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# Significant Approach of Hermite-Hadamard Type Fractional Inequalities for MET-$(p,s)$-Convex Function and its Applications

**Mamoona Siddiq[1], Fazilat Bibi[1], Rana Safder[1], Jamal Salah[2,*], Aseel Smerat[3,4]**

[1]*Department of Mathematics and Statistics, The University of Lahore, Sargodha 40100, Pakista*

[2]*College of Applied and Health Sciences, A'Sharqiyah University, P.O. Box 42, Post Code 400, Ibra, Sultanate of Oman*

[3]*Hourani Center for Applied Scientific Research, Al-Ahliyya Amman University, Amman 19328, Jordan*

[4]*Department of Biosciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai, 602105, India*

[*]*Corresponding author:* damous73@yahoo.com

**Abstract.** There is a very close relation between convexity and fractional operators to improve well-known inequalities. Different types of inequality have been examined, providing new ways in the inequality literature. In this paper, we discuss Hermite-Hadamard extensions by successfully implementation of modified exponential trigonometry (MET)-$(p,s)$-convexity with Riemann–Liouvile fractional operator (RL-FO). Moreover, we extract some corollaries which show the straingth of our main results. Furthermore, we discuss numerical implementation for specfic cases, special values of the parameters, and demonstrate that the derived outcome generalizes numerous inequalities that have been previously found in the literature.

## 1. Introduction

Fractional integral inequalities have been very well investigated in the last several decades due to their extensive applications in analysis, optimization theory, applied mathematics and fractional calculus [1–3]. Fractional operators are very useful to extend classical inequalities and find applications in physics, engineering, control theory, numerical analysis and differential equations [4–6]. In the generalizations of integral inequalities and convex analysis [7–9] The Riemann-Liouville fractional integral operator has been widely applied in particular.

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The Hermite-Hadamard inequality is one of the basic, solid and well-known classical inequalities in convex analysis [10, 11]. Let

$$g : I \subseteq \mathbb{R} \to \mathbb{R}$$

is a convex function on $[k_1, k_2]$, then the Hermite–Hadamard inequality can be expressed as

$$g\left(\frac{k_1+k_2}{2}\right) \leq \frac{1}{k_2-k_1}\int_{k_1}^{k_2} g(x)\,dx \leq \frac{g(k_1)+g(k_2)}{2}.$$

The inequality provides bounds on an interval for the mean value of a convex function , and is a fundamental tool in the theory of inequalities. Since this inequality is of great importance, it has been attempted to get some improvements and generalisations of Hermite-Hadamard type inequalities by several types of convex functions and fractional integral operators, see [12–14].

Lately, many classes of convex functions have been defined and investigated such as s-convex functions, Godunova-Levin convex functions, h-convex functions, trigonometric convex functions, exponentially convex functions, harmonic convex functions and preinvex functions, etc. [15, 16]. For these generalized convex functions, new bounds are expected to be achieved, until so far there are some contributions about the generalization of Hermite-Hadamard-type, Fejr-Hadamard-type, Ostrowski-type and Chebyshev type inequalities, respectively [17–19].

The generalized convexity and fractional calculus have been very active research areas in recent years and their application to approximation theory and optimization, engineering and other areas is apparent. Using a combination of convexity theory and fractional operators, it is possible to obtain much more precise estimates for functions with a non-local and memory-dependent characteristic, which could lead to the development of more effective tools and techniques. They can enhance and improve bounds and analysis which can yield valuable results in diverse areas. In fact, important implications and limits have been demonstrated from these theories together and it is an area that needs to be explored.

The modified exponential trigonometric convexity has recently been proposed by Budak and Grbz in [3]. We generalize the class of modified exponential trigonometric convexity to the parameterized p-convexity and trigonometrical and exponential convexity by combining them together, which results in MET-$(p,s)$-convex functions. The Riemann-Liouville fractional integral operator was applied to some inequalities such as Hermite-Hadamard inequality to extend them to the h-convex, Godunova-Levin convex functions. But there has not been any attempt to explore these inequalities using the new concept MET-$(p,s)$-convexity and also that of h-Godunova-Levin preinvex functions and there is not a result which relatesMET-$(p,s)$-convex functions to these inequalities and new fractional integral inequalities on such convexity have not introduced. This work is a welcome addition to the void. our goal in this paper is to improve some Hermite-Hadamard type inequalities for h-Godunova-Levin preinvex functions in conjunction with MET-$(p,s)$-convexity by utilizing the Riemann-Liouville fractional integral operator. Making use of MET-$(p,s)$-convexity in the already proved inequalities we derive some new Hermite-Hadamard type fractional inequalities. We also make new definitions of lemmas, corollaries and special cases.

We also established new general results which contain previously known inequalities from [26–28] or other papers. The new inequality extends more cases for Hermite-Hadamard type inequality for fractional integral operator in the framework of MET-$(p, s)$-convexity.

## 2. Preliminareis

In this section, we recall some basic definitions and results which will be used throughout the paper.

**Definition 2.1.** *[29,30] A function $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$g(x + (1-\eta)y) \leq \eta(x) + (1-\eta)g(y)$$

*where $x, y \in J$ and $\eta \in [0,1]$.*

**Definition 2.2.** *[31] A set $Z \subseteq \mathbb{R}^n$ is called an invex set with a bifunction*

$$\beta : Z \times Z \to \mathbb{R}^n,$$

*if*

$$a + t\beta(b,a) \in Z$$

*where $a, b \in Z$ and $t \in [0,1]$.*

**Definition 2.3.** *[31] A function $f : K \to \mathbb{R}$ is preinvex w.r.t bi-function $\beta$ if*

$$f(a + t\beta(b,a)) \leq (1-t)f(a) + tf(b) \tag{2.1}$$

*where $a, b \in K$ and $t \in [0,1]$.*

**Definition 2.4.** *[32] Let $J : [0,\infty) \to \mathbb{R}$, then $J$ is a s-convex function in the second sense if*

$$J(\xi + (1-\xi)b) \leq \xi^s J(a) + (1-\xi)^s J(b) \tag{2.2}$$

*where $a, b \geq 0$, $\xi \in [0,1]$, and $s \in (0,1]$.*

**Definition 2.5.** *[15] Let $h : I \subseteq (0,\infty) \to \mathbb{R}$ be real valued function, is said to be a $p$-convex if*

$$h\left((\tau a^p + (1-\tau)b^p)^{\frac{1}{p}}\right) \leq \tau h(a) + (1-\tau)h(b), \tag{2.3}$$

*for all $a, b \in I$, $\tau \in [0,1]$ and $p \neq 0$.*

**Definition 2.6.** *[31] For $a, b \in I$ and $\tau \in (0,1)$, then a function $\nu : I \to \mathbb{R}$ is said to be a Godunova–Levin (G-L) function if it satisfies the following inequality:*

$$\nu(\tau a + (1-\tau)b) \leq \frac{\nu(a)}{\tau} + \frac{\nu(b)}{1-\tau} \tag{2.4}$$

**Definition 2.7.** *[32] For $p \in \mathbb{R} \setminus \{0\}$, $s \in (0,1]$, $a_1, b_1 \in I$ and $\tau \in [0,1]$, then the function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be modified exponential trigonometry (MET)-$(p,s)$-convex function if it satisfies the following inequality:*

$$f\left(\left(\tau\, a_1^p + (1-\tau)\, b_1^p\right)^{\frac{1}{p}}\right) \;\leq\; \sin\left(\frac{\pi\tau}{2}\right)e^{(1-\tau)s}\, f(a_1) \;+\; \cos\left(\frac{\pi\tau}{2}\right)e^{\tau s}\, f(b_1). \tag{2.5}$$

Here the weight functions

$$W_1(t) \;=\; \sin\left(\tfrac{\pi\tau}{2}\right)\cdot e^{(1-\tau)s} \qquad \text{and} \qquad W_2(\tau) \;=\; \cos\left(\tfrac{\pi\tau}{2}\right)\cdot e^{\tau s}$$

combine trigonometric oscillation with exponential growth/decay, controlled by the parameters $p$ and $s$. This creates a rich family of convex-type functions that contains many previously known classes as special cases.

