

Solution Forms and Stability Behavior for Selected Nonlinear Systems of Difference Equations

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Abstract. In this paper, we investigate the qualitative behavior and derivation of explicit closed-form solutions for the fourth-order nonlinear rational difference equation

$$z_{m+1} = \frac{\alpha z_{m-2} z_{m-3}}{\beta z_m + \gamma z_{m-3}}, \quad m = 0, 1, 2, \dots,$$

where the parameters α, β, γ and the initial conditions $z_{-3}, z_{-2}, z_{-1}, z_0$ are non-zero real numbers.

Our analysis begins with the determination of the equilibrium point and the establishment of sufficient conditions for local asymptotic stability using linearization methods. Furthermore, we explore the global attractivity of the equilibrium point using monotonicity and comparison techniques. The boundedness of positive solutions is analyzed, providing criteria under which the sequence remains bounded.

A core contribution of this work is the derivation of exact closed-form representations for the solutions in four special cases involving different sign combinations of the parameters. The results demonstrate that the structure of this higher-order equation leads to rich dynamical outcomes including stability, boundedness, and explicit solution behavior.

1. INTRODUCTION

The study of difference equations and discrete dynamical systems has evolved into a vital branch of mathematics over the last several decades. These equations serve as fundamental tools for modeling complex processes in diverse fields such as biology, economics, physics, ecology, and engineering. Rational difference equations are particularly significant because they frequently appear as discrete-time analogues of differential equations and as mathematical models for nonlinear phenomena.

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The qualitative analysis of nonlinear rational difference equations is a complex task, as even simple recursive forms can exhibit rich dynamical behavior, including stability, periodicity, boundedness, and convergence. Foundational research in this domain was established by Elaydi, Kocic and Ladas, and Kulenovic and Ladas, who developed many of the fundamental techniques used in the study of nonlinear difference equations and discrete dynamical systems [5, 9, 10]. Their work laid the theoretical groundwork for investigating equilibrium points, local stability, global attractivity, periodic solutions, and boundedness properties of higher-order recursive sequences.

In recent years, a significant body of literature has focused on higher-order rational difference equations and systems. Ogul and Simsek [1] investigated the dynamics and explicit expressions of solutions for nonlinear higher-order rational equations involving delayed terms. Kara and Aktas [2], as well as Kara and Yazlik [7, 13], studied solvable systems of difference equations and obtained explicit representations for their solutions. Similar investigations were carried out by Khelifa, Halim, and Berkal [3, 6, 16], who established solution formulas for higher-order systems using algebraic and recursive techniques.

The derivation of explicit solutions for nonlinear rational difference equations remains one of the most challenging problems in the theory of discrete dynamical systems. Several authors have successfully employed transformation methods, analytical iterations, and mathematical induction to derive closed-form solutions for special classes of rational equations and systems. Among the notable contributions are those of Elsayed and his collaborators [11, 15, 19, 21, 24], who obtained exact solution representations and investigated the qualitative behavior of various nonlinear rational systems. Furthermore, Alharbi and Elsayed [8], Al-Rakhami and Elsayed [14], and Althagafi and Ghezal [22, 23] studied several classes of higher-order equations and established important results concerning periodicity, stability, and global dynamics.

Another important aspect of the qualitative theory concerns boundedness and global attractivity. Abo-Zeid [12] investigated the global behavior of a rational difference equation involving delayed terms and established convergence criteria for its solutions. Gumus, Abo-Zeid, and Turk [4] studied the global dynamics of two-dimensional systems of difference equations, while Touafek and Elsayed [18] analyzed periodicity properties for nonlinear systems. More recently, Alofi [17, 20] examined stability and periodicity properties of higher-order rational difference equations, providing new insights into their long-term behavior.

Motivated by these developments, we consider the fourth-order nonlinear rational difference equation

$$z_{m+1} = \frac{\alpha z_{m-2} z_{m-3}}{\beta z_m + \gamma z_{m-3}}, \quad m = 0, 1, 2, \dots, \quad (1.1)$$

where the parameters α, β, γ are real constants and the initial values

$$z_{-3}, \quad z_{-2}, \quad z_{-1}, \quad z_0$$

are arbitrary nonzero real numbers.

