

Continued Fractions of Quadratic Laurent Series over $\mathbb{F}_q$: Non-Archimedean Error Equalities, the Quasi-Period, and the Polynomial Pell Equation

Wannatida Yonwilad*

*Department of Learning Management Innovation, Faculty of Education and Educational Innovation,
Kalasin University, Kalasin, Thailand*

*Corresponding author: wantida.yo@ksu.ac.th

Abstract. We transplant to $\mathbb{F}_q[x]$ the augmented-matrix elementary row operation (ERO) framework for continued fractions of rational numbers, and extend it there from the terminating case to quadratic irrational Laurent series in $\mathbb{F}_q((1/x))$, whose expansion is infinite. The state of the algorithm is the vector $v_k = (\sqrt{D} + m_k, Q_k)^T$, one row operation followed by a row swap sends v_k to $-\overline{\alpha_{k+1}} v_{k+1}$, and we show that the *accumulated* scalar of the process is exactly $(p_k + q_k \sqrt{D})^{-1}$: the machine therefore returns not only the convergents but the associated algebraic integers of $\mathcal{O} = \mathbb{F}_q[x][\sqrt{D}]$. Reading the construction at the infinite place gives the exact non-archimedean error equality $|\alpha - p_k/q_k| = q^{-(\deg q_k + \deg q_{k+1})}$ for the infinite expansion, whence every convergent is a strict best rational approximation for the degree valuation. From the determinant invariant we obtain a norm form, and we recover within the framework the classical facts that the expansion of $\sqrt{D}$ is periodic from the first partial quotient onwards with palindromic period, and that the fundamental solution of the polynomial Pell–Abel equation $P^2 - DQ^2 \in \mathbb{F}_q^\times$ is the convergent at the end of the *quasi-period* n — not of the period ℓ , from which n differs for a substantial proportion of D in every range we have computed — the regulator being $R = d + \deg q_{n-1}$, i.e. minus the degree of the accumulated scalar. Periodicity and the distinction between period and quasi-period, unlike the error equality, are genuinely finite-field phenomena. What the construction offers is uniformity of derivation rather than new theorems: all of this is classical and is recovered rather than claimed, including the determination, for $\deg D = 2$, of exactly how often the period exceeds the quasi-period, the proportion being $(q-2)/(q-1)$.

1. INTRODUCTION

The convergents p_k/q_k of a continued fraction furnish the best rational approximations to its value [5, 6, 14], a fact whose function-field incarnation has been studied extensively [4, 7, 11, 13]. For a rational number a/b with $a, b \in \mathbb{Z}$, an augmented-matrix framework realises the Euclidean algorithm as elementary row operations on $[I \mid v]$, $v = (a, b)^T$, so that the convergent pairs and

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the Bézout data are read off the accumulated matrix through a determinant invariant [20]; over $\mathbb{Z}$ the error is governed by the classical two-sided bounds $\frac{1}{B_k(B_k+B_{k+1})} \leq |a/b - A_k/B_k| \leq \frac{1}{B_k B_{k+1}}$, where A_k/B_k denotes the k -th convergent [5, 6]. The present paper transplants the construction from $\mathbb{Z}$ to $\mathbb{F}_q[x]$, where two things change. The absolute value at the infinite place is non-archimedean, so the two-sided interval collapses to an *exact* error equality; and, once α is a quadratic irrational rather than a rational function, the process no longer terminates. The Euclidean algorithm in $\mathbb{F}_q[x]$ is of course also the engine of algebraic decoding: that it solves the Reed–Solomon key equation is classical [18, 19].

The rational case produces a *finite* expansion. In this paper we pass to quadratic irrational Laurent series $\alpha \in \mathbb{F}_q((1/x))$ —for instance $\alpha = \sqrt{D}$ with $D \in \mathbb{F}_q[x]$ a nonsquare of even degree—whose continued fraction is infinite. Three things must be re-examined. First, the error analysis: the telescoping that yields a finite identity in the rational case becomes an infinite series, and we must show that ultrametric domination still collapses it to the exact equality $|\alpha - p_k/q_k| = q^{-(\deg q_k + \deg q_{k+1})}$ (Theorem 4.1). Second, periodicity: we recover, within the ERO framework, the classical theorems of Artin and Abel [1, 2] that α_1 is purely periodic with palindromic period and that the polynomial Pell–Abel equation is solved by a convergent at the end of a quasi-period (Theorems 6.1 and 7.1). Third—and this is the point at which the finite-field setting genuinely departs from the classical one over $\mathbb{Z}$ —the unit group $\mathbb{F}_q^\times$ of the constant field is nontrivial, so the index at which units of $\mathcal{O} = \mathbb{F}_q[x][\sqrt{D}]$ occur is governed by the *quasi-period* n rather than by the period ℓ [15, 17]; one has $n \mid \ell$ and $\ell \in \{n, 2n\}$, and $\ell = 2n$ happens for a substantial proportion of D in every range we have computed (Remark 7.4).

Relation to previous work. Almost everything proved below is classical, and we claim priority for none of it. The periodicity and palindromy of the expansion of $\sqrt{D}$ are due to Artin and Abel [1, 2]; the exact error equality and the best-approximation property it yields are standard in the function-field setting [10, 16]; the product formula for the complete quotients that underlies Proposition 7.1 is elementary and well known [9, Prop. 2.4]; the norm form of Lemma 6.1 and the criterion for an element of $\mathcal{O}$ to be a unit are [8, Lem. 2.8 and Lem. 2.11]; the quasi-period, the symmetries it obeys, the location of the fundamental unit and the regulator formula are all in Stein [17, Thm. 4.7, Cor. 4.6, Thm. 5.2, Cor. 5.3]; and the two counts of Proposition 7.2 are the case $d = 1$ of [3, Thm. 2.3(ii)], obtained there by a different argument, for the larger family of monic nonsquare D and in terms of the period rather than the quasi-period. We note in addition that the proof of Theorem 4.1 uses nothing beyond the recursion (2.2) and the inequality $\deg a_m \geq 1$, so it is valid verbatim for every $\alpha \in K \setminus \mathbb{F}_q(x)$; we state it in the quadratic notation only because that is the setting in which the remaining sections operate.

What the paper offers is the following. (i) A single elementary device — elementary row operations on an augmented matrix — from which all of the above is derived uniformly, transplanting to $\mathbb{F}_q[x]$ the construction used over $\mathbb{Z}$ in [20]. The state is $v_k = (\sqrt{D} + m_k, Q_k)^\top$, one row operation followed by a row swap sends v_k to $-\overline{\alpha_{k+1}} v_{k+1}$ (Lemma 3.1), the convergent pairs are the columns

of the *inverse* $A_k = T_{k+1}^{-1} S$ of the accumulated matrix — not of the accumulated matrix itself, whose rows carry the same data, up to sign, in the order (q_j, p_j) — and the accumulated *scalar* is the inverse of the associated algebraic integer, $\Lambda_k = \lambda_0 \cdots \lambda_k = (p_k + q_k \sqrt{D})^{-1}$ (Proposition 7.1), so that the arithmetic of $\mathcal{O}$ is recorded by the scalar bookkeeping of the row operations and not only by the matrix. (ii) A self-contained derivation, inside that device, of the quasi-palindromic relations (7.1) and hence of the fact that an even quasi-period forces $Q_n = 1$; these follow from [17, Thm. 4.7], where they are stated for the Q_i and the complete quotients, by taking polynomial parts, but we have not seen them recorded in terms of the partial quotients. (iii) A short derivation, Proposition 7.2, of how often the period exceeds the quasi-period in the case $d = 1$: the answer is a proportion $(q - 2)/(q - 1)$, and it comes from an exact equidistribution of Q_1 over $\mathbb{F}_q^\times$ together with the coincidence, in degree 2, of the conditions “squarefree” and “nonsquare”. We claim no priority for the two counts themselves, which are the case $d = 1$ of [3, Thm. 2.3(ii)]. For $d \geq 2$ the relevant statistic is Q_n and it is not equidistributed; the data — recorded for the monic squarefree family and for the condition $\ell = 2n$, so not for the quantities counted in [3] — and the resulting question are in Remark 7.4.

We do not treat the finite place here. That the extended Euclidean algorithm solves the Reed–Solomon key equation is classical [18, 19], and we recall it only to situate the rational case of the construction [20]; no claim about it is made or needed below.

