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Bi-Univalent Functions Associated with the Neutrosophic Touchard–Caputo Operator

Khalifa Al-Shaqsi[1], Jamal Salah[2], Ahmad Al Azab[2,*]

[1] *Department of Mathematical and Physical Sciences, College of Arts and Sciences, University of Nizwa, Nizwa, Oman*

[2] *College of Applied and Health Sciences, A'Sharqiyah University, Ibra, Oman*

**Corresponding author:* ahmad.alazab@asu.edu.om

Abstract. This paper introduces a novel operator combining the neutrosophic Touchard (Bell polynomial) distribution with the modified Caputo fractional derivative operator, and applies it to study bi-univalent functions. The neutrosophic Touchard distribution extends both the classical Touchard moment series and the neutrosophic Poisson distribution through an indeterminate moment order $n \geq 0$ and a neutrosophic parameter $m_N \in [m_1, m_2]$. Within the new bi-univalent family $\widetilde{\Upsilon}_\Sigma(\alpha, \lambda, \mu, \eta, \delta, m_N, n; e)$, which unifies Bazilevic and μ-pseudo-starlike bi-univalent functions, we establish upper bounds for the coefficients $|d_2|$ and $|d_3|$ and derive the Fekete–Szegő inequality. Corollaries recover known results, including those of Santhiya–Thilagavathi [23] and Porwal [17]. A numerical case study with four figures illustrates how n tightens the bounds and how the neutrosophic interval quantifies model uncertainty.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k, \qquad \zeta \in \mathbb{U} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$. Every $f \in \mathcal{S}$ satisfies the Koebe one-quarter theorem [3] and admits an inverse f^{-1} satisfying $f^{-1}(f(\zeta)) = \zeta$ for $\zeta \in \mathbb{U}$ and $f(f^{-1}(\omega)) = \omega$ for $|\omega| < r_0(f) \geq \frac{1}{4}$, where

$$f^{-1}(\omega) = \omega - a_2\omega^2 - (2a_2^2 - a_3)\omega^3 - \cdots, \qquad |\omega| < r_0(f). \tag{1.2}$$

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The celebrated Bieberbach conjecture [1], later proved by de Branges [2], states that $|a_n| \le n$ for every $f \in \mathcal{S}$, providing the fundamental motivation for coefficient estimates in geometric function theory.

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote the class of all such functions by Σ. The systematic study of bi-univalent functions was revived by Srivastava et al. [4], and has since attracted considerable attention [5–10].

A function $f \in \mathcal{A}$ is called a *Bazilevic function of order* $\lambda \ge 0$ [14] if

$$\mathrm{Re}\left(\frac{\zeta^{1-\lambda} f'(\zeta)}{[f(\zeta)]^{1-\lambda}}\right) > 0, \qquad \zeta \in \mathbb{U}.$$

It is called a *μ-pseudo-starlike function* [15] if

$$\mathrm{Re}\left(\frac{\zeta[f'(\zeta)]^{\mu}}{f(\zeta)}\right) > 0, \qquad \mu \ge 1, \quad \zeta \in \mathbb{U}.$$

The (G, H) Lucas Polynomials. For polynomials $G(x)$ and $H(x)$ with real coefficients, the (G, H) Lucas polynomials $L_{G,H,k}(x)$ satisfy [11,12]

$$L_{G,H,k}(x) = G(x)\, L_{G,H,k-1}(x) + H(x)\, L_{G,H,k-2}(x), \qquad k \ge 2,$$

with generating function

$$\mathcal{M}_{G(x),H(x)}(\zeta) = \sum_{k=2}^{\infty} L_{G,H,k}(x)\, \zeta^k = \frac{2 - G(x)\zeta}{1 - G(x)\zeta - H(x)\zeta^2}. \tag{1.3}$$

Subordination. If the holomorphic functions f and g are in $\mathbb{U}$, then f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(\zeta)| < 1$ such that $f(\zeta) = g(w(\zeta))$. When g is univalent, $f \prec g$ iff $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$; see [13].

Neutrosophic Probability and the Poisson Series. Classical probability distributions operate with precisely known parameters. In many real-world scenarios the rate parameter is itself uncertain, motivating *neutrosophic statistics* [16], wherein a real parameter m is replaced by the neutrosophic number $m_N = m + I \cdot m_I$ with $m_N \in [m_1, m_2]$ and I the indeterminate component. Applications of the Poisson distribution series to analytic functions were initiated by Porwal [17,18] and extended by Murugusundaramoorthy et al. [28]. The resulting neutrosophic Poisson distribution was applied to bi-univalent function theory by Santhiya and Thilagavathi [23] via the series

$$\Psi(\vartheta_N, \zeta) = \zeta + \sum_{k=2}^{\infty} \frac{(\vartheta_N)^{k-1}}{(k-1)!}\, e^{-\vartheta_N}\, \zeta^k.$$

Touchard (Bell) Polynomials. The *Touchard polynomials* [25,26], also known as exponential polynomials [27], arise from the n-th moment of a Poisson(m) random variable X:

$$T_n(m) = E(X^n) = e^{-m} \sum_{k=0}^{\infty} \frac{m^k k^n}{k!}.$$

The associated generating series was introduced into analytic function theory by Al-Shaqsi [24]:

$$F_n(\zeta, m) = \zeta + \sum_{k=2}^{\infty} \frac{m^{k-1}(k-1)^n}{(k-1)!} e^{-m} \zeta^k, \qquad n \geq 0,\ m > 0. \tag{1.4}$$

Setting $n = 0$ in (1.4) recovers the Poisson series of [17].