The main objective of this paper is to investigate the qualitative behavior and explicit solution forms of Eq. (1.1). First, we determine the equilibrium point and establish sufficient conditions for local asymptotic stability through linearization techniques. Next, we study the global attractivity and boundedness properties of the solutions. Furthermore, we derive explicit solution representations for four special cases obtained by considering different sign combinations in the denominator of the equation. These results extend several existing studies in the literature and contribute to the ongoing development of the qualitative theory of higher-order rational difference equations.

The remainder of this paper is organized as follows. In Section 2, the equilibrium point is investigated. Section 3 is devoted to local stability. Section 4 discusses the global attractivity of the equilibrium point. In Section 5, boundedness properties of solutions are established. Section 6 presents explicit solution forms for four special cases of the considered equation. Section 7 contains numerical simulations, while concluding remarks are given in Section 8.

2. EQUILIBRIUM POINT

Consider Eq. (1.1), where the initial conditions

$$z_{-3}, \quad z_{-2}, \quad z_{-1}, \quad z_0$$

are arbitrary positive real numbers.

To determine the equilibrium point, assume that

$$z_{m+1} = z_m = z_{m-2} = z_{m-3} = \bar{z}.$$

Substituting into Eq. (1.1), we obtain

$$\bar{z} = \frac{\alpha \bar{z}^2}{(\beta + \gamma) \bar{z}}.$$

Hence,

$$\bar{z} = \frac{\alpha}{\beta + \gamma} \bar{z}.$$

Therefore,

$$\left(1 - \frac{\alpha}{\beta + \gamma}\right) \bar{z} = 0.$$

Thus,

$$(\beta + \gamma - \alpha) \bar{z} = 0.$$

Consequently, if

$$\alpha \neq \beta + \gamma,$$

then Eq. (1.1) has the unique equilibrium point

$$\boxed{\bar{z} = 0}.$$

On the other hand, if

$$\alpha = \beta + \gamma,$$

then every constant value $\bar{z}$ satisfies the equilibrium equation. Hence, every constant is an equilibrium point. Therefore, throughout this paper, we assume

$$\alpha \neq \beta + \gamma.$$

3. LOCAL STABILITY OF EQ. (1.1)

Theorem 3.1. *Assume that*

$$\alpha(3\beta + \gamma) < (\beta + \gamma)^2.$$

Then the equilibrium point $\bar{z} = 0$ of Eq. (1.1) is locally asymptotically stable.

Proof. Define

$$f(u, v, w) = \frac{\alpha vw}{\beta u + \gamma w},$$

where

$$u = z_m, \quad v = z_{m-2}, \quad w = z_{m-3}.$$

The partial derivatives are

$$\frac{\partial f}{\partial u} = -\frac{\alpha \beta vw}{(\beta u + \gamma w)^2},$$

$$\frac{\partial f}{\partial v} = \frac{\alpha w}{\beta u + \gamma w},$$

and

$$\frac{\partial f}{\partial w} = \frac{\alpha \beta uv}{(\beta u + \gamma w)^2}.$$

Evaluating at the equilibrium point $\bar{z}$, we obtain

$$\frac{\partial f}{\partial u} = -\frac{\alpha \beta}{(\beta + \gamma)^2},$$

$$\frac{\partial f}{\partial v} = \frac{\alpha}{\beta + \gamma},$$

and

$$\frac{\partial f}{\partial w} = \frac{\alpha \beta}{(\beta + \gamma)^2}.$$

Therefore, the linearized equation becomes

$$y_{m+1} = -\frac{\alpha \beta}{(\beta + \gamma)^2} y_m + \frac{\alpha}{\beta + \gamma} y_{m-2} + \frac{\alpha \beta}{(\beta + \gamma)^2} y_{m-3}.$$

Using the sufficient condition

$$\sum |p_i| < 1,$$

we obtain

$$\frac{\alpha \beta}{(\beta + \gamma)^2} + \frac{\alpha}{\beta + \gamma} + \frac{\alpha \beta}{(\beta + \gamma)^2} < 1.$$

Hence,

$$\frac{2\alpha \beta}{(\beta + \gamma)^2} + \frac{\alpha}{\beta + \gamma} < 1.$$

Multiplying by $(\beta + \gamma)^2$, we get

$$2\alpha\beta + \alpha(\beta + \gamma) < (\beta + \gamma)^2.$$

Therefore,

$$\alpha(3\beta + \gamma) < (\beta + \gamma)^2.$$

Hence, the equilibrium point $\bar{z} = 0$ is locally asymptotically stable.

