Int. J. Anal. Appl. (2026), 24:263

International Journal of Analysis and Applications

On Neutrosophic Soft Boolean Rings and Ideals

Gadde Sambasiva Rao[1], Dasari Ramesh[2], Shake Baji[3], Aiyared Iampan[4,*], R. Durga Prasad[5], B. Satyanarayana[6]

[1]*Department of Basic Science and Humanities, Vignan's Lara Institute of Technology and Science, Vadlamudi, Andhra Pradesh 522213, India*

[2]*Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Andhra Pradesh 522302, India*

[3]*Department of Engineering Mathematics, Vasireddy Venkatadri International Technological University, Guntur, Andhra Pradesh 522508, India*

[4]*Department of Mathematics, School of Sciences, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

[5]*Department of Humanities and Sciences, KG Reddy College of Engineering and Technology, Hyderabad, Telangana 501504, India*

[6]*Department of Mathematics, University College of Sciences, Acharya Nagarjuna University, Guntur, Andhra Pradesh 522510, India*

**Corresponding author:* aiyared.ia@up.ac.th

Abstract. In this paper, we study neutrosophic soft sets in the setting of Boolean rings. We introduce the notions of neutrosophic soft Boolean rings and neutrosophic soft ideals over Boolean rings. Several basic operations, including union, intersection, AND, and OR operations, are defined and investigated in this framework. We establish some closure properties of neutrosophic soft Boolean rings and neutrosophic soft ideals under the proposed operations, together with suitable conditions when needed. We also show that every neutrosophic soft ideal of a Boolean ring is a neutrosophic soft Boolean ring. The results provide a basic algebraic framework for studying neutrosophic soft structures associated with Boolean rings.

Received: Jun. 25, 2026.

2020 *Mathematics Subject Classification.* 03E72, 03G05, 06D72.

Key words and phrases. soft set; neutrosophic soft set; Boolean ring; neutrosophic soft Boolean ring; neutrosophic soft ideal; soft algebraic structure.

https://doi.org/10.28924/2291-8639-24-2026-263

ISSN: 2291-8639

1. Introduction

Soft set theory was introduced by Molodtsov [14] as a parameterized mathematical tool for dealing with uncertainty. Since then, this theory has been developed in several directions and has been combined with other uncertainty models. For instance, fuzzy soft sets were studied in [2, 10, 15], generalized fuzzy soft sets were considered in [13, 21], and possibility fuzzy soft sets were introduced in [4]. The notion of intuitionistic fuzzy soft sets was introduced by Maji et al. [11] by combining intuitionistic fuzzy sets with soft set theory. Later, the concept of neutrosophic soft sets was introduced in [12], and related extensions such as intuitionistic neutrosophic soft sets and generalized neutrosophic soft sets were investigated in [6, 7].

The study of algebraic structures in the framework of soft set theory has also received considerable attention. Aktas and Cagman [3] introduced soft groups and compared soft sets with fuzzy sets and rough sets. Further developments include fuzzy soft groups [5], soft rings [1], fuzzy soft rings and fuzzy soft ideals [8, 9], and intuitionistic fuzzy soft rings [22]. These works show that soft set-based models can be used to describe algebraic structures together with parameterized uncertainty.

Boolean rings form an important class of rings in which every element is idempotent. Their simple but rich algebraic structure makes them useful for studying algebraic properties in a clear and concrete way. Recently, fuzzy soft Boolean rings and related structures have been studied in [16–20]. These studies motivate the investigation of Boolean rings in a neutrosophic soft setting, where truth-membership, indeterminacy, and falsity-membership functions are considered simultaneously.

In this paper, we introduce and study neutrosophic soft Boolean rings and neutrosophic soft ideals over Boolean rings. We define basic operations on these structures, including union, intersection, AND, and OR, and examine their related closure properties. The purpose of the paper is to provide a basic algebraic framework for neutrosophic soft structures associated with Boolean rings.

The paper is organized as follows. Section 2 recalls some preliminary notions on soft sets, neutrosophic sets, neutrosophic soft sets, and Boolean rings. In Section 3, we introduce neutrosophic soft Boolean rings and investigate their basic properties under intersection, union, AND, and OR operations. Section 4 is devoted to neutrosophic soft ideals over Boolean rings; in this section, we also establish the relationship between neutrosophic soft ideals and neutrosophic soft Boolean rings and study their closure properties under the proposed operations. Finally, Section 5 gives the conclusion.

2. Preliminaries

In this section, we recall some basic notions needed in the sequel. Throughout this paper, let D be a nonempty universe and let V be a nonempty set of parameters. For a set D, we denote by $\mathcal{P}(D)$ the power set of D. We also use $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for all $a, b \in [0, 1]$.

Definition 2.1. *Let G be a nonempty subset of V. A soft set over D with respect to G is a pair (J, G), where*

$$J : G \to \mathcal{P}(D)$$

is a mapping. For each $e \in G$, the set $J(e)$ is regarded as the set of e-approximate elements of D.