Motivation. Replacing m in (1.4) by a neutrosophic interval $m_N \in [m_1, m_2]$ yields the *neutrosophic Touchard distribution series*, simultaneously generalising: (i) the neutrosophic Poisson series of [23] (set $n = 0$); (ii) the Touchard series of [24] (set $m_N = m$ crisp); (iii) the Poisson series of [17] ($n = 0$, $m_N = m$ crisp). Combining this series with the modified Caputo operator of Salah [19,20] produces the *neutrosophic Touchard–Caputo operator*, studied in this paper.

2. The Neutrosophic Touchard–Caputo Operator

Definition 2.1. *Let $m_N \in [m_1, m_2]$ with $m_1, m_2 > 0$ and $n \geq 0$. The* neutrosophic Touchard distribution series *is*

$$\Psi_n(\zeta, m_N) = \zeta + \sum_{k=2}^{\infty} \frac{(m_N)^{k-1}(k-1)^n}{(k-1)!} e^{-m_N} \zeta^k, \qquad \zeta \in \mathbb{U}. \tag{2.1}$$

Remark 2.1. *(i) $n = 0$:* (2.1) *reduces to the neutrosophic Poisson series of [23]. (ii) $m_N = m$ (crisp):* (2.1) *reduces to $F_n(\zeta, m)$ of [24]. (iii) $n = 0$, $m_N = m$ crisp:* (2.1) *reduces to [17].*

Definition 2.2. *For $f \in \mathcal{A}$, the* neutrosophic Al-Shaqsi operator $\mathcal{I}_n^{m_N} : \mathcal{A} \to \mathcal{A}$ *is defined via the Hadamard product by*

$$\mathcal{I}_n^{m_N} f(\zeta) = \Psi_n(\zeta, m_N) * f(\zeta) = \zeta + \sum_{k=2}^{\infty} \frac{(m_N)^{k-1}(k-1)^n}{(k-1)!} e^{-m_N} a_k \zeta^k. \tag{2.2}$$

Definition 2.3 (Salah–Darus [19,20])**.** *The* modified Caputo derivative operator *of order δ is*

$$I_\delta^\eta f(\zeta) = \frac{\Gamma(2+\eta-\delta)}{\eta-\delta} \zeta^{\delta-\eta} \int_0^\zeta \frac{\Psi_\eta f(\tau)}{(\zeta-\tau)^{\delta+1-\eta}} d\tau, \tag{2.3}$$

where $\eta - 1 < \delta \leq \eta < 2$ and $\Psi_\eta f(\tau) = \Gamma(2-\eta)\, \tau^\eta D_\tau^\eta f(\tau)$. For $f \in \mathcal{A}$:

$$I_\delta^\eta f(\zeta) = \zeta + \sum_{k=2}^{\infty} \frac{[\Gamma(k+1)]^2\, \Gamma(2+\eta-\delta)\, \Gamma(2-\eta)}{\Gamma(k+\eta-\delta+1)\, \Gamma(k-\eta+1)} a_k \zeta^k. \tag{2.4}$$

Definition 2.4. *For $f \in \mathcal{A}$, $m_N \in [m_1, m_2]$, $n \geq 0$, and $\eta - 1 < \delta \leq \eta < 2$, the* neutrosophic Touchard–Caputo operator $\mathcal{M}_{\delta, m_N, n}^\eta : \mathcal{A} \to \mathcal{A}$ *is defined by*

$$\mathcal{M}_{\delta, m_N, n}^\eta f(\zeta) = I_\delta^\eta * \mathcal{I}_n^{m_N} f(\zeta) = \zeta + \sum_{k=2}^{\infty} \Phi_k a_k \zeta^k, \tag{2.5}$$

where the combined weight is

$$\Phi_k = \frac{(m_N)^{k-1}(k-1)^n \cdot k^2\, \Gamma(k)\, \Gamma(2+\eta-\delta)\, \Gamma(2-\eta)}{(k-1)!\, \Gamma(k+\eta-\delta+1)\, \Gamma(k-\eta+1)} e^{-m_N}, \tag{2.6}$$

noting that $k^2\Gamma(k) = [\Gamma(k+1)]^2/\Gamma(k)$*. Using* $\Gamma(z+1) = z\,\Gamma(z)$*, the first two weights simplify to*

