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On Graded Weakly S-2-Prime Ideals

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Abstract. Let G be a group with identity e, let $\mathfrak{T}$ be a commutative G-graded ring, and let $S \subseteq h(\mathfrak{T})$ be a multiplicatively closed subset of $\mathfrak{T}$. In this paper, we introduce and study graded weakly S-2-prime ideals, which extend the notion of graded weakly 2-prime ideals in graded commutative rings. We establish several basic properties and characterizations of these ideals and investigate their relationship with graded S-2-prime ideals. In particular, we examine the role of graded S-double-zeros and obtain conditions under which graded weakly S-2-prime ideals coincide with graded S-2-prime ideals. We further study these ideals in graded idealizations and amalgamated algebras. Several examples are included to illustrate the main concepts and results.

1. Introduction

Throughout this paper, G denotes a group with identity e, and $\mathfrak{T}$ denotes a commutative G-graded ring with identity $1 \neq 0$. Thus

$$\mathfrak{T} = \bigoplus_{g \in G} \mathfrak{T}_g$$

with $\mathfrak{T}_g \mathfrak{T}_h \subseteq \mathfrak{T}_{gh}$ for all $g, h \in G$. An element $a \in \mathfrak{T}$ is called homogeneous if $a \in \mathfrak{T}_g$ for some $g \in G$, and the set of all homogeneous elements of $\mathfrak{T}$ is denoted by

$$h(\mathfrak{T}) = \bigcup_{g \in G} \mathfrak{T}_g.$$

A proper graded ideal of $\mathfrak{T}$ will be referred to simply as a graded ideal. For background on graded rings and graded ideals, we refer the reader to ([5] and [8]).

Let $S \subseteq h(\mathfrak{T})$ be a multiplicatively closed subset of $\mathfrak{T}$; that is, $0 \notin S$, $1 \in S$, and $ab \in S$ for all $a, b \in S$. Throughout the paper, S will denote such a subset.

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Graded prime ideals and their generalizations have been studied extensively in graded commutative algebra. A proper graded ideal D of $\mathfrak{T}$ is called graded prime if, whenever $r, t \in h(\mathfrak{T})$ satisfy $rt \in D$, then $r \in D$ or $t \in D$, see [10].

A further generalization was introduced in [6] under the name of graded 2-prime ideals. A proper graded ideal $\mathfrak{D}$ of $\mathfrak{T}$ is called graded 2-prime if, whenever $r, t \in h(\mathfrak{T})$ satisfy $rt \in \mathfrak{D}$, then $r^2 \in \mathfrak{D}$ or $t^2 \in \mathfrak{D}$.

More recently, graded weakly 2-prime ideals were studied in [3]. A graded ideal $\mathfrak{D}$ of $\mathfrak{T}$ is called a graded weakly 2-prime ideal (GW2P-ideal) if, whenever $r, t \in h(\mathfrak{T})$ satisfy $0 \neq rt \in \mathfrak{D}$, then $r^2 \in \mathfrak{D}$ or $t^2 \in \mathfrak{D}$.

Motivated by these developments, we introduce and study graded weakly S-2-prime ideals, which extend the notion of graded weakly 2-prime ideals in graded commutative rings. We establish several basic properties and characterizations of these ideals and investigate their relationship with graded S-2-prime ideals.

2. Results

Definition 2.1. *Let $\mathfrak{T}$ be a G-graded ring, $S \subseteq h(\mathfrak{T})$ be an m.c.s. of $\mathfrak{T}$, and $\mathfrak{D}$ a proper graded ideal of $\mathfrak{T}$ with $\mathfrak{D} \cap S = \emptyset$. We call $\mathfrak{D}$ a* graded weakly S-2-prime ideal *of $\mathfrak{T}$* (GWS2P-ideal) *associated with s if there exists $s \in S$ such that for all $a, b \in h(\mathfrak{T})$,*

$$0 \neq ab \in \mathfrak{D} \implies sa^2 \in \mathfrak{D} \quad or \quad sb^2 \in \mathfrak{D}.$$

Definition 2.2. *Let $\mathfrak{T} = \bigoplus_{g \in G} \mathfrak{T}_g$ be a G-graded ring, $g \in G$, $S \subseteq \mathfrak{T}_e$ an m.c.s. of $\mathfrak{T}$, and $\mathfrak{D}$ a proper graded ideal of $\mathfrak{T}$ with $\mathfrak{D} \cap S = \emptyset$.*

(i) *$\mathfrak{D}$ is called a g-S-*2-prime ideal *of $\mathfrak{T}$ associated with s if there exists $s \in S$ such that for all $a, b \in \mathfrak{T}_g$,*

$$ab \in \mathfrak{D} \implies sa^2 \in \mathfrak{D} \quad or \quad sb^2 \in \mathfrak{D}.$$

(ii) *$\mathfrak{D}$ is called a g-*weakly S-2-prime ideal *of $\mathfrak{T}$ associated with s if there exists $s \in S$ such that for all $a, b \in \mathfrak{T}_g$,*

$$0 \neq ab \in \mathfrak{D} \implies sa^2 \in \mathfrak{D} \quad or \quad sb^2 \in \mathfrak{D}.$$

Clearly, every graded weakly 2-prime ideal of $\mathfrak{T}$ disjoint from an m.c.s. $S \subseteq h(\mathfrak{T})$ is a GWS2P-ideal of $\mathfrak{T}$. Furthermore, when $S \subseteq U(\mathfrak{T})$, the classes of GW2P-ideals and GWS2P-ideals of $\mathfrak{T}$ coincide.

Example 2.1. *Let $\mathfrak{T} = \mathbb{Z}[X]$ be graded by $\mathfrak{T}_i = \mathbb{Z}X^i$ for $i \geq 0$ and $\mathfrak{T}_i = \{0\}$ otherwise. Set $\mathfrak{D} = 5X\mathfrak{T}$ and $S = \{5^n : n \in \mathbb{N} \cup \{0\}\} \subseteq h(\mathfrak{T})$. Then $\mathfrak{D}$ is a proper graded ideal of $\mathfrak{T}$ and $\mathfrak{D} \cap S = \emptyset$. We show that $\mathfrak{D}$ is a* GWS2P-ideal *of $\mathfrak{T}$ with respect to $s = 5$, but that $\mathfrak{D}$ is not a* GW2P-ideal.

Let $f(X), g(X) \in h(\mathfrak{T})$ such that $0 \neq f(X)g(X) \in \mathfrak{D}$. Since $\mathfrak{D} \subseteq X\mathfrak{T}$ and $X\mathfrak{T}$ is a graded prime ideal of $\mathfrak{T}$ ($\mathfrak{T}/X\mathfrak{T} \cong \mathbb{Z}$), it follows that $f(X) \in X\mathfrak{T}$ or $g(X) \in X\mathfrak{T}$. If $f(X) = Xh(X)$, then $5f^2(X) = 5X^2h^2(X) \in 5X\mathfrak{T} = \mathfrak{D}$. Likewise, if $g(X) \in X\mathfrak{T}$, then $5g^2(X) \in \mathfrak{D}$. Hence $\mathfrak{D}$ is a GWS2P-ideal *of $\mathfrak{T}$ with respect to $s = 5$.*

On the other hand, $0 \neq 5X \in \mathfrak{D}$, *whereas* $25 \notin 5X\mathfrak{T}$ *and* $X^2 \notin 5X\mathfrak{T}$. *Thus the defining condition of a* GW2P-ideal *fails for the homogeneous elements* 5 *and* X. *Therefore,* $\mathfrak{D}$ *is not a* GW2P-ideal.