Definition 2.2. *A single-valued neutrosophic set U on D is a set of the form*

$$U = \{\langle x, \mathfrak{T}_U(x), \mathfrak{I}_U(x), \mathfrak{F}_U(x)\rangle : x \in D\},$$

where

$$\mathfrak{T}_U, \mathfrak{I}_U, \mathfrak{F}_U : D \to [0, 1].$$

For each $x \in D$, the values $\mathfrak{T}_U(x)$, $\mathfrak{I}_U(x)$ and $\mathfrak{F}_U(x)$ denote the truth-membership degree, indeterminacy degree and falsity-membership degree of x, respectively. In particular,

$$0 \leqslant \mathfrak{T}_U(x) + \mathfrak{I}_U(x) + \mathfrak{F}_U(x) \leqslant 3$$

for all $x \in D$.

Definition 2.3. *Let $\mathcal{N}(D)$ denote the collection of all single-valued neutrosophic sets on D, and let G be a nonempty subset of V. A neutrosophic soft set over D with respect to G is a pair (J, G), where*

$$J : G \to \mathcal{N}(D)$$

is a mapping. Thus, for each $e \in G$,

$$J(e) = \left\{\left\langle x, \mathfrak{T}_{J(e)}(x), \mathfrak{I}_{J(e)}(x), \mathfrak{F}_{J(e)}(x)\right\rangle : x \in D\right\}.$$

Definition 2.4. *Let (J, G) and (K, H) be two neutrosophic soft sets over D. We say that (J, G) is a neutrosophic soft subset of (K, H), written*

$$(J, G) \subseteq (K, H),$$

if $G \subseteq H$ and, for all $e \in G$ and $x \in D$,

$$\mathfrak{T}_{J(e)}(x) \leqslant \mathfrak{T}_{K(e)}(x), \quad \mathfrak{I}_{J(e)}(x) \geqslant \mathfrak{I}_{K(e)}(x), \quad \mathfrak{F}_{J(e)}(x) \geqslant \mathfrak{F}_{K(e)}(x).$$

If $(J, G) \subseteq (K, H)$ and $(K, H) \subseteq (J, G)$, then (J, G) and (K, H) are said to be equal, and we write

$$(J, G) = (K, H).$$

Definition 2.5. *Let (J, G) and (K, H) be two neutrosophic soft sets over D. Their union is the neutrosophic soft set*

$$(J, G) \cup (K, H) = (L, G \cup H),$$

where, for all $x \in D$,

$$\mathfrak{T}_{L(e)}(x) = \begin{cases} \mathfrak{T}_{J(e)}(x), & \text{if } e \in G - H, \\ \mathfrak{T}_{K(e)}(x), & \text{if } e \in H - G, \\ \mathfrak{T}_{J(e)}(x) \vee \mathfrak{T}_{K(e)}(x), & \text{if } e \in G \cap H, \end{cases}$$

$$\mathfrak{I}_{L(e)}(x) = \begin{cases} \mathfrak{I}_{J(e)}(x), & \text{if } e \in G - H, \\ \mathfrak{I}_{K(e)}(x), & \text{if } e \in H - G, \\ \mathfrak{I}_{J(e)}(x) \wedge \mathfrak{I}_{K(e)}(x), & \text{if } e \in G \cap H, \end{cases}$$

and

$$\mathfrak{F}_{L(e)}(x) = \begin{cases} \mathfrak{F}_{J(e)}(x), & \text{if } e \in G - H, \\ \mathfrak{F}_{K(e)}(x), & \text{if } e \in H - G, \\ \mathfrak{F}_{J(e)}(x) \wedge \mathfrak{F}_{K(e)}(x), & \text{if } e \in G \cap H. \end{cases}$$

Definition 2.6. *Let (J, G) and (K, H) be two neutrosophic soft sets over D such that $G \cap H \neq \varnothing$. Their intersection is the neutrosophic soft set*

$$(J, G) \cap (K, H) = (L, G \cap H),$$

where, for all $e \in G \cap H$ and $x \in D$,

$$\mathfrak{T}_{L(e)}(x) = \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{K(e)}(x),$$

$$\mathfrak{I}_{L(e)}(x) = \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{K(e)}(x),$$

and

$$\mathfrak{F}_{L(e)}(x) = \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{K(e)}(x).$$

Definition 2.7. *Let (J, G) and (K, H) be two neutrosophic soft sets over D. The AND operation of (J, G) and (K, H) is the neutrosophic soft set*

$$(J, G) \wedge (K, H) = (L, G \times H),$$

where, for all $(e, f) \in G \times H$ and $x \in D$,

$$\mathfrak{T}_{L(e,f)}(x) = \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{K(f)}(x),$$

$$\mathfrak{I}_{L(e,f)}(x) = \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{K(f)}(x),$$

and

$$\mathfrak{F}_{L(e,f)}(x) = \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{K(f)}(x).$$

The OR operation of (J, G) and (K, H) is the neutrosophic soft set

$$(J, G) \vee (K, H) = (M, G \times H),$$

where, for all $(e, f) \in G \times H$ and $x \in D$,

$$\mathfrak{T}_{M(e,f)}(x) = \mathfrak{T}_{J(e)}(x) \vee \mathfrak{T}_{K(f)}(x),$$

$$\mathfrak{I}_{M(e,f)}(x) = \mathfrak{I}_{J(e)}(x) \wedge \mathfrak{I}_{K(f)}(x),$$

and

$$\mathfrak{F}_{M(e,f)}(x) = \mathfrak{F}_{J(e)}(x) \wedge \mathfrak{F}_{K(f)}(x).$$

Definition 2.8. *A ring R is called a Boolean ring if*

$$x^2 = x$$

for all $x \in R$. In a Boolean ring, every element is idempotent. It is well known that every Boolean ring is commutative and satisfies $x + x = 0$ for all $x \in R$.