**Remark 2.1.** *The MET-$(p,s)$-convex function recovers the following special cases:*

- (i) *When $p = 1$ and $s \to 0$: reduces to classical convexity, since $\sin(\pi\tau/2) \approx \tau$ and $\cos(\pi\tau/2) \approx 1-\tau$ near $\tau = 0$.*
- (ii) *When $p = 1$ and $s = 1$: gives the basic MET-convex function.*
- (iii) *The parameter $p \neq 0$ allows the power-mean interpolation $\left(\tau\, a_1^p + (1-\tau)\, b_1^p\right)^{1/p}$ of the arguments, which for $p = 1$ becomes the arithmetic mean $\tau\, a_1 + (1-\tau)\, b_1$.*

Weight Normalization At $\tau = 0$:

$$W_1(0) = 0, \qquad W_2(0) = e^0 = 1,$$

so $f(b_1)$ is recovered. At $\tau = 1$:

$$W_1(1) = e^0 = 1, \qquad W_2(1) = 0,$$

so $f(a_1)$ is recovered. This ensures *endpoint consistency* of the MET-$(p,s)$-convex inequality.

## 3. Main Results

Here, we discuss the generalization of Hermite-Hadamard (H-H) type inequalities by implementation of MET-$(p,s)$-convex function with Reimann-Liouville (R-L) fractional operators, and extract some corollaries. Furthermore, we give some examples and draw their graphs that verify our main results.

**Theorem 3.1.** *Let $f : [\mu_1, \mu_2] \to \mathbb{R}$ be a positive MET-$(p,s)$-convex function on $[\mu_1, \mu_2]$, where $0 < \mu_1 < \mu_2$, $p \in \mathbb{R} \setminus \{0\}$, $s \in (0,1]$, and $f \in L_1[\mu_1, \mu_2]$. Then, for the Riemann–Liouville fractional operators (RL-FO) of order $\eta > 0$, the following H-H inequality holds:*

$$f\left(\left(\frac{\mu_1^p + \mu_2^p}{2}\right)^{\frac{1}{p}}\right)\frac{1}{\eta} \leq \frac{\Gamma(\eta+1)}{(\mu_2-\mu_1)^\eta}\left[x^\eta_{\mu_1^+} f(\mu_2) + x^\eta_{\mu_2^-} f(\mu_1)\right]$$
$$\leq [f(\mu_1) + f(\mu_2)] \int_0^1 \left[\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s} + \cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\right]\theta^{\eta-1} d\theta.$$

*Proof.* Consider the definition of the MET-$(p,s)$-convex function

$$f\left(\left(ta_1^p+(1-t)b_1^p\right)^{\frac{1}{p}}\right)\le \sin\left(\frac{\pi t}{2}\right)e^{(1-t)^s}f(a_1)+\cos\left(\frac{\pi t}{2}\right)e^{t^s}f(b_1). \tag{3.1}$$

After replacing the value $a_1=\theta\mu_1+(1-\theta)\mu_2$ and $b_1=(1-\theta)\mu_1+\theta\mu_2$ and $t=\dfrac{1}{2}$ for $\theta\in[0,1]$ in equation (3.1), we have

$$\begin{aligned} f\left(\left[\frac{(\theta\mu_1+(1-\theta)\mu_2)^p+((1-\theta)\mu_1+\theta\mu_2)^p}{2}\right]^{\frac{1}{p}}\right)&\le \sin\left(\frac{\pi}{4}\right)e^{\left(\frac{1}{2}\right)^s}f(\theta\mu_1+(1-\theta)\mu_2)\\ &+\cos\left(\frac{\pi}{4}\right)e^{\left(\frac{1}{2}\right)^s}f((1-\theta)\mu_1+\theta\mu_2). \end{aligned} \tag{3.2}$$

$$\begin{aligned} &f\left(\left[\frac{(\theta\mu_1+(1-\theta)\mu_2)^p+((1-\theta)\mu_1+\theta\mu_2)^p}{2}\right]^{\frac{1}{p}}\right)\\ &\le \frac{\sqrt{2}}{2}e^{\left(\frac{1}{2}\right)^s}\Big[f(\theta\mu_1+(1-\theta)\mu_2)+f((1-\theta)\mu_1+\theta\mu_2)\Big]. \end{aligned} \tag{3.3}$$

Multiplying both sides by $\theta^{\eta-1}$ and then integrating over $[0,1]$, we obtain the following

$$\begin{aligned} &\int_0^1\theta^{\eta-1}f\left(\left[\frac{(\theta\mu_1+(1-\theta)\mu_2)^p+((1-\theta)\mu_1+\theta\mu_2)^p}{2}\right]^{\frac{1}{p}}\right)d\theta\\ &\le \frac{\sqrt{2}}{2}e^{\left(\frac{1}{2}\right)^s}\int_0^1\theta^{\eta-1}f(\theta\mu_1+(1-\theta)\mu_2)d\theta\\ &+\frac{\sqrt{2}}{2}e^{\left(\frac{1}{2}\right)^s}\int_0^1\theta^{\eta-1}f((1-\theta)\mu_1+\theta\mu_2)d\theta. \end{aligned} \tag{3.4}$$

For making a suitable substitution, taking the value $w=\theta\mu_1+(1-\theta)\mu_2$, $dw=-(\mu_2-\mu_1)d\theta$ and $\theta\to 0$ as $w\to\mu_2$, $\theta\to 1$, as $w\to\mu_1$, we have the first integral term of the equation (3.4)

$$\int_0^1\theta^{\eta-1}f(\theta\mu_1+(1-\theta)\mu_2)d\theta=\frac{1}{(\mu_2-\mu_1)^\eta}\int_{\mu_1}^{\mu_2}(\mu_2-w)^{\eta-1}f(w)\,dw.$$

$$\int_0^1\theta^{\eta-1}f(\theta\mu_1+(1-\theta)\mu_2)d\theta=\frac{\Gamma(\eta)}{(\mu_2-\mu_1)^\eta}J^{\eta}_{\mu_1^+}f(\mu_2). \tag{3.5}$$

Similarly, consider the substitution $t=(1-\theta)\mu_1+\theta\mu_2$ $dt=(\mu_2-\mu_1)d\theta$ and $\theta\to 0$, $t\to\mu_1$, and $\theta\to 1$, $t\to\mu_2$, we have

$$\int_0^1\theta^{\eta-1}f((1-\theta)\mu_1+\theta\mu_2)d\theta=\frac{1}{(\mu_2-\mu_1)^\eta}\int_{\mu_1}^{\mu_2}(t-\mu_1)^{\eta-1}f(t)\,dt. \tag{3.6}$$

$$\int_0^1\theta^{\eta-1}f((1-\theta)\mu_1+\theta\mu_2)d\theta=\frac{\Gamma(\eta)}{(\mu_2-\mu_1)^\eta}J^{\eta}_{\mu_2^-}f(\mu_1). \tag{3.7}$$