□

4. GLOBAL ATTRACTOR OF THE EQUILIBRIUM POINT

Theorem 4.1. *The equilibrium point $\bar{z} = 0$ of Eq. (1.1) is a global attractor if*

$$\alpha \neq \gamma.$$

Proof. Assume that m and M satisfy

$$m = f(M, m, m), \quad M = f(m, M, M).$$

Then

$$m = \frac{\alpha m^2}{\beta M + \gamma m},$$

and

$$M = \frac{\alpha M^2}{\beta m + \gamma M}.$$

Thus,

$$\beta m M + \gamma m^2 = \alpha m^2,$$

and

$$\beta m M + \gamma M^2 = \alpha M^2.$$

Hence,

$$\beta m M = (\alpha - \gamma) m^2,$$

and

$$\beta m M = (\alpha - \gamma) M^2.$$

Therefore,

$$(\alpha - \gamma)(m^2 - M^2) = 0.$$

Hence,

$$(\alpha - \gamma)(m - M)(m + M) = 0.$$

Since

$$\alpha \neq \gamma$$

and

$$m, M > 0,$$

we conclude that

$$m = M.$$

Thus, the equilibrium point is a global attractor. $\square$

5. BOUNDEDNESS OF SOLUTIONS OF EQ. (1.1)

Theorem 5.1. *Every positive solution of Eq. (1.1) is bounded if*

$$\alpha < \gamma.$$

Proof. From Eq. (1.1), we have

$$z_{m+1} = \frac{\alpha z_{m-2} z_{m-3}}{\beta z_m + \gamma z_{m-3}}.$$

Since

$$\beta z_m + \gamma z_{m-3} \geq \gamma z_{m-3},$$

we obtain

$$z_{m+1} \leq \frac{\alpha z_{m-2} z_{m-3}}{\gamma z_{m-3}}.$$

Hence,

$$z_{m+1} \leq \frac{\alpha}{\gamma} z_{m-2}.$$

Consider the comparison equation

$$y_{m+1} = \frac{\alpha}{\gamma} y_{m-2}.$$

If

$$\frac{\alpha}{\gamma} < 1,$$

then the sequence remains bounded.

Therefore, every positive solution of Eq. (1.1) is bounded whenever

$$\alpha < \gamma.$$

$\square$

6. EXPLICIT SOLUTIONS

In this section, we derive explicit solution forms for special cases of Eq. (1.1). Throughout this section, we assume

$$z_{-3} = p, \quad z_{-2} = q, \quad z_{-1} = r, \quad z_0 = s.$$

6.1. **Case I:** $\alpha = 1, \beta = 1, \gamma = 1$. Consider

$$z_{m+1} = \frac{z_{m-2}z_{m-3}}{z_m + z_{m-3}}. \quad (6.1)$$

The first terms are

$$\begin{aligned} z_1 &= \frac{pq}{s+p}, \\ z_2 &= \frac{r(s+p)}{s+2p}, \\ z_3 &= \frac{s(s+2p)}{2s+3p}, \\ z_4 &= \frac{pq(2s+3p)}{(s+p)(3s+5p)}, \\ z_5 &= \frac{r(s+p)(3s+5p)}{(s+2p)(5s+8p)}, \\ z_6 &= \frac{s(s+2p)(5s+8p)}{(2s+3p)(8s+13p)}. \end{aligned}$$

Observe that

$$1, 1, 2, 3, 5, 8, 13, \dots$$

are Fibonacci numbers.

Define

$$P_k = F_{k+2}p + F_{k+1}s,$$

where

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k.$$

Theorem 6.1. For every $n \geq 0$,

$$\begin{aligned} z_{3n+1} &= pq \frac{\prod_{j=0}^{n-1} P_{3j+2}}{\prod_{j=0}^n P_{3j}}, \\ z_{3n+2} &= r \frac{\prod_{j=0}^n P_{3j}}{\prod_{j=0}^n P_{3j+1}}, \end{aligned}$$

and

$$z_{3n+3} = s \frac{\prod_{j=0}^n P_{3j+1}}{\prod_{j=0}^n P_{3j+2}}.$$

Proof. The proof follows by mathematical induction.

For $n = 0$,

$$P_0 = p + s, \quad P_1 = 2p + s, \quad P_2 = 3p + 2s.$$

Hence,

$$z_1 = \frac{pq}{P_0},$$

$$z_2 = \frac{rP_0}{P_1},$$

and

$$z_3 = \frac{sP_1}{P_2}.$$

Thus the formulas hold for $n = 0$.