Remark 2.1. *In Definition 2.6, the parameter set of the ordinary intersection is $G \cap H$. Hence, only parameters belonging to both G and H are considered. If one uses the parameter set $G \cup H$, then the operation should be treated as an extended form of intersection, not as the ordinary intersection.*

3. Neutrosophic Soft Boolean Rings

In this section, we introduce neutrosophic soft Boolean rings and investigate some of their basic properties. Throughout this section, R denotes a Boolean ring.

Definition 3.1. *Let G be a nonempty set of parameters and let (J, G) be a neutrosophic soft set over R. Then (J, G) is called a neutrosophic soft Boolean ring over R if, for each $e \in G$ and all $x, y \in R$, the following conditions hold:*

$$\mathfrak{T}_{J(e)}(x+y) \geqslant \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{J(e)}(y), \qquad \mathfrak{T}_{J(e)}(xy) \geqslant \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{J(e)}(y),$$

$$\mathfrak{I}_{J(e)}(x+y) \leqslant \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{J(e)}(y), \qquad \mathfrak{I}_{J(e)}(xy) \leqslant \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{J(e)}(y),$$

and

$$\mathfrak{F}_{J(e)}(x+y) \leqslant \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{J(e)}(y), \qquad \mathfrak{F}_{J(e)}(xy) \leqslant \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{J(e)}(y).$$

Example 3.1. *Let $R = \{0, o, n, s\}$. Define two binary operations $+$ and $\cdot$ on R by the following tables:*

$+$	0	o	n	s
0	0	o	n	s
o	o	0	s	n
n	n	s	0	o
s	s	n	o	0

$\cdot$	0	o	n	s
0	0	0	0	0
o	0	o	0	o
n	0	0	n	n
s	0	o	n	s

Then $(R, +, \cdot)$ is a Boolean ring. Moreover, the subset

$$A = \{0, o\}$$

is closed under addition and multiplication.

Let $G = \{e_1, e_2, e_3\}$. Define a neutrosophic soft set (J, G) over R as follows, where each ordered triple is written in the form

$$(\mathfrak{T}, \mathfrak{I}, \mathfrak{F}).$$

$$J(e_1) = \left\{ \frac{0}{(0.90, 0.20, 0.10)}, \frac{o}{(0.90, 0.20, 0.10)}, \frac{n}{(0.40, 0.70, 0.60)}, \frac{s}{(0.40, 0.70, 0.60)} \right\},$$

$$J(e_2) = \left\{ \frac{0}{(0.80, 0.25, 0.15)}, \frac{o}{(0.80, 0.25, 0.15)}, \frac{n}{(0.35, 0.65, 0.55)}, \frac{s}{(0.35, 0.65, 0.55)} \right\},$$

and

$$J(e_3) = \left\{ \frac{0}{(0.85, 0.15, 0.20)}, \frac{o}{(0.85, 0.15, 0.20)}, \frac{n}{(0.45, 0.75, 0.65)}, \frac{s}{(0.45, 0.75, 0.65)} \right\}.$$

Since $A = \{0, o\}$ is a subring of R, the truth-membership functions have larger values on A, while the indeterminacy and falsity-membership functions have smaller values on A. Hence, by using the operation tables, one can directly verify that, for each $e_i \in G$, $i = 1, 2, 3$, the functions

$$\mathfrak{T}_{J(e_i)}, \quad \mathfrak{I}_{J(e_i)}, \quad \mathfrak{F}_{J(e_i)}$$

satisfy all inequalities in Definition 3.1. Therefore (J, G) is a neutrosophic soft Boolean ring over R.

Definition 3.2. *Let (J, G) and (K, H) be two neutrosophic soft Boolean rings over R. We say that (J, G) is a neutrosophic soft Boolean subring of (K, H) if*

$$(J, G) \subseteq (K, H)$$

in the sense of Definition 2.4.

Definition 3.3. *Let U and W be two single-valued neutrosophic sets on R. We write*

$$U \preceq W$$

if, for all $x \in R$,

$$\mathfrak{T}_U(x) \leqslant \mathfrak{T}_W(x), \quad \mathfrak{I}_U(x) \geqslant \mathfrak{I}_W(x), \quad \mathfrak{F}_U(x) \geqslant \mathfrak{F}_W(x).$$

We say that U and W are comparable if either

$$U \preceq W$$

or

$$W \preceq U.$$

Theorem 3.1. *Let (J, G) and (K, H) be two neutrosophic soft Boolean rings over R. If $G \cap H \neq \emptyset$, then*

$$(J, G) \cap (K, H)$$

is also a neutrosophic soft Boolean ring over R.