$$\Phi_2 = 4(1+\eta-\delta)(1-\eta)\, m_N\, e^{-m_N}, \tag{2.7}$$

$$\Phi_3 = 2^n(2+\eta-\delta)(2-\eta)(m_N)^2\, e^{-m_N}. \tag{2.8}$$

3. The Bi-Univalent Function Family

Definition 3.1. *Let* $\lambda \geq 0$*,* $\mu \geq 1$*,* $0 \leq \alpha \leq 1$*,* $\eta - 1 < \delta \leq \eta < 2$*,* $m_N \in [m_1, m_2]$*,* $n \geq 0$*, and let* $e : \mathbb{U} \to \mathbb{C}$ *be analytic with* $e(0) = 1$*,* $e(\zeta) = 1 + h_1\zeta + h_2\zeta^2 + \cdots$*. A function* $f \in \Sigma$ *belongs to* $\widetilde{\Upsilon}_\Sigma(\alpha, \lambda, \mu, \eta, \delta, m_N, n;\, e)$ *if the following two subordination conditions hold:*

$$(1-\alpha)\frac{\zeta^{1-\lambda}(\mathcal{M}^{\eta}_{\delta,m_N,n}f(\zeta))'}{(\mathcal{M}^{\eta}_{\delta,m_N,n}f(\zeta))^{1-\lambda}} + \alpha\frac{\zeta(\mathcal{M}^{\eta}_{\delta,m_N,n}f(\zeta))'^{\mu}}{\mathcal{M}^{\eta}_{\delta,m_N,n}f(\zeta)} \prec e(\zeta), \tag{3.1}$$

$$(1-\alpha)\frac{\omega^{1-\lambda}(\mathcal{M}^{\eta}_{\delta,m_N,n}f^{-1}(\omega))'}{(\mathcal{M}^{\eta}_{\delta,m_N,n}f^{-1}(\omega))^{1-\lambda}} + \alpha\frac{\omega(\mathcal{M}^{\eta}_{\delta,m_N,n}f^{-1}(\omega))'^{\mu}}{\mathcal{M}^{\eta}_{\delta,m_N,n}f^{-1}(\omega)} \prec e(\omega), \tag{3.2}$$

where f^{-1} *is given by* (1.2).

Remark 3.1. *(1)* $\alpha = 1$*: reduces to the* μ*-pseudo-bi-starlike family* $\mathcal{N}_\Sigma(\mu, \eta, \delta, m_N, n;\, e)$*. (2)* $\alpha = 0$*: reduces to the bi-Bazilevic family* $\mathcal{C}_\Sigma(\lambda, \eta, \delta, m_N, n;\, e)$*. (3)* $\alpha = \mu = 1$*: reduces to the bi-starlike family* $\mathcal{D}_\Sigma(\eta, \delta, m_N, n;\, e)$*. (4)* $n = 0$*: reduces to the family studied in [23].*

4. Main Results

Throughout we use the notation

$$\begin{aligned} E &= (1+\eta-\delta)(1-\eta)\Big[(1-\alpha)(\lambda+1) + 2\delta(\mu-1)\Big] m_N\, e^{-m_N}, \\ F &= 2^{n-1}(2+\eta-\delta)(2-\eta)\Big[18(1-\alpha)(\lambda+1) + 54\delta(\mu-1)\Big](m_N)^2 e^{-m_N}, \\ J &= (1+\eta-\delta)^2(1-\eta)^2\Big[(1-\alpha)(\lambda^2+31\lambda-32) + 32\delta(\mu^2-3\mu+2)\Big](m_N)^2 e^{-2m_N}. \end{aligned} \tag{4.1}$$

Remark 4.1. *Setting* $n = 0$ *gives* $2^{n-1} = \frac{1}{2}$ *in* F*, which recovers the constants* E, F, J *of [23] (with* $m_N = \vartheta_N$*).*

Theorem 4.1. *Under the hypotheses of Definition 3.1, if* $f \in \widetilde{\Upsilon}_\Sigma(\alpha, \lambda, \mu, \eta, \delta, m_N, n;\, e)$*, then*

$$|d_2| \leq \frac{|h_1|}{4E} \tag{4.2}$$

and

$$|d_3| \leq \min\!\left(\max\left\{\frac{|h_1|}{F}, \left|\frac{h_2}{F} - \frac{J\,h_1^2}{16E^2F}\right|\right\},\ \max\left\{\frac{|h_1|}{F}, \left|\frac{h_2}{F} - \frac{(2F+J)h_1^2}{16E^2F}\right|\right\}\right), \tag{4.3}$$

where E*,* F*,* J *are given by* (4.1).