Assume that they hold for some integer n . Using Eq. (6.1),

$$z_{3n+4} = \frac{z_{3n+1}z_{3n}}{z_{3n+3} + z_{3n}}.$$

Substituting the induction hypothesis yields

$$z_{3n+4} = pq \frac{\prod_{j=0}^n P_{3j+2}}{\prod_{j=0}^{n+1} P_{3j}}.$$

Similarly,

$$z_{3n+5} = r \frac{\prod_{j=0}^{n+1} P_{3j}}{\prod_{j=0}^{n+1} P_{3j+1}},$$

and

$$z_{3n+6} = s \frac{\prod_{j=0}^{n+1} P_{3j+1}}{\prod_{j=0}^{n+1} P_{3j+2}}.$$

Therefore, the formulas are valid for $n + 1$, and the proof is complete.

□

6.2. **Case II:** $\alpha = 1, \beta = 1, \gamma = -1$. Consider

$$z_{m+1} = \frac{z_{m-2}z_{m-3}}{z_m - z_{m-3}}. \quad (6.2)$$

The first terms are

$$\begin{aligned} z_1 &= \frac{pq}{s-p}, \\ z_2 &= \frac{r(s-p)}{2p-s}, \\ z_3 &= \frac{s(2p-s)}{2s-3p}, \\ z_4 &= \frac{pq(2s-3p)}{(s-p)(5p-3s)}. \end{aligned}$$

The expressions

$$p-s, \quad 2p-s, \quad 3p-2s, \quad 5p-3s, \dots$$

follow the Fibonacci pattern.

Define

$$Q_k = F_{k+2}p - F_{k+1}s.$$

Theorem 6.2. For every $n \geq 0$,

$$\begin{aligned} z_{3n+1} &= (-1)^{n+1} pq \frac{\prod_{j=0}^{n-1} Q_{3j+2}}{\prod_{j=0}^n Q_{3j}}, \\ z_{3n+2} &= (-1)^{n+1} r \frac{\prod_{j=0}^n Q_{3j}}{\prod_{j=0}^n Q_{3j+1}}, \end{aligned}$$

and

$$z_{3n+3} = (-1)^{n+1} s \frac{\prod_{j=0}^n Q_{3j+1}}{\prod_{j=0}^n Q_{3j+2}}.$$

Proof. The proof follows by mathematical induction.

For $n = 0$,

$$Q_0 = p-s, \quad Q_1 = 2p-s, \quad Q_2 = 3p-2s.$$

Then

$$z_1 = -\frac{pq}{Q_0},$$

$$z_2 = -\frac{rQ_0}{Q_1},$$

and

$$z_3 = -\frac{sQ_1}{Q_2}.$$

Thus, the formulas hold for $n = 0$.

Assume that they hold for some integer n . Using Eq. (6.2),

$$z_{3n+4} = \frac{z_{3n+1}z_{3n}}{z_{3n+3} - z_{3n}}.$$

From the induction hypothesis,

$$\frac{z_{3n+3}}{z_{3n}} = -\frac{Q_{3n+1}}{Q_{3n+2}}.$$

Hence,

$$z_{3n+3} - z_{3n} = -z_{3n} \left(\frac{Q_{3n+1}}{Q_{3n+2}} + 1 \right) = -z_{3n} \frac{Q_{3n+3}}{Q_{3n+2}}.$$

Therefore,

$$z_{3n+4} = -z_{3n+1} \frac{Q_{3n+2}}{Q_{3n+3}}.$$

Substituting the formula of z_{3n+1} , we get

$$z_{3n+4} = (-1)^{n+2} pq \frac{\prod_{j=0}^n Q_{3j+2}}{\prod_{j=0}^{n+1} Q_{3j}}.$$

Similarly,

$$z_{3n+5} = (-1)^{n+2} r \frac{\prod_{j=0}^{n+1} Q_{3j}}{\prod_{j=0}^{n+1} Q_{3j+1}},$$

and

$$z_{3n+6} = (-1)^{n+2} s \frac{\prod_{j=0}^{n+1} Q_{3j+1}}{\prod_{j=0}^{n+1} Q_{3j+2}}.$$

Therefore, the formulas are valid for $n + 1$, and the proof is complete.