Proof. Let

$$(J, G) \cap (K, H) = (L, G \cap H).$$

Take $e \in G \cap H$ and $x, y \in R$. By Definition 2.6, we have

$$\mathfrak{T}_{L(e)}(x) = \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{K(e)}(x),$$

$$\mathfrak{I}_{L(e)}(x) = \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{K(e)}(x),$$

and

$$\mathfrak{F}_{L(e)}(x) = \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{K(e)}(x).$$

Since (J, G) and (K, H) are neutrosophic soft Boolean rings over R, we obtain

$$\begin{aligned}\mathfrak{T}_{L(e)}(x+y) &= \mathfrak{T}_{J(e)}(x+y) \wedge \mathfrak{T}_{K(e)}(x+y)\\ &\geqslant \big(\mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{J(e)}(y)\big) \wedge \big(\mathfrak{T}_{K(e)}(x) \wedge \mathfrak{T}_{K(e)}(y)\big)\\ &= \big(\mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{K(e)}(x)\big) \wedge \big(\mathfrak{T}_{J(e)}(y) \wedge \mathfrak{T}_{K(e)}(y)\big)\\ &= \mathfrak{T}_{L(e)}(x) \wedge \mathfrak{T}_{L(e)}(y).\end{aligned}$$

Similarly,

$$\mathfrak{T}_{L(e)}(xy) \geqslant \mathfrak{T}_{L(e)}(x) \wedge \mathfrak{T}_{L(e)}(y).$$

For the indeterminacy function, we have

$$\begin{aligned}\mathfrak{I}_{L(e)}(x+y) &= \mathfrak{I}_{J(e)}(x+y) \vee \mathfrak{I}_{K(e)}(x+y)\\ &\leqslant \big(\mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{J(e)}(y)\big) \vee \big(\mathfrak{I}_{K(e)}(x) \vee \mathfrak{I}_{K(e)}(y)\big)\\ &= \big(\mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{K(e)}(x)\big) \vee \big(\mathfrak{I}_{J(e)}(y) \vee \mathfrak{I}_{K(e)}(y)\big)\\ &= \mathfrak{I}_{L(e)}(x) \vee \mathfrak{I}_{L(e)}(y).\end{aligned}$$

Similarly,

$$\mathfrak{I}_{L(e)}(xy) \leqslant \mathfrak{I}_{L(e)}(x) \vee \mathfrak{I}_{L(e)}(y).$$

For the falsity-membership function, we have

$$\begin{aligned}\mathfrak{F}_{L(e)}(x+y) &= \mathfrak{F}_{J(e)}(x+y) \vee \mathfrak{F}_{K(e)}(x+y)\\ &\leqslant \big(\mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{J(e)}(y)\big) \vee \big(\mathfrak{F}_{K(e)}(x) \vee \mathfrak{F}_{K(e)}(y)\big)\\ &= \big(\mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{K(e)}(x)\big) \vee \big(\mathfrak{F}_{J(e)}(y) \vee \mathfrak{F}_{K(e)}(y)\big)\\ &= \mathfrak{F}_{L(e)}(x) \vee \mathfrak{F}_{L(e)}(y).\end{aligned}$$

Similarly,

$$\mathfrak{F}_{L(e)}(xy) \leqslant \mathfrak{F}_{L(e)}(x) \vee \mathfrak{F}_{L(e)}(y).$$

Hence $(L, G \cap H)$ is a neutrosophic soft Boolean ring over R. □

Theorem 3.2. *Let (J, G) and (K, H) be two neutrosophic soft Boolean rings over R. Assume that $J(e)$ and $K(e)$ are comparable for every $e \in G \cap H$. Then*

$$(J, G) \cup (K, H)$$

is a neutrosophic soft Boolean ring over R.

Proof. Let

$$(J, G) \cup (K, H) = (L, G \cup H).$$

Take $e \in G \cup H$ and $x, y \in R$.

If $e \in G - H$, then $L(e) = J(e)$. Hence the required inequalities follow from the fact that (J, G) is a neutrosophic soft Boolean ring over R. If $e \in H - G$, then $L(e) = K(e)$, and the result follows in the same way.

Now suppose that $e \in G \cap H$. By assumption, $J(e)$ and $K(e)$ are comparable. If

$$J(e) \preceq K(e),$$

then the union operation gives $L(e) = K(e)$. If

$$K(e) \preceq J(e),$$

then the union operation gives $L(e) = J(e)$. In either case, $L(e)$ satisfies all inequalities in Definition 3.1. Therefore $(L, G \cup H)$ is a neutrosophic soft Boolean ring over R. □

Remark 3.1. *The comparability assumption in Theorem 3.2 cannot be omitted in general. As in the classical theory of fuzzy algebraic structures, the pointwise union of two neutrosophic soft Boolean rings need not preserve the required closure inequalities.*

Theorem 3.3. *Let (J, G) and (K, H) be two neutrosophic soft Boolean rings over R. Then*

$$(J, G) \wedge (K, H)$$

is a neutrosophic soft Boolean ring over R.

