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Existence and Uniqueness Results for Tripled Systems Involving Sequential Caputo Fractional Derivatives

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Abstract. In this manuscript, we investigated a tripled system governed by nonlinear equations involving sequential Caputo fractional derivatives. These derivatives provide a generalized framework that describes memory and nonlocal effects more effectively than classical differential operators. The main contributions of this work included establishing existence and uniqueness results for the considered system and studying a class of triple equations. The proposed approach demonstrated the efficiency of fractional techniques in extending classical analytical methods. Moreover, an illustrative example was presented to confirm the applicability of the obtained results.

1. Introduction

Classical differential equations can be generalized to fractional differential equations, where the order of differentiation is allowed to be non-integer or fractional. In recent years, fractional differential equations have gained significant attention because of their effectiveness in describing complex phenomena that involve memory and hereditary characteristics, which cannot be adequately captured by traditional integer-order models [1–5]. These equations have found extensive applications in various fields such as physics, engineering, biology, control theory, viscoelasticity, fluid mechanics, and signal processing, where many real-world processes exhibit nonlocal and history-dependent behavior.

Fractional calculus provides a powerful mathematical framework for modeling such phenomena through different definitions of fractional derivatives and integrals. Among the existing formulations, the Caputo fractional derivative [6] is one of the most commonly used due to its compatibility with classical initial and boundary conditions. In particular, the Caputo derivative allows the use

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of physically meaningful initial conditions expressed in terms of integer-order derivatives, making it highly suitable for practical and physical applications. Furthermore, the nonlocal nature of fractional operators enables a more accurate representation of dynamical systems with long-term memory effects and complex interactions. For more details about the applications of fractional calculus, see [7–15].

However, the use of fractional derivatives in these models allows for a better characterization of viscoelastic effects and nonlocal interactions, which are commonly encountered in sophisticated engineering materials and complex dynamical systems. Therefore, fractional differential equations provide a richer mathematical description and offer enhanced predictive capabilities in various applications of applied physics and engineering.

For contextual background, we refer to several works related to the present investigation. For example, Wang and Yang examined in [10] the existence of positive solutions for a nonlinear fourth-order system of the form

$$\begin{cases} \varphi^{(4)}(s) + \gamma_1 \varphi''(s) - \eta_1 \varphi(\zeta) = \ell_1\left(s, \varphi(s), \phi(s)\right),\ s \in V = [0,1], \\ \phi^{(4)}(s) + \gamma_2 \phi''(s) - \eta_2 \phi(\zeta) = \ell_2\left(s, \varphi(s), \phi(s)\right),\ s \in V, \end{cases}$$

via the stipulations

$$\begin{cases} \varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = 0, \\ \phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0, \end{cases}$$

where $\ell_i : V \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, and $\gamma_i, \eta_i \in \mathbb{R}$ such that $\gamma_i \leq 2\pi^2$, $\frac{-\gamma_i^2}{4} \leq \eta_i$, $\frac{\eta_i}{\pi^4} + \frac{\gamma_i}{\pi^4} < 1$, $i = 1, 2$. The authors demonstrated the existence of positive solution results by providing a cone Q in $C(V) \times C(V)$. They built another successful solution outcome by building over a product cone.

In [16], the authors studied the existence and uniqueness of solutions for the following system with sequential Caputo fractional derivatives:

$$\begin{cases} \left({}^cD^{\eta} + \vartheta\, {}^cD^{\eta-1}\right)\zeta(s) = \ell_1\left(s, \zeta(s), \omega(s), I_{0^+}^{r_1}\zeta(s), I_{0^+}^{r_2}\omega(s)\right),\ s \in V, \\ \left({}^cD^{\gamma} + \theta\, {}^cD^{\gamma-1}\right)\omega(s) = \ell_2\left(s, \zeta(s), \omega(s), I_{0^+}^{z_1}\zeta(s), I_{0^+}^{z_2}\omega(s)\right),\ s \in V, \end{cases}$$

under the constraints

$$\begin{cases} \zeta(0) = \zeta'(0) = 0,\ \zeta'(1) = 1,\ \zeta(1) = \int_0^1 \zeta(t)\, dR_1(t) + \int_0^1 \omega(t)\, dR_2(t), \\ \omega(0) = \omega'(0) = 0,\ \omega'(1) = 1,\ \omega(1) = \int_0^1 \zeta(t)\, dM_1(t) + \int_0^1 \omega(t)\, dM_2(t), \end{cases}$$

where $\eta, \gamma \in (3,4]$, $\vartheta, \theta > 0$, $r_1, r_2, z_1, z_2 > 0$, $\ell_1, \ell_2 : V \times \mathbb{R}^4 \to \mathbb{R}$ are continuous functions, $I_{0^+}^{r_1}, I_{0^+}^{r_2}, I_{0^+}^{z_1}, I_{0^+}^{z_2}$ are Riemann-Liouville integrals of orders r_1, r_2, z_1, z_2, and R_1, R_2, M_1, M_2 are Riemann-Stieltjes integrals. These systems have important applications in the biosciences; see [1]. The authors established the existence and uniqueness of solutions for the considered system.

Furthermore, Bensassa et al. investigated in [17] the existence, uniqueness, and Ulam–Hyers stability of the following system:

$$\begin{cases} D^{\eta_1} D^{\eta_2} \varphi(\zeta) = \ell_1\left(\zeta, \varphi(\zeta), \omega(\zeta)\right) + b_1 \hbar_1\left(\zeta, \varphi(\zeta)\right) + c_1 \varpi_1\left(\zeta, D^{\rho}\varphi(\zeta)\right), \\ D^{\gamma_1} D^{\gamma_2} \omega(\zeta) = \ell_2\left(\zeta, \varphi(\zeta), \omega(\zeta)\right) + b_2 \hbar_2\left(\zeta, \omega(\zeta)\right) + c_2 \varpi_2\left(\zeta, D^{\rho}\omega(\zeta)\right), \end{cases}$$

under the stipulations

$$\begin{cases} \varphi(0) = \varphi(1) = b, \;\; \varphi'(0) = \varphi'(1) = 0, \\ \omega(0) = \omega(1) = c, \;\; \omega'(0) = \omega'(1) = 0, \end{cases}$$

where $D^{\eta_i}, D^{\gamma_i}, D^{\rho}$ are CFDs, $\rho \in (0,1]$, $\ell_1, \ell_2 \in C\left([0,1] \times \mathbb{R} \times \mathbb{R}; \mathbb{R}\right)$, $b_1, b_2, c_1, c_2 \in \mathbb{R}$, and $\hbar_1, \hbar_2, \varpi_1, \varpi_2 \in C\left([0,1] \times \mathbb{R}; \mathbb{R}\right)$. Motivated by the aforementioned studies, particularly the work presented in [10], the present paper investigates the following problem involving sequential Caputo fractional derivatives:

$$\begin{cases} D^{\eta_1} D^{\gamma_1} \varphi_1(\zeta) = \backepsilon_1 Z_1\Big(\zeta, \varphi_1(\zeta), \varphi_2(\zeta), \varphi_3(\zeta), D^{\rho-1}\varphi_1(\zeta), D^{\rho}\varphi_1(\zeta)\Big), \\ D^{\eta_2} D^{\gamma_2} \varphi_2(\zeta) = \backepsilon_2 Z_2\Big(\zeta, \varphi_1(\zeta), \varphi_2(\zeta), \varphi_3(\zeta), D^{\rho-1}\varphi_2(\zeta), D^{\rho}\varphi_2(\zeta)\Big), \\ D^{\eta_3} D^{\gamma_3} \varphi_3(\zeta) = \backepsilon_3 Z_3\Big(\zeta, \varphi_1(\zeta), \varphi_2(\zeta), \varphi_3(\zeta), D^{\rho-1}\varphi_3(\zeta), D^{\rho}\varphi_3(\zeta)\Big), \end{cases} \tag{1.1}$$

under conditions

$$\begin{cases} \varphi_j(0) = \varphi_j(1) = b_j \in \mathbb{R}, \\ D^{\gamma_j}\varphi_j(0) = D^{\gamma_j}\varphi_j(1) = 0, \;\; j = 1,2,3, \end{cases} \tag{1.2}$$

where D^{η_j}, D^{γ_j}, and D^{ρ} are Caputo fractional derivatives with orders $\eta_j \in (2,3]$, $\gamma_j \in (0,1]$ and $\rho \in (1,2]$, $j = 1,2,3$, respectively. Moreover, $Z_i : [0,1] \times \mathbb{R}^5 \to \mathbb{R}$ are continuous functions and $\backepsilon_i \in \mathbb{R}^+$, $j = 1,2,3$.