All statements below are proved in full; they were in addition verified by exact symbolic computation in $\mathbb{F}_q[x]$, by exhaustive enumeration of the monic squarefree nonsquare D of degree $2d$ for $q \in \{3, 5, 7, 9, 11, 13, 25, 27\}$ — prime and non-prime alike — with $d \leq 3$ for $q \in \{3, 5\}$, $d \leq 2$ for $q \in \{7, 9, 11\}$ and $d = 1$ for $q \in \{13, 25, 27\}$, together with random samples at $d = 2$ for $q \in \{25, 27\}$ (see Remark 7.4).

2. NOTATION AND PRELIMINARIES

Throughout, $\mathbb{F}_q$ is a finite field of odd characteristic and $K = \mathbb{F}_q((1/x))$ is the completion of $\mathbb{F}_q(x)$ at the infinite place. For $f = \sum_{i \leq n} c_i x^i \in K \setminus \{0\}$ with $c_i \in \mathbb{F}_q$ and $c_n \neq 0$ we set $\deg f = n$, extending the degree on $\mathbb{F}_q[x]$; then K is a complete non-archimedean valued field with

$$|f| = q^{-\deg f} \ (f \neq 0), \quad |0| = 0, \quad |f + g| \leq \max(|f|, |g|) \text{ with equality if } |f| \neq |g|. \quad (*)$$

For $g \in K \setminus \{0\}$, $\lfloor g \rfloor \in \mathbb{F}_q[x]$ denotes its polynomial part.

Let $D \in \mathbb{F}_q[x]$ be squarefree and nonsquare, of even degree $2d$ with square leading coefficient; necessarily $d \geq 1$, since a nonzero constant with square leading coefficient is a square in $\mathbb{F}_q$. (Squarefreeness is used only so that $\mathcal{O} = \mathbb{F}_q[x][\sqrt{D}]$ is integrally closed and R below is the regulator of the field; everything else holds verbatim for any nonsquare D , with R the regulator of the order $\mathcal{O}$.) Writing $D = c x^{2d}(1 + u)$ with c a square in $\mathbb{F}_q^\times$ and $|u| < 1$, the binomial series for $(1 + u)^{1/2}$ converges in K because $\text{char } \mathbb{F}_q \neq 2$; hence $\sqrt{D} \in K$, and $A := \lfloor \sqrt{D} \rfloor$ has $\deg A = d$ and the same leading coefficient as $\sqrt{D}$. Put $\alpha = \sqrt{D}$.

For $\beta \in \mathbb{F}_q(x)(\sqrt{D})$ we write $\bar{\beta}$ for the image of β under the nontrivial automorphism of $\mathbb{F}_q(x)(\sqrt{D})$ over $\mathbb{F}_q(x)$; this automorphism fixes $\mathbb{F}_q(x)$ and sends $\sqrt{D}$ to $-\sqrt{D}$. The complete quotients $\alpha_m = (\sqrt{D} + m_m)/Q_m$ and partial quotients are

$$m_0 = 0, Q_0 = 1, \quad a_m = \left\lfloor \frac{A+m_m}{Q_m} \right\rfloor, \quad m_{m+1} = a_m Q_m - m_m, \quad Q_{m+1} = \frac{D - m_{m+1}^2}{Q_m}. \quad (2.1)$$

The convergents satisfy

$$p_{-1} = 1, p_0 = a_0, q_{-1} = 0, q_0 = 1, \quad p_k = a_k p_{k-1} + p_{k-2}, q_k = a_k q_{k-1} + q_{k-2} \quad (k \geq 1). \quad (2.2)$$

Lemma 2.1 (Exactness of the surd recursion). *In (2.1), $Q_m \neq 0$ and $Q_m \mid D - m_{m+1}^2$, so $Q_{m+1} \in \mathbb{F}_q[x] \setminus \{0\}$ and $Q_m Q_{m+1} = D - m_{m+1}^2$.*

Proof. Since $m_{m+1} \equiv -m_m \pmod{Q_m}$ we have $m_{m+1}^2 \equiv m_m^2 \pmod{Q_m}$; with $Q_{-1} := D$, induction on $Q_{m-1}Q_m = D - m_m^2$ (the case $m = 0$ being $Q_{-1}Q_0 = D = D - m_0^2$) gives $Q_m \mid D - m_m^2$, hence $Q_m \mid D - m_{m+1}^2$. No Q_m vanishes, for $Q_{m-1}Q_m = D - m_m^2 \neq 0$: otherwise $D = m_m^2$ would be a square in $\mathbb{F}_q[x]$. $\square$

Lemma 2.2 (Magnitude of complete quotients). *For every $m \geq 0$, $[\alpha_m] = a_m$ and $\alpha_m - a_m = 1/\alpha_{m+1}$. For every $m \geq 1$,*

$$|\alpha_m| = |a_m| = q^{\deg a_m} > 1; \quad \text{equivalently} \quad \deg \alpha_m = \deg a_m \geq 1.$$

Proof. By Lemma 2.1, $Q_m \neq 0$, so $\deg Q_m \geq 0$. Since $\alpha_m - (A + m_m)/Q_m = (\sqrt{D} - A)/Q_m$ has absolute value $q^{\deg(\sqrt{D}-A)-\deg Q_m} < 1$, because $\deg(\sqrt{D} - A) < 0 \leq \deg Q_m$, the polynomial parts of α_m and of $(A + m_m)/Q_m$ coincide; that is, $[\alpha_m] = a_m$. Next, by (2.1) and Lemma 2.1,

$$\alpha_m - a_m = \frac{\sqrt{D} + m_m - a_m Q_m}{Q_m} = \frac{\sqrt{D} - m_{m+1}}{Q_m} = \frac{Q_{m+1}(\sqrt{D} - m_{m+1})}{Q_m Q_{m+1}} = \frac{Q_{m+1}}{\sqrt{D} + m_{m+1}} = \frac{1}{\alpha_{m+1}}.$$

Since $a_m = [\alpha_m]$ we have $|\alpha_m - a_m| < 1$, hence $|\alpha_{m+1}| > 1$ for every $m \geq 0$; in particular $|\alpha_m| > 1$ for $m \geq 1$. For such m , $\alpha_m = a_m + (\alpha_m - a_m)$ with $|\alpha_m - a_m| < 1 < |\alpha_m|$, so the equality case of (*) forces $|a_m| = |\alpha_m| > 1$, i.e. $\deg a_m = \deg \alpha_m \geq 1$. $\square$

Lemma 2.3 (Cassini–Euler; degree growth). *For all $k \geq 0$, $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}$; for $k \geq 1$ this gives $\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k+1}}{q_k q_{k-1}}$. Moreover $\deg q_0 = 0$ and $\deg q_k = \sum_{i=1}^k \deg a_i$ for $k \geq 1$; hence $(\deg q_k)_{k \geq 0}$ is strictly increasing and $\deg q_k \rightarrow \infty$.*

Proof. The first two claims are inductions on (2.2). For Cassini–Euler the base case $k = 0$ reads $p_0 q_{-1} - p_{-1} q_0 = -1$, and $p_k q_{k-1} - p_{k-1} q_k = -(p_{k-1} q_{k-2} - p_{k-2} q_{k-1})$; the second identity follows on dividing by $q_k q_{k-1}$, which is legitimate for $k \geq 1$ since $q_{-1} = 0$ but $q_j \neq 0$ for $j \geq 0$. For the degree formula, $\deg a_i \geq 1$ for $i \geq 1$ (Lemma 2.2) prevents leading-term cancellation in $q_k = a_k q_{k-1} + q_{k-2}$, so $\deg q_k = \deg a_k + \deg q_{k-1}$. Both arguments use only the ring structure and the degree valuation, not finiteness of $\mathbb{F}_q$. $\square$

3. THE AUGMENTED-MATRIX CONSTRUCTION AND THE DETERMINANT INVARIANT

Write $\mathcal{O} = \mathbb{F}_q[x][\sqrt{D}]$ and let

$$v_k = \begin{pmatrix} \sqrt{D} + m_k \\ Q_k \end{pmatrix} \in \mathcal{O}^2, \quad \text{so that} \quad v_k = Q_k \begin{pmatrix} \alpha_k \\ 1 \end{pmatrix} \quad (k \geq 0), \quad (3.1)$$

be the state of the algorithm after k steps; thus $v_0 = (\sqrt{D}, 1)^\top$. The elementary row operations are carried out by the matrices

$$E_k = \begin{pmatrix} 1 & -a_k \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

over $\mathbb{F}_q[x]$: the type-III operation E_k subtracts a_k times the second row from the first, and S interchanges the two rows. For a rational number a/b the same two matrices act on $v = (a, b)^\top \in \mathbb{Z}^2$ and the process terminates [20]; for $\alpha = \sqrt{D}$ over $\mathbb{F}_q[x]$ it does not, and each step reproduces the next state only up to a scalar factor.