□

6.3. **Case III:** $\alpha = 1, \beta = -1, \gamma = 1$. Consider

$$z_{m+1} = \frac{z_{m-2}z_{m-3}}{z_{m-3} - z_m}. \quad (6.3)$$

The first terms are

$$\begin{aligned} z_1 &= \frac{pq}{p-s}, \\ z_2 &= -\frac{r(p-s)}{s}, \\ z_3 &= \frac{s^2}{p}, \\ z_4 &= \frac{p^2q}{(p-s)^2}, \\ z_5 &= \frac{r(p-s)^2}{s^2}, \\ z_6 &= \frac{s^3}{p^2}. \end{aligned}$$

A power pattern is observed.

Theorem 6.3. For every $n \geq 0$,

$$\begin{aligned} z_{3n+1} &= \frac{p^{n+1}q}{(p-s)^{n+1}}, \\ z_{3n+2} &= (-1)^{n+1} \frac{r(p-s)^{n+1}}{s^{n+1}}, \end{aligned}$$

and

$$z_{3n+3} = \frac{s^{n+2}}{p^{n+1}}.$$

Proof. The proof follows by mathematical induction.

For $n = 0$, the formulas give

$$\begin{aligned} z_1 &= \frac{pq}{p-s}, \\ z_2 &= -\frac{r(p-s)}{s}, \end{aligned}$$

and

$$z_3 = \frac{s^2}{p}.$$

Thus, the formulas hold for $n = 0$.

Assume that they hold for some integer n . Using Eq. (6.3),

$$z_{3n+4} = \frac{z_{3n+1}z_{3n}}{z_{3n} - z_{3n+3}}.$$

From the induction hypothesis,

$$z_{3n} = \frac{s^{n+1}}{p^n},$$

and

$$z_{3n+3} = \frac{s^{n+2}}{p^{n+1}}.$$

Hence,

$$z_{3n} - z_{3n+3} = \frac{s^{n+1}}{p^n} - \frac{s^{n+2}}{p^{n+1}} = \frac{s^{n+1}(p-s)}{p^{n+1}}.$$

Therefore,

$$z_{3n+4} = \frac{\frac{p^{n+1}q}{(p-s)^{n+1}} \cdot \frac{s^{n+1}}{p^n}}{\frac{s^{n+1}(p-s)}{p^{n+1}}} = \frac{p^{n+2}q}{(p-s)^{n+2}}.$$

Similarly,

$$z_{3n+5} = (-1)^{n+2} \frac{r(p-s)^{n+2}}{s^{n+2}},$$

and

$$z_{3n+6} = \frac{s^{n+3}}{p^{n+2}}.$$

Therefore, the formulas are valid for $n+1$, and the proof is complete. □

6.4. Case IV: $\alpha = 1, \beta = -1, \gamma = -1$. Consider

$$z_{m+1} = -\frac{z_{m-2}z_{m-3}}{z_m + z_{m-3}}. \quad (6.4)$$

The first terms are

$$\begin{aligned} z_1 &= -\frac{pq}{p+s}, \\ z_2 &= -\frac{r(p+s)}{s}, \\ z_3 &= \frac{s^2}{p}, \\ z_4 &= \frac{p^2q}{(p+s)^2}, \\ z_5 &= \frac{r(p+s)^2}{s^2}, \\ z_6 &= \frac{s^3}{p^2}. \end{aligned}$$

The same power pattern appears, with $p+s$ replacing $p-s$.

Theorem 6.4. For every $n \geq 0$,

$$\begin{aligned} z_{3n+1} &= (-1)^{n+1} \frac{p^{n+1}q}{(p+s)^{n+1}}, \\ z_{3n+2} &= (-1)^{n+1} \frac{r(p+s)^{n+1}}{s^{n+1}}, \end{aligned}$$

and

$$z_{3n+3} = \frac{s^{n+2}}{p^{n+1}}.$$

Proof. The proof follows by mathematical induction.

For $n = 0$, the formulas give

$$z_1 = -\frac{pq}{p+s},$$

$$z_2 = -\frac{r(p+s)}{s},$$

and

$$z_3 = \frac{s^2}{p}.$$

Thus, the formulas hold for $n = 0$.