Theorem 2.1. *Let* $S \subseteq \mathfrak{T}_e$ *be an m.c.s. of* $\mathfrak{T}$, *let* $g \in G$, *and let* $\mathfrak{D}$ *be a* g*-weakly* S*-2-prime ideal of* $\mathfrak{T}$. *If* $\mathfrak{D}$ *is not a* g*-*S*-2-prime ideal of* $\mathfrak{T}$, *then* $\mathfrak{D}_g^2 = (0)$. *Consequently,* $Grad(\mathfrak{D}_g) \subseteq Grad(0_{\mathfrak{T}})$.

Proof. Assume, to the contrary, that $\mathfrak{D}_g^2 \neq (0)$. We show that $\mathfrak{D}$ is a g-S-2-prime ideal of $\mathfrak{T}$, yielding a contradiction.

Let $a_g, b_g \in \mathfrak{T}_g$ such that $a_g b_g \in \mathfrak{D}_g$.

If $a_g b_g \neq 0$, then, since $\mathfrak{D}$ is g-weakly S-2-prime, there exists $s_e \in S$ such that $s_e a_g^2 \in \mathfrak{D}_g$ or $s_e b_g^2 \in \mathfrak{D}_g$. Suppose now that $a_g b_g = 0$. If $a_g \mathfrak{D}_g \neq 0$, choose $x_g \in \mathfrak{D}_g$ such that $a_g x_g \neq 0$. Then $0 \neq a_g x_g = a_g(x_g + b_g) \in \mathfrak{D}_g$. Since $\mathfrak{D}$ is g-weakly S-2-prime, there exists $s_e \in S$ such that $s_e a_g^2 \in \mathfrak{D}_g$ or $s_e(x_g + b_g)^2 \in \mathfrak{D}_g$. Since $x_g \in \mathfrak{D}_g$, it follows that $s_e a_g^2 \in \mathfrak{D}_g$ or $s_e b_g^2 \in \mathfrak{D}_g$. Similarly, if $b_g \mathfrak{D}_g \neq 0$, then $s_e a_g^2 \in \mathfrak{D}_g$ or $s_e b_g^2 \in \mathfrak{D}_g$. Now assume that $a_g \mathfrak{D}_g = 0$ and $b_g \mathfrak{D}_g = 0$. Since $\mathfrak{D}_g^2 \neq (0)$, there exist $x_g, y_g \in \mathfrak{D}_g$ such that $x_g y_g \neq 0$. Hence $0 \neq (a_g + x_g)(b_g + y_g) = x_g y_g \in \mathfrak{D}_g$. Therefore, there exists $s_e \in S$ such that $s_e(a_g + x_g)^2 \in \mathfrak{D}_g$ or $s_e(b_g + y_g)^2 \in \mathfrak{D}_g$. Since $x_g, y_g \in \mathfrak{D}_g$, it follows that $s_e a_g^2 \in \mathfrak{D}_g$ or $s_e b_g^2 \in \mathfrak{D}_g$.

Thus, $a_g b_g \in \mathfrak{D}_g \implies s_e a_g^2 \in \mathfrak{D}_g$ or $s_e b_g^2 \in \mathfrak{D}_g$. Hence $\mathfrak{D}$ is a g-S-2-prime ideal of $\mathfrak{T}$, contrary to the hypothesis. Therefore, $\mathfrak{D}_g^2 = (0)$. Let $x_g \in Grad(\mathfrak{D}_g)$. Then $x_g^n \in \mathfrak{D}_g$ for some positive integer n. Since $\mathfrak{D}_g^2 = (0)$, we obtain $x_g^{2n} \in \mathfrak{D}_g^2 = (0)$, and hence $x_g \in Grad(0_{\mathfrak{T}})$. Therefore, $Grad(\mathfrak{D}_g) \subseteq Grad(0_{\mathfrak{T}})$. □

Theorem 2.2. *Let* $\varphi : \mathfrak{T} \longrightarrow \tilde{\mathfrak{T}}$ *be a graded ring homomorphism such that* $\varphi(1_{\mathfrak{T}}) = 1_{\tilde{\mathfrak{T}}}$, *and let* $S \subseteq h(\mathfrak{T})$ *be an m.c.s. of* $\mathfrak{T}$. *Then the following statements hold:*

(i) *If* φ *is a graded epimorphism and* $\mathfrak{D}$ *is a GWS2P-ideal of* $\mathfrak{T}$ *with* $\ker(\varphi) \subseteq \mathfrak{D}$, *then* $\varphi(\mathfrak{D})$ *is a GW*$\varphi(S)$*2P-ideal of* $\tilde{\mathfrak{T}}$.

(ii) *If* φ *is a graded monomorphism and* $\tilde{\mathfrak{D}}$ *is a GW*$\varphi(S)$*2P-ideal of* $\tilde{\mathfrak{T}}$ *such that* $0_{\tilde{\mathfrak{T}}} \notin \varphi(S)$, *then* $\varphi^{-1}(\tilde{\mathfrak{D}})$ *is a GWS2P-ideal of* $\mathfrak{T}$.

Proof. (i) Let $\mathfrak{D}$ be a GWS2P-ideal of $\mathfrak{T}$ associated with $s \in S$ and suppose that $\ker(\varphi) \subseteq \mathfrak{D}$. Since φ is a graded epimorphism, $\varphi(\mathfrak{D})$ is a graded ideal of $\tilde{\mathfrak{T}}$ and $\varphi(S)$ is a multiplicatively closed subset of $h(\tilde{\mathfrak{T}})$. Moreover, $0_{\tilde{\mathfrak{T}}} \notin \varphi(S)$; otherwise, there exists $u \in S$ such that $\varphi(u) = 0_{\tilde{\mathfrak{T}}}$. Hence $u \in \ker(\varphi) \subseteq \mathfrak{D}$, contradicting $\mathfrak{D} \cap S = \emptyset$.

Assume that $y \in \varphi(S) \cap \varphi(\mathfrak{D})$. Then $y = \varphi(u) = \varphi(q)$ for some $u \in S$ and $q \in \mathfrak{D}$. Thus $u - q \in \ker(\varphi) \subseteq \mathfrak{D}$, and so $u = (u - q) + q \in \mathfrak{D}$, a contradiction. Therefore, $\varphi(S) \cap \varphi(\mathfrak{D}) = \emptyset$.