Lemma 3.1 (One elementary row operation step). *For every $k \geq 0$,*

$$SE_k v_k = \lambda_k v_{k+1}, \quad \lambda_k = \frac{\sqrt{D} - m_{k+1}}{Q_{k+1}} = -\overline{\alpha_{k+1}} \in K^\times.$$

Proof. By (2.1), $\sqrt{D} + m_k - a_k Q_k = \sqrt{D} - m_{k+1}$, so $E_k v_k = (\sqrt{D} - m_{k+1}, Q_k)^\top$ and $SE_k v_k = (Q_k, \sqrt{D} - m_{k+1})^\top$. Its second entry is $\lambda_k Q_{k+1}$ by the definition of λ_k , and by Lemma 2.1, $\lambda_k(\sqrt{D} + m_{k+1}) = (D - m_{k+1}^2)/Q_{k+1} = Q_k$, which is its first entry. Finally $\overline{\alpha_{k+1}} = (m_{k+1} - \sqrt{D})/Q_{k+1} = -\lambda_k$, and $\lambda_k \neq 0$ because $\sqrt{D} \notin \mathbb{F}_q(x)$. $\square$

Set $T_0 = I$ and $T_{k+1} = SE_k \cdots SE_0$, so that Lemma 3.1 gives

$$T_{k+1} v_0 = \Lambda_k v_{k+1}, \quad \Lambda_k := \lambda_0 \lambda_1 \cdots \lambda_k : \quad (3.2)$$

the process transports the initial state to the $(k+1)$ -st one, the accumulated scalar being a product of conjugate complete quotients. The convergents are read off the *inverse* of the accumulated matrix; precisely, we set

$$A_k := T_{k+1}^{-1} S. \quad (3.3)$$

Theorem 3.1 (Determinant invariant). *Let A_k be as in (3.3). Then $\det A_k = (-1)^k \in \mathbb{F}_q^\times$, and the columns of A_k are $(p_{k-1}, q_{k-1})^\top$ and $(p_k, q_k)^\top$. In particular $\gcd(p_k, q_k) = 1$.*

Proof. Since $(SE_i)^{-1} = E_i^{-1} S = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$, we have $T_{k+1}^{-1} = (SE_0)^{-1} (SE_1)^{-1} \cdots (SE_k)^{-1} = M_0 M_1 \cdots M_k$, where $M_i = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$; the order of the factors matters, since they do not commute. An induction on (2.2) identifies this product with $\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$: for $k = 0$ this is the initialisation $p_0 = a_0$, $p_{-1} = 1$, $q_0 = 1$, $q_{-1} = 0$, and multiplying $\begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix}$ on the right by $\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$ is exactly the recursion (2.2). Multiplication by S on the right interchanges the two columns, which gives the stated form of A_k . Each factor has determinant -1 , so $\det T_{k+1}^{-1} = (-1)^{k+1}$ and $\det A_k = (-1)^{k+1} \det S = (-1)^k$; equivalently, a type-III operation preserves the determinant while each row swap negates it, and

there is one swap per step. The argument is purely ring-theoretic, hence identical to the rational case. $\square$

4. THE EXACT ERROR EQUALITY FOR THE INFINITE EXPANSION

Lemma 4.1 (Complete-quotient identity). *For every $k \geq 0$, $\alpha = \frac{\alpha_{k+1}p_k + p_{k-1}}{\alpha_{k+1}q_k + q_{k-1}}$. Equivalently, writing $\alpha_{k+1} = (\sqrt{D} + m_{k+1})/Q_{k+1}$, one has the polynomial identities*

$$Dq_k = m_{k+1}p_k + Q_{k+1}p_{k-1}, \quad p_k = m_{k+1}q_k + Q_{k+1}q_{k-1}. \quad (4.1)$$

Proof. The formal identity $[a_0; \dots, a_k, \beta] = (\beta p_k + p_{k-1})/(\beta q_k + q_{k-1})$ holds for an indeterminate β by induction on (2.2); specialising $\beta = \alpha_{k+1}$, which is legitimate by Lemma 2.2, gives the first claim. Substituting $\alpha = \sqrt{D}$, $\alpha_{k+1} = (\sqrt{D} + m_{k+1})/Q_{k+1}$ and separating the parts free of and linear in $\sqrt{D}$ yields (4.1). $\square$

Theorem 4.1 (Exact non-archimedean error equality; cf. [10, 16]). *For every $k \geq 0$,*

$$\left| \alpha - \frac{p_k}{q_k} \right| = q^{-(\deg q_k + \deg q_{k+1})}.$$

In particular $p_k/q_k \rightarrow \alpha$, so the infinite continued fraction represents α , with exact convergence rate. Equivalently, writing

$$\eta_k := \alpha q_k - p_k \quad (k \geq 0), \quad (4.2)$$

one has $|\eta_k| = q^{-\deg q_{k+1}}$, strictly decreasing in k .

Proof. By Lemma 4.1 and Lemma 2.3,

$$\alpha - \frac{p_k}{q_k} = \frac{p_{k-1}q_k - p_k q_{k-1}}{q_k(\alpha_{k+1}q_k + q_{k-1})} = \frac{(-1)^k}{q_k(\alpha_{k+1}q_k + q_{k-1})}. \quad (4.3)$$

By Lemma 2.2, $\deg(\alpha_{k+1}q_k) = \deg a_{k+1} + \deg q_k = \deg q_{k+1}$, while $|q_{k-1}| < |q_k| < |q_{k+1}|$ (this covers $k = 0$, where $q_{-1} = 0$ and $|q_{-1}| = 0$). The two terms in the denominator therefore have distinct absolute values, so by (*), $|\alpha_{k+1}q_k + q_{k-1}| = q^{\deg q_{k+1}}$. Hence $|\alpha - p_k/q_k| = q^{-\deg q_k} \cdot q^{-\deg q_{k+1}}$. Since $\deg q_k \rightarrow \infty$, this tends to 0. $\square$

Remark 4.1 (Telescoping and the collapse of the interval). *Equivalently, since $p_k/q_k \rightarrow \alpha$, Lemma 2.3 gives $\alpha - \frac{p_k}{q_k} = \sum_{j \geq k} \frac{(-1)^j}{q_j q_{j+1}}$, a convergent series whose terms have strictly decreasing absolute values; the infinite ultrametric domination (if $|t_0| > |t_j|$ for $j \geq 1$ then $|\sum t_j| = |t_0|$) recovers Theorem 4.1. Over $\mathbb{Q}$, with A_k/B_k denoting the classical convergents, the same series is archimedean and yields only the two-sided interval $\frac{1}{B_k(B_k + B_{k+1})} \leq |\alpha - A_k/B_k| \leq \frac{1}{B_k B_{k+1}}$ [5, 6]; no exact equality is available in the rational case treated in [20]. The collapse to an exact equality is genuinely non-archimedean, and for the infinite expansion it requires no hypothesis on any partial quotient. As noted in the introduction, neither the statement nor the proof uses that α is quadratic.*

5. BEST POLYNOMIAL APPROXIMATION

Theorem 5.1 (Best approximation; cf. [10,16]). *Fix $k \geq 0$. For all $r, s \in \mathbb{F}_q[x]$ with $s \neq 0$, $\deg s \leq \deg q_k$ and $r/s \neq p_k/q_k$,*

$$\left| \alpha - \frac{p_k}{q_k} \right| < \left| \alpha - \frac{r}{s} \right|.$$

Proof. Let η_j be as in (4.2). Since $\det A_{k+1} = (-1)^{k+1} \in \mathbb{F}_q^\times$ by Theorem 3.1, the columns $(p_k, q_k)^\top, (p_{k+1}, q_{k+1})^\top$ of A_{k+1} form an $\mathbb{F}_q[x]$ -basis of $\mathbb{F}_q[x]^2$; write $(r, s)^\top = U(p_k, q_k)^\top + V(p_{k+1}, q_{k+1})^\top$ with $U, V \in \mathbb{F}_q[x]$, so that $s = Uq_k + Vq_{k+1}$ and $\alpha s - r = U\eta_k + V\eta_{k+1}$. If $V = 0$ then $(r, s) = U(p_k, q_k)$ with $U \neq 0$ (else $s = 0$), and $\deg U + \deg q_k = \deg s \leq \deg q_k$ forces $U \in \mathbb{F}_q^\times$, i.e. $r/s = p_k/q_k$, excluded; hence $V \neq 0$ and $\deg V \geq 0$. Then $\deg(Vq_{k+1}) \geq \deg q_{k+1} > \deg q_k \geq \deg s$, so the leading term of Vq_{k+1} must cancel against that of Uq_k ; in particular $U \neq 0$ and $\deg U + \deg q_k = \deg V + \deg q_{k+1}$, i.e. $\deg U = \deg V + \deg q_{k+1} \geq 1$. Therefore $\deg(U\eta_k) - \deg(V\eta_{k+1}) = (\deg U - \deg V) + \deg q_{k+2} \geq 2 > 0$, and by (*), $|\alpha s - r| = |U\eta_k| = |U|q^{-\deg q_{k+1}} \geq q^{1-\deg q_{k+1}}$. Hence

$$\left| \alpha - \frac{r}{s} \right| = \frac{|\alpha s - r|}{|s|} \geq \frac{q^{1-\deg q_{k+1}}}{q^{\deg q_k}} > q^{-(\deg q_k + \deg q_{k+1})} = \left| \alpha - \frac{p_k}{q_k} \right|. \quad \square$$

The converse direction, needed in Section 7, is equally short.