It is worth emphasizing that the problem considered in this paper is more general than those investigated in the aforementioned studies. In particular, the inclusion of Caputo fractional derivatives in the initial conditions distinguishes our work from most existing results. This feature introduces additional realism and complexity, thereby enhancing the physical relevance and applicability of the model. Moreover, such conditions are consistent with observable initial data and are well suited for efficient numerical implementations, allowing a more accurate description of memory effects and nonlocal characteristics of the system. Furthermore, the considered model involves three general nonlinear functions, Z_1, Z_2, and Z_3, which significantly extend the scope of previously studied problems and provide a broader framework for analyzing complex dynamical behaviors. This generalized setting allows the treatment of more sophisticated phenomena arising in applied physics and engineering, leading to deeper analytical insights and improved predictive capabilities.

The present manuscript focused on the study of a tripled system described by nonlinear equations with sequential Caputo fractional derivatives. These fractional operators are well suited for capturing hereditary and nonlocal phenomena that cannot be adequately represented by classical derivatives. The paper derived existence and uniqueness conditions for the investigated system and analyzed related triple equations. The developed methodology illustrated the importance of fractional calculus in broadening the framework of classical differential analysis. An example was also provided to support the applicability of the theoretical results.

2. Preliminaries

In this section, we recall several basic definitions and lemmas from fractional calculus, which will be used throughout the paper. These results are available in [4].

Definition 2.1. *For any function $\varpi \in C(V)$, $V = [0,1]$, the fractional integral of order η is described as*

$$I^{\eta}\varpi(s) = \frac{1}{\Gamma(\eta)}\int_0^s (s-r)^{\eta-1}\varpi(r)\,dr,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. *For the function $\varpi \in C^m(V)$, the CFD of order η is given by*

$$D^{\eta}\varpi(s) = \frac{1}{\Gamma(m-\eta)}\int_0^s (s-r)^{m-\eta-1}\varpi^{(m)}(r)\,dr,\ m = [\eta]+1,$$

where $[\eta]$ represents the integer part of η.

Lemma 2.1. *Assume that $\eta > 0$, the solution of the equation*

$$D^{\eta}\varpi(s) = 0,\ s \in V$$

is written as

$$\varpi(s) = k_0 + k_1 s + k_2 s^2 + \cdots + k_{m-1}s^{m-1},$$

where $k_v \in \mathbb{R}$, $v = 0,1,2,\cdots,m-1$, $m = [\eta]+1$.

Lemma 2.2. *For any $\eta > 0$, we have*

$$I^{\eta}_{0^+}D^{\eta}\varpi(s) = \varpi(s) + k_0 + k_1 s + k_2 s^2 + \cdots + k_{m-1}s^{m-1},$$

where $k_v \in \mathbb{R}$, $v = 0,1,2,\cdots,m-1$, $m = [\eta]+1$.

Remark 2.1. *For the function $\varpi \in C(V)$, we have the following results:*

- (i) *For each $\eta,\gamma > 0$ and for all $s \in V$, $I^{\eta}I^{\gamma}\varpi(s) = I^{\eta+\gamma}\varpi(s)$;*
- (ii) *According to Definition 2.2 and the property (i), if $0 < \eta \leq \gamma$, we get $D^{\eta}I^{\gamma}\varpi(s) = I^{\eta-\gamma}\varpi(s)$;*
- (iii) *If $\eta = \gamma$ in (ii), we have $D^{\eta}I^{\gamma}\varpi(s) = \varpi(s)$.*

It is very important to present the equivalent integral equation to the problem (1.1).

Lemma 2.3. *Assume that $R_1, R_2, R_3 \in C(V,\mathbb{R})$, $\eta_j \in (2,3]$, $\gamma_j \in (0,1]$, and $\eth_j > 0$, $j = 1,2,3$. The solution of the problem*

$$\begin{cases} D^{\eta_1}D^{\gamma_1}\varphi_1(\zeta) = \eth_1 R_1(\zeta), \\ D^{\eta_2}D^{\gamma_2}\varphi_2(\zeta) = \eth_2 R_2(\zeta), \\ D^{\eta_3}D^{\gamma_3}\varphi_3(\zeta) = \eth_3 R_3(\zeta), \end{cases} \tag{2.1}$$

under conditions

$$\begin{cases} \varphi_j(0) = \varphi_j(1) = b_j \in \mathbb{R}, \\ D^{\lambda_j}\varphi_j(0) = D^{\lambda_j}\varphi_j(1) = 0, \end{cases}$$

is equivalent to

$$\begin{cases} \varphi_1(\zeta) = b_1 + \eth_1 I^{\eta_1+\gamma_1} R_1(\zeta) \\ +\left(\frac{2\eth_1}{\gamma_1\Gamma(\eta_1)}\int_0^1 (1-r)^{\eta_1-1} R_1(r)\,dr - \frac{\eth_1\Gamma(\gamma_1+3)}{\gamma_1\Gamma(\eta_1+\gamma_1)}\int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\,dr\right)\zeta^{\gamma_1+1} \\ +\left(\frac{\eth_1\Gamma(\gamma_1+3)}{\gamma_1\Gamma(\eta_1+\gamma_1)}\int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\,dr - \frac{\eth_1(\gamma_1+2)}{\gamma_1\Gamma(\eta_1)}\int_0^1 (1-r)^{\eta_1-1} R_1(r)\,dr\right)\zeta^{\gamma_1+2}, \\ \varphi_2(\zeta) = b_2 + \eth_2 I^{\eta_2+\gamma_2} R_2(\zeta) \\ +\left(\frac{2\eth_2}{\gamma_2\Gamma(\eta_2)}\int_0^1 (1-r)^{\eta_2-1} R_2(r)\,dr - \frac{\eth_2\Gamma(\gamma_2+3)}{\gamma_2\Gamma(\eta_2+\gamma_2)}\int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\,dr\right)\zeta^{\gamma_2+1} \\ +\left(\frac{\eth_2\Gamma(\gamma_2+3)}{\gamma_2\Gamma(\eta_2+\gamma_2)}\int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\,dr - \frac{\eth_2(\gamma_2+2)}{\gamma_2\Gamma(\eta_2)}\int_0^1 (1-r)^{\eta_2-1} R_2(r)\,dr\right)\zeta^{\gamma_2+2}, \\ \varphi_3(\zeta) = b_3 + \eth_3 I^{\eta_3+\gamma_3} R_3(\zeta) \\ +\left(\frac{2\eth_3}{\gamma_3\Gamma(\eta_3)}\int_0^1 (1-r)^{\eta_3-1} R_3(r)\,dr - \frac{\eth_3\Gamma(\gamma_3+3)}{\gamma_3\Gamma(\eta_3+\gamma_3)}\int_0^1 (1-r)^{\eta_3+\gamma_3-1} R_3(r)\,dr\right)\zeta^{\gamma_3+1} \\ +\left(\frac{\eth_3\Gamma(\gamma_3+3)}{\gamma_3\Gamma(\eta_3+\gamma_3)}\int_0^1 (1-r)^{\eta_3+\gamma_3-1} R_3(r)\,dr - \frac{\eth_3(\gamma_3+2)}{\gamma_3\Gamma(\eta_3)}\int_0^1 (1-r)^{\eta_3-1} R_3(r)\,dr\right)\zeta^{\gamma_3+2}. \end{cases} \tag{2.2}$$

