\kappa_3)}f)(0)\right)\ \kappa_2+\left(c_1+(\mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0)\right)(\{1\})=0$$

By setting the coefficients equal to zero, we obtain c_1 and c_2 :

$$c_1=-(\mathbb{I}_{(\kappa_2)}\ \frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0),\ c_2=-(\mathbb{I}_{(\kappa_3)}f)(0).$$

So by substituting c_1 and c_2 in the relation (4.13) and using derivation we complete the proof. □

Remark 4.2. *The formula* (4.10) *can be expressed through the projector operator in the following form:*

$$\begin{aligned}(P_{2L}f)(t)&=f(t)-(\mathbb{I}_{(\kappa_0)}\ {}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}I^1f)(t)\\&=(\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}I^1f)(0)\ (\kappa_0*\kappa_1)(t)\\&\quad+(\mathbb{I}_{(\kappa_3)}I^1f)(0)\ \left(\frac{d}{dt}\ (\kappa_0*\kappa_1*\kappa_2)\right)(t).\end{aligned}\tag{4.15}$$

Explicit projector formulas play an essential role in the analysis of fractional differential equations. Through these formulas, differential problems can be reduced to certain integral equations. Moreover, the coefficients appearing in the projector representation also indicate the initial conditions associated with the corresponding fractional differential equations; in particular, formula (4.15) *provides this structure for fractional differential equations with second-level GFDs* (*for an operational approach for computing the solutions of fractional differential equations involving 1st level GFDs, see* [19]).

5. Application

In this part of the paper, we want to find a formula for the Laplace transform of the second-level GFDs. In the next step, we discuss the fractional relaxation equation with these operators. In what follows, the procedure is carried out assuming an additional condition

$$(\kappa_0 * \kappa_1 * \kappa_2)(0) = 0.$$

From the previous subsection, we have the representation

$$\begin{aligned}(\mathbb{I}_{(\kappa_0)}\ {}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}f)(t) &= (\kappa_0 * {}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}f)(t)\\ &= f'(t) - (\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0)(\kappa_0 * \kappa_1)(t)\\ &- (\mathbb{I}_{(\kappa_3)}f)(0)\ \left(\frac{d}{dt}\ (\kappa_0 * \kappa_1 * \kappa_2)\right)(t),\end{aligned} \tag{5.1}$$

Now we apply the Laplace transform to (5.1). Based on the convolution theorem of the Laplace transform, the following relations hold:

$$\begin{aligned}(\mathcal{L}\kappa_0)(s).(\mathcal{L}\ {}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}f)(s) = s(\mathcal{L}f)(s) - f(0) - (\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0)(\mathcal{L}\kappa_0)(s)\\ (\mathcal{L}\kappa_1)(s) - s(\mathbb{I}_{(\kappa_3)}f)(0)(\mathcal{L}\kappa_0)(s)(\mathcal{L}\kappa_1)(s)(\mathcal{L}\kappa_2)(s)\end{aligned}$$

Thus, we arrive at the representation

$$\begin{aligned}(\mathcal{L}\ {}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}f)(s) = \frac{s(\mathcal{L}f)(s) - f(0)}{(\mathcal{L}\kappa_0)(s)} - (\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\ \mathbb{I}_{(\kappa_3)}f)(0)(\mathcal{L}\kappa_1)(s)\\ - s(\mathbb{I}_{(\kappa_3)}f)(0)(\mathcal{L}\kappa_1)(s)(\mathcal{L}\kappa_2)(s)\end{aligned} \tag{5.2}$$

Remark 5.1. *In order to state the following theorem, we impose the following two conditions:*

- $(A1)$ *Each kernel* κ_i *for* $i = 0, 1, 2, 3$ *belongs to* $C_{-1}(0, \infty)$ *and is of the exponential order* α_i*, it means that there exist constants* $M_i > 0$, $\alpha_i \in \mathbb{R}$ *and* $T \geqslant 0$ *such that*

$$|\kappa_i(t)| \leqslant M_i e^{\alpha_i t}, \quad \forall t \geqslant T,$$

  This ensures $(\mathcal{L}\kappa_i)(s)$ *exist and are analytic in* $\Re s > \max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$.
- $(A2)$ *Let* $c > \alpha$*, where* α *is the exponential order from* $(A1)$*. We require that*

$$\inf_{w\in\mathbb{R}} |c + iw + \lambda(\mathcal{L}\kappa_0)(c + iw)| > 0,$$

  this means on the Bromwich contour $\Re(s) = c$*, the denominator never equals zero, consequently* $\dfrac{1}{s + \lambda(\mathcal{L}\kappa_0)(s)}$ *is analytic and bounded.*