Assume that they hold for some integer n . Using Eq. (6.4),

$$z_{3n+4} = \frac{z_{3n+1}z_{3n}}{-z_{3n+3} - z_{3n}}.$$

From the induction hypothesis,

$$z_{3n} = \frac{s^{n+1}}{p^n},$$

and

$$z_{3n+3} = \frac{s^{n+2}}{p^{n+1}}.$$

Hence,

$$-z_{3n+3} - z_{3n} = -\frac{s^{n+2}}{p^{n+1}} - \frac{s^{n+1}}{p^n} = -\frac{s^{n+1}(p+s)}{p^{n+1}}.$$

Therefore,

$$z_{3n+4} = (-1)^{n+2} \frac{p^{n+2}q}{(p+s)^{n+2}}.$$

Similarly,

$$z_{3n+5} = (-1)^{n+2} \frac{r(p+s)^{n+2}}{s^{n+2}},$$

and

$$z_{3n+6} = \frac{s^{n+3}}{p^{n+2}}.$$

Therefore, the formulas are valid for $n + 1$, and the proof is complete.

□

7. NUMERICAL SIMULATIONS

In this section, numerical simulations are presented to illustrate the theoretical results obtained in the previous sections. All computations were performed using MATLAB.

7.1. Local Stability. Consider Eq. (1.1) with

$$\alpha = 0.2, \quad \beta = 1, \quad \gamma = 1,$$

and initial values

$$z_{-3} = 0.5, \quad z_{-2} = 0.7, \quad z_{-1} = 0.6, \quad z_0 = 0.8.$$

Figure 1 illustrates the numerical behavior of the solution.

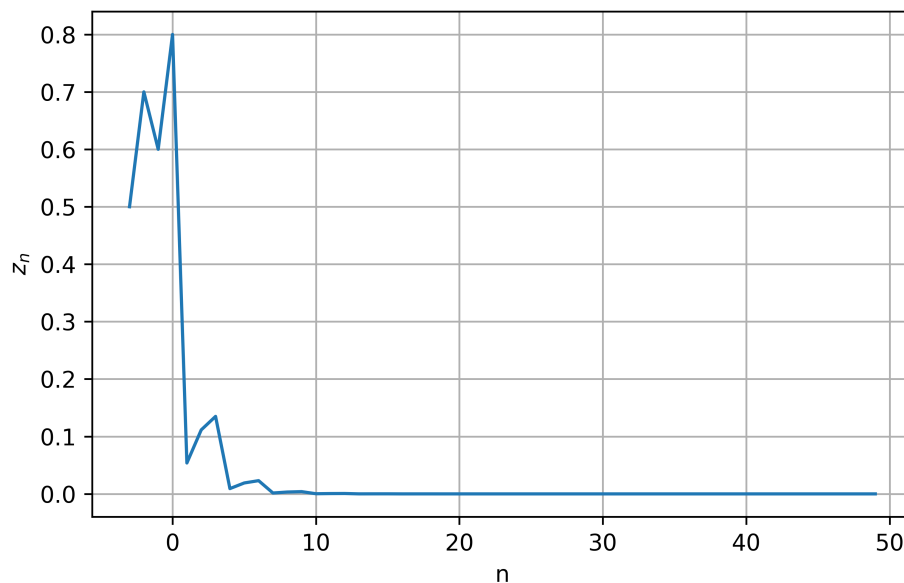


FIGURE 1. Numerical illustration of the local asymptotic stability of the equilibrium point.

7.2. **Boundedness.** Consider Eq. (1.1) with

$$\alpha = 0.5, \quad \beta = 1, \quad \gamma = 2,$$

and initial values

$$z_{-3} = 3, \quad z_{-2} = 2, \quad z_{-1} = 4, \quad z_0 = 5.$$

Figure 2 shows that the generated solution remains bounded.