Proof. Applying $I^{\eta_j+\gamma_j}$, $(j=1,2,3)$ on (2.1) and using Lemma 2.2, we have

$$\begin{cases} \varphi_1(\zeta) = \eth_1 I^{\eta_1+\gamma_1} R_1(\zeta) + \frac{k_0\zeta^{\gamma_1}}{\Gamma(\gamma_1+1)} + \frac{k_1\zeta^{\gamma_1+1}}{\Gamma(\gamma_1+2)} + \frac{k_2\zeta^{\gamma_1+2}}{\Gamma(\gamma_1+3)} + k_3, \\ \varphi_2(\zeta) = \eth_2 I^{\eta_2+\gamma_2} R_2(\zeta) + \frac{c_0\zeta^{\gamma_2}}{\Gamma(\gamma_2+1)} + \frac{c_1\zeta^{\gamma_2+1}}{\Gamma(\gamma_2+2)} + \frac{c_2\zeta^{\gamma_2+2}}{\Gamma(\gamma_2+3)} + c_3, \\ \varphi_3(\zeta) = \eth_3 I^{\eta_3+\gamma_3} R_3(\zeta) + \frac{c_0\zeta^{\gamma_3}}{\Gamma(\gamma_3+1)} + \frac{c_1\zeta^{\gamma_3+1}}{\Gamma(\gamma_3+2)} + \frac{c_2\zeta^{\gamma_3+2}}{\Gamma(\gamma_3+3)} + a_3. \end{cases} \tag{2.3}$$

Because $\varphi_j(0) = b_j$ $(j=1,2,3)$, we conclude that $k_3 = b_1\, c_3 = b_2, a_3 = b_3$. Applying D^{γ_j} $(j=1,2,3)$ on (2.3), we have

$$\begin{cases} D^{\gamma_1}\varphi_1(\zeta) = \frac{\eth_1}{\Gamma(\eta_1)}\int_0^\zeta (\zeta-r)^{\eta_1-1} R_1(r)\,dr + k_0 + k_1\zeta + k_2\zeta^2, \\ D^{\gamma_2}\varphi_2(\zeta) = \frac{\eth_2}{\Gamma(\eta_2)}\int_0^\zeta (\zeta-r)^{\eta_2-1} R_2(r)\,dr + c_0 + c_1\zeta + c_2\zeta^2, \\ D^{\gamma_3}\varphi_3(\zeta) = \frac{\eth_3}{\Gamma(\eta_3)}\int_0^\zeta (\zeta-r)^{\eta_3-1} R_3(r)\,dr + a_0 + a_1\zeta + a_2\zeta^2. \end{cases} \tag{2.4}$$

Letting $\zeta = 0$ in (2.3), we have $k_0 = c_0 = a_0 = 0$. Taking $\zeta = 1$ in (2.3) and (2.4), we obtain that

$$\begin{cases} \frac{\eth_1}{\Gamma(\eta_1+\gamma_1)}\int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\,dr + \frac{k_1}{\Gamma(\gamma_1+2)} + \frac{2k_2}{\Gamma(\gamma_1+3)} = 0, \\ \frac{\eth_1}{\Gamma(\eta_1)}\int_0^1 (1-r)^{\eta_1-1} R_1(r)\,dr + k_1 + k_2 = 0, \\ \frac{\eth_2}{\Gamma(\eta_2+\gamma_2)}\int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\,dr + \frac{c_1}{\Gamma(\gamma_2+2)} + \frac{2c_2}{\Gamma(\gamma_2+3)} = 0, \\ \frac{\eth_2}{\Gamma(\eta_2)}\int_0^1 (1-r)^{\eta_2-1} R_2(r)\,dr + c_1 + c_2 = 0, \\ \frac{\eth 3}{\Gamma(\eta_3+\gamma_3)}\int_0^1 (1-r)^{\eta_3+\gamma_3-1} R_3(r)\,dr + \frac{a_1}{\Gamma(\gamma_3+2)} + \frac{2a_2}{\Gamma(\gamma_3+3)} = 0, \\ \frac{\eth_3}{\Gamma(\eta_3)}\int_0^1 (1-r)^{\eta_3-1} R_3(r)\,dr + a_1 + a_2 = 0. \end{cases}$$

After solving the above equations, we have

$$\begin{cases} k_1 = -\frac{\eth_1\Gamma(\gamma_1+3)}{\gamma_1\Gamma(\eta_1+\gamma_1)}\int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\,dr + \frac{2\eth_1}{\gamma_1\Gamma(\eta_1)}\int_0^1 (1-r)^{\eta_1-1} R_1(r)\,dr, \\ k_2 = \frac{\eth_1\Gamma(\gamma_1+3)}{\gamma_1\Gamma(\eta_1+\gamma_1)}\int_0^1 (1-r)^{\eta_1+\gamma_1-1} R_1(r)\,dr - \frac{\eth_1(\gamma_1+2)}{\gamma_1\Gamma(\eta_1)}\int_0^1 (1-r)^{\eta_1-1} R_1(r)\,dr, \\ c_1 = -\frac{\eth_2\Gamma(\gamma_2+3)}{\gamma_2\Gamma(\eta_2+\gamma_2)}\int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\,dr + \frac{2\eth_2}{\gamma_2\Gamma(\eta_2)}\int_0^1 (1-r)^{\eta_2-1} R_2(r)\,dr, \\ c_2 = \frac{\eth_2\Gamma(\gamma_2+3)}{\gamma_2\Gamma(\eta_2+\gamma_2)}\int_0^1 (1-r)^{\eta_2+\gamma_2-1} R_2(r)\,dr - \frac{\eth_2(\gamma_2+2)}{\gamma_2\Gamma(\eta_2)}\int_0^1 (1-r)^{\eta_2-1} R_2(r)\,dr, \\ a_1 = -\frac{\eth_3\Gamma(\gamma_3+3)}{\gamma_3\Gamma(\eta_3+\gamma_3)}\int_0^1 (1-r)^{\eta_3+\gamma_3-1} R_3(r)\,dr + \frac{2\eth_3}{\gamma_3\Gamma(\eta_3)}\int_0^1 (1-r)^{\eta_3-1} R_3(r)\,dr, \\ a_2 = \frac{\eth_3\Gamma(\gamma_3+3)}{\gamma_3\Gamma(\eta_3+\gamma_3)}\int_0^1 (1-r)^{\eta_3+\gamma_3-1} R_3(r)\,dr - \frac{\eth_3(\gamma_3+2)}{\gamma_3\Gamma(\eta_3)}\int_0^1 (1-r)^{\eta_3-1} R_3(r)\,dr. \end{cases}$$

Substituting from the values of k_j, c_j and a_j $(j = 1, 2, 3)$ in (2.3), we get (2.2). ◻

3. Existence and uniqueness theorem

In this part, we demonstrate how the fixed point (FP) method is used to establish the existence and uniqueness of a solution to our problem.

We define the Banach space

$$\Xi = \left\{ \varphi \in C\left(V, \mathbb{R}\right),\ D^{\rho-1}\varphi \in C\left(V, \mathbb{R}\right),\ \text{and}\ D^{\rho-1}\varphi \in C\left(V, \mathbb{R}\right) \right\}$$

under the norm

$$\left\|\varphi\right\|_{\Xi} = \left\|\varphi\right\|_{\infty} + \left\|D^{\rho-1}\varphi\right\|_{\infty} + \left\|D^{\rho-1}\varphi\right\|_{\infty},$$

where

$$\left\|\varphi\right\|_{\infty} = \sup_{\zeta\in V}\left|\varphi(\zeta)\right|,\ \left\|D^{\rho-1}\varphi\right\|_{\infty} = \sup_{\zeta\in V}\left|D^{\rho-1}\varphi(\zeta)\right|,\ \text{and}\ \left\|D^{\rho}\varphi\right\|_{\infty} = \sup_{\zeta\in V}\left|D^{\rho}\varphi(\zeta)\right|,\ V = [0, 1].$$

Consider the space $\Xi^3 = \Xi \times \Xi \times \Xi$ defined on the norm

$$\left\|(\varphi, \omega, \mu)\right\|_{\Xi^3} = \left\|\varphi\right\|_{\Xi} + \|\omega\|_{\Xi} + \left\|\mu\right\|_{\Xi}.$$

Clearly, $\left(\Xi^3, \|.\|\right)$ is a Banach space.