Let $a', b' \in h(\tilde{\mathfrak{T}})$ such that $0_{\tilde{\mathfrak{T}}} \neq a'b' \in \varphi(\mathfrak{D})$. Since φ is surjective, there exist $a, b \in h(\mathfrak{T})$ such that $\varphi(a) = a'$ and $\varphi(b) = b'$. Then $0_{\tilde{\mathfrak{T}}} \neq \varphi(ab) = a'b' \in \varphi(\mathfrak{D})$. Hence there exists $q \in \mathfrak{D}$ such that $\varphi(ab) = \varphi(q)$. Therefore $ab - q \in \ker(\varphi) \subseteq \mathfrak{D}$, which implies that $ab \in \mathfrak{D}$. Since $\varphi(ab) \neq 0_{\tilde{\mathfrak{T}}}$, we have $ab \neq 0_{\mathfrak{T}}$. As $\mathfrak{D}$ is a GWS2P-ideal, it follows that $sa^2 \in \mathfrak{D}$ or $sb^2 \in \mathfrak{D}$. Applying φ, we obtain $\varphi(s)(a')^2 = \varphi(sa^2) \in \varphi(\mathfrak{D})$ or $\varphi(s)(b')^2 = \varphi(sb^2) \in \varphi(\mathfrak{D})$. Hence $\varphi(\mathfrak{D})$ is a GW$\varphi(S)$2P-ideal of $\tilde{\mathfrak{T}}$.

(ii) Let $\tilde{\mathfrak{D}}$ be a GW$\varphi(S)$2P-ideal of $\tilde{\mathfrak{T}}$ associated with $\varphi(s) \in \varphi(S)$, where $s \in S$. Since $\tilde{\mathfrak{D}}$ is graded, $\varphi^{-1}(\tilde{\mathfrak{D}})$ is a graded ideal of $\mathfrak{T}$. Assume that $r \in \varphi^{-1}(\tilde{\mathfrak{D}}) \cap S$. Then $\varphi(r) \in \tilde{\mathfrak{D}} \cap \varphi(S)$, a contradiction. Therefore $\varphi^{-1}(\tilde{\mathfrak{D}}) \cap S = \emptyset$.

Let $a, b \in h(\mathfrak{T})$ such that $0_{\mathfrak{T}} \neq ab \in \varphi^{-1}(\tilde{\mathfrak{D}})$. Since φ is injective, $0_{\tilde{\mathfrak{T}}} \neq \varphi(ab) = \varphi(a)\varphi(b) \in \tilde{\mathfrak{D}}$. Since $\tilde{\mathfrak{D}}$ is a GW$\varphi(S)$2P-ideal, $\varphi(s)(\varphi(a))^2 \in \tilde{\mathfrak{D}}$ or $\varphi(s)(\varphi(b))^2 \in \tilde{\mathfrak{D}}$. Equivalently, $\varphi(sa^2) \in \tilde{\mathfrak{D}}$ or $\varphi(sb^2) \in \tilde{\mathfrak{D}}$. Hence $sa^2 \in \varphi^{-1}(\tilde{\mathfrak{D}})$ or $sb^2 \in \varphi^{-1}(\tilde{\mathfrak{D}})$. Therefore $\varphi^{-1}(\tilde{\mathfrak{D}})$ is a GWS2P-ideal of $\mathfrak{T}$. □

Let $S \subseteq h(\mathfrak{T})$ be an m.c.s. of $\mathfrak{T}$, and let $P \subseteq \mathfrak{D}$ be graded ideals of $\mathfrak{T}$ such that $\mathfrak{D} \cap S = \emptyset$. Define $\bar{S} = \{s + P : s \in S\}$. Then $\bar{S}$ is an m.c.s. of $\mathfrak{T}/P$. Moreover, $(\mathfrak{D}/P) \cap \bar{S} = \emptyset$.

Corollary 2.1. *Let $S \subseteq h(\mathfrak{T})$ be an m.c.s. of $\mathfrak{T}$. Then:*

(i) *Let $J \subseteq K$ be graded ideals of $\mathfrak{T}$. If K is a GWS2P-ideal of $\mathfrak{T}$, then K/J is a graded weakly $\bar{S}$-2-prime ideal of $\mathfrak{T}/J$.*

(ii) *Let $\mathfrak{T}$ be a graded subring of $\tilde{\mathfrak{T}}$. If $\tilde{K}$ is a GWS2P-ideal of $\tilde{\mathfrak{T}}$, then $\tilde{K} \cap \mathfrak{T}$ is a GWS2P-ideal of $\mathfrak{T}$.*

Proof. (i) Let $\varphi : \mathfrak{T} \longrightarrow \mathfrak{T}/J$ be defined by $\varphi(r) = r + J$. Then φ is a graded epimorphism with $\ker(\varphi) = J \subseteq K$. Since $\bar{S} = \{s + J : s \in S\}$ is an m.c.s. of $\mathfrak{T}/J$, Theorem 2.2(i) implies that $\varphi(K) = K/J$ is a graded weakly $\bar{S}$-2-prime ideal of $\mathfrak{T}/J$.

(ii) Consider the inclusion homomorphism $h : \mathfrak{T} \longrightarrow \tilde{\mathfrak{T}}$ defined by $h(r) = r$. Then h is a graded monomorphism. Since $\tilde{K}$ is a GWS2P-ideal of $\tilde{\mathfrak{T}}$, Theorem 2.2(ii) yields that $h^{-1}(\tilde{K}) = \tilde{K} \cap \mathfrak{T}$ is a GWS2P-ideal of $\mathfrak{T}$.

□

Theorem 2.3. *Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be G-graded rings, and let $\mathfrak{T} = \mathfrak{T}_1 \times \mathfrak{T}_2$. Let $S = S_1 \times S_2$, where $S_1 \subseteq (\mathfrak{T}_1)_e$ and $S_2 \subseteq (\mathfrak{T}_2)_e$ are m.c.s.'s of $\mathfrak{T}_1$ and $\mathfrak{T}_2$, respectively. Suppose that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are nonzero proper graded ideals of $\mathfrak{T}_1$ and $\mathfrak{T}_2$, respectively. Then the following statements are equivalent:*

(i) *$\mathfrak{D} = \mathfrak{D}_1 \times \mathfrak{D}_2$ is an e-weakly S-2-prime ideal of $\mathfrak{T}$.*

(ii) *Either $\mathfrak{D}_1$ is an e-S_1-2-prime ideal of $\mathfrak{T}_1$ and $S_2 \cap \mathfrak{D}_2 \neq \emptyset$, or $\mathfrak{D}_2$ is an e-S_2-2-prime ideal of $\mathfrak{T}_2$ and $S_1 \cap \mathfrak{D}_1 \neq \emptyset$.*

(iii) *$\mathfrak{D} = \mathfrak{D}_1 \times \mathfrak{D}_2$ is an e-S-2-prime ideal of $\mathfrak{T}$.*

Proof. $(i) \Rightarrow (ii)$. Assume that $\mathfrak{D} = \mathfrak{D}_1 \times \mathfrak{D}_2$ is an e-weakly S-2-prime ideal of $\mathfrak{T}$ associated with $s = (s_{1e}, s_{2e}) \in S$.