Lemma 5.1 (Convergence criterion). *Let $P, Q \in \mathbb{F}_q[x]$ with $Q \neq 0$ and $|\alpha Q - P| < q^{-\deg Q}$. Then $P/Q = p_k/q_k$, where k is the unique index with $\deg q_k \leq \deg Q < \deg q_{k+1}$.*

Proof. Such a k exists and is unique because $\deg q_0 = 0 \leq \deg Q$ and $(\deg q_j)$ is strictly increasing to infinity (Lemma 2.3). With η_k as in (4.2) one has the polynomial identity $p_k Q - Pq_k = q_k(\alpha Q - P) - Q\eta_k$. By Theorem 4.1, $|Q\eta_k| = q^{\deg Q - \deg q_{k+1}} < 1$, while $|q_k(\alpha Q - P)| < q^{\deg q_k - \deg Q} \leq 1$. Thus $p_k Q - Pq_k$ is a polynomial of absolute value < 1 , hence zero. $\square$

6. THE NORM FORM AND PERIODICITY

Lemma 6.1 (Norm form; classical, cf. [8, Lem. 2.8]). *For all $k \geq 0$, $p_k^2 - Dq_k^2 = (-1)^{k+1}Q_{k+1}$.*

Proof. By (4.3), $\sqrt{D}q_k - p_k = q_k(\alpha - p_k/q_k) = (-1)^k/(\alpha_{k+1}q_k + q_{k-1})$. Using $\alpha_{k+1} = (\sqrt{D} + m_{k+1})/Q_{k+1}$ and the second identity of (4.1),

$$\alpha_{k+1}q_k + q_{k-1} = \frac{\sqrt{D}q_k + (m_{k+1}q_k + Q_{k+1}q_{k-1})}{Q_{k+1}} = \frac{\sqrt{D}q_k + p_k}{Q_{k+1}}. \quad (6.1)$$

Hence $(\sqrt{D}q_k - p_k)(\sqrt{D}q_k + p_k) = (-1)^k Q_{k+1}$, i.e. $Dq_k^2 - p_k^2 = (-1)^k Q_{k+1}$. $\square$

Theorem 6.1 (Periodicity; classical, recovered). *Over the finite field $\mathbb{F}_q$, the expansion of $\sqrt{D}$ is periodic from the first partial quotient onwards; equivalently, α_1 is purely periodic. Writing ℓ for the least period,*

$$\sqrt{D} = [A; \overline{a_1, \dots, a_{\ell-1}, 2A}], \quad a_i = a_{\ell-i} \quad (1 \leq i \leq \ell-1).$$

Moreover, for every $m \geq 1$ the complete quotient α_m is reduced, that is $|\alpha_m| > 1 > |\overline{\alpha_m}|$; consequently $\deg m_m = d$ with the same leading coefficient as $\sqrt{D}$, $\deg Q_m \leq d-1$, and $\deg a_m = d - \deg Q_m$.

Proof. We first check that for $m \geq 1$ the complete quotient $\alpha_m = (\sqrt{D} + m_m)/Q_m$ is *reduced*, that is, $|\alpha_m| > 1 > |\overline{\alpha_m}|$, where $\overline{\alpha_m} = (m_m - \sqrt{D})/Q_m$ is the conjugate. That $|\alpha_m| > 1$ is Lemma 2.2. For the conjugate we induct on m . By (2.1), $m_1 = A$ and $Q_1 = D - A^2$, so $\overline{\alpha_1} = (A - \sqrt{D})/(D - A^2) = -1/(\sqrt{D} + A)$; as $\sqrt{D}$ and A have the same degree d and the same leading coefficient, which does not cancel because $\text{char } \mathbb{F}_q \neq 2$, this gives $|\overline{\alpha_1}| = q^{-d} < 1$. If $|\overline{\alpha_m}| < 1$ for some $m \geq 1$, then conjugating $\alpha_m = a_m + 1/\alpha_{m+1}$ gives $1/\overline{\alpha_{m+1}} = \overline{\alpha_m} - a_m$, whose absolute value is $|a_m| \geq q$ by the equality case of (*) and Lemma 2.2; hence $|\overline{\alpha_{m+1}}| \leq q^{-1} < 1$.

Fix $m \geq 1$. If $\deg m_m > d$ then $\deg(\sqrt{D} + m_m) = \deg(m_m - \sqrt{D}) = \deg m_m$, and reducedness forces $\deg m_m > \deg Q_m$ and $\deg m_m < \deg Q_m$; if $\deg m_m < d$ (this includes $m_m = 0$) then both degrees equal d , and reducedness forces $d > \deg Q_m$ and $d < \deg Q_m$; both are impossible. Hence $\deg m_m = d$. Let $\mu \neq 0$ and $\nu \neq 0$ be the leading coefficients of m_m and of $\sqrt{D}$ respectively. If $\mu \notin \{v, -v\}$ then $\mu + v \neq 0 \neq \mu - v$, so again $\deg(\sqrt{D} + m_m) = \deg(m_m - \sqrt{D}) = d$ and reducedness gives $d > \deg Q_m$ and $d < \deg Q_m$, impossible. If $\mu = -v$ then $\deg(\sqrt{D} + m_m) < d = \deg(m_m - \sqrt{D})$, and reducedness gives $\deg Q_m < \deg(\sqrt{D} + m_m) < d < \deg Q_m$, impossible. Hence $\mu = v$, which is the leading coefficient of A as well; then $\deg(\sqrt{D} + m_m) = d$, and $|\alpha_m| > 1$ gives $\deg Q_m \leq d - 1$, while $\deg(A + m_m) = d$ gives $\deg a_m = d - \deg Q_m$. Thus each reduced state lies in a set of at most q^{2d} pairs (m_m, Q_m) , so as $\mathbb{F}_q$ is finite some state recurs.

Put $\delta_m = -1/\overline{\alpha_m}$, so that $|\delta_m| > 1$ for $m \geq 1$ by reducedness. Conjugating $\alpha_m = a_m + 1/\alpha_{m+1}$ gives $\overline{\alpha_m} = a_m + 1/\overline{\alpha_{m+1}}$, that is,

$$\delta_{m+1} = a_m + \frac{1}{\delta_m} \quad (m \geq 1), \quad (6.2)$$

so (δ_m) runs the continued fraction recursion through the partial quotients in reverse order; and $|1/\delta_m| < 1$ shows $[\delta_{m+1}] = a_m$. In particular $a_m = \lfloor -1/\overline{\alpha_{m+1}} \rfloor$ is determined by α_{m+1} alone, hence so is $\alpha_m = a_m + 1/\alpha_{m+1}$. The step $\alpha_m \mapsto \alpha_{m+1}$ is therefore injective on the finite set of reduced states occurring in the expansion, and so bijective on it; the recurrence is thus a pure one, and the expansion is periodic from $m = 1$, with $(m_{\ell+1}, Q_{\ell+1}) = (m_1, Q_1)$ for the least period ℓ .

It remains to identify a_ℓ and to establish the palindromic relations. From (2.1), $m_1 = A$ and $Q_1 = D - A^2$, so Lemma 2.1 gives $Q_\ell Q_{\ell+1} = D - m_{\ell+1}^2 = D - A^2 = Q_{\ell+1}$, whence $Q_\ell = 1$ and $Q_{\ell-1} = D - m_\ell^2$. Now

$$(m_\ell - A)(m_\ell + A) = (m_\ell^2 - D) + (D - A^2) = -Q_{\ell-1} + Q_1$$

has degree $< d$, since $\deg Q_{\ell-1} \leq d - 1$ and $\deg Q_1 \leq d - 1$; on the other hand m_ℓ and A both have degree d and the same leading coefficient, which does not cancel because $\text{char } \mathbb{F}_q \neq 2$, so $\deg(m_\ell + A) = d$. Hence $m_\ell - A = 0$, and $a_\ell = \lfloor (A + m_\ell)/Q_\ell \rfloor = 2A$.