Now, in order to apply the technique of the fixed-point, we consider the operator $\mho = (\mho_1, \mho_2, \mho_3)$ such that $\mho : \Xi^3 \to \Xi^3$. This operator can be written as

$$\mho\left(\varphi(\zeta), \omega(\zeta), \mu\left(\zeta\right)\right) = \sum_{j=1}^{3} \mho_j\left(\varphi(\zeta), \omega(\zeta), \mu\left(\zeta\right)\right),$$

where

$$\begin{aligned}
& \mho_1\left(\varphi(\zeta), \omega(\zeta), \mu\left(\zeta\right)\right) \\
= \ & b_1 + \Game_1 I^{\eta_1+\gamma_1} Z_1\left(\zeta, \varphi(\zeta), \omega(\zeta), \mu\left(\zeta\right), D^{\rho-1}\varphi(\zeta), D^{\rho}\varphi(\zeta)\right) \\
& + \left(\begin{array}{r} \frac{2\Game_1}{\gamma_1\Gamma(\eta_1)} \int_0^1 \left(1-r\right)^{\eta_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu\left(\zeta\right), D^{\rho-1}\varphi(r), D^{\rho}\omega(r)\right) dr \\ -\frac{\Game_1\Gamma(\gamma_1+3)}{\gamma_1\Gamma(\eta_1+\gamma_1)} \int_0^1 \left(1-r\right)^{\eta_1+\gamma_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu\left(\zeta\right), D^{\rho-1}\varphi(r), D^{\rho}\omega(r)\right) dr \end{array} \right) \zeta^{\gamma_1+1} \\
& + \left(\begin{array}{r} \frac{\Game_1\Gamma(\gamma_1+3)}{\gamma_1\Gamma(\eta_1+\gamma_1)} \int_0^1 \left(1-r\right)^{\eta_1+\gamma_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu\left(\zeta\right), D^{\rho-1}\varphi(r), D^{\rho}\omega(r)\right) dr \\ -\frac{\Game_1(\gamma_1+2)}{\gamma_1\Gamma(\eta_1)} \int_0^1 \left(1-r\right)^{\eta_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu\left(\zeta\right), D^{\rho-1}\varphi(r), D^{\rho}\omega(r)\right) dr \end{array} \right) \zeta^{\gamma_1+2},
\end{aligned}$$

$$\begin{aligned}
& \mho_2\left(\varphi(\zeta), \omega(\zeta), \mu\left(\zeta\right)\right) \\
= \ & b_2 + \Game_2 I^{\eta_2+\gamma_2} Z_2\left(\zeta, \varphi(\zeta), \omega(\zeta), \mu\left(\zeta\right), D^{\rho-1}\omega(\zeta), D^{\rho}\omega(\zeta)\right) \\
& + \left(\begin{array}{r} \frac{2\Game_2}{\gamma_2\Gamma(\eta_2)} \int_0^1 \left(1-r\right)^{\eta_2-1} Z_2\left(r, \varphi(r), \omega(r), \mu\left(\zeta\right), D^{\rho-1}\omega(r), D^{\rho}\omega(r)\right) dr \\ -\frac{\Game_2\Gamma(\gamma_2+3)}{\gamma_2\Gamma(\eta_2+\gamma_2)} \int_0^1 \left(1-r\right)^{\eta_2+\gamma_2-1} Z_2\left(r, \varphi(r), \omega(r), \mu\left(\zeta\right), D^{\rho-1}\omega(r), D^{\rho}\omega(r)\right) dr \end{array} \right) \zeta^{\gamma_2+1} \\
& + \left(\begin{array}{r} \frac{\Game_2\Gamma(\gamma_2+3)}{\gamma_2\Gamma(\eta_2+\gamma_2)} \int_0^1 \left(1-r\right)^{\eta_2+\gamma_2-1} Z_2\left(r, \varphi(r), \omega(r), \mu\left(\zeta\right), D^{\rho-1}\omega(r), D^{\rho}\omega(r)\right) dr \\ -\frac{\Game_2(\gamma_2+2)}{\gamma_2\Gamma(\eta_2)} \int_0^1 \left(1-r\right)^{\eta_1-1} Z_2\left(r, \varphi(r), \omega(r), \mu\left(\zeta\right), D^{\rho-1}\omega(r), D^{\rho}\omega(r)\right) dr \end{array} \right) \zeta^{\gamma_2+2},
\end{aligned}$$

and

$$
\begin{aligned}
&\mho_3\left(\varphi(\zeta),\omega(\zeta),\mu\left(\zeta\right)\right)\\
=\;& b_3+\eth_3 I^{\eta_3+\gamma_3}Z_3\left(\zeta,\varphi(\zeta),\omega(\zeta),\mu\left(\zeta\right),D^{\rho-1}\mu\left(\zeta\right),D^{\rho}\mu\left(\zeta\right)\right)\\
&+\left(\begin{array}{c}\frac{2\eth_3}{\gamma_3\Gamma(\eta_3)}\int_0^1\left(1-r\right)^{\eta_3-1}Z_3\left(r,\varphi(r),\omega(r),\mu\left(\zeta\right),D^{\rho-1}\mu\left(\zeta\right),D^{\rho}\mu\left(\zeta\right)\right)dr\\
-\frac{\eth_3\Gamma(\gamma_3+3)}{\gamma_3\Gamma(\eta_3+\gamma_3)}\int_0^1\left(1-r\right)^{\eta_3+\gamma_3-1}Z_3\left(r,\varphi(r),\omega(r),\mu\left(\zeta\right),D^{\rho-1}\mu\left(\zeta\right),D^{\rho}\mu\left(\zeta\right)\right)dr\end{array}\right)\zeta^{\gamma_3+1}\\
&+\left(\begin{array}{c}\frac{\eth_3\Gamma(\gamma_3+3)}{\gamma_3\Gamma(\eta_3+\gamma_3)}\int_0^1\left(1-r\right)^{\eta_3+\gamma_3-1}Z_3\left(r,\varphi(r),\omega(r),\mu\left(\zeta\right),D^{\rho-1}\mu\left(\zeta\right),D^{\rho}\mu\left(\zeta\right)\right)dr\\
-\frac{\eth_3(\gamma_3+2)}{\gamma_3\Gamma(\eta_3)}\int_0^1\left(1-r\right)^{\eta_3-1}Z_3\left(r,\varphi(r),\omega(r),\mu\left(\zeta\right),D^{\rho-1}\mu\left(\zeta\right),D^{\rho}\mu\left(\zeta\right)\right)dr\end{array}\right)\zeta^{\gamma_3+2}.
\end{aligned}
$$

Hence, the FP of the operator $\mho$ is equivalent to the solution of the system (1.1).

Next, we need the following condition:

(A1) The function $Z_j : [0,1]\times\mathbb{R}^5 \to \mathbb{R}$ are continuous and there exists a matrix of positive functions M_{lj}, $l = 1,2,3,4,5$, $j = 1,2,3$ such that

$$
\begin{aligned}
&\left|Z_j\left(\zeta,\varphi,\omega,\mu,\delta,\varrho\right)-Z_j\left(\zeta,\widetilde{\varphi},\widetilde{\omega},\widetilde{\mu},\widetilde{\delta},\widetilde{\varrho}\right)\right|\\
\leq\;& M_{1j}(\zeta)\left|\varphi-\widetilde{\varphi}\right|+M_{2j}(\zeta)\left|\omega-\widetilde{\omega}\right|+M_{3j}(\zeta)\left|\mu-\widetilde{\mu}\right|+M_{4j}(\zeta)\left|\delta-\widetilde{\delta}\right|+M_{5j}(\zeta)\left|\varrho-\widetilde{\varrho}\right|,
\end{aligned}
$$

with $m_{lj}=\max\left\{M_{lj}\right\}$, for all $\zeta\in V$ and $(\varphi,\omega,\mu,\delta,\varrho),\left(\widetilde{\varphi},\widetilde{\omega},\widetilde{\mu},\widetilde{\delta},\widetilde{\varrho}\right)\in\mathbb{R}^5$.