Let $(a_e, b_e) \in \mathfrak{D} \cap \mathfrak{T}_e$ with $(a_e, b_e) \neq (0, 0)$. Then $(a_e, b_e) = (a_e, 1)(1, b_e) \in \mathfrak{D}$. Since $\mathfrak{D}$ is an e-weakly S-2-prime ideal, $(s_{1e}, s_{2e})(a_e, 1)^2 \in \mathfrak{D}$ or $(s_{1e}, s_{2e})(1, b_e)^2 \in \mathfrak{D}$. Hence, $(s_{1e}a_e^2, s_{2e}) \in \mathfrak{D}$ or $(s_{1e}, s_{2e}b_e^2) \in \mathfrak{D}$. Therefore, $s_{2e} \in \mathfrak{D}_2$ or $s_{1e} \in \mathfrak{D}_1$, and consequently $S_2 \cap \mathfrak{D}_2 \neq \emptyset$ or $S_1 \cap \mathfrak{D}_1 \neq \emptyset$.

Assume that $S_2 \cap \mathfrak{D}_2 \neq \emptyset$. Since $S \cap \mathfrak{D} = \emptyset$, it follows that $S_1 \cap \mathfrak{D}_1 = \emptyset$. Let $x_e, y_e \in (\mathfrak{T}_1)_e$ such that $x_e y_e \in \mathfrak{D}_1$. Choose $m_e \in S_2 \cap \mathfrak{D}_2$. Then $0 \neq (x_e, m_e)(y_e, 1) \in \mathfrak{D}$. Since $\mathfrak{D}$ is an e-weakly S-2-prime ideal, $(s_{1e}x_e^2, s_{2e}m_e^2) \in \mathfrak{D}$ or $(s_{1e}y_e^2, s_{2e}) \in \mathfrak{D}$. Thus, $s_{1e}x_e^2 \in \mathfrak{D}_1$ or $s_{1e}y_e^2 \in \mathfrak{D}_1$. Hence, $\mathfrak{D}_1$ is an e-S_1-2-prime ideal of $\mathfrak{T}_1$.

The proof for the case $S_1 \cap \mathfrak{D}_1 \neq \emptyset$ is analogous.

$(ii) \Rightarrow (iii)$. Assume that $\mathfrak{D}_1$ is an e-S_1-2-prime ideal of $\mathfrak{T}_1$ associated with $s_{1e} \in S_1$ and that $S_2 \cap \mathfrak{D}_2 \neq \emptyset$.

Let $(a_e, b_e), (c_e, d_e) \in \mathfrak{T}_e$ such that $(a_e, b_e)(c_e, d_e) \in \mathfrak{D}$. Then $a_e c_e \in \mathfrak{D}_1$. Since $\mathfrak{D}_1$ is an e-S_1-2-prime ideal, $s_{1e}a_e^2 \in \mathfrak{D}_1$ or $s_{1e}c_e^2 \in \mathfrak{D}_1$.

Choose $s_{2e} \in S_2 \cap \mathfrak{D}_2$. Then $s_{2e}b_e^2, s_{2e}d_e^2 \in \mathfrak{D}_2$. Hence, $(s_{1e}a_e^2, s_{2e}b_e^2) \in \mathfrak{D}$ or $(s_{1e}c_e^2, s_{2e}d_e^2) \in \mathfrak{D}$. Therefore, $(s_{1e}, s_{2e})(a_e, b_e)^2 \in \mathfrak{D}$ or $(s_{1e}, s_{2e})(c_e, d_e)^2 \in \mathfrak{D}$. Thus, $\mathfrak{D}$ is an e-S-2-prime ideal of $\mathfrak{T}$.

The argument is similar when $\mathfrak{D}_2$ is an e-S_2-2-prime ideal of $\mathfrak{T}_2$ and $S_1 \cap \mathfrak{D}_1 \neq \emptyset$.

$(iii) \Rightarrow (i)$. This follows directly from the definitions. □

Let $\mathfrak{D}$ be a graded ideal of $\mathfrak{T}$. Following [9], define $G\text{-}Z_{\mathfrak{D}}[\mathfrak{T}] = \{z \in h(\mathfrak{T}) : zt \in \mathfrak{D}$ for some $t \in h(\mathfrak{T}) \setminus \mathfrak{D}\}$. If $\mathfrak{D} = (0)$, then $G\text{-}Z_{\mathfrak{D}}[\mathfrak{T}]$ coincides with the set of all homogeneous zero divisors of $\mathfrak{T}$. In this case, we write $G\text{-}Zd[\mathfrak{T}] = G\text{-}Z_{(0)}[\mathfrak{T}]$. The set of all homogeneous regular elements of $\mathfrak{T}$, namely, the homogeneous non-zero divisors of $\mathfrak{T}$, is denoted by $G\text{-}reg(\mathfrak{T}) = h(\mathfrak{T}) \setminus G\text{-}Zd[\mathfrak{T}]$.

Theorem 2.4. *Let $S, S' \subseteq h(\mathfrak{T})$ be m.c.s.'s of $\mathfrak{T}$, and let $\mathfrak{D}$ be a graded ideal of $\mathfrak{T}$. If $\mathfrak{D}$ is a graded weakly S'-2-prime ideal of $\mathfrak{T}$ such that $\mathfrak{D} \cap S = \emptyset$, then $S^{-1}\mathfrak{D}$ is a graded weakly $S^{-1}S'$-2-prime ideal of $S^{-1}\mathfrak{T}$. Conversely, if $S^{-1}\mathfrak{D}$ is a graded weakly $S^{-1}S'$-2-prime ideal of $S^{-1}\mathfrak{T}$, $G\text{-}Z_{\mathfrak{D}}[\mathfrak{T}] \cap S = \emptyset$, and $S \subseteq G\text{-}reg(\mathfrak{T})$, then $\mathfrak{D}$ is a graded weakly S'-2-prime ideal of $\mathfrak{T}$.*

Proof. Assume that $\mathfrak{D}$ is a graded weakly S'-2-prime ideal of $\mathfrak{T}$ associated with $s' \in S'$. Since $\mathfrak{D} \cap S = \emptyset$, we have $S^{-1}\mathfrak{D} \cap S^{-1}S' = \emptyset$.

Let $0 \neq \frac{\alpha}{s_1}\frac{\beta}{s_2} = \frac{\alpha\beta}{s_1 s_2} \in S^{-1}\mathfrak{D}$, where $\alpha, \beta \in h(\mathfrak{T})$ and $s_1, s_2 \in S$. Then there exists $s_3 \in S$ such that $0 \neq s_3\alpha\beta \in \mathfrak{D}$. Since $\mathfrak{D}$ is a graded weakly S'-2-prime ideal, $s'(s_3\alpha)^2 \in \mathfrak{D}$ or $s'\beta^2 \in \mathfrak{D}$. Hence $\frac{s'}{1}\left(\frac{\alpha}{s_1}\right)^2 = \frac{s'}{1}\left(\frac{s_3\alpha}{s_3 s_1}\right)^2 \in S^{-1}\mathfrak{D}$ or $\frac{s'}{1}\left(\frac{\beta}{s_2}\right)^2 \in S^{-1}\mathfrak{D}$. Therefore $S^{-1}\mathfrak{D}$ is a graded weakly $S^{-1}S'$-2-prime ideal of $S^{-1}\mathfrak{T}$.