Putting the values of (3.5) and (3.7) in (3.4), we have

$$f\left(\left(\frac{\mu_1^p+\mu_2^p}{2}\right)^{\frac{1}{p}}\right)\int_0^1 \theta^{\eta-1}d\theta \leq \frac{\Gamma(\eta)}{(\mu_2-\mu_1)^{\eta}}\left[J^{\eta}_{\mu_1^+}f(\mu_2)+J^{\eta}_{\mu_2^-}f(\mu_1)\right], \tag{3.8}$$

$$f\left(\left(\frac{\mu_1^p+\mu_2^p}{2}\right)^{\frac{1}{p}}\right)\frac{1}{\eta} \leq \frac{\Gamma(\eta+1)}{(\mu_2-\mu_1)^{\eta}}\left[J^{\eta}_{\mu_1^+}f(\mu_2)+J^{\eta}_{\mu_2^-}f(\mu_1)\right]. \tag{3.9}$$

Now, again consider the definition of MET-$(p,s)$-convexity

$$f\left(\left(\theta\mu_1^p+(1-\theta)\mu_2^p\right)^{\frac{1}{p}}\right) \leq \sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}f(\mu_1)+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}f(\mu_2). \tag{3.10}$$

After multiplication by $\theta^{\eta-1}$ and then integrating over $[0,1]$, we have the following

$$\begin{aligned}&\int_0^1 \theta^{\eta-1}f\left(\left(\theta\mu_1^p+(1-\theta)\mu_2^p\right)^{\frac{1}{p}}\right)d\theta\\ &\leq f(\mu_1)\int_0^1 \sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}\theta^{\eta-1}d\theta+f(\mu_2)\int_0^1 \cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\theta^{\eta-1}d\theta.\end{aligned} \tag{3.11}$$

After simplification, we have

$$\begin{aligned}&\frac{\Gamma(\eta+1)}{(\mu_2-\mu_1)^{\eta}}\left[J^{\eta}_{\mu_1^+}f(\mu_2)+J^{\eta}_{\mu_2^-}f(\mu_1)\right]\\ &\leq [f(\mu_1)+f(\mu_2)]\int_0^1\left[\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\right]\theta^{\eta-1}d\theta.\end{aligned} \tag{3.12}$$

Finally, combining inequalities (3.9) and (3.12), we have the required result. □

**Example.** *To verify the theorem 3.1 Consider the function*

$$f(z)=e^z,$$

*with the parameters*

$$\mu_1=1,\qquad \mu_2=3,\qquad p=1,\qquad s=1,\qquad \eta=1.$$

*First, compute the generalized midpoint:*

$$\left(\frac{\mu_1^p+\mu_2^p}{2}\right)^{\frac{1}{p}}=\left(\frac{1+3}{2}\right)=2.$$

*Hence,*

$$L.H.S.=f(2)=e^2\approx 7.389.$$

*Now,*

$$f(\mu_1)+f(\mu_2)=e+e^3\approx 22.803.$$

*The integral term becomes*

$$I=\int_0^1\left[\sin\left(\frac{\pi\theta}{2}\right)e^{1-\theta}+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta}\right]d\theta.$$

*Thus, the obtained inequality takes the form*

$$e^2 \leq \frac{\Gamma(2)}{2(3-1)}\left[J^1_{1^+}(f\circ\phi)(3)+J^1_{3^-}(f\circ\phi)(1)\right]$$

$$\leq (e+e^3)\int_0^1\left[\sin\left(\frac{\pi\theta}{2}\right)e^{1-\theta}+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta}\right]d\theta.$$

*Therefore, the inequality is satisfied for the chosen parameters.*

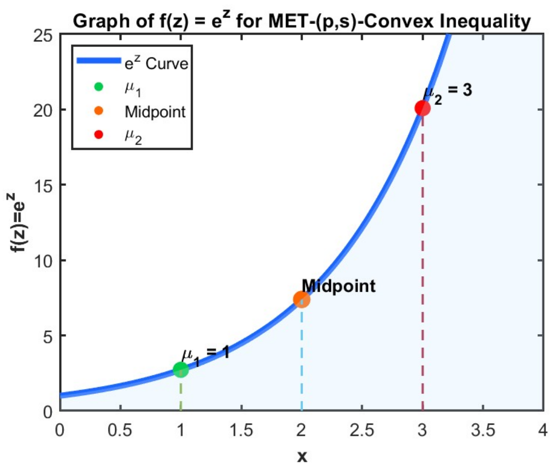



Figure 1. 2D graph

**Corollary 3.1.** *Replacing the value $\eta = 1$ in theorem 3.1, then obtain the inequality for MET-$(p,s)$-convex functions as follows:*

$$f\left(\left(\frac{\mu_1^p+\mu_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(x)\,dx$$

$$\leq [f(\mu_1)+f(\mu_2)]\int_0^1\left[\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\right]d\theta.$$

**Corollary 3.2.** *By putting the value $s = 1$ in theorem 3.1, then the inequality reduces to*

$$f\left(\left(\frac{\mu_1^p+\mu_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Gamma(\eta+1)}{(\mu_2-\mu_1)^\eta}\left[x^\eta_{\mu_1^+}f(\mu_2)+x^\eta_{\mu_2^-}f(\mu_1)\right]$$

$$\leq [f(\mu_1)+f(\mu_2)]\int_0^1\left[\sin\left(\frac{\pi\theta}{2}\right)e^{1-\theta}+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta}\right]\theta^{\eta-1}d\theta.$$

**Corollary 3.3.** *Choosing $p = 1$ in theorem 3.1, we get the Hermite–Hadamard type inequality for MET-$(1,s)$-convex functions given by*

$$f\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{\Gamma(\eta+1)}{(\mu_2-\mu_1)^\eta}\left[x^\eta_{\mu_1^+}f(\mu_2)+x^\eta_{\mu_2^-}f(\mu_1)\right]$$

$$\leq [f(\mu_1)+f(\mu_2)]\int_0^1\left[\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\right]\theta^{\eta-1}d\theta.$$

**Corollary 3.4.** *Setting the parameters $\eta = 1$ and $p = 1$ in 3.1, then we obtain the classical integral inequality*

$$f\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(x)\,dx$$
$$\leq [f(\mu_1)+f(\mu_2)]\int_0^1 \left[\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s} + \cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\right]d\theta.$$

**Corollary 3.5.** *For $s = 1$ and $p = 1$, then theorem 3.1 is reduce as follows:*

$$f\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} f(x)\,dx$$
$$\leq [f(\mu_1)+f(\mu_2)]\int_0^1 \left[\sin\left(\frac{\pi\theta}{2}\right)e^{1-\theta} + \cos\left(\frac{\pi\theta}{2}\right)e^{\theta}\right]d\theta.$$

**Lemma 3.1.** *Let $f : x = [\mu_1, \mu_1 + \zeta(\mu_2, \mu_1)] \to \mathbb{R}$ be a differentiable function such that $f' \in L_1[\mu_1, \mu_1 + \zeta(\mu_2, \mu_1)]$, where $x$ is an open invex set with respect to $\zeta : x \times x \to \mathbb{R}$, and $\zeta(\mu_2, \mu_1) > 0$.*

*Assume further that $|f'|$ is a MET-$(p, s)$-convex function on $x$. Then, for fractional integrals of order $\eta > 0$, the following inequality holds:*

$$\left|\frac{f(\mu_1)+f(\mu_1+\zeta(\mu_2,\mu_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\mu_2,\mu_1)}\left[x^\eta_{\mu_1^+}f(\mu_1+\zeta(\mu_2,\mu_1)) + x^\eta_{(\mu_1+\zeta(\mu_2,\mu_1))^-}f(\mu_1)\right]\right|$$
$$\leq \frac{\zeta(\mu_2,\mu_1)}{2}\int_0^1 \left|\theta^\eta-(1-\theta)^\eta\right| \times\left[\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}|f'(\mu_1)| + \cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}|f'(\mu_2)|\right]d\theta.$$