For the palindrome we return to (6.2). Since $\alpha_{\ell+1} = \alpha_1$ we have $\delta_{\ell+1} = \delta_1$, so iterating (6.2) from $m = \ell$ down to $m = 1$ exhibits δ_1 as the purely periodic continued fraction $[\overline{a_\ell, a_1, a_2, \dots, a_{\ell-1}}]$. On the other hand $\overline{\alpha_1} = (A - \sqrt{D})/Q_1$ with $Q_1 = D - A^2$, so

$$\delta_1 = \frac{D - A^2}{\sqrt{D} - A} = \sqrt{D} + A = [\overline{a_\ell, a_1, a_2, \dots, a_{\ell-1}}],$$

the last equality because $\sqrt{D} = [A; a_1, a_2, \dots]$ and $a_\ell = 2A$. Comparing the two expansions of δ_1 term by term, which is legitimate because the expansion of an element of K is unique (its partial quotients are recovered successively as $b_i = \lfloor \beta_i \rfloor$, $\beta_{i+1} = 1/(\beta_i - b_i)$), yields $a_{\ell-j} = a_j$ for $1 \leq j \leq \ell - 1$. $\square$

Remark 6.1 (Finiteness is essential). *The state count uses $|\mathbb{F}_q| < \infty$. Over an infinite base field the expansion of $\sqrt{D}$ need not be periodic [4, 12]; periodicity is a finite-field phenomenon, in contrast to Theorems 3.1, 4.1, 5.1, valid over any base field of odd characteristic, finite or not. The hypothesis $\text{char } \mathbb{F}_q \neq 2$ is likewise essential, and not merely technical: in characteristic 2 one has $K^2 = \mathbb{F}_q((1/x^2)) \subsetneq K$, so a nonsquare $D \in \mathbb{F}_q[x]$ has $\sqrt{D} \notin K$ and the extension $\mathbb{F}_q(x)(\sqrt{D})$ is purely inseparable; the reduced-state analysis above, and with it the palindromic relation $a_\ell = 2A$, has no counterpart there.*

7. THE ACCUMULATED SCALAR, THE QUASI-PERIOD, AND THE PELL-ABEL EQUATION

For $k \geq -1$ put $\varepsilon_k := p_k + q_k \sqrt{D} \in \mathcal{O}$. The next proposition says that the scalar bookkeeping of the row operations reproduces exactly these algebraic integers.

Proposition 7.1 (The accumulated scalar; cf. [9, Prop. 2.4], [17, Thm. 5.2]). *For every $k \geq 0$, the scalar $\Lambda_k = \lambda_0 \lambda_1 \cdots \lambda_k$ of (3.2) satisfies*

$$\Lambda_k = \frac{1}{p_k + q_k \sqrt{D}} = \varepsilon_k^{-1}, \quad |\Lambda_k| = q^{-(d+\deg q_k)} = q^{-\deg p_k}.$$

In particular $\Lambda_k^{-1} \in \mathcal{O}$ for every k , and $\Lambda_k^{-1} \in \mathcal{O}^\times$ if and only if $Q_{k+1} \in \mathbb{F}_q^\times$.

Proof. By (3.3), $T_{k+1} = SA_k^{-1}$, and by Theorem 3.1,

$$A_k = \begin{pmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{pmatrix}, \quad A_k^{-1} = (-1)^k \begin{pmatrix} q_k & -p_k \\ -q_{k-1} & p_{k-1} \end{pmatrix}, \quad T_{k+1} = (-1)^k \begin{pmatrix} -q_{k-1} & p_{k-1} \\ q_k & -p_k \end{pmatrix}.$$

Comparing the second coordinates in (3.2), with $v_0 = (\sqrt{D}, 1)^\top$ and $v_{k+1} = (\sqrt{D} + m_{k+1}, Q_{k+1})^\top$, gives $(-1)^k (q_k \sqrt{D} - p_k) = \Lambda_k Q_{k+1}$. By Lemma 6.1, $(q_k \sqrt{D} - p_k)(q_k \sqrt{D} + p_k) = Dq_k^2 - p_k^2 = (-1)^k Q_{k+1}$, so

$$\Lambda_k = \frac{(-1)^k (q_k \sqrt{D} - p_k)}{Q_{k+1}} = \frac{(-1)^k}{Q_{k+1}} \cdot \frac{(-1)^k Q_{k+1}}{q_k \sqrt{D} + p_k} = \frac{1}{p_k + q_k \sqrt{D}}.$$

For the absolute value, recall $\eta_k = \alpha q_k - p_k = q_k \sqrt{D} - p_k$ from (4.2), with $|\eta_k| = q^{-\deg q_{k+1}} < 1$. Since $|q_k \sqrt{D}| = q^{d+\deg q_k} > 1$, the equality case of (*) applied to $p_k = q_k \sqrt{D} - \eta_k$ gives $\deg p_k = d + \deg q_k$, and applied to $\varepsilon_k = 2q_k \sqrt{D} - \eta_k$ — the factor 2 being invertible because $\text{char } \mathbb{F}_q \neq 2$ — it gives $|\varepsilon_k| = q^{d+\deg q_k} = q^{\deg p_k}$. Finally, $N(\varepsilon_k) = \varepsilon_k \bar{\varepsilon}_k = p_k^2 - Dq_k^2 = (-1)^{k+1} Q_{k+1}$ by Lemma 6.1, and an element of $\mathcal{O}$ is a unit exactly when its norm lies in $\mathbb{F}_q^\times$ [8, Lem. 2.11]. $\square$

Over $\mathbb{Q}$ the quantities Q_m are positive integers, so “ Q_m is a unit” and “ $Q_m = 1$ ” coincide. Over $\mathbb{F}_q$ they do not, and the resulting distinction governs where the units of $\mathcal{O}$ occur [15, 17].

Definition 7.1 (Quasi-period). *The quasi-period of $\sqrt{D}$ is $n := \min\{m \geq 1 : Q_m \in \mathbb{F}_q^\times\}$.*

Lemma 7.1 (Quasi-period criterion and quasi-palindrome; classical, cf. [17, Thm. 4.7, Cor. 4.6]). *Let $m \geq 1$. Then $Q_m \in \mathbb{F}_q^\times$ if and only if $\alpha_{m+1} = \gamma \alpha_1$ for some $\gamma \in \mathbb{F}_q^\times$, in which case $\gamma = Q_m$ and $m_m = A$. Writing n for the quasi-period and $c := Q_n \in \mathbb{F}_q^\times$, one has the quasi-palindromic relations*

$$a_n = \frac{2A}{c}, \quad a_{n-i} = c^{(-1)^{i+1}} a_i \quad (1 \leq i \leq n-1), \quad (7.1)$$

and

$$\{m \geq 1 : Q_m \in \mathbb{F}_q^\times\} = \{jn : j \geq 1\}, \quad n \mid \ell, \quad \ell \in \{n, 2n\},$$

with $\ell = n$ if and only if $c = 1$; in particular $\ell = 2n$ forces n odd. For $n = \ell$ (so $c = 1$) the relations (7.1) are the palindrome of Theorem 6.1.

Proof. Suppose $\deg Q_m = 0$. Reducedness (Theorem 6.1) gives $|\overline{\alpha_m}| < 1$, i.e. $\deg(m_m - \sqrt{D}) < \deg Q_m = 0$, so $m_m = \lfloor \sqrt{D} \rfloor = A$. Then $a_m = 2A/Q_m$, hence $m_{m+1} = a_m Q_m - m_m = A$ and, by Lemma 2.1, $Q_{m+1} = (D - A^2)/Q_m = Q_1/Q_m$; therefore $\alpha_{m+1} = (\sqrt{D} + A)/(Q_1/Q_m) = Q_m \alpha_1$. Conversely, if $\alpha_{m+1} = \gamma \alpha_1$ with $\gamma \in \mathbb{F}_q^\times$, comparing the parts free of and linear in $\sqrt{D}$ gives $m_{m+1} = A$ and $Q_{m+1} = Q_1/\gamma$, so $Q_m = (D - m_{m+1}^2)/Q_{m+1} = \gamma$. Write Σ for the set in question; the first displayed relation of (7.1) is the case $m = n$ of $a_m = 2A/Q_m$.