We now present the theorem concerning the existence and uniqueness of the solution.

Theorem 3.1. *Assume that condition (A_1) holds. Then, the system (1.1) under assumptions (2.1) admits a unique solution, provided that $\Omega_i<1$ where*

$$
\Omega_j=B_j\max\left\{\begin{array}{c}\left(m_{1j}+m_{4j}m_{1j}+m_{5j}m_{1j}\right),\\ \left(m_{2j}+m_{4j}m_{2j}+m_{5j}m_{2j}\right),\\ \left(m_{3j}+m_{4j}m_{3j}+m_{5j}m_{3j}\right)\end{array}\right\},
$$

for $j=1,2,3$ and

$$
\begin{aligned}
B_j \;=\;& \eth_j\left\{\left(\frac{\gamma_j+2\Gamma\left(\gamma_j+3\right)}{\gamma_j\Gamma\left(\eta_j+\gamma_j\right)}+\frac{\gamma_j+4}{\gamma_j\Gamma\left(\eta_j\right)}\right)+\frac{1}{\Gamma\left(\eta_j+\gamma_j-\rho+2\right)}\right.\\
&+\frac{\Gamma\left(\gamma_j+2\right)}{\gamma_j\Gamma\left(\eta_j+1\right)\Gamma\left(\gamma_j-\rho+4\right)}\left[2\left(\gamma_j-\rho+3\right)+\left(\gamma_j+2\right)^2\right]\\
&+\frac{\Gamma\left(\gamma_j+2\right)\Gamma\left(\gamma_j+3\right)}{\gamma_j\Gamma\left(\gamma_j+\eta_j+1\right)\Gamma\left(\gamma_j-\rho+4\right)}\left[2\gamma_j-\rho+5\right]+\frac{1}{\Gamma\left(\eta_j+\gamma_j-\rho+1\right)}\\
&+\frac{\Gamma\left(\gamma_j+2\right)}{\gamma_j\Gamma\left(\eta_j+1\right)\Gamma\left(\gamma_j-\rho+3\right)}\left[2\left(\gamma_j-\rho+2\right)+\left(\gamma_j+2\right)^2\right]\\
&\left.+\frac{\Gamma\left(\gamma_j+2\right)\Gamma\left(\gamma_j+3\right)}{\gamma_j\Gamma\left(\gamma_j+\eta_j+1\right)\Gamma\left(\gamma_j-\rho+3\right)}\left[2\gamma_j-\rho+4\right]\right\}.
\end{aligned}
$$

Proof. It is enough to prove that the mapping $\mho$ is a Banach contraction principle to finish the proof.

Assume that $(\varphi_1, \omega_1, \mu_1)$ and $(\varphi_2, \omega_2, \mu_2)$ be arbitrary elements in Ξ^3. Then for each $\zeta \in [0,1]$ and $j = 1,2,3$, we have

$$
\begin{aligned}
& \left|\mho_1\left(\varphi_1(\zeta), \omega_1(\zeta), \omega_1(\zeta), \mu_1(\zeta)\right) - \mho_1\left(\varphi_2(\zeta), \omega_2(\zeta), \mu_2(\zeta)\right)\right| \\
\leq\; & \frac{\eth_1}{\Gamma(\eta_1+\gamma_1)} \left| \int_0^\zeta (\zeta - r)^{\eta_1+\gamma_1-1} Z_1\left(r, \varphi_1(r), \omega_1(r), \mu_1(\zeta), D^{\rho-1}\varphi_1(r), D^\rho\varphi_1(r)\right)\right. \\
& \left. - Z_1\left(r, \varphi_2(r), \omega_2(r), \mu_2(\zeta), D^{\rho-1}\varphi_2(r), D^\rho\varphi_2(r)\right)\right| dr \\
& + \frac{2\eth_1}{\gamma_1\Gamma(\eta_1)} \left| \int_0^1 (\zeta - r)^{\eta_1-1} Z_1\left(r, \varphi_1(r), \omega_1(r), \mu_1(\zeta), D^{\rho-1}\varphi_1(r), D^\rho\varphi_1(r)\right)\right. \\
& \left. - Z_1\left(r, \varphi_2(r), \omega_2(r), \mu_2(\zeta), D^{\rho-1}\varphi_2(r), D^\rho\varphi_2(r)\right)\right| dr \\
& + \frac{\eth_1\Gamma(\gamma_1+3)}{\gamma_1\Gamma(\eta_1+\gamma_1)} \left| \int_0^1 (\zeta - r)^{\eta_1+\gamma_1-1} Z_1\left(r, \varphi_1(r), \omega_1(r), \mu_1(\zeta), D^{\rho-1}\varphi_1(r), D^\rho\varphi_1(r)\right)\right. \\
& \left. - Z_1\left(r, \varphi_2(r), \omega_2(r), \mu_2(\zeta), D^{\rho-1}\varphi_2(r), D^\rho\varphi_2(r)\right)\right| dr \\
& + \frac{\eth_1(\gamma_1+2)}{\gamma_1\Gamma(\eta_1)} \left| \int_0^1 (\zeta - r)^{\eta_1-1} Z_1\left(r, \varphi_1(r), \omega_1(r), \mu_1(\zeta), D^{\rho-1}\varphi_1(r), D^\rho\varphi_1(r)\right)\right. \\
& \left. - Z_1\left(r, \varphi_2(r), \omega_2(r), \mu_2(\zeta), D^{\rho-1}\varphi_2(r), D^\rho\varphi_2(r)\right)\right| dr \\
\leq\; & \frac{\eth_1\gamma_1 + 2\eth_1\Gamma(\gamma_1+3)}{\gamma_1\Gamma(\eta_1+\gamma_1)} \left| \int_0^1 (\zeta - r)^{\eta_1+\gamma_1-1} Z_1\left(r, \varphi_1(r), \omega_1(r), \mu_1(\zeta), D^{\rho-1}\varphi_1(r), D^\rho\varphi_1(r)\right)\right. \\
& \left. - Z_1\left(r, \varphi_2(r), \omega_2(r), \mu_2(\zeta), D^{\rho-1}\varphi_2(r), D^\rho\varphi_2(r)\right)\right| dr \\
& + \frac{\eth_1(\gamma_1+4)}{\gamma_1\Gamma(\eta_1)} \left| \int_0^1 (\zeta - r)^{\eta_1-1} Z_1\left(r, \varphi_1(r), \omega_1(r), \mu_1(\zeta), D^{\rho-1}\varphi_1(r), D^\rho\varphi_1(r)\right)\right. \\
& \left. - Z_1\left(r, \varphi_2(r), \omega_2(r), \mu_2(\zeta), D^{\rho-1}\varphi_2(r), D^\rho\varphi_2(r)\right)\right| dr.
\end{aligned}
$$

From (A_1), we have

$$
\begin{aligned}
& \left\|\mho_1\left(\varphi_1, \omega_1, \mu_1\right) - \mho_j\left(\varphi_2, \omega_2, \mu_2\right)\right\|_\infty \\
\leq\; & \left(\frac{\eth_1\gamma_1 + 2\eth_1\Gamma(\gamma_1+3)}{\gamma_1\Gamma(\eta_1+\gamma_1)} + \frac{\eth_1(\gamma_1+4)}{\gamma_1\Gamma(\eta_1)}\right)\Big[M_{1j}\left\|\varphi_1 - \varphi_2\right\|_\infty + M_{2j}\left\|\omega_1 - \omega_2\right\|_\infty \\
& + M_{31}\left\|\mu_1 - \mu_2\right\|_\infty + M_{41}\left\|D^{\rho-1}\varphi_1 - D^{\rho-1}\varphi_2\right\|_\infty + M_{51}\left\|D^\rho\varphi_1 - D^\rho\varphi_2\right\|_\infty\Big].
\end{aligned} \tag{3.1}
$$

