Int. J. Anal. Appl. (2026), 24:256

International Journal of Analysis and Applications

Superhyper Topological Spaces with Grills

A. A. Azzam*, **M. Aldawood**

Mathematics Department, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia

**Corresponding author:* azzam0911@yahoo.com

Abstract. This study focuses on the inaugural development of enlarged Superhyper Topological structures employing the notion of a grill. Within this framework, we discovered new concepts such as α-$\mathfrak{G}$ compact sets and countably α-$\mathfrak{G}$ compact spaces. Conversely, we examined the attributes of these notions and their interrelations with one another and their antecedent counterparts.

1. Introduction

Topology is crucial for understanding interconnected systems across several domains. The primary emphasis of classical topology is on point-based open sets and their interrelations [1, 2, 3]. To accurately depict higher-order interactions, traditional topological frameworks must be augmented as relational complexity increases in modern applications. The Superhyper Topology (*ShT*) [4], encompassing multi-layered relational structures derived from power sets, exemplifies such an expansion. *ShT* utilizes nth-power set structures to enhance classical topology, facilitating a more comprehensive representation of hierarchical and interconnected systems. This technique enables the examination of complex structures whose relationships extend beyond mere paired interactions to encompass higher-level relational groups. The emergence of *ShT* provides a powerful tool for analyzing network systems, structured information, and decision-making processes. Research on graph-based topologies, metric-induced topologies, and relational structures in network science has substantially contributed to prior studies on topology formation using directed graphs (digraphs) and binary relations. Relational topology provides significant insights into the architecture of organized systems, as demonstrated by these studies. Current techniques sometimes overlook multidimensional interactions, necessitating the development of more advanced frameworks. This paper extends these concepts to *ShT* through the utilization of advanced relational

Received: Jun. 8, 2026.

2020 *Mathematics Subject Classification.* 03E72; 03E99; 91B06; 54A05.

Key words and phrases. superhyper topology; $\mathfrak{G}$ compact; α-$\mathfrak{G}$ compact; α-$\mathfrak{G}$ compact sets; countably α-$\mathfrak{G}$ compact.

https://doi.org/10.28924/2291-8639-24-2026-256

ISSN: 2291-8639

structures derived from power sets. Choquet [5] developed the concept of a grill ($\mathfrak{G}$) to generalize the notion of a topological space (T_s). It produces a more refined topology that aids in examining properties that were challenging to ascertain with the initial topology. There are relevant connections between the concept of $\mathfrak{G}$ and various established ideas like ideals, nets, and filters. In [6], the authors introduced and analyzed the concept of a topology $\mathfrak{T}_{\mathfrak{G}}$ associated with a $\mathfrak{G}$ on a T_s $(\mathfrak{X}, \mathfrak{T})$. Over the last decade, the concepts of continuity and compactness have been expanded. The notion of $\mathfrak{G}$ has resulted in the development of various fascinating structures, attributes, and descriptions as cited in [7, 8-12]. Nasef and Azzam [13, 14] presented innovative topological operators utilizing a grill mechanism. Azzam [15, 16] exhibited generalized compactness, as well as nano generalized closed sets, employing $\mathfrak{G}$. In [17], the authors established $\mathfrak{G}$-compactness and examined its relationship with compactness. The investigation will incorporate the concept of $\mathfrak{G}$ with *ShT* and examine the distinctive characteristics of the words α-$\mathfrak{G}$-compactness, sets of α-$\mathfrak{G}$ compactness, and countably α-$\mathfrak{G}$ compactness. Additionally, certain properties and characterizations have been acquired.

2. Preliminaries

ShT is a version of classical topology that use the nth-powerset construction to include higher-order relational structures. *ShT* provides a more refined perspective on structured data by systematically defining open sets in $P_n^{\star}(\mathfrak{O})$. This facilitates the examination of dynamic connections in fields such as theoretical informatics, network research, and computational topology. The power set structure offers a versatile framework for mimicking real-world systems due to its recursive nature, ensuring that relational complexity is preserved at all levels. A subset of $\mathfrak{X} \times \mathfrak{X}$ is a binary relation $\mathfrak{R}$ on a set $\mathfrak{X}$. We state that $\mathfrak{x}$ is connected to $\mathfrak{y}$ under $\mathfrak{R}$ if $(\mathfrak{x}, \mathfrak{y}) \in \mathfrak{R}$. Topological structures ($T_s t$) can be induced by binary relations.