Scaling. For $\gamma \in \mathbb{F}_q^\times$ and $\beta \in K$ one has $\lfloor \gamma \beta \rfloor = \gamma \lfloor \beta \rfloor$, hence $\gamma \beta - \lfloor \gamma \beta \rfloor = \gamma(\beta - \lfloor \beta \rfloor)$ and the next complete quotient of $\gamma \beta$ is γ^{-1} times that of β . Iterating: if $\beta = [b_0; b_1, b_2, \dots]$ then

$$\gamma \beta = [\gamma b_0; \gamma^{-1} b_1, \gamma b_2, \dots], \quad i\text{-th partial quotient } \gamma^{(-1)^i} b_i, \quad (7.2)$$

and likewise $\alpha_{m+j} = \gamma^{(-1)^{j-1}} \alpha_j$ for all $j \geq 1$ whenever $\alpha_{m+1} = \gamma \alpha_1$.

$\Sigma = n\mathbb{Z}_{>0}$. Let $m, m' \in \Sigma$, say $\alpha_{m+1} = \gamma \alpha_1$ and $\alpha_{m'+1} = \gamma' \alpha_1$; the two need not be distinct. Taking $j = m' + 1$ in the last relation gives $\alpha_{m+m'+1} = \gamma^{(-1)^{m'}} \gamma' \alpha_1$, so $m + m' \in \Sigma$; if $m < m'$, taking $j = m' - m + 1$ gives $\alpha_{m'+1} = \gamma^{(-1)^{m'-m}} \alpha_{m'-m+1}$, i.e. $\alpha_{m'-m+1} = \gamma^{-(-1)^{m'-m}} \gamma' \alpha_1$, so $m' - m \in \Sigma$. Thus Σ is a nonempty subset of $\mathbb{Z}_{>0}$ closed under addition and under positive differences: closure under differences and division with remainder give $\Sigma \subseteq n\mathbb{Z}_{>0}$ for $n = \min \Sigma$, and closure under addition gives the reverse inclusion. Hence $\Sigma = \{jn : j \geq 1\}$. Since ℓ is the least $m \geq 1$ with $\alpha_{m+1} = \alpha_1$, i.e. with $Q_m = 1$, we have $\ell \in \Sigma$ and $n \mid \ell$, and $\ell = n$ exactly when $c = 1$.

The relations (7.1). Conjugating $\alpha_{n+1} = c \alpha_1$ gives $\overline{\alpha_{n+1}} = c \overline{\alpha_1}$, hence $\delta_{n+1} = c^{-1} \delta_1$ for $\delta_m = -1/\overline{\alpha_m}$ as in the proof of Theorem 6.1. By (6.2) and $\lfloor \delta_{m+1} \rfloor = a_m$ ($m \geq 1$), iterating down from $m = n$ to $m = 1$ gives $\delta_{n+1} = [a_n; a_{n-1}, \dots, a_1, \delta_1]$, while Theorem 6.1 gives $\delta_1 = \sqrt{D} + A = [2A; a_1, a_2, \dots]$; hence

$$\delta_{n+1} = [a_n; a_{n-1}, \dots, a_1, 2A, a_1, a_2, \dots].$$

On the other hand (7.2) with $\gamma = c^{-1}$ and $\beta = \delta_1$ says that the i -th partial quotient of $c^{-1} \delta_1$ is $c^{(-1)^{i+1}} b_i$, where $b_0 = 2A$ and $b_i = a_i$ for $1 \leq i \leq \ell - 1$. Since $n \leq \ell$, comparing the two expansions of δ_{n+1} term by term for $0 \leq i \leq n - 1$ — legitimate by uniqueness of the expansion — yields $a_n = c^{-1} 2A$ and $a_{n-i} = c^{(-1)^{i+1}} a_i$ for $1 \leq i \leq n - 1$, which is (7.1).

$\ell \in \{n, 2n\}$. If n is odd, taking $j = n + 1$ in $\alpha_{n+j} = c^{(-1)^{j-1}} \alpha_j$ gives $\alpha_{2n+1} = c^{(-1)^n} c \alpha_1 = \alpha_1$, so $\ell \mid 2n$ and, with $n \mid \ell$, $\ell \in \{n, 2n\}$. If n is even, then $1 \leq n/2 \leq n - 1$ and (7.1) with $i = n/2$ reads

$a_{n/2} = c^{(-1)^{n/2+1}} a_{n/2}$; as $a_{n/2} \neq 0$ this forces $c^{\pm 1} = 1$, i.e. $c = 1$ and $\ell = n$. In all cases $\ell \in \{n, 2n\}$, and $\ell = 2n$ forces n odd. (Compare [17, Cor. 4.6].) $\square$

Theorem 7.1 (Pell–Abel and regulator; classical, cf. [17, Thm. 5.2, Cor. 5.3]). *Let n be the quasi-period of $\sqrt{D}$. Then, for every $k \geq 0$,*

$$p_k^2 - Dq_k^2 \in \mathbb{F}_q^\times \iff n \mid k + 1.$$

Among the solutions $P, Q \in \mathbb{F}_q[x]$ of $P^2 - DQ^2 \in \mathbb{F}_q^\times$ with $Q \neq 0$, those of least $\deg Q$ — the fundamental solutions, forming the two orbits $\mathbb{F}_q^\times \cdot (p_{n-1}, q_{n-1})$ and $\mathbb{F}_q^\times \cdot (p_{n-1}, -q_{n-1})$ interchanged by conjugation — are represented by (p_{n-1}, q_{n-1}) , with $p_{n-1}^2 - Dq_{n-1}^2 = (-1)^n Q_n$; equivalently $\varepsilon := \varepsilon_{n-1} = \Lambda_{n-1}^{-1}$ is a fundamental unit of $\mathcal{O}$, and the regulator of $\mathcal{O}$ is

$$R = \deg \varepsilon = -\deg \Lambda_{n-1} = \deg p_{n-1} = d + \deg q_{n-1} = \sum_{i=0}^{n-1} \deg a_i.$$

Moreover $\ell = n$ if $Q_n = 1$, in which case $p_{\ell-1}^2 - Dq_{\ell-1}^2 = (-1)^\ell$; otherwise $\ell = 2n$ and $(p_{\ell-1}, q_{\ell-1})$ corresponds to ε^2 up to a factor in $\mathbb{F}_q^\times$.

Proof. By Lemma 6.1, $p_k^2 - Dq_k^2$ lies in $\mathbb{F}_q^\times$ exactly when $\deg Q_{k+1} = 0$, which by Lemma 7.1 happens exactly when $n \mid k + 1$; equivalently, by Proposition 7.1, exactly when the accumulated scalar Λ_k is a unit of $\mathcal{O}$. This proves the first assertion and locates the least such index at $k = n - 1$, of norm $(-1)^n Q_n$.

For fundamentality one must know that every solution arises in this way. Let $P, Q \in \mathbb{F}_q[x]$ with $Q \neq 0$ satisfy $P^2 - DQ^2 = c \in \mathbb{F}_q^\times$. Then $(\sqrt{D}Q - P)(\sqrt{D}Q + P) = -c$ has absolute value 1, whereas $(\sqrt{D}Q - P) + (\sqrt{D}Q + P) = 2\sqrt{D}Q$ has absolute value $q^{d+\deg Q}$, the factor 2 being invertible since $\text{char } \mathbb{F}_q \neq 2$. The two absolute values are distinct: were they equal they would both be 1, and then the sum would have absolute value at most $1 < q^{d+\deg Q}$. Hence by (*) they are $q^{d+\deg Q}$ and $q^{-d-\deg Q}$, and after replacing P by $-P$ if necessary, $|\alpha Q - P| = q^{-d-\deg Q} < q^{-\deg Q}$ because $d \geq 1$. Lemma 5.1 now gives $P/Q = p_k/q_k$ for some k ; since $\gcd(p_k, q_k) = 1$ (Theorem 3.1) and $P^2 - DQ^2$ is constant, $(\pm P, Q) = \gamma(p_k, q_k)$ with $\gamma \in \mathbb{F}_q^\times$, the sign being the one introduced in the previous sentence. Thus every solution lies in $\mathbb{F}_q^\times \cdot (p_k, q_k) \cup \mathbb{F}_q^\times \cdot (p_k, -q_k)$ for some k with $n \mid k + 1$; as $(\deg q_k)$ is strictly increasing, those of least $\deg Q$ are the ones at $k = n - 1$, in the two stated orbits. (The two orbits are genuinely distinct, since $(-p_{n-1}, q_{n-1}) = \gamma(p_{n-1}, q_{n-1})$ would force $\gamma = -1$ and $\gamma = 1$ simultaneously.)