Proof. Since $f \in \widetilde{\Upsilon}_\Sigma$, there exist holomorphic Schwarz functions $\chi, \varphi : \mathbb{U} \to \mathbb{U}$ with $\chi(0) = \varphi(0) = 0$, written as

$$\chi(\zeta) = b_1\zeta + b_2\zeta^2 + \cdots, \tag{4.4}$$

$$\varphi(\omega) = l_1\omega + l_2\omega^2 + \cdots, \tag{4.5}$$

satisfying $|b_n| \le 1$ and $|l_n| \le 1$ for all n by the Schwarz lemma. Expanding (3.1)–(3.2) using (2.5) and comparing coefficients of ζ^1, ζ^2 and ω^1, ω^2 yields:

$$4E\,d_2 = h_1 b_1, \qquad F\,d_3 + J\,d_2^2 = h_1 b_2 + h_2 b_1^2, \tag{4.6}$$

$$-4E\,d_2 = h_1 l_1, \quad F(2d_2^2 - d_3) + J\,d_2^2 = h_1 l_2 + h_2 l_1^2. \tag{4.7}$$

Bound for $|d_2|$. From (4.6) and $|b_1| \le 1$: $4E|d_2| \le |h_1|$, giving (4.2).

First bound for $|d_3|$. Solving (4.6) for d_3: $Fd_3 = h_1 b_2 + (h_2 - Jh_1^2/16E^2)b_1^2$. The sharp inequality $|b_2 - \gamma b_1^2| \le \max\{1, |\gamma|\}$ for all $\gamma \in \mathbb{C}$ [3,29] gives

$$|d_3| \le \max\left\{\frac{|h_1|}{F}, \left|\frac{h_2}{F} - \frac{Jh_1^2}{16E^2F}\right|\right\}. \tag{4.8}$$

Second bound for $|d_3|$. From (4.7): $-Fd_3 = h_1 l_2 + (h_2 - (2F+J)h_1^2/16E^2)l_1^2$, giving

$$|d_3| \le \max\left\{\frac{|h_1|}{F}, \left|\frac{h_2}{F} - \frac{(2F+J)h_1^2}{16E^2F}\right|\right\}. \tag{4.9}$$

Taking the minimum of (4.8) and (4.9) yields (4.3). □

Theorem 4.2. *Under the hypotheses of Theorem 4.1, for any* $\beta \in \mathbb{C}$*:*

$$|d_3 - \beta d_2^2| \le \frac{|h_1|}{F} \min\left(\max\left\{1, \left|\frac{h_2}{h_1} - \frac{(J-\beta F)\,h_1}{16E^2}\right|\right\},\ \max\left\{1, \left|\frac{h_2}{h_1} - \frac{(2F+J-\beta F)\,h_1}{16E^2}\right|\right\}\right). \tag{4.10}$$

Proof. From (4.6): $d_3 - \beta d_2^2 = (h_1/F)[b_2 + (h_2/h_1 - (J-\beta F)h_1/16E^2)b_1^2]$. Applying [3,29] gives the first estimate. From (4.7): $d_3 - \beta d_2^2 = (h_1/F)[l_2 + (h_2/h_1 - (2F+J-\beta F)h_1/16E^2)l_1^2]$. The second estimate follows similarly. Taking the minimum yields (4.10). □

5. Corollaries

Taking $e(\zeta) = \mathcal{M}_{G(x),H(x)}(\zeta) - 1$ (equation (1.3)) gives $h_1 = G(x)$ and $h_2 = G^2(x) + 2H(x)$.

Corollary 5.1. *If* $f \in \widetilde{\Upsilon}_\Sigma(\alpha, \lambda, \mu, \eta, \delta, m_N, n;\ \mathcal{M}_{G(x),H(x)} - 1)$*, then*

$$|d_2| \le \frac{|G(x)|}{4E}, \tag{5.1}$$

$$\begin{aligned}|d_3| \le \min\Bigg(&\max\left\{\frac{|G(x)|}{F}, \left|\frac{G^2(x)+2H(x)}{F} - \frac{JG^2(x)}{16E^2F}\right|\right\},\\ &\max\left\{\frac{|G(x)|}{F}, \left|\frac{G^2(x)+2H(x)}{F} - \frac{(2F+J)G^2(x)}{16E^2F}\right|\right\}\Bigg),\end{aligned} \tag{5.2}$$

$$\begin{aligned}|d_3 - \beta d_2^2| \le \frac{|G(x)|}{F} \min\Bigg(&\max\left\{1, \left|\frac{G^2(x)+2H(x)}{G(x)} - \frac{(J-\beta F)G(x)}{16E^2}\right|\right\},\\ &\max\left\{1, \left|\frac{G^2(x)+2H(x)}{G(x)} - \frac{(2F+J-\beta F)G(x)}{16E^2}\right|\right\}\Bigg),\end{aligned} \tag{5.3}$$

for all admissible parameters, where E*,* F*,* J *are given by* (4.1) *and* $\mathcal{M}_{G(x),H(x)}$ *by* (1.3).