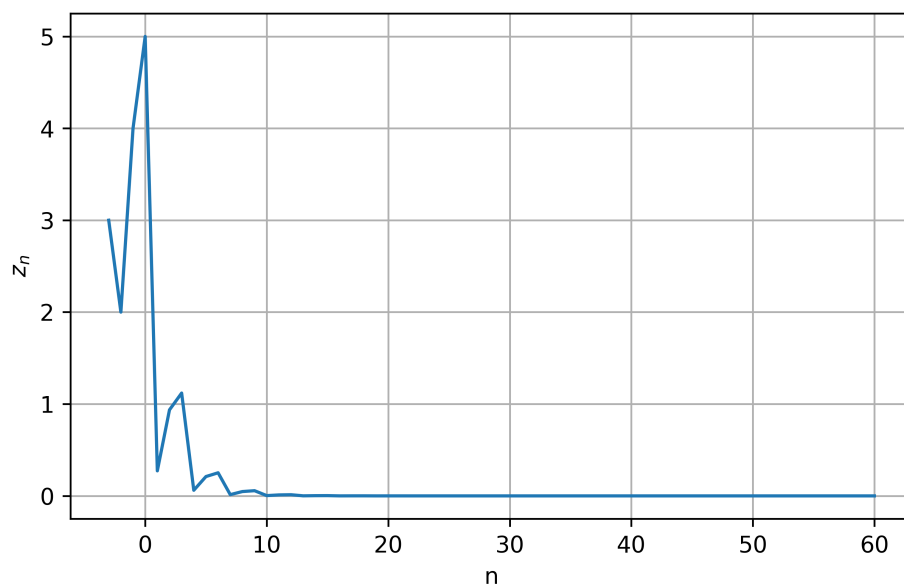


FIGURE 2. Numerical illustration showing the boundedness of positive solutions.

7.3. Special Cases. Figures 3–6 show the numerical behavior of the four special cases discussed in Section 6.

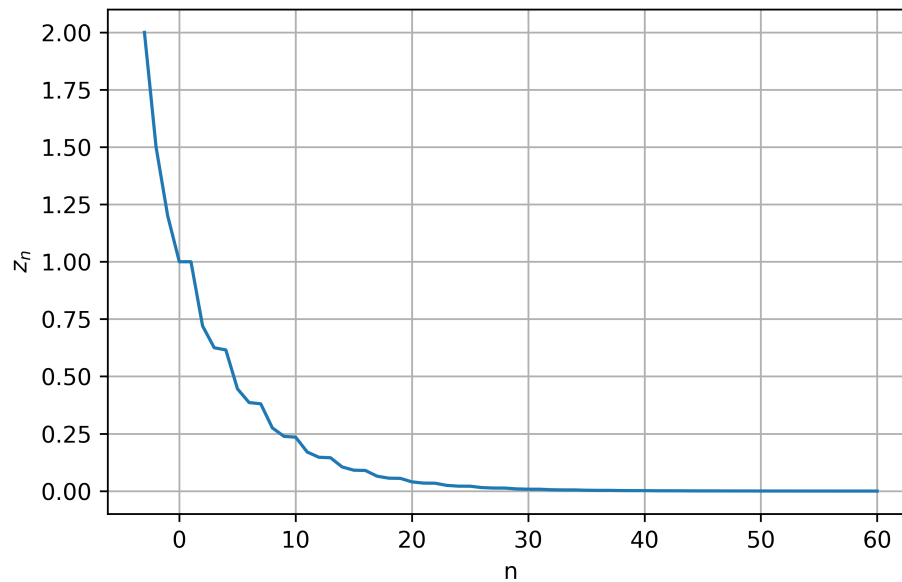


FIGURE 3. Numerical behavior of Case I.

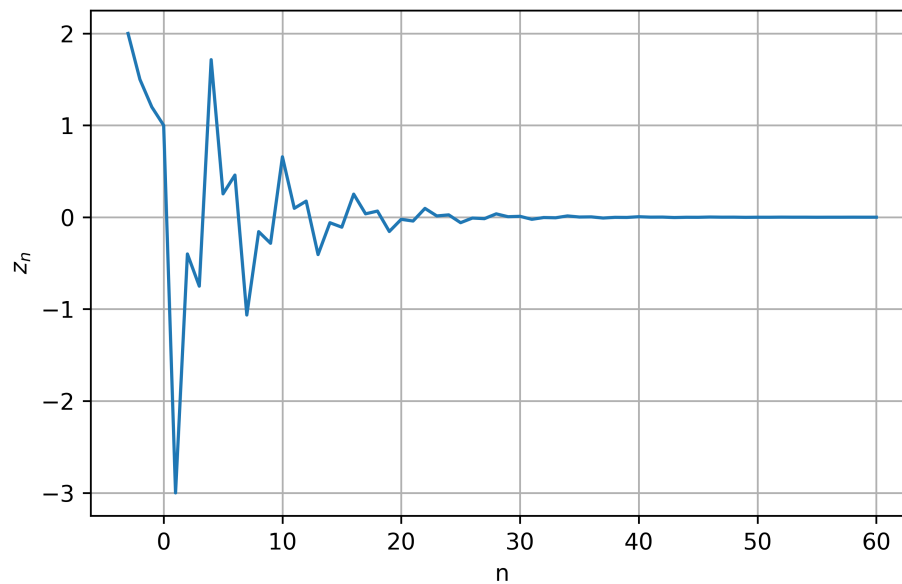


FIGURE 4. Numerical behavior of Case II.