We now analyze the general fractional relaxation equation with the second-level GFDs and prove the following result:

Theorem 5.1. *The fractional relaxation equation*

$$({}_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}y)(x) = -\lambda y(x),\ \lambda > 0,\ x > 0 \tag{5.3}$$

with the initial conditions

$$y_0 = y(0),\ y_1 = (\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_3)}y)(0),\ y_2 = (\mathbb{I}_{(\kappa_3)}y)(0) \tag{5.4}$$

under the assumptions $(A1)$ *and* $(A2)$ *has a unique solution, which is represented by*

$$y(x) = \mathcal{L}^{-1}\left[\frac{1}{s+\lambda(\mathcal{L}\kappa_0)(s)}\left(y_0 + \sum_{j=1}^{2} y_j s^{j-1}\prod_{i=0}^{j}(\mathcal{L}\kappa_i)(s)\right)\right] \tag{5.5}$$

where $\mathcal{L}$ *denotes the Laplace transform.*

Proof. According to the formula (5.2), we have

$$\frac{s(\mathcal{L}y)(s)-y_0}{(\mathcal{L}\kappa_0)(s)} - \sum_{j=1}^{2} y_j s^{j-1}\prod_{i=1}^{j}(\mathcal{L}\kappa_i)(s) + \lambda(\mathcal{L}y)(s) = 0$$

By simplifying, we arrive at

$$(s+\lambda(\mathcal{L}\kappa_0)(s))(\mathcal{L}y)(s) = y_0 + (\mathcal{L}\kappa_0)(s)\sum_{j=1}^{2} y_j s^{j-1}\prod_{i=1}^{j}(\mathcal{L}\kappa_i)(s)$$

The solution in the Laplace domain is as follows:

$$(\mathcal{L}y)(s) = \frac{1}{s+\lambda(\mathcal{L}\kappa_0)(s)}\left(y_0 + \sum_{j=1}^{2} y_j s^{j-1}\prod_{i=0}^{j}(\mathcal{L}\kappa_i)(s)\right)$$

Under the conditions $(A1)$ and $(A2)$, the Laplace transform is invertible. Now, we employ the inverse Laplace transform and immediately arrive at the formula (5.5) for the solution to equation (5.3) with the conditions (5.4). □

Remark 5.2. *Now we aim to obtain the solution of the equation* (5.3) *by considering*

$$\kappa_0(t) = h_\alpha(t),\ \kappa_1(t) = h_{\beta_1}(t),\ \kappa_2(t) = h_{\beta_2}(t),\ \kappa_3(t) = h_{1-\alpha-\beta_1-\beta_2}(t) \tag{5.6}$$

Before finding the solutions, we need to find the inverse Laplace transform of the $\dfrac{1}{s+\frac{\lambda}{s^\alpha}}$*:*

$$\frac{1}{s+\frac{\lambda}{s^\alpha}} = \frac{1}{s(1+\frac{\lambda}{s^{\alpha+1}})} = \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n}{s^{n(\alpha+1)}} = \sum_{n=0}^{\infty}(-1)^n\lambda^n\frac{1}{s^{n(\alpha+1)+1}}$$

By applying the inverse Laplace transform, we have

$$\mathcal{L}^{-1}\left[\frac{1}{s+\frac{\lambda}{s^\alpha}}\right] = \sum_{n=0}^{\infty}(-1)^n\lambda^n\frac{t^{n(\alpha+1)}}{\Gamma(n(\alpha+1)+1)} = E_{\alpha+1,1}(-\lambda t^{\alpha+1}).$$