Corollary 5.2 ($\alpha = 1$: μ-pseudo-bi-starlike family). *With constants*

$$E_1 = (1+\eta-\delta)(1-\eta)\cdot 2\delta(\mu-1)\cdot m_N e^{-m_N},$$
$$F_1 = 2^{n-1}(2+\eta-\delta)(2-\eta)\cdot 54\delta(\mu-1)\cdot (m_N)^2 e^{-m_N},$$
$$J_1 = (1+\eta-\delta)^2(1-\eta)^2\cdot 32\delta(\mu^2-3\mu+2)\cdot (m_N)^2 e^{-2m_N},$$

the bounds (4.2)–(4.10) *hold with E, F, J replaced by E_1, F_1, J_1.*

Corollary 5.3 ($\alpha = 0$: bi-Bazilevic family). *With constants*

$$E_0 = (1+\eta-\delta)(1-\eta)(\lambda+1)\cdot m_N e^{-m_N},$$
$$F_0 = 2^{n-1}(2+\eta-\delta)(2-\eta)\cdot 18(\lambda+1)\cdot (m_N)^2 e^{-m_N},$$
$$J_0 = (1+\eta-\delta)^2(1-\eta)^2(\lambda^2+31\lambda-32)\cdot (m_N)^2 e^{-2m_N},$$

the bounds (4.2)–(4.10) *hold with E, F, J replaced by E_0, F_0, J_0.*

6. Numerical Case Study

Parameter Setup. We fix $\alpha = 0$, $\lambda = 1$, $\mu = 2$, $\eta = 0.8$, $\delta = 0.5$, $m_N \in [2, 5]$, and $e(\zeta) = (1+\zeta)/(1-\zeta)$, giving $h_1 = h_2 = 2$. The constraint $\eta - 1 < \delta \le \eta < 2$ reads $-0.2 < 0.5 \le 0.8 < 2$ (satisfied). With these values $\lambda^2 + 31\lambda - 32 = 0$ and $\mu^2 - 3\mu + 2 = 0$, so $J_0 = 0$. The reduced constants from Corollary 5.3 are

$$E_0 = 0.52\, m_N e^{-m_N}, \quad F_0 = 2^{n-1}\cdot 99.36\,(m_N)^2 e^{-m_N}.$$

Computed Bounds.

n	2^{n-1}	$\lvert d_2\rvert \le (m_N = 2)$	$\lvert d_2\rvert \le (m_N = 5)$	$\lvert d_3\rvert \le (m_N = 2)$
0	0.50	3.56	28.54	0.0370
1	1.00	3.56	28.54	0.0185
2	2.00	3.56	28.54	0.0093
3	4.00	3.56	28.54	0.0046

The neutrosophic interval gives interval-valued bounds: $|d_2| \in [3.56, 28.54]$ and $|d_3|^{(n=2)} \in [0.0093, 0.060]$. Each unit increase in n halves the $|d_3|$ bound; $|d_2|$ is independent of n.

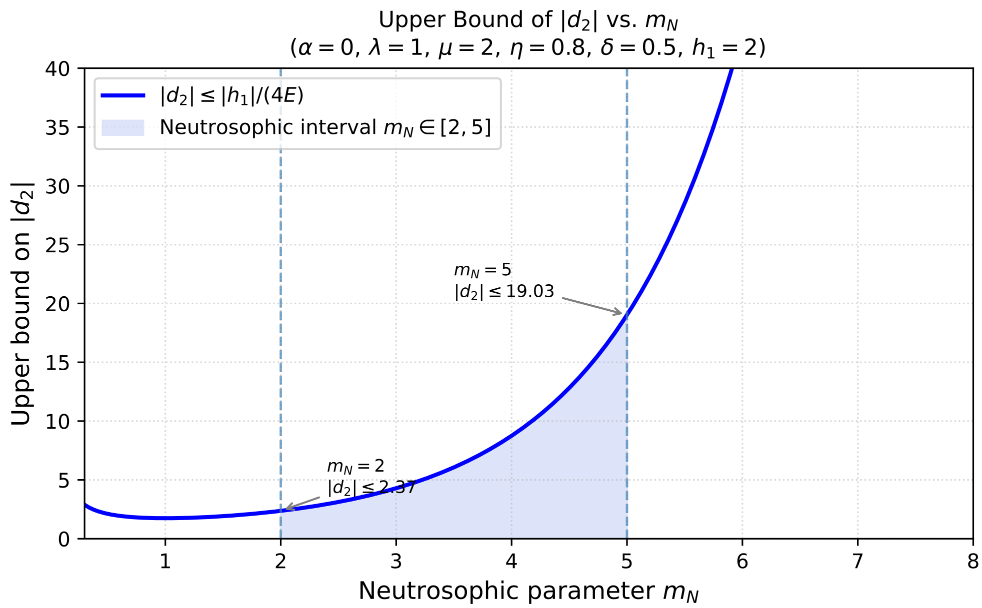

FIGURE 1. Upper bound on $|d_2|$ vs. the neutrosophic parameter m_N. The bound is independent of n; the shaded region marks $m_N \in [2, 5]$.