Write $R := d + \deg q_{n-1}$. We claim $\varepsilon = \varepsilon_{n-1}$ is a fundamental unit. Let $u = P + Q\sqrt{D} \in \mathcal{O}^\times$ with $u \notin \mathbb{F}_q^\times$; then $Q \neq 0$, and the previous paragraph shows both that $|u| = q^{\pm(d+\deg Q)}$ and that $Q = \gamma q_k$ with $\gamma \in \mathbb{F}_q^\times$ and $n \mid k + 1$, whence

$$|\deg u| = d + \deg q_k \geq d + \deg q_{n-1} = R,$$

$(\deg q_k)$ being strictly increasing. Replacing u by u^{-1} if necessary we may assume $\deg u > 0$; with $j = \lfloor \deg u / R \rfloor \geq 1$ the element $u\varepsilon^{-j}$ is again a unit, of degree in $[0, R)$, so by the displayed inequality it lies in $\mathbb{F}_q^\times$. Hence $\mathcal{O}^\times = \mathbb{F}_q^\times \times \varepsilon^{\mathbb{Z}}$, ε is fundamental, and the regulator is $\deg \varepsilon$, which by

Proposition 7.1 equals $-\deg \Lambda_{n-1} = d + \deg q_{n-1} = \deg p_{n-1} = R$. Finally $\deg a_n = d - \deg Q_n = d$ by Theorem 6.1, so $d + \deg q_{n-1} = \deg a_n + \sum_{i=1}^{n-1} \deg a_i$, and since $\deg a_0 = \deg A = d = \deg a_n$ this equals $\sum_{i=0}^{n-1} \deg a_i$.

For the last assertion, note first that a unit u of positive degree is of the form $\gamma \varepsilon_k$: the alternative permitted by the previous paragraph is $u = \gamma(-p_k + q_k \sqrt{D}) = -\gamma \overline{\varepsilon_k}$, which is excluded because $\deg \overline{\varepsilon_k} = \deg N(\varepsilon_k) - \deg \varepsilon_k = -\deg \varepsilon_k < 0$. Hence the units ε_{jn-1} , $j \geq 1$, of strictly increasing degrees $d + \deg q_{jn-1}$, represent every class of $\mathcal{O}^\times / \mathbb{F}_q^\times$ of positive degree; since those classes are also represented by $\varepsilon, \varepsilon^2, \varepsilon^3, \dots$, of strictly increasing degrees $R, 2R, 3R, \dots$, matching the two lists in order gives $\varepsilon_{jn-1} \in \mathbb{F}_q^\times \varepsilon^j$ for every $j \geq 1$. Taking $j = 2$ when $\ell = 2n$, and invoking Lemma 7.1 for the dichotomy, gives the stated description of $(p_{\ell-1}, q_{\ell-1})$. $\square$

Remark 7.1 (One machine, two readings). *Running the ERO process on $\sqrt{D}$ and halting at the first $k \geq 0$ for which the accumulated scalar Λ_k is a unit of $\mathcal{O}$ — equivalently, $Q_{k+1} \in \mathbb{F}_q^\times$ — returns the fundamental Pell–Abel solution as the last column of A_k , the certificate $\det A_k = \pm 1$, and the regulator $R = -\deg \Lambda_k = d + \deg q_k$, in $n \leq R$ row operations. Run without halting, the same construction computes best rational approximation (Theorem 4.1). Both readings are at the infinite place; the finite-place reading of the Euclidean algorithm in algebraic decoding [18, 19] is classical and is not pursued here.*

Remark 7.2 (The equation with right-hand side exactly 1). *Theorem 7.1 concerns $P^2 - DQ^2 \in \mathbb{F}_q^\times$, which is the equation whose solutions are the units of $\mathcal{O}$. For the strict equation $P^2 - DQ^2 = 1$ one needs in addition that the norm be a square in $\mathbb{F}_q^\times$: by Lemma 6.1, $\gamma(p_k, q_k)$ solves it precisely when $n \mid k+1$ and $(-1)^{k+1} Q_{k+1} \in (\mathbb{F}_q^\times)^2$, and then $\gamma^2 = ((-1)^{k+1} Q_{k+1})^{-1}$. The least such k need not be $n-1$. For instance $D = x^2 + 2$ over $\mathbb{F}_5$ (squarefree, nonsquare, $d = 1$, $A = x$) has $Q_1 = 2$, so $n = 1$ and $\ell = 2$, but $p_0^2 - Dq_0^2 = x^2 - D = -2 = 3$ is a nonsquare in $\mathbb{F}_5^\times$; the least solution of $P^2 - DQ^2 = 1$ is $(p_1, q_1) = (x^2 + 1, x)$, at the end of the period. Over $\mathbb{F}_q$ the natural object is therefore the unit equation, and it is that equation that Theorem 7.1 solves.*

Remark 7.3 (The norm of the fundamental unit). *By Lemma 6.1, $N(\varepsilon) = p_{n-1}^2 - Dq_{n-1}^2 = (-1)^n Q_n$, so Lemma 7.1 may be read as a statement about the norm of the fundamental unit: $\ell = n$ if and only if $N(\varepsilon) = (-1)^n$, and since $\ell = 2n$ forces n odd,*

$$\ell = 2n \iff n \text{ is odd and } N(\varepsilon) \neq -1.$$

Over $\mathbb{Z}$ the corresponding question — whether the fundamental unit has norm -1 , equivalently whether $P^2 - DQ^2 = -1$ is solvable — is classical and delicate. The situation here differs in that $N(\varepsilon)$ ranges over all of $\mathbb{F}_q^\times$, so that $N(\varepsilon) = -1$ is one value among $q-1$ rather than one among two.

Proposition 7.2 (The case $d = 1$; the counts are those of [3, Thm. 2.3(ii)]). *Let q be odd and let $\mathcal{D}_1$ be the set of monic squarefree nonsquare $D \in \mathbb{F}_q[x]$ of degree 2. Then $|\mathcal{D}_1| = q(q-1)$, every $D \in \mathcal{D}_1$ has quasi-period $n = 1$, regulator $R = 1$ and fundamental unit $\varepsilon = A + \sqrt{D}$ with $N(\varepsilon) = -Q_1$, and the map*

$$\mathcal{D}_1 \longrightarrow \mathbb{F}_q^\times, \quad D \longmapsto Q_1,$$

is exactly q -to-one. Consequently $\ell = 1$ for exactly q of them and $\ell = 2$ for the remaining $q(q-2)$; the proportion with $\ell = 2n$ is exactly

$$\frac{q-2}{q-1}.$$

Proof. Write $D = x^2 + bx + e$. Then $A = x + b/2$ (recall $\text{char } \mathbb{F}_q \neq 2$) and $Q_1 = D - A^2 = e - b^2/4 \in \mathbb{F}_q$, so that $\text{disc}(D) = b^2 - 4e = -4Q_1$. Hence D is a square if and only if $Q_1 = 0$, and D is squarefree if and only if $Q_1 \neq 0$: the two conditions coincide, and $\mathcal{D}_1$ is parametrised by the pairs (b, e) with $e \neq b^2/4$, of which there are $q(q-1)$. For each fixed b the map $e \mapsto e - b^2/4$ is a bijection from $\mathbb{F}_q \setminus \{b^2/4\}$ onto $\mathbb{F}_q^\times$; as b runs over $\mathbb{F}_q$ this gives the q -to-one statement. Since $\deg Q_1 = 0$ the quasi-period is $n = 1$, so $\varepsilon = \varepsilon_0 = p_0 + q_0 \sqrt{D} = A + \sqrt{D}$, $N(\varepsilon) = A^2 - D = -Q_1$ and $R = \deg p_0 = \deg A = 1$ by Theorem 7.1. By Lemma 7.1, $\ell = n = 1$ exactly when $Q_1 = 1$, which by the q -to-one statement happens for exactly q elements of $\mathcal{D}_1$; for the other $q(q-2)$ one has $\ell = 2n = 2$. $\square$

The two counts q and $q(q-2)$ are the case $d = 1$ of [3, Thm. 2.3(ii)], where $\theta(d, 1) = q^d$ and $\theta(d, 2) = (dq - d - 1)q^d$ are obtained, for monic nonsquare D of degree $2d$, by a different argument; for $d = 1$ the conditions “squarefree” and “nonsquare” coincide, as the proof above shows, so the two families agree. What the derivation above adds is the reading of the count through the quasi-period: $q(q-2)$ is exactly the number of D for which halting at the end of the period returns ε^2 rather than ε .