Let us now find the solutions of the equation (5.3) *by considering* (5.6). *For the first term, we should find the inverse Laplace transform of* $\dfrac{(\mathcal{L}\kappa_0)(\mathcal{L}\kappa_1)}{s+\lambda(\mathcal{L}\kappa_0)}$*:*

$$\frac{(\mathcal{L}\kappa_0)(\mathcal{L}\kappa_1)}{s+\lambda(\mathcal{L}\kappa_0)} = \frac{\dfrac{1}{s^{\alpha}}.\dfrac{1}{s^{\beta_1}}}{s+\dfrac{\lambda}{s^{\alpha}}} = \frac{1}{s^{\alpha+\beta_1+1}}.\frac{1}{1+\dfrac{\lambda}{s^{\alpha+1}}} = \sum_{n=0}^{\infty}(-\lambda)^n\frac{1}{s^{n(\alpha+1)+\alpha+\beta_1+1}}$$

So

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{(\mathcal{L}\kappa_0)(\mathcal{L}\kappa_1)}{s+\lambda(\mathcal{L}\kappa_0)}\right] &= \sum_{n=0}^{\infty}(-\lambda)^n\frac{t^{n(\alpha+1)+\alpha+\gamma_1}}{\Gamma(n(\alpha+1)+\alpha+\gamma_1+1)}\\ &= t^{\alpha+\beta_1}E_{\alpha+1,\alpha+\beta_1+1}(-\lambda t^{\alpha+1})\end{aligned}$$

With the same operations for the second term, we arrive at the formula:

$$\mathcal{L}^{-1}\left[\frac{s(\mathcal{L}\kappa_0)(\mathcal{L}\kappa_1)(\mathcal{L}\kappa_2)}{s+\lambda(\mathcal{L}\kappa)}\right] = t^{\alpha+\beta_1+\beta_2-1}E_{\alpha+1,\alpha+\beta_1+\beta_2}(-\lambda t^{\alpha+1})$$

Thus, the solution of the equation (5.3) *should take the form of a relation below:*

$$\begin{aligned}y(t) &= y(0)E_{\alpha+1,1}(-\lambda t^{\alpha+1}) + y_1 t^{\alpha+\beta_1}E_{\alpha+1,\alpha+\beta_1+1}(-\lambda t^{\alpha+1})\\ &\quad + y_2 t^{\alpha+\beta_1+\beta_2-1}E_{\alpha+1,\alpha+\beta_1+\beta_2}(-\lambda t^{\alpha+1}).\end{aligned}$$

This answer is similar to that of the fractional relaxation equation, as stated in [21].

We now consider the damped oscillator equation [27] within our general fractional framework in the following examples:

Example 5.1. *For the kernels in* $\mathcal{L}_1^2$ *we examine the damped oscillator equation with the second-level GFDs given below:*

$$_{2L}\mathbb{D}_{(\kappa_1,\kappa_2,\kappa_3)}u(t) + \gamma\, u'(t) + w^2\, u(t) = f(t), \quad t>0. \tag{5.7}$$

where the forcing term $f(t)$ *and the solution* $u(t)$ *must belong to the space* $C_{-1,(\kappa_1,\kappa_2,\kappa_3)}(0,\infty)$. *By applying Laplace transform to the both sides of* (5.7) *we have*

$$\frac{pU(p)-u(0)}{K_0(p)} - A\,K_1(p) - p\,B\,K_1(p)K_2(p) + \gamma(p\,U(p)-u_0) + w^2\,U(p) = F(p)$$

where

$$\begin{gathered}A := (\mathbb{I}_{(\kappa_2)}\frac{d}{dt}\,\mathbb{I}_{(\kappa_3)}u)(0), \quad B := (\mathbb{I}_{(\kappa_3)}u)(0),\\ \mathcal{L}\{f(t)\}(p) = F(p), \qquad \mathcal{L}\{\kappa_j(t)\}(p) = K_j(p), \qquad \mathcal{L}\{u(t)\}(p) = U(p).\end{gathered}$$

Rearranging the terms

$$U(p) = \frac{F(p)+\gamma u_0 + \dfrac{u_0}{K_0(p)} + A\,K_1(p) + p\,B\,K_1(p)K_2(p)}{\dfrac{p}{K_0(p)} + \gamma\,p + w^2}.$$

The denominator of the Laplace-domain solution,

$$\Phi(p) = \frac{p}{K_0(p)} + \gamma\, p + w^2,$$

must not vanish in the half-plane where the Laplace transforms of u, F, *and the kernels are defined. specifically, we require* $\Phi(p) \neq 0$ *for all* p *with* $\Re(p) \geq 0$. *The solution is obtained by taking the inverse Laplace transform of the expression* $U(p)$.