*Proof.* Consider the fractional equation

$$\frac{f(\mu_1)+f(\mu_1+\zeta(\mu_2,\mu_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\mu_2,\mu_1)}\left[x^\eta_{\mu_1^+}f(\mu_1+\zeta(\mu_2,\mu_1)) + x^\eta_{(\mu_1+\zeta(\mu_2,\mu_1))^-}f(\mu_1)\right]$$
$$= \frac{\zeta(\mu_2,\mu_1)}{2}\int_0^1 [\theta^\eta-(1-\theta)^\eta]\,f'(\mu_1+\theta\zeta(\mu_2,\mu_1))\,d\theta. \tag{3.13}$$

Taking the modulus on both sides and using the triangle inequality in equation (3.13), we obtain

$$\left|\frac{f(\mu_1)+f(\mu_1+\zeta(\mu_2,\mu_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\mu_2,\mu_1)}\left[x^\eta_{\mu_1^+}f(\mu_1+\zeta(\mu_2,\mu_1)) + x^\eta_{(\mu_1+\zeta(\mu_2,\mu_1))^-}f(\mu_1)\right]\right|$$
$$\leq \frac{\zeta(\mu_2,\mu_1)}{2}\int_0^1 \left|\theta^\eta-(1-\theta)^\eta\right||f'(\mu_1+\theta\zeta(\mu_2,\mu_1))|\,d\theta. \tag{3.14}$$

Since $|f'|$ is a MET-$(p, s)$-convex function

$$|f'(\mu_1+\theta\zeta(\mu_2,\mu_1))| \leq \sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}|f'(\mu_1)| \quad + \cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}|f'(\mu_2)|. \tag{3.15}$$

Substituting (3.14) into (3.15), we get

$$\left|\frac{f(\mu_1)+f(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta^{\eta}(\mu_2,\mu_1)}\left[x^{\eta}_{\mu_1^+}f(\mu_1+\zeta(\mu_2,\mu_1))+x^{\eta}_{(\mu_1+\zeta(\mu_2,\mu_1))^-}f(\mu_1)\right]\right|$$
$$\leq \frac{\zeta(\mu_2,\mu_1)}{2}\int_0^1\left|\theta^{\eta}-(1-\theta)^{\eta}\right|\left[\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}|f'(\mu_1)|+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}|f'(\mu_2)|\right]d\theta.$$

□

**Theorem 3.2.** *Suppose a mapping $\mathfrak{R}: x=[\mu_1,\mu_1+\zeta(\mu_2,\mu_1)]\to(0,\infty)$ with $x\in\mathbb{R}$, be a differentiable function on $x$, and let $|\mathfrak{R}'|$ be an MET-$(p,s)$-convex function on $x$. Then the following inequality holds for fractional integrals:*

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta^{\eta}(\mu_2,\mu_1)}\left[x^{\eta}_{\mu_1^+}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))+x^{\eta}_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{R}(\mu_1)\right]\right|$$
$$\leq\frac{\zeta(\mu_2,\mu_1)}{2}\left[|\mathfrak{R}'(\mu_1)|A+|\mathfrak{R}'(\mu_2)|B\right],$$

*where*

$$A=\int_0^1|\theta^{\eta}-(1-\theta)^{\eta}|\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}\,d\theta,$$
$$B=\int_0^1|\theta^{\eta}-(1-\theta)^{\eta}|\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\,d\theta.$$

*Proof.* By using lemma 3.1, we obtain

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta^{\eta}(\mu_2,\mu_1)}\left[J^{\eta}_{\mu_1^+}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))+J^{\eta}_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{R}(\mu_1)\right]\right|$$
$$=\frac{\zeta(\mu_2,\mu_1)}{2}\left|\int_0^1\left[\theta^{\eta}-(1-\theta)^{\eta}\right]\mathfrak{R}'\left(\mu_1+\theta\zeta(\mu_2,\mu_1)\right)\,d\theta\right|. \tag{3.16}$$

Using the inequality

$$\left|\int_a^b f(\theta)\,d\theta\right|\leq\int_a^b|f(\theta)|\,d\theta, \tag{3.17}$$

we have

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta^{\eta}(\mu_2,\mu_1)}\left[J^{\eta}_{\mu_1^+}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))+J^{\eta}_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{R}(\mu_1)\right]\right|$$
$$\leq\frac{\zeta(\mu_2,\mu_1)}{2}\int_0^1\left|\theta^{\eta}-(1-\theta)^{\eta}\right|\left|\mathfrak{R}'\left(\mu_1+\theta\zeta(\mu_2,\mu_1)\right)\right|d\theta. \tag{3.18}$$

Since $|\mathfrak{R}'|$ is a positive MET-$(p,s)$-convex function, therefore for every $\theta\in[0,1]$, we have

$$\left|\mathfrak{R}'\left(\mu_1+\theta\zeta(\mu_2,\mu_1)\right)\right|\leq\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}|\mathfrak{R}'(\mu_1)|+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}|\mathfrak{R}'(\mu_2)|. \tag{3.19}$$

Substituting inequality (3.19) into (3.18) , we get

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta^{\eta}(\mu_2,\mu_1)}\left[J^{\eta}_{\mu_1^+}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))+J^{\eta}_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{R}(\mu_1)\right]\right|$$
$$\leq \frac{\zeta(\mu_2,\mu_1)}{2}\int_0^1 \left|\theta^{\eta}-(1-\theta)^{\eta}\right|\left[\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}|\mathfrak{R}'(\mu_1)|+\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}|\mathfrak{R}'(\mu_2)|\right]d\theta. \tag{3.20}$$

Distributing the integral over the summation, we obtain

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta^{\eta}(\mu_2,\mu_1)}\left[J^{\eta}_{\mu_1^+}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))+J^{\eta}_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{R}(\mu_1)\right]\right|$$
$$\leq \frac{\zeta(\mu_2,\mu_1)}{2}|\mathfrak{R}'(\mu_1)|\int_0^1 \left|\theta^{\eta}-(1-\theta)^{\eta}\right|\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}d\theta$$
$$+\frac{\zeta(\mu_2,\mu_1)}{2}|\mathfrak{R}'(\mu_2)|\int_0^1 \left|\theta^{\eta}-(1-\theta)^{\eta}\right|\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}d\theta. \tag{3.21}$$

$$A_{\eta}=\int_0^1 \left|\theta^{\eta}-(1-\theta)^{\eta}\right|\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}\,d\theta, \tag{3.22}$$

and

$$B_{\eta}=\int_0^1 \left|\theta^{\eta}-(1-\theta)^{\eta}\right|\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\,d\theta. \tag{3.23}$$

Using (3.22) and (3.23) in (3.21), we arrive at

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta^{\eta}(\mu_2,\mu_1)}\left[J^{\eta}_{\mu_1^+}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))+J^{\eta}_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{R}(\mu_1)\right]\right|$$
$$\leq \frac{\zeta(\mu_2,\mu_1)}{2}\left[|\mathfrak{R}'(\mu_1)|A_{\eta}+|\mathfrak{R}'(\mu_2)|B_{\eta}\right]. \tag{3.24}$$

□

**Corollary 3.6.** *By replacing the value $\eta=1$ in theorem 3.2, then we obtain:*

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{1}{\zeta(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\zeta(\mu_2,\mu_1)}\mathfrak{R}(\theta)\,d\theta\right|$$
$$\leq \frac{\zeta(\mu_2,\mu_1)}{2}\left[|\mathfrak{R}'(\mu_1)|A_1+|\mathfrak{R}'(\mu_2)|B_1\right],$$

*where*

$$A_1=\int_0^1 |1-2\theta|\,\sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}\,d\theta,\quad B_1=\int_0^1 |1-2\theta|\,\cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\,d\theta.$$