Proof. Let

$$(J, G) \wedge (K, H) = (L, G \times H).$$

Take $(e, f) \in G \times H$ and $x, y \in R$. By Definition 2.7, we have

$$\mathfrak{T}_{L(e,f)}(x) = \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{K(f)}(x),$$

$$\mathfrak{I}_{L(e,f)}(x) = \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{K(f)}(x),$$

and

$$\mathfrak{F}_{L(e,f)}(x) = \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{K(f)}(x).$$

For the truth-membership function, we obtain

$$\begin{aligned}\mathfrak{T}_{L(e,f)}(x + y) &= \mathfrak{T}_{J(e)}(x + y) \wedge \mathfrak{T}_{K(f)}(x + y)\\ &\geqslant \big(\mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{J(e)}(y)\big) \wedge \big(\mathfrak{T}_{K(f)}(x) \wedge \mathfrak{T}_{K(f)}(y)\big)\\ &= \mathfrak{T}_{L(e,f)}(x) \wedge \mathfrak{T}_{L(e,f)}(y).\end{aligned}$$

Similarly,

$$\mathfrak{T}_{L(e,f)}(xy) \geqslant \mathfrak{T}_{L(e,f)}(x) \wedge \mathfrak{T}_{L(e,f)}(y).$$

For the indeterminacy function, we have

$$\begin{aligned}\mathfrak{I}_{L(e,f)}(x + y) &= \mathfrak{I}_{J(e)}(x + y) \vee \mathfrak{I}_{K(f)}(x + y)\\ &\leqslant \big(\mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{J(e)}(y)\big) \vee \big(\mathfrak{I}_{K(f)}(x) \vee \mathfrak{I}_{K(f)}(y)\big)\\ &= \mathfrak{I}_{L(e,f)}(x) \vee \mathfrak{I}_{L(e,f)}(y).\end{aligned}$$

Similarly,

$$\mathfrak{I}_{L(e,f)}(xy) \leqslant \mathfrak{I}_{L(e,f)}(x) \vee \mathfrak{I}_{L(e,f)}(y).$$

For the falsity-membership function, we obtain

$$\begin{aligned}\mathfrak{F}_{L(e,f)}(x+y) &= \mathfrak{F}_{J(e)}(x+y) \vee \mathfrak{F}_{K(f)}(x+y)\\ &\leqslant \big(\mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{J(e)}(y)\big) \vee \big(\mathfrak{F}_{K(f)}(x) \vee \mathfrak{F}_{K(f)}(y)\big)\\ &= \mathfrak{F}_{L(e,f)}(x) \vee \mathfrak{F}_{L(e,f)}(y).\end{aligned}$$

Similarly,

$$\mathfrak{F}_{L(e,f)}(xy) \leqslant \mathfrak{F}_{L(e,f)}(x) \vee \mathfrak{F}_{L(e,f)}(y).$$

Therefore $(L, G \times H)$ is a neutrosophic soft Boolean ring over R. □

Theorem 3.4. *Let (J, G) and (K, H) be two neutrosophic soft Boolean rings over R. Assume that $J(e)$ and $K(f)$ are comparable for every $(e, f) \in G \times H$. Then*

$$(J, G) \vee (K, H)$$

is a neutrosophic soft Boolean ring over R.

Proof. Let

$$(J, G) \vee (K, H) = (M, G \times H).$$

Take $(e, f) \in G \times H$. By the comparability assumption, either

$$J(e) \preceq K(f)$$

or

$$K(f) \preceq J(e).$$

If $J(e) \preceq K(f)$, then the definition of the OR operation gives

$$M(e, f) = K(f).$$

If $K(f) \preceq J(e)$, then the definition of the OR operation gives

$$M(e, f) = J(e).$$

In either case, $M(e, f)$ satisfies all the conditions in Definition 3.1. Therefore

$$(J, G) \vee (K, H)$$

is a neutrosophic soft Boolean ring over R. □

4. Neutrosophic Soft Ideals over Boolean Rings

In this section, we introduce neutrosophic soft ideals over Boolean rings and study some of their basic closure properties. Throughout this section, R denotes a Boolean ring.

Definition 4.1. *Let G be a nonempty set of parameters and let (J, G) be a neutrosophic soft set over R. Then (J, G) is called a neutrosophic soft ideal of R if, for each $e \in G$ and all $x, y \in R$, the following conditions hold:*

$$\mathfrak{T}_{J(e)}(x+y) \geqslant \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{J(e)}(y), \qquad \mathfrak{T}_{J(e)}(xy) \geqslant \mathfrak{T}_{J(e)}(x) \vee \mathfrak{T}_{J(e)}(y),$$

$$\mathfrak{I}_{J(e)}(x+y) \leqslant \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{J(e)}(y), \qquad \mathfrak{I}_{J(e)}(xy) \leqslant \mathfrak{I}_{J(e)}(x) \wedge \mathfrak{I}_{J(e)}(y),$$

and

$$\mathfrak{F}_{J(e)}(x+y) \leqslant \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{J(e)}(y), \qquad \mathfrak{F}_{J(e)}(xy) \leqslant \mathfrak{F}_{J(e)}(x) \wedge \mathfrak{F}_{J(e)}(y).$$

Proposition 4.1. *Every neutrosophic soft ideal of R is a neutrosophic soft Boolean ring over R.*

Proof. Let (J, G) be a neutrosophic soft ideal of R. Then, for each $e \in G$ and all $x, y \in R$, the additive conditions in Definition 3.1 follow directly from Definition 4.1. For the truth-membership function, Definition 4.1 gives

$$\mathfrak{T}_{J(e)}(xy) \geqslant \mathfrak{T}_{J(e)}(x) \vee \mathfrak{T}_{J(e)}(y).$$

Since

$$\mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{J(e)}(y) \leqslant \mathfrak{T}_{J(e)}(x) \vee \mathfrak{T}_{J(e)}(y),$$

we obtain

$$\mathfrak{T}_{J(e)}(xy) \geqslant \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{J(e)}(y).$$