8. CONCLUSION

In this paper, we studied the qualitative behavior and explicit solution forms of a fourth-order nonlinear rational difference equation. We determined the equilibrium point and established

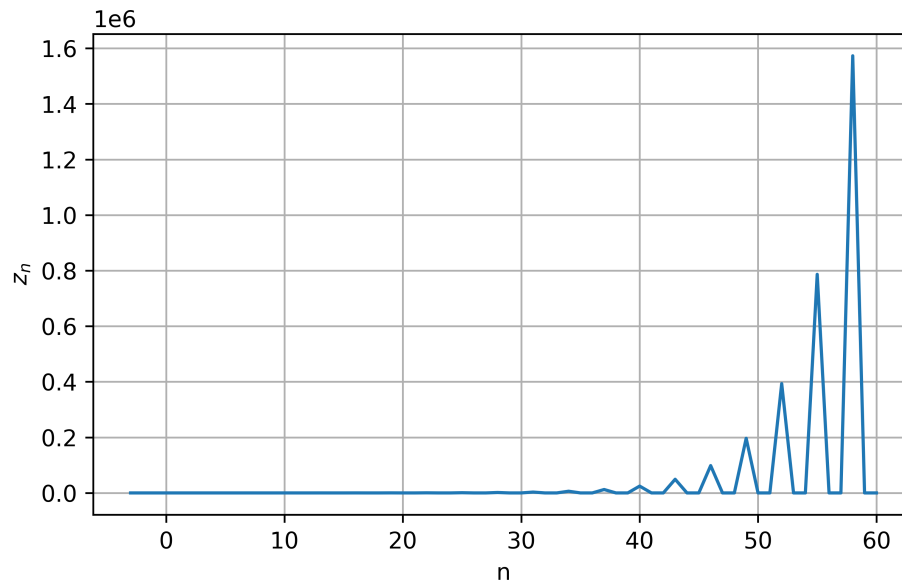


FIGURE 5. Numerical behavior of Case III.

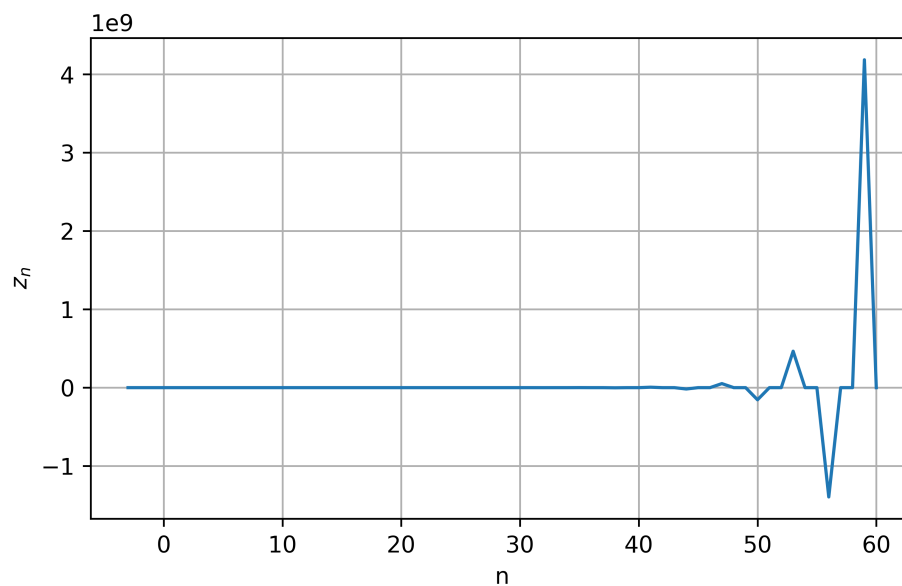


FIGURE 6. Numerical behavior of Case IV.

sufficient conditions for the local asymptotic stability of the equilibrium point. In addition, the global attractivity of the equilibrium point was investigated under a suitable condition on the parameters. The boundedness of positive solutions was also discussed by using comparison arguments. Furthermore, four special cases were considered and their explicit solutions were derived by observing the patterns of the first terms and applying mathematical induction. Numerical simulations were included to support the theoretical results concerning stability, boundedness,

and the behavior of the special cases. The results obtained in this paper contribute to the qualitative theory of higher-order rational difference equations and may be extended to more general nonlinear recursive equations and systems.

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