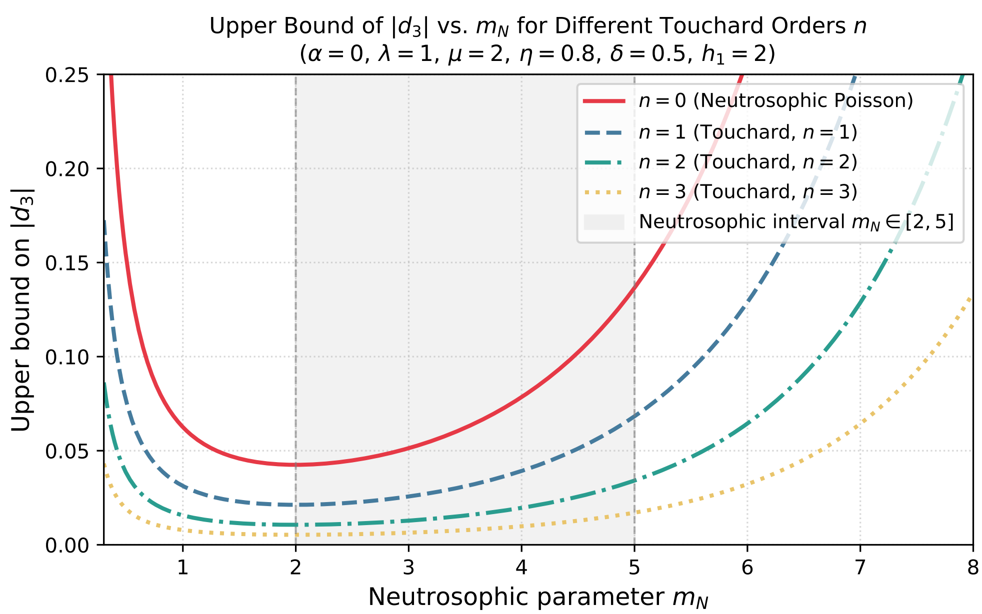

FIGURE 2. Upper bound on $|d_3|$ vs. m_N for $n = 0, 1, 2, 3$. The case $n = 0$ is the neutrosophic Poisson baseline [23]; each unit increase in n halves the bound.

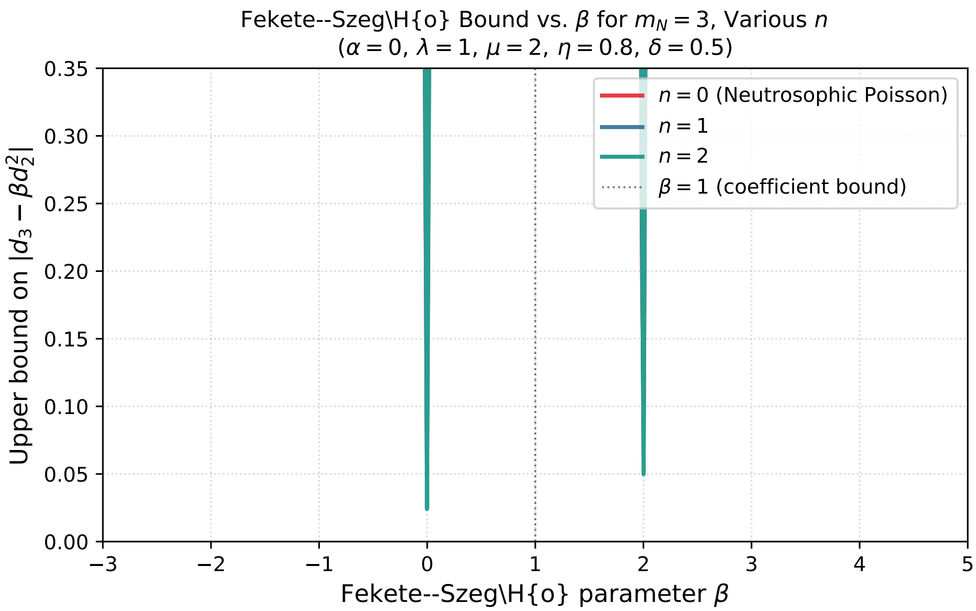

FIGURE 3. Fekete–Szegő bound $|d_3 - \beta d_2^2|$ vs. real β for $m_N = 3$, $n = 0, 1, 2$ (with $J_0 = 0$). Higher n gives uniformly smaller bounds.