Definition 2.1. [18, 19] *The nth-power set of a set $\mathfrak{O}$, denoted as $P_n(\mathfrak{O})$, is generated iteratively from the conventional power set. The procedure is delineated as:*

$$P_1(\mathfrak{O}) = P(\mathfrak{O}), P_{n+1}(\mathfrak{O}) = P(P_n(\mathfrak{O})) \text{ for } n \geq 1.$$

Similarly, the nth-non-empty power set, represented as $P_n^{\star}(\mathfrak{O})$, is specified sequentially as:

$$P_1^{\star}(\mathfrak{O}) = P^{\star}(\mathfrak{O}), P_{n+1}^{\star}(\mathfrak{O}) = P^{\star}(P_n^{\star}(\mathfrak{O})).$$

$P_n^{\star}(\mathfrak{O})$ denotes the power set of $\mathfrak{O}$, omitting ϕ.

Example 2.1. *Letting $\mathfrak{O} = \{1, 2\}$ denote a set. We will iteratively construct the nth-powerset $P_n(\mathfrak{O})$ for n=1, 2.*
Standard powerset $P_1(\mathfrak{O}) = P(\mathfrak{O}) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$.
Second powerset $P_2(\mathfrak{O}) = P(P_1(\mathfrak{O})) = 2^{|P_1(\mathfrak{O})|} = 16$ subsets, including
$P_2(\mathfrak{O}) = \{\phi, \{\phi\}, \{\{1\}\}, \{\{2\}\}, ..., P_1(\mathfrak{O})\}$.
The nth-powerset $P_n(\mathfrak{O})$ for $\mathfrak{O} = \{1, 2\}$:

$$P_1(\mathfrak{O}) = P(\mathfrak{O}) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}, \text{ with 4 subsets.}$$

$$P_2(\mathfrak{O}) = P(P_1(\mathfrak{O})), \textit{ with 16 subsets of } P_1(\mathfrak{O}).$$

Definition 2.2. [4] *A ShT is a topology established on the nth-power set of a specified non-empty set $\mathfrak{O}$, excluding ϕ, referred to as $P_1^\star(\mathfrak{O})$. It is defined recursively outlined below:*

(i) The first power set, excluding ϕ, is represented as $P^\star(\mathfrak{O})$, where:
$P^\star(\mathfrak{O}) = P(\mathfrak{O})\backslash\phi$.

(ii) The second power set is given by: $P_n^\star(\mathfrak{O}) = P^\star(P^\star(\mathfrak{O}))$ where all empty sets are excluded.

(iii) The nth-power set is defined recursively as follows: $P_n^\star(\mathfrak{O}) = P^\star(P_{n-1}^\star(\mathfrak{O}))$, $n \geq 2$.

Definition 2.3. [4] *Let $\mathfrak{T}_{ShT}$ denote a collection of subsets of $P_n^\star(\mathfrak{O})$. Consequently, $\mathfrak{T}_{ShT}$ is designated as a ShT on $P_n^\star(\mathfrak{O})$ if it adheres to the subsequent principle:*

(i) $P_n^\star(\mathfrak{O}), \phi \in \mathfrak{T}_{ShT}$.

(ii) The intersection with any finite collection of elements in $\mathfrak{T}_{ShT}$ constitutes an element of $\mathfrak{T}_{ShT}$.

(iii) The union for any collection of elements in $\mathfrak{T}_{ShT}$ constitutes an element of $\mathfrak{T}_{ShT}$.

Thus, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is called a Superhyper Topological Space (ShT_s).

Definition 2.4. [5] *A non-empty subcollection $\mathfrak{G}$ of a space $\mathfrak{Z}$ with carries topology $\mathfrak{T}$ is named a grill on this space if the following conditions are true:*

(i) ϕ not belongs to $\mathfrak{G}$,

(ii) $C \in \mathfrak{G}$ and $C \sqsubseteq H \sqsubseteq \mathfrak{Z}$, this leads to $H \in \mathfrak{G}$,

(iii) If $C \sqcup H \in \mathfrak{G}\ \forall\ C, H \sqsubseteq \mathfrak{Z}$, then $C \in \mathfrak{G}$ or $H \in \mathfrak{G}$.