For the indeterminacy function, Definition 4.1 gives

$$\mathfrak{I}_{J(e)}(xy) \leqslant \mathfrak{I}_{J(e)}(x) \wedge \mathfrak{I}_{J(e)}(y) \leqslant \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{J(e)}(y).$$

Similarly,

$$\mathfrak{F}_{J(e)}(xy) \leqslant \mathfrak{F}_{J(e)}(x) \wedge \mathfrak{F}_{J(e)}(y) \leqslant \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{J(e)}(y).$$

Hence all multiplicative conditions in Definition 3.1 also hold. Therefore (J, G) is a neutrosophic soft Boolean ring over R. □

Example 4.1. *Let $R = \{0, o, n, s\}$. Define two binary operations $+$ and $\cdot$ on R by*

$+$	0	o	n	s
0	0	o	n	s
o	o	0	s	n
n	n	s	0	o
s	s	n	o	0

$\cdot$	0	o	n	s
0	0	0	0	0
o	0	o	0	o
n	0	0	n	n
s	0	o	n	s

Then $(R, +, \cdot)$ is a Boolean ring. The subset

$$A = \{0, o\}$$

is an ideal of R.

Let $G = \{e_1, e_2\}$. Define a neutrosophic soft set (J, G) over R as follows:

$$J(e_1) = \left\{ \frac{0}{(0.90, 0.20, 0.10)}, \frac{o}{(0.90, 0.20, 0.10)}, \frac{n}{(0.40, 0.70, 0.60)}, \frac{s}{(0.40, 0.70, 0.60)} \right\},$$

and

$$J(e_2) = \left\{ \frac{0}{(0.80, 0.10, 0.20)}, \frac{o}{(0.80, 0.10, 0.20)}, \frac{n}{(0.30, 0.60, 0.70)}, \frac{s}{(0.30, 0.60, 0.70)} \right\}.$$

Here each ordered triple is written in the form

$$(\mathfrak{T}, \mathfrak{I}, \mathfrak{F}).$$

Since A is an ideal of R, a direct verification shows that all conditions in Definition 4.1 are satisfied. Hence (J, G) is a neutrosophic soft ideal of R.

Theorem 4.1. *Let (J, G) and (K, H) be two neutrosophic soft ideals of R. If $G \cap H \neq \varnothing$, then*

$$(J, G) \cap (K, H)$$

is also a neutrosophic soft ideal of R.

Proof. Let

$$(J, G) \cap (K, H) = (L, G \cap H).$$

Take $e \in G \cap H$ and $x, y \in R$. By Definition 2.6,

$$\mathfrak{T}_{L(e)}(x) = \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{K(e)}(x),$$

$$\mathfrak{I}_{L(e)}(x) = \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{K(e)}(x),$$

and

$$\mathfrak{F}_{L(e)}(x) = \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{K(e)}(x).$$

For the truth-membership function, we have

$$\begin{aligned}
\mathfrak{T}_{L(e)}(x + y) &= \mathfrak{T}_{J(e)}(x + y) \wedge \mathfrak{T}_{K(e)}(x + y) \\
&\geqslant \big(\mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{J(e)}(y)\big) \wedge \big(\mathfrak{T}_{K(e)}(x) \wedge \mathfrak{T}_{K(e)}(y)\big) \\
&= \mathfrak{T}_{L(e)}(x) \wedge \mathfrak{T}_{L(e)}(y).
\end{aligned}$$

Also,

$$\begin{aligned}
\mathfrak{T}_{L(e)}(xy) &= \mathfrak{T}_{J(e)}(xy) \wedge \mathfrak{T}_{K(e)}(xy) \\
&\geqslant \big(\mathfrak{T}_{J(e)}(x) \vee \mathfrak{T}_{J(e)}(y)\big) \wedge \big(\mathfrak{T}_{K(e)}(x) \vee \mathfrak{T}_{K(e)}(y)\big) \\
&\geqslant \big(\mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{K(e)}(x)\big) \vee \big(\mathfrak{T}_{J(e)}(y) \wedge \mathfrak{T}_{K(e)}(y)\big) \\
&= \mathfrak{T}_{L(e)}(x) \vee \mathfrak{T}_{L(e)}(y).
\end{aligned}$$

For the indeterminacy function, we obtain

$$\begin{aligned}
\mathfrak{I}_{L(e)}(x + y) &= \mathfrak{I}_{J(e)}(x + y) \vee \mathfrak{I}_{K(e)}(x + y) \\
&\leqslant \big(\mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{J(e)}(y)\big) \vee \big(\mathfrak{I}_{K(e)}(x) \vee \mathfrak{I}_{K(e)}(y)\big) \\
&= \mathfrak{I}_{L(e)}(x) \vee \mathfrak{I}_{L(e)}(y).
\end{aligned}$$

Moreover,

$$\begin{aligned}\mathfrak{I}_{L(e)}(xy) &= \mathfrak{I}_{J(e)}(xy) \vee \mathfrak{I}_{K(e)}(xy)\\ &\leqslant \big(\mathfrak{I}_{J(e)}(x) \wedge \mathfrak{I}_{J(e)}(y)\big) \vee \big(\mathfrak{I}_{K(e)}(x) \wedge \mathfrak{I}_{K(e)}(y)\big)\\ &\leqslant \big(\mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{K(e)}(x)\big) \wedge \big(\mathfrak{I}_{J(e)}(y) \vee \mathfrak{I}_{K(e)}(y)\big)\\ &= \mathfrak{I}_{L(e)}(x) \wedge \mathfrak{I}_{L(e)}(y).\end{aligned}$$