Conversely, assume that $S^{-1}\mathfrak{D}$ is a graded weakly $S^{-1}S'$-2-prime ideal of $S^{-1}\mathfrak{T}$ associated with $\frac{s'}{s} \in S^{-1}S'$. Let $0 \neq \alpha\beta \in \mathfrak{D}$ for some $\alpha, \beta \in h(\mathfrak{T})$. Since $S \subseteq G\text{-}reg(\mathfrak{T})$, $0 \neq \frac{\alpha}{1}\frac{\beta}{1} \in S^{-1}\mathfrak{D}$. Hence $\left(\frac{s'}{s}\right)\left(\frac{\alpha}{1}\right)^2 \in S^{-1}\mathfrak{D}$ or $\left(\frac{s'}{s}\right)\left(\frac{\beta}{1}\right)^2 \in S^{-1}\mathfrak{D}$. Therefore there exists $s_1 \in S$ such that $s_1 s'\alpha^2 \in \mathfrak{D}$ or $s_1 s'\beta^2 \in \mathfrak{D}$. Since $G\text{-}Z_{\mathfrak{D}}[\mathfrak{T}] \cap S = \emptyset$, it follows that $s'\alpha^2 \in \mathfrak{D}$ or $s'\beta^2 \in \mathfrak{D}$. Thus $\mathfrak{D}$ is a graded weakly S'-2-prime ideal of $\mathfrak{T}$. □

Theorem 2.5. *Let $S \subseteq h(\mathfrak{T})$ be an m.c.s. of $\mathfrak{T}$, and let $\mathfrak{D}$ be a graded ideal of $\mathfrak{T}$ such that $\mathfrak{D} \cap S = \emptyset$. Then the following statements are equivalent:*

(i) *$\mathfrak{D}$ is a GWS2P-ideal of $\mathfrak{T}$ associated with $s \in S$.*

(ii) *For every $r \in h(\mathfrak{T})$ with $r^2 \notin (\mathfrak{D} :_{\mathfrak{T}} s)$, $(\mathfrak{D} :_{\mathfrak{T}} r) \cap h(\mathfrak{T}) \subseteq (0 :_{\mathfrak{T}} r) \cup \{t \in h(\mathfrak{T}) : st^2 \in \mathfrak{D}\}$.*

Proof. $(i) \Rightarrow (ii)$. Assume that $\mathfrak{D}$ is a GWS2P-ideal of $\mathfrak{T}$ associated with $s \in S$. Let $r, t \in h(\mathfrak{T})$ such that $r^2 \notin (\mathfrak{D} :_{\mathfrak{T}} s)$ and $t \in (\mathfrak{D} :_{\mathfrak{T}} r)$. Then $tr \in \mathfrak{D}$.

If $tr = 0$, then $t \in (0 :_{\mathfrak{T}} r)$. Suppose that $tr \neq 0$. Since $0 \neq tr \in \mathfrak{D}$ and $sr^2 \notin \mathfrak{D}$, the definition of a GWS2P-ideal yields $st^2 \in \mathfrak{D}$. Hence $t \in \{x \in h(\mathfrak{T}) : sx^2 \in \mathfrak{D}\}$. Therefore $(\mathfrak{D} :_{\mathfrak{T}} r) \cap h(\mathfrak{T}) \subseteq (0 :_{\mathfrak{T}} r) \cup \{t \in h(\mathfrak{T}) : st^2 \in \mathfrak{D}\}$.

$(ii) \Rightarrow (i)$. Let $a, b \in h(\mathfrak{T})$ such that $0 \neq ab \in \mathfrak{D}$ and assume that $sa^2 \notin \mathfrak{D}$. Then $a^2 \notin (\mathfrak{D} :_{\mathfrak{T}} s)$ and $b \in (\mathfrak{D} :_{\mathfrak{T}} a)$. Since $ab \neq 0$, we have $b \notin (0 :_{\mathfrak{T}} a)$. Applying (ii) with $r = a$, it follows that $b \in \{t \in h(\mathfrak{T}) : st^2 \in \mathfrak{D}\}$. Hence $sb^2 \in \mathfrak{D}$. Therefore $\mathfrak{D}$ is a GWS2P-ideal of $\mathfrak{T}$. □

Let $\mathfrak{T}$ be a G-graded ring and let $S \subseteq h(\mathfrak{T})$ be an m.c.s. of $\mathfrak{T}$. Recall that a proper graded ideal D of $\mathfrak{T}$ is called a graded S-2-prime ideal (briefly, a GS2P-ideal) associated with $s \in S$ if, for all $\alpha, \beta \in h(\mathfrak{T})$ with $\alpha\beta \in D$, one has $s\alpha^2 \in D$ or $s\beta^2 \in D$; see [4].

Definition 2.3. *Let $S \subseteq h(\mathfrak{T})$ be an m.c.s. of $\mathfrak{T}$, and let $\mathfrak{D}$ be a GWS2P-ideal of $\mathfrak{T}$ associated with $s \in S$.*

(i) *Let $a, b \in h(\mathfrak{T})$. The pair (a, b) is called a graded S-double-zero of $\mathfrak{D}$ if $ab = 0$, $sa^2 \notin \mathfrak{D}$, and $sb^2 \notin \mathfrak{D}$.*

(ii) *Let W and J be graded ideals of $\mathfrak{T}$ such that $WJ \subseteq \mathfrak{D}$. If no pair $(a, b) \in (W \cap h(\mathfrak{T})) \times (J \cap h(\mathfrak{T}))$ is a graded S-double-zero of $\mathfrak{D}$, then $\mathfrak{D}$ is said to be free of graded S-double-zeros with respect to WJ.*

Remark 2.1. *A GWS2P-ideal of $\mathfrak{T}$ with no graded S-double-zeros is necessarily a GS2P-ideal. Consequently, every GWS2P-ideal that is not GS2P admits a graded S-double-zero.*

For a graded ideal D of $\mathfrak{T}$, let $D_{[m]} = \langle a^m : a \in h(\mathfrak{T}) \cap D \rangle$ denote the graded ideal generated by the m-th powers of homogeneous elements of D. Then $D_{[m]} \subseteq D^m \subseteq D$, with equality when $m = 1$ (see [1]).