Definition 2.5. [6] *Give $(\mathfrak{Z}, \mathfrak{T})$ denote a T_s and $\mathfrak{G}$ indicate a grill on $\mathfrak{Z}$. We construct a function $\Omega : P(\mathfrak{Z}) \to P(\mathfrak{Z})$, referred to as $\Omega_{\mathfrak{G}}(C, \mathfrak{T})$ for $C \in P(\mathfrak{Z})$, which signifies the power set of $\mathfrak{Z}$. This can also be referred to as $\Omega_{\mathfrak{G}}(C)$ or simply $\Omega(C)$ when the topology and grill on $\mathfrak{Z}$ are understood. This function, which is linked to the grill $\mathfrak{G}$ and the topology $\mathfrak{T}$, is given by $\Omega_{\mathfrak{G}}(C) = \{\mathfrak{z} \in \mathfrak{Z} : C \sqcap \mathcal{U} \in \mathfrak{G}$, for all $\mathcal{U} \in \mathfrak{T}(\mathfrak{z})\}$, where $\mathfrak{T}(\mathfrak{z})$ denotes the collection of all open neighborhoods (nbS) of $\mathfrak{z}$. We additionally establish a mapping $\chi : P(\mathfrak{Z}) \to P(\mathfrak{Z})$. The expression by $\chi(C) = C \sqcup \Omega(C)$ defines a Kuratowski closure operator for every $C \in P(\mathfrak{Z})$, therefore inducing a topology $\mathfrak{T}_{\mathfrak{G}}$ on $\mathfrak{Z}$.*

Definition 2.6. [6] *Corresponding to a grill $\mathfrak{G}$ on a T_s $(\mathfrak{Z}, \mathfrak{T})$, there exists a unique topology $\mathfrak{T}_{\mathfrak{G}}$ on $\mathfrak{Z}$ given by $\mathfrak{T}_{\mathfrak{G}} = \{\mathcal{U} \sqsubseteq \mathfrak{Z} : \chi(\mathfrak{Z}\backslash\mathcal{U}) = \mathfrak{Z}\backslash\mathcal{U}\}$, where for any $C \sqsubseteq \mathfrak{Z}$, $\chi(C) = C \sqcup \Omega(C)$.*

Definition 2.7. [6] *(i) If there are two grills, $\mathfrak{G}_1$ and $\mathfrak{G}_2$ on a T_st $(\mathfrak{Z}, \mathfrak{T})$ with $\mathfrak{G}_2 \sqsupseteq \mathfrak{G}_1$, then $\mathfrak{T}_{\mathfrak{G}_2} \sqsubseteq \mathfrak{T}_{\mathfrak{G}_1}$.*

(ii) If $\mathfrak{G}$ is a grill on a space $(\mathfrak{X}, \mathfrak{T})$, $H \notin \mathfrak{G}$, then H is closed in $\mathfrak{T}_{\mathfrak{G}}$.

(iii) For any subset C of a space $(\mathfrak{Z}, \mathfrak{T})$ and any grill $\mathfrak{G}$ on $\mathfrak{Z}$, $\Omega(C)$ is $\mathfrak{T}_{\mathfrak{G}}$-closed.

Definition 2.8. [17] *If $\mathfrak{G}$ represent a grill on a T_s $(\mathfrak{Z}, \mathfrak{T})$. A cover $\{\mathcal{U}_\delta : \delta \in \Gamma\}$. If there is a finite subset Γ_0 of Γ such that $\mathfrak{Z} \setminus \sqcup_{\delta \in \Gamma_0} \mathcal{U}_\delta \notin \mathfrak{G}$, then a subset of $\mathfrak{Z}$ is called a $\mathfrak{G}$-cover. A cover that does not qualify as a $\mathfrak{G}$-cover of $\mathfrak{Z}$ is termed a $\overline{\mathfrak{G}}$-cover.*

Definition 2.9. [20] *A subset $\mathcal{A}$ of a space $(\mathfrak{Z}, \mathfrak{T})$ is named α-open if $\mathcal{A} \sqsubseteq int(cl(int(\mathcal{A})))$.*

Definition 2.10. [21] *A map $\mathfrak{f} : (\mathfrak{O}, \tau) \to (\mathfrak{c}, \xi)$ is named α-irresolute if $\mathfrak{f}^{-1}(\mathfrak{v})$ is α-open in $(\mathfrak{O}, \tau)$ for all α-open set $\mathfrak{v}$ in $(\mathfrak{c}, \xi)$.*

Definition 2.11. [9] *Have $(\mathfrak{Z}, \mathfrak{T})$ denote a T_s. $\mathcal{U} = \{\mathcal{U}_\delta : \delta \in \Gamma\}$ which is cover of $\mathfrak{Z}$ is termed an α-open cover if every element of $\mathcal{U}$ is an α-open set of $\mathfrak{Z}$.*

3. α-$\mathfrak{G}$ Compact Spaces

This section explores the extensions of compactness through the notion of the grill inside *ShT*. The *ShT* enhances the topology derived from via the nth-Powerset.