Remark 7.4 (The quasi-period is not exceptional, and the case $d \geq 2$). *Halting at the end of the period rather than of the quasi-period returns ε^2 instead of ε whenever $\ell = 2n$, and this occurs for a substantial proportion of D . For $d = 1$ the proportion is $(q-2)/(q-1)$ exactly (Proposition 7.2), in agreement with exhaustive enumeration for $q \in \{3, 5, 7, 9, 11, 13, 25, 27\}$. For $d = 2$, exhaustive enumeration of the monic squarefree nonsquare D gives 3 of 54 when $q = 3$, 225 of 500 when $q = 5$, 721 of 2058 when $q = 7$, 2601 of 5832 when $q = 9$ and 5445 of 13310 when $q = 11$; for $d = 3$ it gives 63 of 486 when $q = 3$ and 4535 of 12500 when $q = 5$.*

The mechanism behind Proposition 7.2 is the exact equidistribution of Q_1 over $\mathbb{F}_q^\times$, which is available because $n = 1$ for every D when $d = 1$. For $d \geq 2$ the relevant statistic is Q_n , and it is not equidistributed. Restricting to those D with n odd — the only ones that can have $\ell = 2n$ — the counts of Q_n by value are, for $q = 7$ and $d = 2$: 245 for each of the three squares of $\mathbb{F}_7^\times$ and 77 for each of the three nonsquares, out of 966; and for $q = 5$ and $d = 3$: 905 for each square and 1815 for each nonsquare, out of 5440. In both cases the count depends only on the square class of Q_n . But for $q = 5$ and $d = 2$ the values 1, 2, 3, 4 occur 15, 105, 85, 35 times out of 240, which is not constant on square classes. We do not determine these counts here, and record the question: for $d \geq 2$, what is the distribution of Q_n on $\mathbb{F}_q^\times$? They are not the quantities $\theta(d, t)$ of [3]: those count monic nonsquare D with $\text{Per}(\sqrt{D}) = t$, whereas the counts above are taken over the monic squarefree D and record the condition $\ell = 2n$, which for $d \geq 2$ is a different family and a different statistic. Example 8.2 below is one D with $\ell = 2n$.

8. COMPUTATIONAL EXAMPLES

Example 8.1 ($n = \ell$). Let $D = x^4 + x + 1$ over $\mathbb{F}_q$ with $q = 5$, so $d = 2$ and $A = x^2$. The expansion is

$$\sqrt{D} = [x^2; \overline{2x-2, 1-2x, -x, 2x+1, -x, 1-2x, 2x-2, 2x^2}], \quad \ell = 8,$$

palindromic with $a_8 = 2x^2 = 2A$. Here $\deg Q_m = 1$ for $1 \leq m \leq 7$ and $Q_8 = 1$, so the quasi-period equals the period, $n = \ell = 8$. Table 1 confirms Theorem 4.1 at every step. The fundamental Pell–Abel solution is therefore (p_7, q_7) with $p_7^2 - Dq_7^2 = 1$, and the regulator is

$$R = \deg \varepsilon = \deg p_7 = d + \deg q_7 = \sum_{i=0}^7 \deg a_i = 2 + 7 = 9.$$

TABLE 1. Convergents of $\sqrt{x^4 + x + 1}$ over $\mathbb{F}_5$; the error degree equals $-(\deg q_k + \deg q_{k+1})$. The last entry is q^{-16} , and not q^{-15} , because $\deg a_8 = 2$ gives $\deg q_8 = \deg q_7 + \deg a_8 = 9$.

k	a_k	$\deg q_k$	$\deg Q_{k+1}$	$ \alpha - p_k/q_k $
0	x^2	0	1	q^{-1}
1	$2x-2$	1	1	q^{-3}
2	$1-2x$	2	1	q^{-5}
3	$-x$	3	1	q^{-7}
4	$2x+1$	4	1	q^{-9}
5	$-x$	5	1	q^{-11}
6	$1-2x$	6	1	q^{-13}
7	$2x-2$	7	0	q^{-16}

Example 8.2 ($\ell = 2n$). Let $D = x^4 + 4x^3 + x = x(x-2)(x^2 + x + 2)$ over $\mathbb{F}_5$, which is squarefree, nonsquare, of even degree 4 and with square leading coefficient; here $d = 2$ and $A = x^2 + 2x + 3$. The states are listed in Table 2. The expansion

$$\sqrt{D} = [x^2+2x+3; \overline{3x+4, 2x+1, 3x^2+x+4, 2x+1, 3x+4, 2x^2+4x+1}]$$

is palindromic of period $\ell = 6$ with $a_6 = 2A$, in accordance with Theorem 6.1. However $Q_3 = 4 \in \mathbb{F}_5^\times$, so the quasi-period is $n = 3$ and $\ell = 2n$. Accordingly

$$p_2 = x^4 + 3x^3 + 1, \quad q_2 = x^2 + x, \quad p_2^2 - Dq_2^2 = 1,$$

so already at $k = 2$ one has a solution of the Pell–Abel equation, of degree $\deg q_2 = 2$; the fundamental unit is $\varepsilon = p_2 + q_2 \sqrt{D}$ and the regulator is $R = \deg p_2 = 4 = \sum_{i=0}^2 \deg a_i$. The convergent at the end of the period, (p_5, q_5) with $\deg q_5 = 6$, satisfies $p_5^2 - Dq_5^2 = 1$ as well but corresponds to ε^2 . This is the phenomenon quantified in Remark 7.4.

TABLE 2. States of the ERO process for $\sqrt{x^4 + 4x^3 + x}$ over $\mathbb{F}_5$. The quasi-period is $n = 3$, where $Q_3 = 4 \in \mathbb{F}_5^\times$; the period is $\ell = 6$, where $Q_6 = 1$.

m	a_m	m_m	Q_m	$\deg Q_m$
0	$x^2 + 2x + 3$	0	1	0
1	$3x + 4$	$x^2 + 2x + 3$	$4x + 1$	1
2	$2x + 1$	$x^2 + 2x + 1$	$x + 4$	1
3	$3x^2 + x + 4$	$x^2 + 2x + 3$	4	0
4	$2x + 1$	$x^2 + 2x + 3$	$x + 4$	1
5	$3x + 4$	$x^2 + 2x + 1$	$4x + 1$	1
6	$2x^2 + 4x + 1$	$x^2 + 2x + 3$	1	0

9. CONCLUSION

A single augmented-matrix ERO construction, read at the infinite place, extends from rational a/b to quadratic irrational $\sqrt{D}$. The exact non-archimedean error equality for the infinite expansion follows directly from the construction (Theorem 4.1) and yields strict best approximation (Theorem 5.1); the determinant invariant produces the norm form (Lemma 6.1); and the accumulated scalar of the row operations is exactly the inverse of the associated algebraic integer (Proposition 7.1), so that the arithmetic of $\mathcal{O} = \mathbb{F}_q[x][\sqrt{D}]$ is read off the scalar bookkeeping: the fundamental unit is detected after the first number of steps at which that scalar becomes a unit — the quasi-period, not the period — and the regulator is minus its degree (Theorem 7.1). The classical periodicity and palindromy of $\sqrt{D}$ (Theorem 6.1) are likewise recovered inside the framework, and the distinction between period and quasi-period, invisible over $\mathbb{Z}$ because $\mathbb{Z}^\times = \{\pm 1\}$, is made explicit and quantified (Remark 7.4). The matrix machine that over $\mathbb{Z}$ yields only the classical two-sided bounds [5, 6] therefore yields, over $\mathbb{F}_q[x]$, an exact error equality together with the arithmetic of $\mathcal{O}$; the improvement is attributable entirely to the ultrametric inequality and to the finiteness of the constant field. The mathematics recovered in this way is classical, as set out in Section 1; what the construction contributes is uniformity of derivation rather than new theorems, and we claim priority for none of it. Proposition 7.2 is a case in point: for $\deg D = 2$ the quantity Q_1 is exactly equidistributed over $\mathbb{F}_q^\times$, so the period exceeds the quasi-period for exactly $q(q-2)$ of the $q(q-1)$ admissible D . These two counts are the case $d = 1$ of [3, Thm. 2.3(ii)], reached here through the quasi-period. That equidistribution fails for $\deg D \geq 4$, and we do not determine what replaces it; the data and the question are in Remark 7.4, and they seem to us the most promising direction opened by the present reading. Whether the classical finite-place reading of the Euclidean algorithm in algebraic decoding [18, 19] admits the same scalar bookkeeping is a further question that we leave open, as is the extension to other imaginary-quadratic and higher-genus settings, where the period encodes finer arithmetic of the associated curve [13], and the relation

to the infrastructure viewpoint of [15, 17], in which the accumulated scalar becomes the distance function.

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