Example 5.2. *For the kernels in* $\mathcal{L}_1^1$, *we can also consider the following equation*

$$\frac{d^2}{dt^2}u(t) + \gamma\ {}_{1L}\mathbb{D}_{(\kappa_1,\kappa_2)}u(t) + w^2\, u(t) = f(t), \quad t > 0. \tag{5.8}$$

Here, $u(t)$ *and* $f(t)$ *must lie in the natural domain* $C^1_{-1,(\kappa_2)}(0,\infty)$. *Again by applying the Laplace transform we have*

$$p^2U(p) - p\, u_0 - u_0 + \gamma\left[\frac{U(p)}{K_0(p)} - C\, K_1(p)\right] + w^2\, U(p) = F(p)$$

where

$$C := (\mathbb{I}_{(\kappa_2)}u)(0),$$

Therefore, we arrive at the solution as below:

$$u(t) = \mathcal{L}^{-1}\left\{\frac{F(p) + p\, u_0 + u_0 + \gamma\, C\, K_1(p)}{\gamma\dfrac{1}{K_0(p)} + p^2 + w^2}\right\}.$$

In addition, the spectral function

$$\Psi(p) = \gamma\frac{1}{K_0(p)} + p^2 + w^2$$

must satisfy

$$\Psi(p) \neq 0 \qquad for\ \Re(p) \geq 0.$$

6. Conclusions

In this work, we introduced the second-level general fractional derivatives and studied their main properties. By combining three GFIs with two first-order derivatives, the resulting operator extends the structure of the 1st level GFDs and provides a framework for describing systems influenced by multiple memory mechanisms. Within this setting, we established the first and second fundamental theorems of FC and showed that these results remain valid for the second-level construction. We also solved a fractional relaxation equation involving the second-level GFDs, illustrating how this operator naturally appears in applications.

Compared with the 1st level GFDs, the second-level operators offer more flexibility, since they involve three different kernel functions and can describe systems influenced by more than two types of memory effects. Therefore, fractional differential equations involving second-level GFDs

are suitable for mathematical models in which a single kernel is insufficient to capture the full behavior of the process. These models can be employed in several significant applications, including fractional relaxation equations, fractional diffusion equations with two memory kernels, distributed-order models with two dominant orders, viscoelastic systems with dual hereditary components, and fractional kinetic equations with nested memory structures.
Overall, the second-level construction presented in this paper provides a foundation for further developments in general fractional calculus. Possible directions include studying nth-level GFDs, analyzing their mapping properties, exploring operational calculus for the second-level GFDs, and investigating diverse fractional differential equations involving these operators.

7. Further Research

A natural direction for further research is to develop n-th level GFDs by extending the four-kernel construction to an $(n+2)$-tuple of kernels satisfying the Sonin condition. Formally, one can consider a sequence of kernels whose convolution reproduces the constant function $\{1\}$, and then compose the corresponding integral operators and n-th level GFDs in a structured way. This structured composition of kernels and operators provides a natural analogue of the first- and second-level constructions and suggests a definition for an n-th level operator.
However, a rigorous extension of this idea is not straightforward and requires addressing several technical issues. In particular, one must determine precise assumptions on the kernels to ensure that all intermediate convolutions are well-defined. One must also verify that the resulting operators act continuously on the chosen function spaces and that the corresponding inverse operators exist. Additional difficulties arise due to the integrability and regularity requirements needed to guarantee that the higher-level compositions behave analogously to the first- and second-level cases. A systematic analysis of these obstacles-especially the mapping property and the role of the underlying $\mathcal{L}_1$-based spaces-lies beyond the scope of the present paper and could be pursued in future work.

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