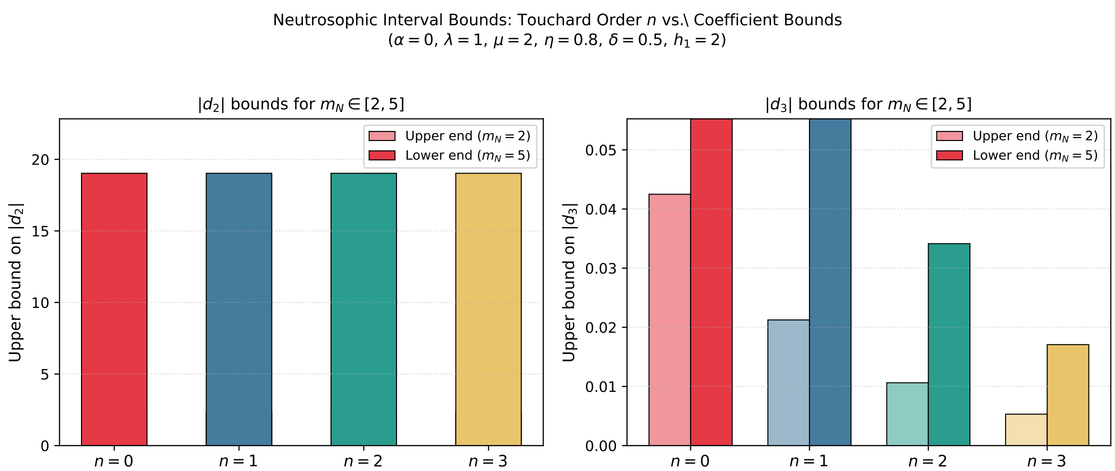

FIGURE 4. Interval bounds on $|d_2|$ (left) and $|d_3|$ (right) for $n = 0, 1, 2, 3$ and $m_N \in [2, 5]$. Faded bars: $m_N = 2$; solid bars: $m_N = 5$.

7. Special Cases Recovering Known Results

Parameter choice	Family reduces to	Reference
$n = 0$	Neutrosophic Poisson + Caputo	[23]
$m_N = m$ crisp, any n	Touchard + Caputo	New (this paper)
$n = 0$, $m_N = m$ crisp	Poisson + Caputo	[17]
$n = 0$, Ruscheweyh op.	Poisson + Ruscheweyh	[22]
$n = 0$, no Caputo	Neutrosophic + Chebyshev	[21]
$\alpha = \mu = 1$	Bi-starlike neutrosophic Touchard	New corollary
$\alpha = 0$	Bi-Bazilevic neutrosophic Touchard	Corollary 5.3

8. Conclusion

We introduced the neutrosophic Touchard–Caputo operator $\mathcal{M}^{\eta}_{\delta,m_N,n}$, generalising the neutrosophic Poisson–Caputo operator of [23], and applied it to define the bi-univalent family $\widetilde{\Upsilon}_{\Sigma}(\alpha,\lambda,\mu,\eta,\delta,m_N,n;\, e)$. We proved sharp upper bounds for $|d_2|$ (Theorem 4.1), upper bounds for $|d_3|$ (Theorem 4.1), and the Fekete–Szegő inequality (Theorem 4.2). All results of [17,21–24] are recovered as special cases. Four figures illustrate the effect of the Touchard order n on the bounds and the propagation of neutrosophic uncertainty.

Open problems for future work include: (1) estimation of the Hankel determinant $H_2(2) = a_2a_4 - a_3^2$; (2) Faber polynomial coefficient estimates for $|d_k|$, $k \geq 4$; (3) study of $\mathcal{M}^{\eta}_{\delta,m_N,n}$ for harmonic bi-univalent functions; (4) a q-analogue of the neutrosophic Touchard distribution.

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References

[1] L. Bieberbach, Uber die Koeffizienten derjenigen Potenzreihen Welche eine Schlichte Abbildung des Einheitskreises Vermitteln, Reimer, 1916.

[2] L. de Branges, A Proof of the Bieberbach Conjecture, Acta Math. 154 (1985), 137–152. https://doi.org/10.1007/BF02392821.

[3] P.L. Duren, Univalent Functions, Springer, 1983.

[4] H.M. Srivastava, A.K. Mishra, P. Gochhayat, Certain Subclasses of Analytic and Bi-Univalent Functions, Appl. Math. Lett. 23 (2010), 1188–1192. https://doi.org/10.1016/j.aml.2010.05.009.

[5] B. Frasin, M. Aouf, New Subclasses of Bi-Univalent Functions, Appl. Math. Lett. 24 (2011), 1569–1573. https://doi.org/10.1016/j.aml.2011.03.048.

[6] S.P. Goyal, P. Goswami, Estimate for Initial Maclaurin Coefficients of Bi-Univalent Functions for a Class Defined by Fractional Derivatives, J. Egypt. Math. Soc. 20 (2012), 179–182. https://doi.org/10.1016/j.joems.2012.08.020.

[7] H.M. Srivastava, D. Bansal, Coefficient Estimates for a Subclass of Analytic and Bi-Univalent Functions, J. Egypt. Math. Soc. 23 (2015), 242–246. https://doi.org/10.1016/j.joems.2014.04.002.