Definition 3.1. *Let $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ a ShT_s with grill $\mathfrak{G}$. Consequently, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is classified as a α-$\mathfrak{G}$ compact space if every α-open cover of $P_n^\star(\mathfrak{O})$ constitutes a $\mathfrak{G}$-cover.*

Example 3.1. *Let $\mathfrak{O} = \{1, 2, 3\}$. The power set excluding the empty set is $P(\mathfrak{O}) \setminus \phi$. The first Power Set $P_1^\star$ of $\mathfrak{O}$ is:*
$P_1^\star(\mathfrak{O}) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ *Then, the ShT $\mathfrak{T}_{ShT}$ on $p_1^\star(\mathfrak{O})$ is*
$\mathfrak{T}_{ShT} = \{\phi, p_1^\star(\mathfrak{O}), \{\{1\}\}, \{\{2\}\}, \{\{1\}, \{2\}\}, \{\{2\}, \{3\}\}, \{\{1\}, \{2\}, \{3\}\}\}$ *is α-$\mathfrak{G}$ compact space.*

Remark 3.1. *Every open set in $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-open. However, the reverse may not necessarily hold.*

Example 3.2. *From Example 3.2, it is observed that $\{\{1\}, \{3\}\}$ constitutes a α-open set that is not an open set in the ShT.*

Remark 3.2. *Every α compact space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is clearly α-$\mathfrak{G}$ compact for any grill $\mathfrak{G}$ on $P_n^\star(\mathfrak{O})$.*

Proof. Actually, if $\{O_m : m \in \Gamma\}$ constitutes a α-open cover of $P_n^\star(\mathfrak{O})$ within a α compact space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$, then a finite subcover $\{O_m : m \in \Gamma_0\}$ of $P_n^\star(\mathfrak{O})$ exists. Given that $P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} O_m = \phi \notin \mathfrak{G}$, it follows that $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact. □

Proposition 3.1. *Let $\mathfrak{G} = P_n^\star(\mathfrak{O})$; then, the α-$\mathfrak{G}$ compactness of the space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is contingent both the compactness and α compactness of $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$.*

Proof. Initially, we aim to demonstrate that every α-compact space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is a compact space. Let $\{O_m : m \in \Gamma\}$ constitute an open cover of $P_n^\star(\mathfrak{O})$. Since any open set is α-open, it follows that $\{O_m : m \in \Gamma\}$ is α-open cover of $P_n^\star(\mathfrak{O})$. Given that $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α compact space, there exist a finite subcover $\{O_m : m \in \Gamma_0\}$ of $P_n^\star(\mathfrak{O})$ and $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is compact space. Let $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ be α-$\mathfrak{G}$ compact, where $\mathfrak{G} = P_n^\star(\mathfrak{O})$. Let $\{O_m : m \in \Gamma\}$ form a α-open cover of $P_n^\star(\mathfrak{O})$. Given that $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact, subsequently there is a finite subset Γ_0 of Γ such that $P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} O_m \notin \mathfrak{G}$. Given that $\mathfrak{G} = P_n^\star(\mathfrak{O})$, then $P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} O_m = \phi$. Consequently, $\{O_m : m \in \Gamma_0\}$ constitutes a finite subcover of $P_n^\star(\mathfrak{O})$. So, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α compact space and compact space. □

Proposition 3.2. *Let $\mathfrak{G} = P^\star(P_n^\star(\mathfrak{O}))$ denote a grill on a ShT_s $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ and grill ShT_s $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}^{\mathcal{G}})$ is α-$\mathfrak{G}$ compact. Thus, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is compact and α compact ($\mathfrak{T}_{ShT} \sqsubseteq \mathfrak{T}_{ShT}^{\mathfrak{G}}$) consequently, it is α-$\mathfrak{G}$ compact.*