Theorem 2.6. *Let $S \subseteq h(\mathfrak{T})$ be an m.c.s. of $\mathfrak{T}$, and let $\mathfrak{D}$ be a GWS2P-ideal of $\mathfrak{T}$ associated with $s \in S$. Suppose that P is a proper graded ideal of $\mathfrak{T}$ such that $aP \subseteq \mathfrak{D}$ for some $a \in h(\mathfrak{T})$. If (a, p) is not a graded S-double-zero of $\mathfrak{D}$ for every $p \in P$, and $sa^2 \notin \mathfrak{D}$, then $sP_{[2]} \subseteq \mathfrak{D}$.*

Proof. Suppose that $sP_{[2]} \nsubseteq \mathfrak{D}$. Then there exists $p \in P \cap h(\mathfrak{T})$ such that $sp^2 \notin \mathfrak{D}$. Since $aP \subseteq \mathfrak{D}$, we have $ap \in \mathfrak{D}$. If $ap \neq 0$, then $sa^2 \in \mathfrak{D}$ or $sp^2 \in \mathfrak{D}$, a contradiction. Hence $ap = 0$. Since (a, p) is not a graded S-double-zero of $\mathfrak{D}$ and $sa^2 \notin \mathfrak{D}$, it follows that $sp^2 \in \mathfrak{D}$, again a contradiction. Therefore, $sP_{[2]} \subseteq \mathfrak{D}$. □

Theorem 2.7. *Let $S \subseteq h(\mathfrak{T})$ be an m.c.s. of $\mathfrak{T}$, and let $\mathfrak{D}$ be a GWS2P-ideal of $\mathfrak{T}$ associated with $s \in S$. Suppose that A and B are proper graded ideals of $\mathfrak{T}$ satisfying $0 \neq AB \subseteq \mathfrak{D}$. If $\mathfrak{D}$ is free of graded S-double-zeros with respect to AB, then either $sA_{[2]} \subseteq \mathfrak{D}$ or $sB_{[2]} \subseteq \mathfrak{D}$.*

Proof. Assume that $sA_{[2]} \nsubseteq \mathfrak{D}$. Then there exists $a \in A \cap h(\mathfrak{T})$ such that $sa^2 \notin \mathfrak{D}$. Since $AB \subseteq \mathfrak{D}$, we have $aB \subseteq \mathfrak{D}$. Moreover, because $\mathfrak{D}$ is free of graded S-double-zeros with respect to AB, the pair (a, b) is not a graded S-double-zero of $\mathfrak{D}$ for every $b \in B \cap h(\mathfrak{T})$. Therefore, by Theorem 2.6, it follows that $sB_{[2]} \subseteq \mathfrak{D}$. Hence either $sA_{[2]} \subseteq \mathfrak{D}$ or $sB_{[2]} \subseteq \mathfrak{D}$. □

Theorem 2.8. *Let G be an abelian group, $\mathfrak{T}$ a G-graded ring, M a G-graded $\mathfrak{T}$-module, $S \subseteq h(\mathfrak{T})$ an m.c.s. of $\mathfrak{T}$, and $\mathfrak{D}$ a proper graded ideal of $\mathfrak{T}$ with $\mathfrak{D} \cap S = \emptyset$. Then the following statements are equivalent:*

(i) *$\mathfrak{D}(+)M$ is a graded weakly $(S(+)M)$-2-prime ideal of $\mathfrak{T}(+)M$.*

(ii) *$\mathfrak{D}(+)M$ is a graded weakly $(S(+)0)$-2-prime ideal of $\mathfrak{T}(+)M$.*

(iii) *$\mathfrak{D}$ is a graded weakly S-2-prime ideal of $\mathfrak{T}$ associated with some $s \in S$, and whenever $a, b \in h(\mathfrak{T})$ satisfy $ab = 0$, $sa^2 \notin \mathfrak{D}$, and $sb^2 \notin \mathfrak{D}$, one has $a, b \in Ann_{\mathfrak{T}}(M)$.*

Proof. $(i) \Rightarrow (iii)$. Assume that $\mathfrak{D}(+)M$ is a graded weakly $(S(+)M)$-2-prime ideal of $\mathfrak{T}(+)M$ associated with $(s,m) \in S(+)M$.

Let $a,b \in h(\mathfrak{T})$ with $0 \neq ab \in \mathfrak{D}$. Then $(0,0) \neq (a,0)(b,0) \in \mathfrak{D}(+)M$. Hence, $(s,m)(a,0)^2 \in \mathfrak{D}(+)M$ or $(s,m)(b,0)^2 \in \mathfrak{D}(+)M$. Therefore, $(sa^2, ma^2) \in \mathfrak{D}(+)M$ or $(sb^2, mb^2) \in \mathfrak{D}(+)M$, which yields $sa^2 \in \mathfrak{D}$ or $sb^2 \in \mathfrak{D}$. Thus $\mathfrak{D}$ is a graded weakly S-2-prime ideal of $\mathfrak{T}$. Now let $a_g, b_g \in h(\mathfrak{T})$ satisfy $a_g b_g = 0$, $sa_g^2 \notin \mathfrak{D}$, and $sb_g^2 \notin \mathfrak{D}$. Suppose that $a_g \notin Ann_{\mathfrak{T}}(M)$. Then there exists $m'_g \in M \cap h(M)$ such that $a_g m'_g \neq 0$. Consequently, $(0,0) \neq (a_g,0)(b_g,m'_g) \in \mathfrak{D}(+)M$. Since $\mathfrak{D}(+)M$ is graded weakly $(S(+)M)$-2-prime, $(s,m)(a_g,0)^2 \in \mathfrak{D}(+)M$ or $(s,m)(b_g,m'_g)^2 \in \mathfrak{D}(+)M$. In either case, the first component belongs to $\mathfrak{D}$, and hence $sa_g^2 \in \mathfrak{D}$ or $sb_g^2 \in \mathfrak{D}$, a contradiction. Therefore $a_g \in Ann_{\mathfrak{T}}(M)$. Similarly, $b_g \in Ann_{\mathfrak{T}}(M)$.

$(iii) \Rightarrow (ii)$. Let $(0,0) \neq (a,m)(b,m') \in \mathfrak{D}(+)M$, where $(a,m),(b,m') \in h(\mathfrak{T}(+)M)$.

If $ab \neq 0$, then $0 \neq ab \in \mathfrak{D}$, and since $\mathfrak{D}$ is graded weakly S-2-prime, we have $sa^2 \in \mathfrak{D}$ or $sb^2 \in \mathfrak{D}$. Hence $(s,0)(a,m)^2 \in \mathfrak{D}(+)M$ or $(s,0)(b,m')^2 \in \mathfrak{D}(+)M$.

Assume now that $ab = 0$. If both $sa^2 \notin \mathfrak{D}$ and $sb^2 \notin \mathfrak{D}$, then by (iii), $a,b \in Ann_{\mathfrak{T}}(M)$. It follows that $(a,m)(b,m') = (0,0)$, contrary to assumption. Therefore, $sa^2 \in \mathfrak{D}$ or $sb^2 \in \mathfrak{D}$, and again $(s,0)(a,m)^2 \in \mathfrak{D}(+)M$ or $(s,0)(b,m')^2 \in \mathfrak{D}(+)M$. Thus $\mathfrak{D}(+)M$ is a graded weakly $(S(+)0)$-2-prime ideal of $\mathfrak{T}(+)M$.