Using the Caputo fractional derivative, we have

$$
\begin{aligned}
& D^{\rho-1}\left(\mho_1\left(\varphi(\zeta), \omega(\zeta), \mu(\zeta)\right)\right) \\
=\; & \eth_1 I^{\eta_1+\gamma_1-\rho+1} Z_1\left(\zeta, \varphi(\zeta), \omega(\zeta), \mu(\zeta), D^{\rho-1}\varphi(\zeta), D^\rho\varphi(\zeta)\right) \\
& + \frac{\eth_1(\gamma_1+2)}{\Gamma(\gamma_1-\rho+3)} \zeta^{\gamma_1-\rho+2}\left(\frac{2}{\gamma_1\Gamma(\eta_1)} \int_0^1 (1-r)^{\eta_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu(\zeta), D^{\rho-1}\varphi(r), D^\rho\varphi(\zeta)\right) dr\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Gamma\left(\gamma_1+3\right)}{\gamma_1 \Gamma\left(\eta_1+\gamma_1\right)} \int_0^1(1-r)^{\eta_1+\gamma_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu(\zeta), D^{\rho-1} \varphi(r), D^\rho \varphi(\zeta)\right) dr\Bigg) \\
&+\frac{\eth_1\left(\gamma_1+3\right)}{\Gamma\left(\gamma_1-\rho+4\right)} \zeta^{\gamma_1-\rho+3} \\
&\times\Bigg(\frac{\Gamma\left(\gamma_1+3\right)}{\gamma_1 \Gamma\left(\eta_1+\gamma_1\right)} \int_0^1(1-r)^{\eta_1+\gamma_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu(\zeta), D^{\rho-1} \varphi(r), D^\rho \varphi(\zeta)\right) dr \\
&-\frac{\gamma_1+2}{\gamma_1 \Gamma\left(\eta_1\right)} \int_0^1(1-r)^{\eta_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu(\zeta), D^{\rho-1} \varphi(r), D^\rho \varphi(\zeta)\right) dr\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
& D^\rho\left(\mho_1(\varphi(\zeta), \omega(\zeta), \mu(\zeta))\right) \\
= \; & \eth_1 I^{\eta_1+\gamma_1-\rho} Z_1\left(\zeta, \varphi(\zeta), \omega(\zeta), \mu(\zeta), D^{\rho-1} \varphi(\zeta), D^\rho \omega(\zeta)\right) \\
&+\frac{\eth_1\left(\gamma_1+2\right)}{\Gamma\left(\gamma_1-\rho+2\right)} \zeta^{\gamma_1-\rho+1}\Bigg(\frac{2}{\gamma_1 \Gamma\left(\eta_1\right)} \int_0^1(1-r)^{\eta_1-1} Z_j\left(r, \varphi(r), \omega(r), \mu(\zeta), D^{\rho-1} \varphi(r), D^\rho \varphi(r)\right) dr \\
&-\frac{\Gamma\left(\gamma_1+3\right)}{\gamma_1 \Gamma\left(\eta_1+\gamma_1\right)} \int_0^1(1-r)^{\eta_1+\gamma_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu(\zeta), D^{\rho-1} \varphi(r), D^\rho \varphi(r)\right) dr\Bigg) \\
&+\frac{\eth_1\left(\gamma_1+3\right)}{\Gamma\left(\gamma_1-\rho+4\right)} \zeta^{\gamma_1-\rho+2} \\
&\times\Bigg(\frac{\Gamma\left(\gamma_1+3\right)}{\gamma_1 \Gamma\left(\eta_1+\gamma_1\right)} \int_0^1(1-r)^{\eta_1+\gamma_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu(\zeta), D^{\rho-1} \varphi(r), D^\rho \varphi(r)\right) dr \\
&-\frac{\gamma_1+2}{\gamma_1 \Gamma\left(\eta_1\right)} \int_0^1(1-r)^{\eta_1-1} Z_1\left(r, \varphi(r), \omega(r), \mu(\zeta), D^{\rho-1} \varphi(r), D^\rho \varphi(r)\right) dr\Bigg).
\end{aligned}
$$

By using the hypothesis (A_1), and similar to (3.1), we get

$$
\begin{aligned}
& \left\|D^{\rho-1} \mho_j\left(\varphi_1, \omega_1, \mu_1\right)-D^{\rho-1} \mho_j\left(\varphi_2, \omega_2, \mu_2\right)\right\|_{\infty} \\
\leq \; & \left(\frac{\eth_1}{\Gamma\left(\eta_1+\gamma_1-\rho+2\right)}+\frac{\eth_1 \Gamma\left(\gamma_1+2\right)}{\gamma_1 \Gamma\left(\eta_1+1\right) \Gamma\left(\gamma_1-\rho+4\right)}\left[2\left(\gamma_1-\rho+3\right)+\left(\gamma_1+2\right)^2\right]\right. \\
&\left.+\frac{\eth_1 \Gamma\left(\gamma_1+2\right) \Gamma\left(\gamma_1+3\right)}{\gamma_1 \Gamma\left(\gamma_1+\eta_1+1\right) \Gamma\left(\gamma_1-\rho+4\right)}\left[2 \gamma_1-\rho+5\right]\right) \\
&\times\Big(M_{11}\left\|\varphi_1-\varphi_2\right\|_{\infty}+M_{21}\left\|\omega_1-\omega_2\right\|_{\infty}+M_{31}\left\|\omega_1-\omega_2\right\|_{\infty} \\
&+M_{41}\left\|D^{\rho-1} \varphi_1-D^{\rho-1} \varphi_2\right\|_{\infty}+M_{51}\left\|D^\rho \varphi_1-D^\rho \varphi_2\right\|_{\infty}\Big) .
\end{aligned} \tag{3.2}
$$

Replacing $\rho-1$ with ρ in (3.2) except in the term within the large parentheses, one has

$$
\begin{aligned}
& \left\|D^\rho \mho_1\left(\varphi_1, \omega_1, \mu_1\right)-D^\rho \mho_1\left(\varphi_2, \omega_2, \mu_2\right)\right\|_{\infty} \\
\leq \; & \left(\frac{\eth_1}{\Gamma\left(\eta_1+\gamma_1-\rho+1\right)}+\frac{\eth_1 \Gamma\left(\gamma_1+2\right)}{\gamma_1 \Gamma\left(\eta_1+1\right) \Gamma\left(\gamma_1-\rho+3\right)}\left[2\left(\gamma_1-\rho+2\right)+\left(\gamma_1+2\right)^2\right]\right. \\
&\left.+\frac{\eth_1 \Gamma\left(\gamma_1+2\right) \Gamma\left(\gamma_1+3\right)}{\gamma_1 \Gamma\left(\gamma_1+\eta_1+1\right) \Gamma\left(\gamma_1-\rho+3\right)}\left[2 \gamma_1-\rho+4\right]\right) \\
&\times\Big(M_{11}\left\|\varphi_1-\varphi_2\right\|_{\infty}+M_{21}\left\|\omega_1-\omega_2\right\|_{\infty}+M_{31}\left\|\mu_1-\mu_2\right\|_{\infty}
\end{aligned}
$$

$$M_{41}\left\|D^{\rho-1}\varphi_1-D^{\rho-1}\varphi_2\right\|_\infty+M_{51}\left\|D^{\rho}\varphi_1-D^{\rho}\varphi_2\right\|_\infty\Big). \tag{3.3}$$