[8] M. Caglar, E. Deniz, H.M. Srivastava, Second Hankel Determinant for Certain Subclasses of Bi-Univalent Functions, Turk. J. Math. 41 (2017), 694–706. https://doi.org/10.3906/mat-1602-25.

[9] H.M. Srivastava, S.S. Eker, S.G. Hamidi, J.M. Jahangiri, Faber Polynomial Coefficient Estimates for Bi-univalent Functions Defined by the Tremblay Fractional Derivative Operator, Bull. Iran. Math. Soc. 44 (2018), 149–157. https://doi.org/10.1007/s41980-018-0011-3.

[10] A.K. Wanas, A.L. Alina, Applications of Horadam Polynomials on Bazilevic Bi-Univalent Function Satisfying Subordinate Conditions, J. Phys.: Conf. Ser. 1294 (2019), 032003. https://doi.org/10.1088/1742-6596/1294/3/032003.

[11] A. Lupas, A Guide of Fibonacci and Lucas Polynomials, Octogon Math. Mag. 7 (1999), 2–12.

[12] G. Lee, M. Asci, Some Properties of the (p,q)-Fibonacci and (p,q)-Lucas Polynomials, J. Appl. Math. 2012 (2012), 264842. https://doi.org/10.1155/2012/264842.

[13] S.S. Miller, P.T. Mocanu, Differential Subordinations: Theory and Applications, Marcel Dekker, 2000.

[14] R. Singh, On Bazilevič Functions, in: Proceedings of the American Mathematical Society, American Mathematical Society (AMS), 1973, pp. 261–271. https://doi.org/10.1090/S0002-9939-1973-0311887-9.

[15] K.O. Babalola, On λ-Pseudo-Starlike Functions, J. Class. Anal. 3 (2013), 137–147. https://doi.org/10.7153/jca-03-12.

[16] R. Alhabib, M.M. Ranna, H. Farah, A.A. Salama, Some Neutrosophic Probability Distributions, Neutrosoph. Sets Syst. 22 (2018), 30–37.

[17] S. Porwal, An Application of a Poisson Distribution Series on Certain Analytic Functions, J. Complex Anal. 2014 (2014), 1–3. https://doi.org/10.1155/2014/984135.

[18] S. Porwal, M. Kumar, A Unified Study on Starlike and Convex Functions Associated with Poisson Distribution Series, Afr. Mat. 27 (2016), 1021–1027. https://doi.org/10.1007/s13370-016-0398-z.

[19] J. Salah, A Note on the Modified Caputo's Fractional Calculus Derivative Operator, Far East J. Math. Sci. 100 (2016), 609–615. https://doi.org/10.17654/ms100040609.

[20] J. Salah, M. Darus, A Subclass of Uniformly Convex Functions Associated with a Fractional Calculus Operator Involving Caputo's Fractional Differentiation, Acta Univ. Apulensis Math. Inform. 24 (2010), 295–306.

[21] A.T. Oladipo, Bounds for Poisson and Neutrosophic Poisson Distributions Associated with Chebyshev Polynomials, Palestine J. Math. 10 (2021), 169–174.

[22] A.K. Wanas, J. Sokół, Applications of Poisson Distribution and Ruscheweyh Derivative Operator for Bi-Univalent Functions, Kragujevac J. Math. 48 (2024), 89–97. https://doi.org/10.46793/KgJMat2401.089W.

[23] S. Santhiya, K. Thilagavathi, Applications of the Neutrosophic Poisson Distribution for Bi-Univalent Functions Involving the Modified Caputo's Derivative Operator, Fractal Fract. 7 (2023), 35. https://doi.org/10.3390/fractalfract7010035.

[24] K. Al-Shaqsi, On Inclusion Results of Certain Subclasses of Analytic Functions Associated with Generating Function, AIP Conf. Proc. 1830 (2017), 070030. https://doi.org/10.1063/1.4980979.

[25] J. Touchard, Sur les Cycles des Substitutions, Acta Math. 70 (1939), 243–297. https://doi.org/10.1007/BF02547349.

[26] E.T. Bell, Exponential Polynomials, Ann. Math. 35 (1934), 258–277. https://doi.org/10.2307/1968431.

[27] Y. Simsek, Special Numbers on Analytic Functions, Appl. Math. 05 (2014), 1091–1098. https://doi.org/10.4236/am.2014.57102.

[28] G. Murugusundaramoorthy, K. Vijaya, S. Porwal, Some Inclusion Results of Certain Subclass of Analytic Functions Associated with Poisson Distribution Series, Hacettepe J. Math. Stat. 45 (2016), 1101–1107. https://doi.org/10.15672/hjms.20164513110.

[29] F.R. Keogh, E.P. Merkes, A Coefficient Inequality for Certain Classes of Analytic Functions, in: Proceedings of the American Mathematical Society, American Mathematical Society (AMS), 1969, pp. 8–12. https://doi.org/10.1090/S0002-9939-1969-0232926-9.