$(ii) \Rightarrow (i)$. Since $(S(+)0) \subseteq (S(+)M)$, every graded weakly $(S(+)0)$-2-prime ideal of $\mathfrak{T}(+)M$ is automatically a graded weakly $(S(+)M)$-2-prime ideal. Hence (i) follows. □

Theorem 2.9. *Let G be an abelian group, $\mathfrak{T}$ a G-graded ring, M a G-graded $\mathfrak{T}$-module, $S \subseteq h(\mathfrak{T})$ an m.c.s. of $\mathfrak{T}$, and $\mathfrak{D}$ a proper graded ideal of $\mathfrak{T}$ with $\mathfrak{D} \cap S = \emptyset$. Then the following statements are equivalent:*

- (i) *$\mathfrak{D}(+)M$ is a graded weakly $(S(+)0)$-2-prime ideal of $\mathfrak{T}(+)M$.*
- (ii) *$\mathfrak{D}(+)M$ is a graded weakly $(S(+)M)$-2-prime ideal of $\mathfrak{T}(+)M$.*
- (iii) *$\mathfrak{D}$ is a graded weakly S-2-prime ideal of $\mathfrak{T}$, and for every graded S-double-zero (a,b) of $\mathfrak{D}$, one has $aM = 0 = bM$.*

Proof. The result follows immediately from Theorem 2.8. □

It follows that every GWS2P-ideal of $\mathfrak{T}$ is GS2P-ideal if and only if (0) is a GS2P-ideal of $\mathfrak{T}$.

Recall that a graded commutative ring $\mathfrak{T}$ with identity $1_{\mathfrak{T}}$ is called a graded S-integral domain if there exists $s \in S$ such that, for all $a,b \in h(\mathfrak{T})$ with $ab = 0$, one has $sa = 0$ or $sb = 0$, where $S \subseteq h(\mathfrak{T})$ is an m.c.s. of $\mathfrak{T}$.

Theorem 2.10. *Let $\mathfrak{T}$ be a graded S-integral domain, M a G-graded $\mathfrak{T}$-module, and $S \subseteq h(\mathfrak{T})$ an m.c.s. of $\mathfrak{T}$. Then a graded ideal $\mathfrak{D}$ of $\mathfrak{T}(+)M$ is a graded weakly $(S(+)0)$-2-prime ideal of $\mathfrak{T}(+)M$ if and only if it is a graded $(S(+)0)$-2-prime ideal of $\mathfrak{T}(+)M$.*

Proof. It suffices to show that the zero graded ideal of $\mathfrak{T}(+)M$ is a graded $(S(+)0)$-2-prime ideal.

Since $0 \notin S$, we have $((0,0)) \cap (S(+)0) = \emptyset$. Let $(\alpha,a),(\beta,b) \in h(\mathfrak{T}(+)M)$ satisfy $(\alpha,a)(\beta,b) = (0,0)$. Then $(\alpha\beta, \alpha b + \beta a) = (0,0)$, and hence $\alpha\beta = 0$. Since $\mathfrak{T}$ is a graded S-integral domain, there exists $s \in S$ such that $s\alpha = 0$ or $s\beta = 0$. If $s\alpha = 0$, then $(s,0)(\alpha,a)^2 = (s,0)(\alpha^2, 2\alpha a) = (s\alpha^2, 2s\alpha a) = (0,0)$. Similarly, if $s\beta = 0$, then $(s,0)(\beta,b)^2 = (0,0)$. Therefore, the zero graded ideal of $\mathfrak{T}(+)M$ is a graded $(S(+)0)$-2-prime ideal of $\mathfrak{T}(+)M$. The conclusion now follows from the fact that every graded weakly $(S(+)0)$-2-prime ideal is graded $(S(+)0)$-2-prime whenever the zero graded ideal is graded $(S(+)0)$-2-prime. □

Recall that a graded ring $\mathfrak{T}$ is said to be graded von Neumann regular if, for every $a \in h(\mathfrak{T})$, there exists $b \in h(\mathfrak{T})$ such that $a = a^2 b$; see [8].

Theorem 2.11. *Let $S \subseteq h(\mathfrak{T})$ be an m.c.s. of $\mathfrak{T}$.*

(i) *If $\mathfrak{T}$ is a graded S-integral domain, then every GWS2P-ideal of $\mathfrak{T}$ is a GS2P-ideal.*
(ii) *Conversely, if $\mathfrak{T}$ is a graded von Neumann regular ring and every GWS2P-ideal of $\mathfrak{T}$ is GS2P-ideal, then $\mathfrak{T}$ is a graded S-integral domain.*

Proof. (i) Let $\mathfrak{D}$ be a GWS2P-ideal of $\mathfrak{T}$ associated with $s \in S$, and let $a, b \in h(\mathfrak{T})$ satisfy $ab \in \mathfrak{D}$.

If $ab \neq 0$, then $sa^2 \in \mathfrak{D}$ or $sb^2 \in \mathfrak{D}$ by the definition of a graded weakly S-2-prime ideal. If $ab = 0$, then, since $\mathfrak{T}$ is a graded S-integral domain, we have $sa = 0$ or $sb = 0$. Consequently, $sa^2 = 0 \in \mathfrak{D}$ or $sb^2 = 0 \in \mathfrak{D}$. Therefore, $\mathfrak{D}$ is a GS2P-ideal of $\mathfrak{T}$.

(ii) Assume that $\mathfrak{T}$ is graded von Neumann regular and that every GWS2P-ideal of $\mathfrak{T}$ is GS2P-ideal. Since (0) is always a GWS2P-ideal, it follows that (0) is a GS2P-ideal of $\mathfrak{T}$.

Let $a, b \in h(\mathfrak{T})$ with $ab = 0$. Since (0) is GS2P, we have $sa^2 = 0$ or $sb^2 = 0$. Hence $a \in Grad((0 :_{\mathfrak{T}} s))$ or $b \in Grad((0 :_{\mathfrak{T}} s))$. By [2, Proposition 8],

$$Grad((0 :_{\mathfrak{T}} s)) = (0 :_{\mathfrak{T}} s)$$

because $\mathfrak{T}$ is graded von Neumann regular. Therefore, $a \in (0 :_{\mathfrak{T}} s)$ or $b \in (0 :_{\mathfrak{T}} s)$, and thus $sa = 0$ or $sb = 0$. Consequently, $\mathfrak{T}$ is a graded S-integral domain. □

Let V and W be commutative graded rings, let J be a graded ideal of W, and let $\varphi : V \to W$ be a graded ring homomorphism. The amalgamation of V with W along J with respect to φ is denoted by $V \bowtie^{\varphi} J$. If I is a graded ideal of V, then $I \bowtie^{\varphi} J = \{(i, \varphi(i) + j) \mid i \in I,\ j \in J\}$ is a graded ideal of $V \bowtie^{\varphi} J$ [7]. Let $S \subseteq h(V)$ be an m.c.s. of V, and define $S \bowtie^{\varphi} 0 = \{(s, \varphi(s)) \mid s \in S\}$. Then $S \bowtie^{\varphi} 0$ is an m.c.s. of $V \bowtie^{\varphi} J$.