The proof for the falsity-membership function is similar to the proof for the indeterminacy function. Thus,

$$\mathfrak{F}_{L(e)}(x+y) \leqslant \mathfrak{F}_{L(e)}(x) \vee \mathfrak{F}_{L(e)}(y)$$

and

$$\mathfrak{F}_{L(e)}(xy) \leqslant \mathfrak{F}_{L(e)}(x) \wedge \mathfrak{F}_{L(e)}(y).$$

Therefore $(L, G \cap H)$ is a neutrosophic soft ideal of R. $\square$

Theorem 4.2. *Let (J, G) and (K, H) be two neutrosophic soft ideals of R. Assume that $J(e)$ and $K(e)$ are comparable for every $e \in G \cap H$. Then*

$$(J, G) \cup (K, H)$$

is a neutrosophic soft ideal of R.

Proof. Let

$$(J, G) \cup (K, H) = (L, G \cup H).$$

Let $e \in G \cup H$.

If $e \in G - H$, then $L(e) = J(e)$. Since (J, G) is a neutrosophic soft ideal of R, $L(e)$ satisfies all the conditions in Definition 4.1. If $e \in H - G$, then $L(e) = K(e)$, and the result follows in the same way.

Now let $e \in G \cap H$. By the comparability assumption, either

$$J(e) \preceq K(e)$$

or

$$K(e) \preceq J(e).$$

If $J(e) \preceq K(e)$, then the union operation gives

$$L(e) = K(e).$$

If $K(e) \preceq J(e)$, then the union operation gives

$$L(e) = J(e).$$

In both cases, $L(e)$ satisfies all the conditions in Definition 4.1. Therefore $(L, G \cup H)$ is a neutrosophic soft ideal of R. $\square$

Remark 4.1. *The comparability condition in Theorem 4.2 is needed in general. Without this condition, the union of two neutrosophic soft ideals may fail to satisfy the required inequalities in Definition 4.1.*

Theorem 4.3. *Let (J, G) and (K, H) be two neutrosophic soft ideals of R. Then*

$$(J, G) \wedge (K, H)$$

is a neutrosophic soft ideal of R.

Proof. Let

$$(J, G) \wedge (K, H) = (L, G \times H).$$

Take $(e, f) \in G \times H$ and $x, y \in R$. By Definition 2.7,

$$\mathfrak{T}_{L(e,f)}(x) = \mathfrak{T}_{J(e)}(x) \wedge \mathfrak{T}_{K(f)}(x),$$

$$\mathfrak{I}_{L(e,f)}(x) = \mathfrak{I}_{J(e)}(x) \vee \mathfrak{I}_{K(f)}(x),$$

and

$$\mathfrak{F}_{L(e,f)}(x) = \mathfrak{F}_{J(e)}(x) \vee \mathfrak{F}_{K(f)}(x).$$

Using the same argument as in the proof of Theorem 4.1, we obtain

$$\mathfrak{T}_{L(e,f)}(x+y) \geqslant \mathfrak{T}_{L(e,f)}(x) \wedge \mathfrak{T}_{L(e,f)}(y),$$

$$\mathfrak{T}_{L(e,f)}(xy) \geqslant \mathfrak{T}_{L(e,f)}(x) \vee \mathfrak{T}_{L(e,f)}(y),$$

$$\mathfrak{I}_{L(e,f)}(x+y) \leqslant \mathfrak{I}_{L(e,f)}(x) \vee \mathfrak{I}_{L(e,f)}(y),$$

$$\mathfrak{I}_{L(e,f)}(xy) \leqslant \mathfrak{I}_{L(e,f)}(x) \wedge \mathfrak{I}_{L(e,f)}(y),$$

$$\mathfrak{F}_{L(e,f)}(x+y) \leqslant \mathfrak{F}_{L(e,f)}(x) \vee \mathfrak{F}_{L(e,f)}(y),$$

and

$$\mathfrak{F}_{L(e,f)}(xy) \leqslant \mathfrak{F}_{L(e,f)}(x) \wedge \mathfrak{F}_{L(e,f)}(y).$$

Therefore $(L, G \times H)$ is a neutrosophic soft ideal of R. □

Theorem 4.4. *Let (J, G) and (K, H) be two neutrosophic soft ideals of R. Assume that $J(e)$ and $K(f)$ are comparable for every $(e, f) \in G \times H$. Then*

$$(J, G) \vee (K, H)$$

is a neutrosophic soft ideal of R.

Proof. Let

$$(J, G) \vee (K, H) = (M, G \times H).$$

Take $(e, f) \in G \times H$. By the comparability assumption, either

$$J(e) \preceq K(f)$$

or

$$K(f) \preceq J(e).$$

If $J(e) \preceq K(f)$, then the definition of the OR operation gives

$$M(e, f) = K(f).$$

If $K(f) \preceq J(e)$, then the definition of the OR operation gives

$$M(e, f) = J(e).$$

In either case, $M(e, f)$ satisfies all the conditions in Definition 4.1. Therefore $(M, G \times H)$ is a neutrosophic soft ideal of R. □

5. Conclusion

In this paper, we introduced neutrosophic soft Boolean rings and neutrosophic soft ideals over Boolean rings. Basic operations, including union, intersection, AND, and OR operations, were considered in this setting. We investigated the closure properties of these structures under the proposed operations and identified suitable conditions for preserving them when required.