It follows from (3.1)-(3.3) and the definition of the norm on Ξ that

$$\begin{aligned}
&\left\|\mho_1\left(\varphi_1,\omega_1,\mu_1\right)-\mho_2\left(\varphi_2,\omega_2,\mu_2\right)\right\|_\infty\\
\leq\ & \eth_1\left\{\left(\frac{\eth_1\gamma_1+2\eth_1\Gamma\left(\gamma_1+3\right)}{\gamma_1\Gamma\left(\eta_1+\gamma_1\right)}+\frac{\eth_1\left(\gamma_1+4\right)}{\gamma_1\Gamma\left(\eta_1\right)}\right)+\frac{1}{\Gamma\left(\eta_1+\gamma_1-\rho+2\right)}\right.\\
&+\frac{\Gamma\left(\gamma_1+2\right)}{\gamma_1\Gamma\left(\eta_1+1\right)\Gamma\left(\gamma_1-\rho+4\right)}\left[2\left(\gamma_1-\rho+3\right)+\left(\gamma_1+2\right)^2\right]\\
&+\frac{\Gamma\left(\gamma_1+2\right)\Gamma\left(\gamma_1+3\right)}{\gamma_1\Gamma\left(\gamma_1+\eta_1+1\right)\Gamma\left(\gamma_1-\rho+4\right)}\left[2\gamma_1-\rho+5\right]+\frac{1}{\Gamma\left(\eta_1+\gamma_1-\rho+1\right)}\\
&+\frac{\Gamma\left(\gamma_1+2\right)}{\gamma_1\Gamma\left(\eta_1+1\right)\Gamma\left(\gamma_1-\rho+3\right)}\left[2\left(\gamma_1-\rho+2\right)+\left(\gamma_1+2\right)^2\right]\\
&\left.+\frac{\Gamma\left(\gamma_1+2\right)\Gamma\left(\gamma_1+3\right)}{\gamma_1\Gamma\left(\gamma_1+\eta_1+1\right)\Gamma\left(\gamma_1-\rho+3\right)}\left[2\gamma_1-\rho+4\right]\right\}\\
&\times\Big[\left(M_{11}+M_{41}M_{11}+M_{51}M_{11}\right)\left\|\varphi_1-\varphi_2\right\|_\infty+\left(M_{21}+M_{41}M_{21}+M_{51}M_{21}\right)\left\|\omega_1-\omega_2\right\|_\infty\\
&+\left(M_{31}+M_{41}M_{31}+M_{51}M_{31}\right)\left\|\mu_1-\mu_2\right\|_\infty\\
\leq\ & B_1\Big\{\left(M_{11}+M_{41}M_{11}+M_{51}M_{11}\right)\left\|\varphi_1-\varphi_2\right\|_\infty\\
&+\left(M_{21}+M_{41}M_{21}+M_{51}M_{21}\right)\left\|\omega_1-\omega_2\right\|_\infty\\
&+\left(M_{31}+M_{41}M_{31}+M_{51}M_{31}\right)\left\|\mu_1-\mu_2\right\|_\infty\Big\}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
&\left\|\mho_2\left(\varphi_1,\omega_1,\mu_1\right)-\mho_2\left(\varphi_2,\omega_2,\mu_2\right)\right\|_\infty\\
\leq\ & B_2\Big\{\left(M_{12}+M_{42}M_{12}+M_{52}M_{12}\right)\left\|\varphi_1-\varphi_2\right\|_\infty\\
&+\left(M_{22}+M_{42}M_{22}+M_{52}M_{22}\right)\left\|\omega_1-\omega_2\right\|_\infty\\
&+\left(M_{32}+M_{42}M_{32}+M_{52}M_{32}\right)\left\|\mu_1-\mu_2\right\|_\infty\Big\}.
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\mho_3\left(\varphi_1,\omega_1,\mu_1\right)-\mho_3\left(\varphi_2,\omega_2,\mu_2\right)\right\|_\infty\\
\leq\ & B_3\Big\{\left(M_{13}+M_{43}M_{13}+M_{53}M_{13}\right)\left\|\varphi_1-\varphi_2\right\|_\infty\\
&+\left(M_{23}+M_{43}M_{23}+M_{53}M_{23}\right)\left\|\omega_1-\omega_2\right\|_\infty\\
&+\left(M_{33}+M_{43}M_{33}+M_{53}M_{33}\right)\left\|\mu_1-\mu_2\right\|_\infty\Big\}.
\end{aligned}$$

Generaly, we can write

$$\begin{aligned}
&\left\|\mho_j\left(\varphi_1,\omega_1,\mu_1\right)-\mho_j\left(\varphi_2,\omega_2,\mu_2\right)\right\|_\infty\\
\leq\ & B_j\Big\{\left(M_{1j}+M_{4j}M_{1j}+M_{5j}M_{1j}\right)\left\|\varphi_1-\varphi_2\right\|_\infty\\
&+\left(M_{2j}+M_{4j}M_{2j}+M_{5j}M_{2j}\right)\left\|\omega_1-\omega_2\right\|_\infty
\end{aligned}$$

$$+\left(M_{3j}+M_{4j}M_{3j}+M_{5j}M_{3j}\right)\left\|\mu_1-\mu_2\right\|_\infty\Big\}.$$

By the definition of the norm on Ξ^3 and Ω_1, Ω_2, and Ω_3, we have

$$\left\|\mho\left(\varphi_1,\omega_1,\mu_1\right)-\mho\left(\varphi_2,\omega_2,\mu_2\right)\right\|_{\Xi^3}\le\Omega_j\left\|(\varphi_1,\omega_1,\mu_1)-(\varphi_2,\omega_2,\mu_2)\right\|.$$

Since $\Omega_j<1$, it follows that $\mho$ is a contraction mapping. Consequently, $\mho$ admits a unique fixed point, which corresponds to the unique solution of system (1.1). □

4. Supportive example

In this section, we present an example to support Theorem 3.1

Example 4.1. *Assume that the following system:*

$$\begin{cases} D^{\eta_1}D^{\gamma_1}\varphi_1(\zeta)=\eth_1 Z_1\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),\varphi_3(\zeta),D^{\rho-1}\varphi_1(\zeta),D^{\rho}\varphi_1(\zeta)\right),\\ D^{\eta_2}D^{\gamma_2}\varphi_2(\zeta)=\eth_2 Z_2\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),\varphi_3(\zeta)D^{\rho-1}\varphi_2(\zeta),D^{\rho}\varphi_2(\zeta)\right),\\ D^{\eta_3}D^{\gamma_3}\varphi_3(\zeta)=\eth_3 Z_3\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),\varphi_3(\zeta)D^{\rho-1}\varphi_3(\zeta),D^{\rho}\varphi_3(\zeta)\right),\\ \varphi_1(0)=\varphi_1(1)=\varphi_2(0)=\varphi_2(1)=\varphi_3(0)=\varphi_3(1)=0,\\ D^{\gamma_1}\varphi_1(0)=D^{\gamma_1}\varphi_1(1)=D^{\gamma_2}\varphi_2(0)=D^{\gamma_2}\varphi_2(1)=D^{\gamma_3}\varphi_3(0)=D^{\gamma_3}\varphi_3(1)=0. \end{cases}\tag{4.1}$$

We set

$$\eta_1=\frac{11}{4},\ \gamma_1=\frac{5}{6},\ \eta_2=\frac{7}{3},\ \gamma_2=\frac{2}{3},\ \eta_3=\frac{5}{2},\ \gamma_3=\frac{1}{4},\ \eth_1=\eth_2=\eth_3=1,\ \textit{and}\ \rho=\frac{3}{2}.$$

Moreover,

$$\begin{aligned} &Z_1\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),\varphi_3(\zeta),D^{\rho-1}\varphi_1(\zeta),D^{\rho}\varphi_1(\zeta)\right)\\ =\ &\frac{1}{200e^{\zeta}}\varphi_1(\zeta)+\left(\frac{\sin(1+\zeta)}{6(5+\zeta)}\right)\varphi_2(\zeta)+\left(\frac{\tan(\zeta)}{8(6+\zeta^2)}\right)\varphi_3(\zeta)\\ &+\left(\frac{\ln(1+\zeta)}{30(\zeta+10)}\right)D^{\frac{1}{2}}\varphi_1(\zeta)+\frac{1}{300e^{\zeta}}D^{\frac{3}{2}}\varphi_1(\zeta), \end{aligned}$$

$$\begin{aligned} &Z_2\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),\varphi_3(\zeta),D^{\rho-1}\varphi_2(\zeta),D^{\rho}\varphi_2(\zeta)\right)\\ =\ &\frac{\ln(1+\zeta)}{60}\varphi_1(\zeta)+\left(\frac{\tan(1+\zeta)}{400(1+e^{\zeta})}\right)\varphi_2(\zeta)+\left(\frac{\cos(\zeta)}{50(2+\zeta)}\right)\varphi_3(\zeta)\\ &+\left(\frac{\sin(\zeta)}{4(4+\zeta)}\right)D^{\frac{1}{2}}\varphi_2(\zeta)+\frac{1}{150e^{4\zeta}}D^{\frac{3}{2}}\varphi_2(\zeta), \end{aligned}$$

and

$$\begin{aligned} &Z_3\left(\zeta,\varphi_1(\zeta),\varphi_2(\zeta),\varphi_3(\zeta),D^{\rho-1}\varphi_2(\zeta),D^{\rho}\varphi_2(\zeta)\right)\\ =\ &\frac{\zeta}{120}\varphi_1(\zeta)+\left(\frac{e^{-\zeta}}{130(1+e^{\zeta})}\right)\varphi_2(\zeta)+\left(\frac{\tan^{-1}(\zeta)}{70(2+\zeta)}\right)\varphi_3(\zeta)\\ &+\left(\frac{\cos(\zeta)}{18(1+\zeta^3)}\right)D^{\frac{1}{2}}\varphi_3(\zeta)+\frac{1}{210\sqrt{\zeta}}D^{\frac{3}{2}}\varphi_3(\zeta), \end{aligned}$$