Theorem 2.12. *Let $f : V \longrightarrow W$ be a graded ring homomorphism, let I be a graded ideal of W, and let $S \subseteq h(V)$ be an m.c.s. of V. If $\mathfrak{D} \bowtie^{f} I$ is a graded weakly $(S \bowtie^{f} 0)$-2-prime ideal of $V \bowtie^{f} I$, then $\mathfrak{D}$ is a GWS2P-ideal of V.*

Proof. Assume that $\mathfrak{D} \bowtie^{f} I$ is a graded weakly $(S \bowtie^{f} 0)$-2-prime ideal of $V \bowtie^{f} I$ associated with $(s, f(s)) \in S \bowtie^{f} 0$.

Let $a_g, b_h \in h(V)$ such that $0 \neq a_g b_h \in \mathfrak{D}$. Then

$$(0,0) \neq (a_g, f(a_g))(b_h, f(b_h)) = (a_g b_h, f(a_g b_h)) \in \mathfrak{D} \bowtie^{f} I.$$

Since $\mathfrak{D} \bowtie^f I$ is graded weakly $(S \bowtie^f 0)$-2-prime, either

$$(s, f(s))(a_g, f(a_g))^2 \in \mathfrak{D} \bowtie^f I$$

or

$$(s, f(s))(b_h, f(b_h))^2 \in \mathfrak{D} \bowtie^f I.$$

Hence,

$$(sa_g^2, f(sa_g^2)) \in \mathfrak{D} \bowtie^f I$$

or

$$(sb_h^2, f(sb_h^2)) \in \mathfrak{D} \bowtie^f I.$$

Therefore, $sa_g^2 \in \mathfrak{D}$ or $sb_h^2 \in \mathfrak{D}$, showing that $\mathfrak{D}$ is a GWS2P-ideal of V. □

Theorem 2.13. *Let $f : V \to W$ be a graded ring homomorphism, let I be a graded ideal of W, and let $S \subseteq h(V)$ be an m.c.s. of V. Suppose that $\mathfrak{D}$ is a graded weakly S-2-prime ideal of V which is not a graded S-2-prime ideal. If $f(a)I = 0 = f(b)I$ for every graded S-double-zero (a, b) of $\mathfrak{D}$, and $I^2 = 0$, then $\mathfrak{D} \bowtie^f I$ is a graded weakly $(S \bowtie^f 0)$-2-prime ideal of $V \bowtie^f I$.*

Proof. Let $(s, f(s)) \in S \bowtie^f 0$, where $s \in S$ is associated with $\mathfrak{D}$. Suppose that

$$(0, 0) \neq (a_g, f(a_g) + i_g)(b_h, f(b_h) + j_h) \in \mathfrak{D} \bowtie^f I,$$

where $a_g, b_h \in h(V)$ and $i_g, j_h \in I \cap h(W)$.

Since the first component of the product belongs to $\mathfrak{D}$, we have $a_g b_h \in \mathfrak{D}$.

Case 1. $a_g b_h \neq 0$.

Since $\mathfrak{D}$ is a graded weakly S-2-prime ideal of V, $sa_g^2 \in \mathfrak{D}$ or $sb_h^2 \in \mathfrak{D}$. Hence

$$(s, f(s))(a_g, f(a_g) + i_g)^2 \in \mathfrak{D} \bowtie^f I$$

or

$$(s, f(s))(b_h, f(b_h) + j_h)^2 \in \mathfrak{D} \bowtie^f I.$$

Case 2. $a_g b_h = 0$.

Assume that $sa_g^2 \notin \mathfrak{D}$ and $sb_h^2 \notin \mathfrak{D}$. Then (a_g, b_h) is a graded S-double-zero of $\mathfrak{D}$. By hypothesis, $f(a_g)I = 0 = f(b_h)I$. Therefore,

$$\begin{aligned}
&(a_g, f(a_g) + i_g)(b_h, f(b_h) + j_h) \\
&= (a_g b_h,\ f(a_g b_h) + f(a_g) j_h + f(b_h) i_g + i_g j_h) \\
&= (0, 0),
\end{aligned}$$

since $a_g b_h = 0$, $f(a_g)I = 0 = f(b_h)I$, and $I^2 = 0$. This contradicts the assumption that the product is nonzero. Hence $sa_g^2 \in \mathfrak{D}$ or $sb_h^2 \in \mathfrak{D}$.

Therefore,

$$(s, f(s))(a_g, f(a_g) + i_g)^2 \in \mathfrak{D} \bowtie^f I$$

or

$$(s, f(s))(b_h, f(b_h) + j_h)^2 \in \mathfrak{D} \bowtie^f I.$$

Thus $\mathfrak{D} \bowtie^f I$ is a graded weakly $(S \bowtie^f 0)$-2-prime ideal of $V \bowtie^f I$. □

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References

[1] R. Abdullah, R. Abu-Dawwas, S. Koç, Ü. Tekir, E. Yıldız, On Graded Strongly Quasi Primary Ideals, Moroc. J. Algebra Geom. Appl. 1 (2022), 409–417.

[2] A.S. Alshehry, On Graded S-Primary Ideals, Mathematics 9 (2021), 2637. https://doi.org/10.3390/math9202637.

[3] A.S. Alshehry, R. Abu-Dawwas, B. Hawary, Graded Weakly Strongly Quasi-Primary Ideals over Commutative Graded Rings, Mathematics 12 (2024), 2857. https://doi.org/10.3390/math12182857.

[4] K. Al-Zoubi, R. Abu Zuraiq, On Graded S-2-Prime Ideals, Submitted.

[5] K. Al-Zoubi, M. Al-Azaizeh, On Graded Weakly 2-Absorbing Primary Submodules, Vietnam J. Math. 47 (2019), 297–307. https://doi.org/10.1007/s10013-018-0321-z.

[6] M. Bataineh, R. Abu-Dawwas, On Graded 2-Prime Ideals, Mathematics 9 (2021), 493. https://doi.org/10.3390/math9050493.

[7] F.Z. Guissi, H. Kim, N. Mahdou, Graded Amalgamated Algebras along an Ideal, J. Algebra Appl. 23 (2024), 2450116. https://doi.org/10.1142/S0219498824501160.

[8] C. Nastasescu, F. Van Oystaeyen, Methods of Graded Rings, Springer Berlin Heidelberg, 2004. https://doi.org/10.1007/b94904.

[9] R.N. Uregen, Ü. Tekir, K.P. Shum, S. Koç, On Graded 2-Absorbing Quasi Primary Ideals, Southeast Asian Bull. Math. 43 (2019), 601–613. https://zbmath.org/7156775.

[10] M. Refai, K. Al-Zoubi, On Graded Primary Ideals, Turk. J. Math. 28 (2004), 217–230.