We also established that every neutrosophic soft ideal of a Boolean ring is a neutrosophic soft Boolean ring. This shows that the proposed notion of neutrosophic soft ideals is naturally related to the corresponding neutrosophic soft Boolean ring structure. The results provide a basic framework for further investigation of neutrosophic soft algebraic structures associated with Boolean rings and related classes of rings.

Acknowledgment: This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2027, FF70).

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] U. Acar, F. Koyuncu, B. Tanay, Soft Sets and Soft Rings, Comput. Math. Appl. 59 (2010), 3458–3463. https://doi.org/10.1016/j.camwa.2010.03.034.

[2] B. Ahmad, A. Kharal, On Fuzzy Soft Sets, Adv. Fuzzy Syst. 2009 (2009), 586507. https://doi.org/10.1155/2009/586507.

[3] H. Aktaş, N. Çağman, Soft Sets and Soft Groups, Inf. Sci. 177 (2007), 2726–2735. https://doi.org/10.1016/j.ins.2006.12.008.

[4] S. Alkhazaleh, A.R. Salleh, N. Hassan, Possibility Fuzzy Soft Set, Adv. Decis. Sci. 2011 (2011), 1–18. https://doi.org/10.1155/2011/479756.

[5] A. Aygünoğlu, H. Aygün, Introduction to Fuzzy Soft Groups, Comput. Math. Appl. 58 (2009), 1279–1286. https://doi.org/10.1016/j.camwa.2009.07.047.

[6] S. Broumi, Generalized Neutrosophic Soft Set, Int. J. Comput. Sci. Eng. Inf. Technol. 3 (2013), 17–30. https://doi.org/10.5121/ijcseit.2013.3202.

[7] S. Broumi, F. Smarandache, Intuitionistic Neutrosophic Soft Set, J. Inf. Comput. Sci. 8 (2013), 130–140. https://doi.org/10.5281/zenodo.2861554.

[8] J. Ghosh, B. Dinda, T.K. Samanta, Fuzzy Soft Rings and Fuzzy Soft Ideals, Int. J. Pure Appl. Sci. Technol. 2 (2011), 66–74.

[9] E. İnan, M.A. Öztürk, Fuzzy Soft Rings and Fuzzy Soft Ideals, Neural Comput. Appl. 21 (2012), 1–8. https://doi.org/10.1007/s00521-011-0550-5.

[10] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy Soft Sets, J. Fuzzy Math. 9 (2001), 589–602.

[11] P.K. Maji, R. Biswas, A.R. Roy, Intuitionistic Fuzzy Soft Sets, J. Fuzzy Math. 9 (2001), 677–692.

[12] P.K. Maji, Neutrosophic Soft Set, Ann. Fuzzy Math. Inform. 5 (2013), 157–168.

[13] P. Majumdar, S.K. Samanta, Generalised Fuzzy Soft Sets, Comput. Math. Appl. 59 (2010), 1425–1432. https://doi.org/10.1016/j.camwa.2009.12.006.

[14] D. Molodtsov, Soft Set Theory—First Results, Comput. Math. Appl. 37 (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5.

[15] T.J. Neog, D.K. Sut, On Fuzzy Soft Complement and Related Properties, Int. J. Energy Inf. Commun. 3 (2012), 23–34.

[16] G.S. Rao, D. Ramesh, A. Iampan, B. Satyanarayana, Fuzzy Soft Boolean Rings, Int. J. Anal. Appl. 21 (2023), 60. https://doi.org/10.28924/2291-8639-21-2023-60.

[17] G.S. Rao, D. Ramesh, B. Satyanarayana, $(\in, \in \vee q_k)$-Fuzzy Soft Boolean Near Rings, Asia Pac. J. Math. 10 (2023), 50. https://doi.org/10.28924/APJM/10-50.

[18] G.S. Rao, D. Ramesh, A. Iampan, B. Satyanarayana, Fuzzy Soft Boolean Near-Rings and Idealistic Fuzzy Soft Boolean Near-Rings, ICIC Express Lett. 18 (2024), 677–684. https://doi.org/10.24507/icicel.18.07.677.

[19] G.S. Rao, D. Ramesh, A. Iampan, B. Satyanarayana, P. Rajani, Intuitionistic Fuzzy Soft Boolean Rings, Int. J. Anal. Appl. 23 (2025), 43. https://doi.org/10.28924/2291-8639-23-2025-43.

[20] G.S. Rao, D. Ramesh, A. Iampan, G.V. Lakshmi, B. Satyanarayana, $(\in, \in \vee q_k)$-Intuitionistic Fuzzy Soft Boolean Near-Rings, Int. J. Anal. Appl. 23 (2025), 91. https://doi.org/10.28924/2291-8639-23-2025-91.

[21] H.L. Yang, Notes on Generalised Fuzzy Soft Sets, J. Math. Res. Expo. 31 (2011), 567–570.

[22] Z.-M. Zhang, Intuitionistic Fuzzy Soft Rings, Int. J. Fuzzy Syst. 14 (2012), 420–433.