**Corollary 3.7.** *If $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$, then theorem 3.2 yields:*

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_2)}{2}-\frac{\Gamma(\eta+1)}{2(\mu_2-\mu_1)^{\eta}}\left[x^{\eta}_{\mu_1}\mathfrak{R}(\mu_2)+x^{\eta}_{\mu_2}\mathfrak{R}(\mu_1)\right]\right| \tag{3.25}$$

$$\leq \frac{\mu_2-\mu_1}{2}\left[|\mathfrak{R}'(\mu_1)|A_2+|\mathfrak{R}'(\mu_2)|B_2\right], \tag{3.26}$$

*where*

$$A_2 = \int_0^1 |\theta^\eta - (1-\theta)^\eta| \sin\Big(\frac{\pi\theta}{2}\Big) e^{(1-\theta)^s}\, d\theta, \quad B_2 = \int_0^1 |\theta^\eta - (1-\theta)^\eta| \cos\Big(\frac{\pi\theta}{2}\Big) e^{\theta^s}\, d\theta. \tag{3.27}$$

**Corollary 3.8.** *If $h(\theta) = 1/\theta$, then $|\mathfrak{R}|$ is convex, and we obtain:*

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_2)}{2} - \frac{\Gamma(\eta+1)}{2(\mu_2-\mu_1)^\eta}\left[x^\eta_{\mu_1}\mathfrak{R}(\mu_2) + x^\eta_{\mu_2}\mathfrak{R}(\mu_1)\right]\right| \leq \frac{\mu_2-\mu_1}{2}\Big[|\mathfrak{R}'(\mu_1)|A_3 + |\mathfrak{R}'(\mu_2)|B_3\Big],$$

*where*

$$A_3 = \int_0^1 \frac{|\theta^\eta - (1-\theta)^\eta|}{\theta} \sin\Big(\frac{\pi\theta}{2}\Big) e^{(1-\theta)^s}\, d\theta, \quad B_3 = \int_0^1 \frac{|\theta^\eta - (1-\theta)^\eta|}{\theta} \cos\Big(\frac{\pi\theta}{2}\Big) e^{\theta^s}\, d\theta.$$

**Theorem 3.3.** *Suppose that $\mathfrak{R} : x = [\mu_1, \mu_1 + \zeta(\mu_2,\mu_1)] \to (0,\infty)$ with $x \in \mathbb{R}$, be a differentiable real valued function on $x$. Also, suppose that $|\mathfrak{R}'|^q$ is an MET-$(p,s)$-convex function on $x$ with $p > 1$ and $q = \frac{p}{p-1}$. For fractional integral, we find*

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta(\mu_2,\mu_1)^\eta}\left[x^\eta_{\mu_1}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1)) + x^\eta_{(\mu_1+\zeta(\mu_2,\mu_1))}\mathfrak{R}(\mu_1)\right]\right|$$
$$\leq \frac{\zeta(\mu_2,\mu_1)}{2}\Big(|\mathfrak{R}'(\mu_1)|^q + |\mathfrak{R}'(\mu_2)|^q\Big)^{\frac{1}{q}}\left(\int_0^1 |\theta^\eta - (1-\theta)^\eta|^p\, d\theta\right)^{\frac{1}{p}} (A+B)^{\frac{1}{q}},$$

*where*

$$A = \int_0^1 \sin\Big(\frac{\pi\theta}{2}\Big) e^{(1-\theta)^s}\, d\theta, \quad B = \int_0^1 \cos\Big(\frac{\pi\theta}{2}\Big) e^{\theta^s}\, d\theta.$$

*Proof.* Using Lemma 3.1, we have

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\mu_2,\mu_1)}\left[J^\eta_{\mu_1^+}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1)) + J^\eta_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{R}(\mu_1)\right]\right|$$
$$= \frac{\zeta(\mu_2,\mu_1)}{2}\left|\int_0^1 \Big[\theta^\eta - (1-\theta)^\eta\Big]\mathfrak{R}'\Big(\mu_1 + \theta\zeta(\mu_2,\mu_1)\Big) d\theta\right|. \tag{3.28}$$

Applying the modulus inequality

$$\left|\int_a^b f(\theta)\, d\theta\right| \leq \int_a^b |f(\theta)|\, d\theta,$$

to (3.28), we obtain

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\mu_2,\mu_1)}\left[J^\eta_{\mu_1^+}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1)) + J^\eta_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{R}(\mu_1)\right]\right|$$
$$\leq \frac{\zeta(\mu_2,\mu_1)}{2}\int_0^1 \Big|\theta^\eta - (1-\theta)^\eta\Big| \Big|\mathfrak{R}'\Big(\mu_1 + \theta\zeta(\mu_2,\mu_1)\Big)\Big|\, d\theta. \tag{3.29}$$

Now applying Hölder's integral inequality in equation (3.29) with conjugate exponents $p > 1$ and

$$q = \frac{p}{p-1}, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

we get

$$\int_0^1 \left|\theta^\eta - (1-\theta)^\eta\right| \left|\mathfrak{N}'\Big(\mu_1 + \theta\zeta(\mu_2,\mu_1)\Big)\right| d\theta \le \left(\int_0^1 \left|\theta^\eta - (1-\theta)^\eta\right|^p d\theta\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left|\mathfrak{N}'\Big(\mu_1 + \theta\zeta(\mu_2,\mu_1)\Big)\right|^q d\theta\right)^{\frac{1}{q}}. \tag{3.30}$$

Substituting (3.30) into (3.29), we arrive at

$$\left|\frac{\mathfrak{N}(\mu_1) + \mathfrak{N}(\mu_1 + \zeta(\mu_2,\mu_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\mu_2,\mu_1)}\left[J^\eta_{\mu_1^+}\mathfrak{N}(\mu_1 + \zeta(\mu_2,\mu_1)) + J^\eta_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{N}(\mu_1)\right]\right|$$
$$\le \frac{\zeta(\mu_2,\mu_1)}{2}\left(\int_0^1 \left|\theta^\eta - (1-\theta)^\eta\right|^p d\theta\right)^{\frac{1}{p}} \times \left(\int_0^1 \left|\mathfrak{N}'\Big(\mu_1 + \theta\zeta(\mu_2,\mu_1)\Big)\right|^q d\theta\right)^{\frac{1}{q}}. \tag{3.31}$$

Since $|\mathfrak{N}'|^q$ is an MET-$(p,s)$-convex function, therefore

$$\left|\mathfrak{N}'\Big(\mu_1 + \theta\zeta(\mu_2,\mu_1)\Big)\right|^q \le \sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}|\mathfrak{N}'(\mu_1)|^q + \cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}|\mathfrak{N}'(\mu_2)|^q. \tag{3.32}$$

Integrating both sides of (3.32) over $[0,1]$, we obtain

$$\int_0^1 \left|\mathfrak{N}'\Big(\mu_1 + \theta\zeta(\mu_2,\mu_1)\Big)\right|^q d\theta \le |\mathfrak{N}'(\mu_1)|^q \int_0^1 \sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}d\theta + |\mathfrak{N}'(\mu_2)|^q \int_0^1 \cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}d\theta. \tag{3.33}$$

Putting the values

$$A = \int_0^1 \sin\left(\frac{\pi\theta}{2}\right)e^{(1-\theta)^s}\,d\theta, \tag{3.34}$$

and

$$B = \int_0^1 \cos\left(\frac{\pi\theta}{2}\right)e^{\theta^s}\,d\theta. \tag{3.35}$$

Using (3.34) and (3.35) in (3.33), we get

$$\int_0^1 \left|\mathfrak{N}'\Big(\mu_1 + \theta\zeta(\mu_2,\mu_1)\Big)\right|^q d\theta \le A|\mathfrak{N}'(\mu_1)|^q + B|\mathfrak{N}'(\mu_2)|^q. \tag{3.36}$$