Clearly, for all $\zeta \in [0,1]$ *and* $\varphi, \omega, \mu, \delta, \varrho, \widetilde{\varphi}, \widetilde{\omega}, \widetilde{\mu}, \widetilde{\delta}, \widetilde{\varrho} \in \mathbb{R}$, *we have*

$$\begin{aligned} & \left| Z_1(\zeta, \varphi, \omega, \mu, \delta, \varrho) - Z_1\left(\zeta, \widetilde{\varphi}, \widetilde{\omega}, \widetilde{\mu}, \widetilde{\delta}, \widetilde{\varrho}\right) \right| \\ \leq\ & \frac{1}{200 e^{\zeta}} \left|\varphi - \widetilde{\varphi}\right| + \frac{e^{\zeta}}{6(5+\zeta)} \left|\omega - \widetilde{\omega}\right| + \frac{1}{8(6+\zeta^2)} \left|\mu - \widetilde{\mu}\right| \\ & + \frac{\zeta}{30(\zeta+10)} \left|\delta - \widetilde{\delta}\right| + \frac{1}{300 e^{\zeta}} \left|\varrho - \widetilde{\varrho}\right|. \end{aligned}$$

Thus, $M_{11}(\zeta) = \frac{1}{200e^{\zeta}}$, $M_{21}(\zeta) = \frac{e^{\zeta}}{6(5+\zeta)}$, $M_{31}(\zeta) = \frac{1}{8(6+\zeta^2)}$, $M_{41}(\zeta) = \frac{\zeta}{30(\zeta+10)}$, *and* $M_{51}(\zeta) = \frac{1}{300e^{\zeta}}$.

Also,

$$\begin{aligned} & \left| Z_2(\zeta, \varphi, \omega, \mu, \delta, \varrho) - Z_2\left(\zeta, \widetilde{\varphi}, \widetilde{\omega}, \widetilde{\mu}, \widetilde{\delta}, \widetilde{\varrho}\right) \right| \\ \leq\ & \frac{\zeta}{60} \left|\varphi - \widetilde{\varphi}\right| + \frac{\zeta}{400(1+e^{\zeta})} \left|\omega - \widetilde{\omega}\right| + \frac{\zeta}{50(2+\zeta)} \left|\mu - \widetilde{\mu}\right| \\ & + \frac{\zeta}{4(4+\zeta)} \left|\delta - \widetilde{\delta}\right| + \frac{1}{150 e^{4\zeta}} \left|\varrho - \widetilde{\varrho}\right|. \end{aligned}$$

Thus, $M_{12}(\zeta) = \frac{\zeta}{60}$, $M_{22}(\zeta) = \frac{\zeta}{400(1+e^{\zeta})}$, $M_{32}(\zeta) = \frac{\zeta}{50(2+\zeta)}$, $M_{42}(\zeta) = \frac{\zeta}{4(4+\zeta)}$, *and* $M_{52}(\zeta) = \frac{1}{150e^{4\zeta}}$.

Furthermore,

$$\begin{aligned} & \left| Z_3(\zeta, \varphi, \omega, \mu, \delta, \varrho) - Z_3\left(\zeta, \widetilde{\varphi}, \widetilde{\omega}, \widetilde{\mu}, \widetilde{\delta}, \widetilde{\varrho}\right) \right| \\ \leq\ & \frac{\zeta}{120} \left|\varphi - \widetilde{\varphi}\right| + \frac{1}{130(1+e^{\zeta})} \left|\omega - \widetilde{\omega}\right| + \frac{\zeta}{70(2+\zeta)} \left|\mu - \widetilde{\mu}\right| \\ & + \frac{\zeta}{18(1+\zeta^3)} \left|\delta - \widetilde{\delta}\right| + \frac{1}{210\sqrt{\zeta}} \left|\varrho - \widetilde{\varrho}\right|. \end{aligned}$$

Thus, $M_{13}(\zeta) = \frac{\zeta}{120}$, $M_{23}(\zeta) = \frac{1}{130(1+e^{\zeta})}$, $M_{33}(\zeta) = \frac{\zeta}{70(2+\zeta)}$, $M_{43}(\zeta) = \frac{\zeta}{18(1+\zeta^3)}$, *and* $M_{53}(\zeta) = \frac{1}{210\sqrt{\zeta}}$.

Since $m_{lj} = \sup_{\zeta\in[0,1]} \left\{M_{lj}\right\}$, *we have*

$$\begin{cases} m_{11} = \frac{1}{200},\ m_{21} = \frac{e}{36},\ m_{31} = \frac{1}{56},\ m_{41} = \frac{1}{330},\ m_{51} = \frac{1}{300e}, \\ m_{12} = \frac{1}{60},\ m_{22} = \frac{1}{400},\ m_{32} = \frac{1}{100},\ m_{42} = \frac{1}{16},\ m_{52} = \frac{1}{150}, \\ m_{13} = \frac{1}{120},\ m_{23} = \frac{1}{130},\ m_{33} = \frac{1}{140},\ m_{43} = \frac{1}{18},\ m_{53} = \frac{1}{210}. \end{cases}$$

By simple calculations, and the definition of B_j $(j = 1,2,3)$, *we can write* $B_1 \approx 18.752$, $B_2 \approx 29.5855$, *and* $B_3 \approx 27.475$.

Additionally,

$$\begin{aligned} \Omega_1 &\approx 18.752 \max \left\{ \begin{array}{c} m_{11} + m_{41}m_{11} + m_{51}m_{11}, \\ (m_{21} + m_{41}m_{21} + m_{51}m_{21}), \\ (m_{31} + m_{41}m_{31} + m_{51}m_{31}) \end{array} \right\} \\ &\approx 18.752 \max\{0.05032, 0.01712, 0.01797\} \approx 0.94360 < 1, \end{aligned}$$

$$\Omega_2 \approx 29.5855 \max \left\{ \begin{array}{c} (m_{12} + m_{42}m_{12} + m_{52}m_{12}), \\ (m_{22} + m_{42}m_{22} + m_{52}m_{22}), \\ (m_{32} + m_{42}m_{32} + m_{52}m_{32}) \end{array} \right\}$$

$$\approx \quad 29.5855 \max\{0.03133, 0.02370, 0.01884\} \approx 0.92691 < 1,$$

and

$$\begin{aligned}\Omega_3 &\approx 27.475 \max\left\{\begin{array}{c}(m_{13} + m_{43}m_{13} + m_{53}m_{13}),\\ (m_{23} + m_{43}m_{23} + m_{53}m_{23}),\\ (m_{33} + m_{43}m_{33} + m_{53}m_{33})\end{array}\right\}\\ &\approx 27.475 \max\{0.03243, 0.03397, 0.02173\} \approx 0.93333 < 1.\end{aligned}$$

Hence, $\Omega_j < 1$, $j = 1, 2, 3$. *Therefore, all requirements of Theorem 3.1 are fulfilled. Then, the problem (4.1) has at least one solution.*

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