Since

$$A|\mathfrak{N}'(\mu_1)|^q + B|\mathfrak{N}'(\mu_2)|^q \le (A+B)\Big(|\mathfrak{N}'(\mu_1)|^q + |\mathfrak{N}'(\mu_2)|^q\Big),$$

it follows from (3.36) that

$$\int_0^1 \left|\mathfrak{N}'\Big(\mu_1 + \theta\zeta(\mu_2,\mu_1)\Big)\right|^q d\theta \le (A+B)\Big(|\mathfrak{N}'(\mu_1)|^q + |\mathfrak{N}'(\mu_2)|^q\Big). \tag{3.37}$$

Taking the power $\frac{1}{q}$ on both sides of (3.37), we obtain

$$\left(\int_0^1 \left|\mathfrak{R}'\Big(\mu_1+\theta\zeta(\mu_2,\mu_1)\Big)\right|^q d\theta\right)^{\frac{1}{q}} \le (A+B)^{\frac{1}{q}}\Big(|\mathfrak{R}'(\mu_1)|^q+|\mathfrak{R}'(\mu_2)|^q\Big)^{\frac{1}{q}}. \tag{3.38}$$

Finally, substituting (3.38) into (3.31), we obtain

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{\Gamma(\eta+1)}{2\zeta^{\eta}(\mu_2,\mu_1)}\left[J^{\eta}_{\mu_1^+}\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))+J^{\eta}_{(\mu_1+\zeta(\mu_2,\mu_1))^-}\mathfrak{R}(\mu_1)\right]\right|$$
$$\le \frac{\zeta(\mu_2,\mu_1)}{2}\Big(|\mathfrak{R}'(\mu_1)|^q+|\mathfrak{R}'(\mu_2)|^q\Big)^{\frac{1}{q}} \quad \times\left(\int_0^1 \left|\theta^{\eta}-(1-\theta)^{\eta}\right|^p d\theta\right)^{\frac{1}{p}}(A+B)^{\frac{1}{q}}. \tag{3.39}$$

This completes the proof. □

**Corollary 3.9.** *In this corollary, there are two cases:*

- *If we take the value $\eta=1$ in theorem 3.3, then have the following inequality:*

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{1}{\zeta(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\zeta(\mu_2,\mu_1)}\mathfrak{R}(\theta)\,d\theta\right|$$
$$\le \frac{\zeta(\mu_2,\mu_1)}{2(p+1)^{1/p}}\Big(|\mathfrak{R}'(\mu_1)|^q+|\mathfrak{R}'(\mu_2)|^q\Big)^{1/q}(A_4+B_4)^{1/q}, \tag{3.40}$$

*where*

$$A_4=\int_0^1 \sin\Big(\frac{\pi\theta}{2}\Big)e^{(1-\theta)^s}\,d\theta, \quad B_4=\int_0^1 \cos\Big(\frac{\pi\theta}{2}\Big)e^{\theta^s}\,d\theta.$$

- *If $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$ and $h(\theta)=1/\theta^s$ in theorem 3.3, then we obtain:*

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_2)}{2}-\frac{1}{(\mu_2-\mu_1)}\int_{\mu_1}^{\mu_2}\mathfrak{R}(\theta)\,d\theta\right|$$
$$\le \frac{(\mu_2-\mu_1)}{2(p+1)^{1/p}}\Big(|\mathfrak{R}'(\mu_1)|^q+|\mathfrak{R}'(\mu_2)|^q\Big)^{1/q}\frac{1}{s+1}. \tag{3.41}$$

**Corollary 3.10.** *If $h(\theta)=\theta^s$ in theorem 3.3, then we obtain:*

$$\left|\frac{\mathfrak{R}(\mu_1)+\mathfrak{R}(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{1}{\zeta(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\zeta(\mu_2,\mu_1)}\mathfrak{R}(\theta)\,d\theta\right|$$
$$\le \frac{\zeta(\mu_2,\mu_1)}{2(p+1)^{1/p}}\Big(|\mathfrak{R}'(\mu_1)|^{\frac{p}{p-1}}+|\mathfrak{R}'(\mu_2)|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}(A_5+B_5),$$

*where*

$$A_5=\int_0^1 \sin\Big(\frac{\pi\theta}{2}\Big)e^{(1-\theta)^s}\,d\theta, \quad B_5=\int_0^1 \cos\Big(\frac{\pi\theta}{2}\Big)e^{\theta^s}\,d\theta.$$

**Theorem 3.4.** *Suppose that $\mathfrak{R} : x = [\mu_1, \mu_1 + \zeta(\mu_2, \mu_1)] \to (0, \infty)$ with $x \in \mathbb{R}$, be a differentiable real valued function on $x$. Also, suppose that $|\mathfrak{R}'|^q$ is an MET-$(p,s)$-convex function on $x$ with $p > 1$ and $q = \frac{p}{p-1}$. Then for fractional integral, we have*

$$\left|\frac{\mathfrak{R}(\mu_1) + \mathfrak{R}(\mu_1 + \zeta(\mu_2, \mu_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\mu_2, \mu_1)}\left[x^\eta_{\mu_1}\mathfrak{R}(\mu_1 + \zeta(\mu_2, \mu_1)) + x^\eta_{(\mu_1+\zeta(\mu_2,\mu_1))}\mathfrak{R}(\mu_1)\right]\right|$$
$$\leq \frac{\zeta(\mu_2, \mu_1)}{2}\Big(|\mathfrak{R}'(\mu_1)|^q + |\mathfrak{R}'(\mu_2)|^q\Big)^{1/q}\left(\int_0^1 |\theta^\eta - (1-\theta)^\eta|^p \, d\theta\right)^{1/p} (A+B)^{1/q}, \tag{3.42}$$

*where*

$$A = \int_0^1 \sin\Big(\frac{\pi\theta}{2}\Big)e^{(1-\theta)^s}\, d\theta, \quad B = \int_0^1 \cos\Big(\frac{\pi\theta}{2}\Big)e^{\theta^s}\, d\theta.$$

*Proof.* Using lemma 3.1, we have

$$\left|\frac{\mathfrak{R}(\mu_1) + \mathfrak{R}(\mu_1 + \zeta(\mu_2, \mu_1))}{2} - \frac{\Gamma(\eta+1)}{2\zeta^\eta(\mu_2, \mu_1)}\left[x^\eta_{\mu_1}\mathfrak{R}(\mu_1 + \zeta(\mu_2, \mu_1)) + x^\eta_{(\mu_1+\zeta(\mu_2,\mu_1))}\mathfrak{R}(\mu_1)\right]\right|$$
$$= \frac{\zeta(\mu_2, \mu_1)}{2}\int_0^1 |\theta^\eta - (1-\theta)^\eta||\mathfrak{R}'(\mu_1 + \theta\zeta(\mu_2, \mu_1))|\, d\theta. \tag{3.43}$$

Applying Hölder's inequality, we have

$$\leq \frac{\zeta(\mu_2, \mu_1)}{2}\left(\int_0^1 |\theta^\eta - (1-\theta)^\eta|^p\, d\theta\right)^{1/p}\left(\int_0^1 |\mathfrak{R}'(\mu_1 + \theta\zeta(\mu_2, \mu_1))|^q\, d\theta\right)^{1/q}. \tag{3.44}$$

Since $|\mathfrak{R}'|^q$ is MET-$(p,s)$-convex, we have

$$|\mathfrak{R}'(\mu_1 + \theta\zeta(\mu_2, \mu_1))|^q \leq \sin\Big(\frac{\pi\theta}{2}\Big)e^{(1-\theta)^s}|\mathfrak{R}'(\mu_1)|^q + \cos\Big(\frac{\pi\theta}{2}\Big)e^{\theta^s}|\mathfrak{R}'(\mu_2)|^q. \tag{3.45}$$

Integrating over $[0,1]$, we obtain

$$\int_0^1 |\mathfrak{R}'(\mu_1 + \theta\zeta(\mu_2, \mu_1))|^q\, d\theta \leq (|\mathfrak{R}'(\mu_1)|^q + |\mathfrak{R}'(\mu_2)|^q)(A+B). \tag{3.46}$$

Substituting this into Hölder's inequality gives the required result. □

**Corollary 3.11.** *If $\eta = 1$ in theorem 3.4, then we obtain:*

$$\left|\frac{\mathfrak{R}(\mu_1) + \mathfrak{R}(\mu_1 + \zeta(\mu_2, \mu_1))}{2} - \frac{1}{\zeta(\mu_2, \mu_1)}\int_{\mu_1}^{\mu_1+\zeta(\mu_2,\mu_1)} \mathfrak{R}(\theta)\, d\theta\right|$$
$$\leq \frac{\zeta(\mu_2, \mu_1)}{2}\Big(|\mathfrak{R}'(\mu_1)|^q + |\mathfrak{R}'(\mu_2)|^q\Big)^{1/q}(A_6 + B_6)^{1/q}, \tag{3.47}$$

*where*

$$A_6 = \int_0^1 \sin\Big(\frac{\pi\theta}{2}\Big)e^{(1-\theta)^s}\, d\theta, \quad B_6 = \int_0^1 \cos\Big(\frac{\pi\theta}{2}\Big)e^{\theta^s}\, d\theta. \tag{3.48}$$

**Corollary 3.12.** *If $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$, $h(\theta)=\frac{1}{\theta}$, $q=1$, and $\eta=1$ in theorem 3.4, then we obtain:*

$$\left|\frac{\Re(\mu_1)+\Re(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Re(x)\,dx\right|\le\frac{\mu_2-\mu_1}{8}\Big(|\Re'(\mu_1)|+|\Re'(\mu_2)|\Big)(A_7+B_7),$$

*where*

**Corollary 3.13.** *If $\eta=1$ and $h(\theta)=\theta^s$ in theorem 3.4, then we obtain:*

$$\left|\frac{\Re(\mu_1)+\Re(\mu_1+\zeta(\mu_2,\mu_1))}{2}-\frac{1}{\zeta(\mu_2,\mu_1)}\int_{\mu_1}^{\mu_1+\zeta(\mu_2,\mu_1)}\Re(\theta)\,d\theta\right|$$
$$\le\frac{\zeta(\mu_2,\mu_1)}{4}\Big(|\Re'(\mu_1)|^q+|\Re'(\mu_2)|^q\Big)^{1/q}\left|\frac{2^{s+1}-2s}{(s-2)(s-1)}\right|^{1/q}(A_8+B_8),$$

*where*

$$A_8=\int_0^1\sin\Big(\frac{\pi\theta}{2}\Big)e^{(1-\theta)^s}\,d\theta,\quad B_8=\int_0^1\cos\Big(\frac{\pi\theta}{2}\Big)e^{\theta^s}\,d\theta.$$

**Corollary 3.14.** *If $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$, $h(\theta)=\frac{1}{\theta}$, and $q=1$ in theorem 3.4, then we obtain:*

$$\left|\frac{\Re(\mu_1)+\Re(\mu_2)}{2}-\frac{\Gamma(\eta+1)}{2(\mu_2-\mu_1)^\eta}\left[x^{\eta}_{\mu_1^+}\Re(\mu_1)+x^{\eta}_{\mu_2^-}\Re(\mu_2)\right]\right|$$
$$\le\frac{\mu_2-\mu_1}{2(\eta+1)}\Big(1-\frac{1}{2^\eta}\Big)\Big(|\Re'(\mu_1)|+|\Re'(\mu_2)|\Big)(A_9+B_9),\ \textit{where}$$
$$A_9=\int_0^1\sin\Big(\frac{\pi\theta}{2}\Big)e^{(1-\theta)^s}\,d\theta,\quad B_9=\int_0^1\cos\Big(\frac{\pi\theta}{2}\Big)e^{\theta^s}\,d\theta.$$

## 4. Numerical Application

### Example 1.

This example will verify the theorem 3.2. Consider the mapping
$\Re(x)=e^{-2ax}\cos(x)$, where the parameters are chosen as
$\mu_1=0,\quad \mu_2=1,\quad a=1,\quad \eta=1,\quad s=1$, and
$\zeta(\mu_2,\mu_1)=\mu_2-\mu_1=1$, so that the interval reduces to $x\in[0,1]$.
The function values at the endpoints are given by
$\Re(0)=e^0\cos(0)=1$, $\Re(1)=e^{-2}\cos(1)\approx 0.0731$.
Hence, $\frac{\Re(0)+N(1)}{2}\approx 0.53655$.
The derivative of $\Re(x)$ is $\Re'(x)=e^{-2x}(-2\cos x-\sin x)$, which gives
$|\Re'(0)|=2,\quad |\Re'(1)|\approx 0.260$.
Therefore, the right-hand side of the inequality becomes

$$\frac{\zeta(\mu_2,\mu_1)}{2}\left[|\Re'(\mu_1)|A+|\Re'(\mu_2)|B\right]=\frac{1}{2}\left[2A+0.260B\right]=A+0.13B.$$

Since $A>0$ and $B>0$, the right-hand side provides a positive upper bound. The exponential decay term $e^{-2ax}$ ensures that $|\Re'(x)|$ remains bounded, which validates the inequality for the chosen parameter set

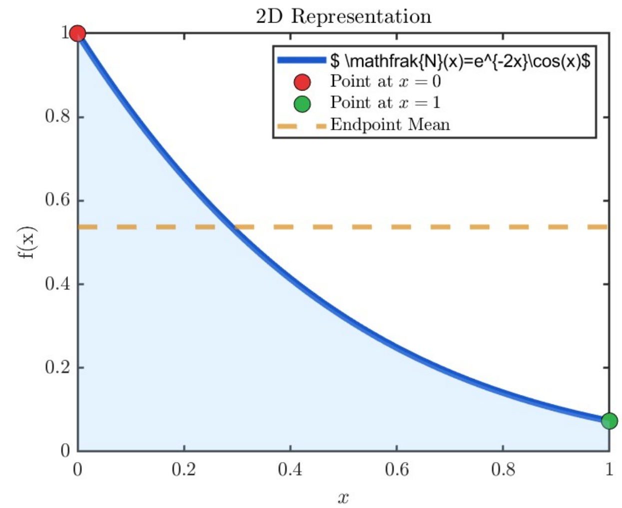



FIGURE 2. 2D grap

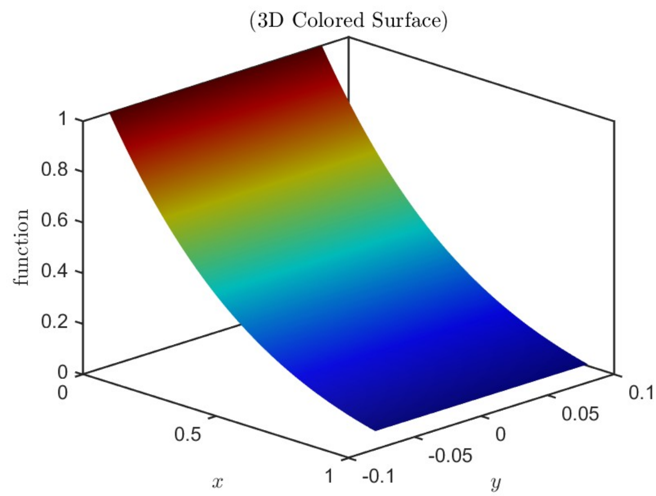



FIGURE 3. 3D graph

EXAMPLE 2.

For validation of theorem 3.3 we take a numerical example. Consider the function
$\mathfrak{N}(t) = t^2$,
defined on the interval
$[\mu_1, \mu_2] = [1, 2]$.
Choose
$\eta = 1, \qquad s = 1, \qquad \zeta(\mu_2, \mu_1) = \mu_2 - \mu_1 = 1.$
Then
$\mathfrak{N}'(t) = 2t.$

Hence,

$|\mathfrak{N}'(t)| = 2t,$

which is positive and convex on $[1,2]$. Therefore, the assumptions of the obtained inequality are satisfied.

Now,

$\mathfrak{N}(1) = 1, \qquad \mathfrak{N}(2) = 4.$

Thus,

$$\begin{aligned}\frac{\mathfrak{N}(1)+\mathfrak{N}(2)}{2} &= \frac{1+4}{2}\\ &= \frac{5}{2} = 2.5.\end{aligned}$$

Next, evaluate the integral term:

$$\begin{aligned}\int_1^2 t^2\,dt &= \left[\frac{t^3}{3}\right]_1^2\\ &= \frac{8-1}{3}\\ &= \frac{7}{3}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2-1}\int_1^2 t^2\,dt &= \frac{7}{3}\\ &\approx 2.3333.\end{aligned}$$

Hence, the left-hand side of the inequality becomes

$$\begin{aligned}\left|\frac{\mathfrak{N}(1)+\mathfrak{N}(2)}{2} - \frac{1}{2-1}\int_1^2 t^2\,dt\right| &= |2.5 - 2.3333|\\ &= 0.1667.\end{aligned}$$

Now,For Right han side

$$|\mathfrak{N}'(1)| = 2, \qquad |\mathfrak{N}'(2)| = 4.$$

The constants are given by

$$A_1 = \int_0^1 |1-2\theta| \sin\left(\frac{\pi\theta}{2}\right) e^{1-\theta}\,d\theta,$$

and

$$B_1 = \int_0^1 |1-2\theta| \cos\left(\frac{\pi\theta}{2}\right) e^{\theta}\,d\theta.$$

Using numerical computations, we obtain

$$A_1 \approx 0.4542, \qquad B_1 \approx 0.7135.$$

Therefore,

$$\begin{aligned}\frac{1}{2}\left[2A_1+4B_1\right] &= \frac{1}{2}\left[2(0.4542)+4(0.7135)\right]\\ &= \frac{1}{2}(0.9084+2.8540)\\ &= 1.8812.\end{aligned}$$

Thus,

$$0.1667 \leq 1.8812.$$

Hence, the obtained inequality is verified.

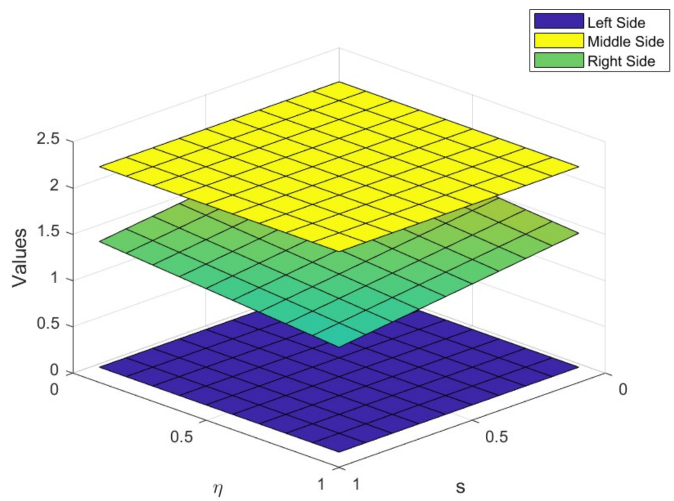
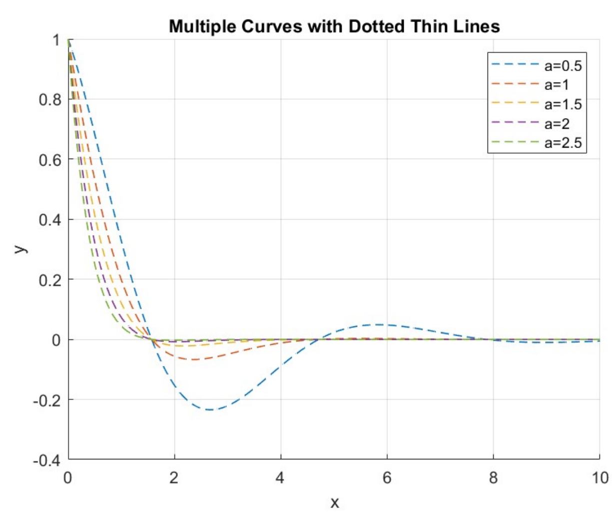



FIGURE 4. 3D surface of R.H.S



| Case | Parameter $a$ | Decay Rate | Oscillation Strength | Graph Behavior |
|---|---|---|---|---|
| 1 | 0.5 | Slow decay | Strong oscillations | Curve remains visible for a long range of $x$ |
| 2 | 1 | Moderate decay | Moderate oscillations | Gradual damping of the signal |
| 3 | 1.5 | Faster decay | Reduced oscillations | Solution approaches zero more quickly |
| 4 | 2 | Strong decay | Weak oscillations | Rapid loss of amplitude |
| 5 | 2.5 | Very strong decay | Very weak oscillations | Curve vanishes quickly near small $x$ |

Table 1. decay and oscillation behavior

## 5. Discussion

The study below shows how the parameter $a$ affects the the function $y = e^{-ax}\cos(x)$. The damping characteristics of the system is found to be controlled by the parameter $a$.

When $a$ is small (e.g., $a = 0.5$), the exponential decay term $e^{-ax}$ decays slowly, and the oscillations can be larger over a larger domain. Thus, the solution is sustained oscillatory.

The higher the value of $a$, the more amplitude will decay, and the more quickly, as shown in the graph. When $a = 2$ and $a = 2.5$ the oscillations decrease rapidly and the solution tends to 0 in a shorter time. We can observe that Smaller values of $a$ lead to slow damping and larger values lead to fast damping and quick stabilization of the system.

From a physical perspective, $a$ is called a damping coefficient, and is interpreted as a loss of energy in the system.

## 6. Conclusion

In this paper, we considered the specific class of convexity; named as MET-$(p,s)$-convexity and discussed its implementation to the Hermite-Hadamard type inequalities. Moreover. we developed lemma that helpful to obtained the extension and generalization of more inequalities, and extract some corollaries from our main results. Our results clearly showed that MET-$(p,s)$-convexity is a convenient and better generalization in the derivation of fractional inequalities.

Theorems 3.2 and 3.3 in fact, generalize the results for the MET-$(p,s)$-convexity, based on the aid of trigonometric weight functions and exponential modulating functions. Many of the inequalities are further shown to be generalizable to the weighted and modulated method.

The corollaries 3.1 to 3.14 follow immediately from the above theorems as particular cases. We notice that some of the existing integral inequalities such as those of Dragomir–Agarwal [8], Mihai [33], Mudassar [34], Almutairi–Klaman [1] etc., can be recovered if we select $h(\theta) = 1/\theta, h(\theta) = \theta^s, \eta = 1$, etc.

We point out that the addition of MET-$(p,s)$-convexity to exponential–trigonometric modulation can further improve many of the established fractional integral inequalities.

From the above considerations, we conclude that the concept of MET-$(p, s)$-convexity is very useful and also can be applicable in fractional inequalities and it provides a new horizon for research in fractional inequalities and optimization theory.

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