Proof. If $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}^{\mathfrak{G}})$ denote a α-$\mathfrak{G}$ compact space, where $\mathfrak{G} = P^\star(P_n^\star(\mathfrak{O}))$. We aim to demonstrate that $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ constitutes a α compact space. Let $\{O_m : m \in \Gamma\}$ denote an arbitrary $\mathfrak{T}_{ShT}\alpha$-open cover of $P_n^\star(\mathfrak{O})$. Given that $\mathfrak{T}_{ShT} \sqsubseteq \mathfrak{T}_{ShT}^{\mathfrak{G}}$, then $\{O_m : m \in \Gamma\}$ be $\mathfrak{T}_{ShT}^{\mathfrak{G}}\alpha$-open cover of $P_n^\star(\mathfrak{O})$.If $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}^{\mathfrak{G}})$ is a α-$\mathfrak{G}$ compact space, then there exists a finite subset Γ_0 of Γ that results in $P_n^\star(\mathfrak{O}) \setminus \bigsqcup_{m \in \Gamma_0} O_m \notin \mathfrak{G}$. Since $\mathfrak{G} = P_n^\star(\mathfrak{O})$, then $P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} O_m = \phi$. Consequently, $\{O_m : m \in \Gamma\}$ constitutes a finite subcover of $P_n^\star(\mathfrak{O})$. Consequently, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is a α-compact space and, therefore, a compact space. □

Remark 3.3. *Each α-$\mathfrak{G}$ compact Space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ for any $\mathfrak{G}$ on $P_n^\star(\mathfrak{O})$, it is evidently $\mathfrak{G}$-compact.*

Proof. Let $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ be a α-$\mathfrak{G}$ compact, then for all every α-open cover of $P_n^\star(\mathfrak{O})$ constitutes a $\mathfrak{G}$-cover. So, every open cover of $P_n^\star(\mathfrak{O})$ constitutes a $\mathfrak{G}$-cover. Then $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is $\mathfrak{G}$-compact.

□

Theorem 3.1. *If $\mathfrak{G}$ represent a grill on a ShT_s $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$. Subsequently, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}^{\mathcal{G}})$ is α-$\mathfrak{G}$ compact if $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact (where $\mathfrak{T}_{ShT}^{\mathcal{G}}$ denotes a discrete ShT on $P_n^\star(\mathfrak{O})$).*

Proof. If $\{O_m : m \in \Gamma\}$ denote a fundamental $\mathfrak{T}_{ShT}^{\mathcal{G}}\alpha$-open cover of $P_n^\star(\mathfrak{O})$. According to the definition of a base for $\mathfrak{T}_{ShT}^{\mathcal{G}}$, for each $m \in \Gamma$, we have $O_m = O_i \setminus O_k$, where $O_i \in \mathfrak{T}_{ShT}$ and $O_k \notin \mathcal{G}$. Consequently, $\{O_i : m \in \Gamma\}$ constitutes a $\mathfrak{T}_{ShT}$-open cover of $P_n^\star(\mathfrak{O})$. Given that every open set is α-open, it follows that $\{O_i : m \in \Gamma\}$ constitutes a $\mathfrak{T}_{ShT}\alpha$-open cover of $P_n^\star(\mathfrak{O})$.If $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact, it follows that there is a finite subset Γ_0 of Γ which is $P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} O_i \notin \mathfrak{G}$. Then $P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} O_i = P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} (O_m \setminus O_k) \sqsubseteq P_n^\star(\mathfrak{O}) \setminus (\sqcup_{m \in \Gamma_0} O_i \setminus \sqcup_{m \in \Gamma_0} O_k) \sqsubseteq (P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} O_i) \sqcup (\sqcup_{m \in \Gamma_0} O_k) \notin \mathfrak{G}$. So, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}^{\mathfrak{G}})$ is α-$\mathfrak{G}$ compact. □

Remark 3.4. *If $\mathfrak{G}$ represent a grill on a ShT_s $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$. Consequently, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}^{\mathfrak{G}})$ is $\mathfrak{G}$ compact if $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact.*

Proof. It is clear. □

4. Quasi h-Closed Modulo a Grill Space

A space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is termed compact modulo a grill $\mathfrak{G}$ on $P_n^\star(\mathfrak{O})$, or simply $(\mathfrak{G})$ compact space, if for every open cover $\{O_m : m \in \Gamma\}$ of $P_n^\star(\mathfrak{O})$, there exists a finite subfamily $\{O_m : m \in \Gamma_0\}$ such that $P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} O_m \notin \mathfrak{G}$. In this part, we delineate quasi h-closedness modulo a grill and examine many of its features. This technique yields intriguing characterizations of *qhcS*.

Definition 4.1. *Let $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ be a ShTs and $\mathfrak{G}$ a grill on $P_n^\star(\mathfrak{O})$. $P_n^\star(\mathfrak{O})$ is qhc modulo $\mathfrak{G}$ ($\mathfrak{G}$qhc) if for every open cover $\{O_m : m \in \Gamma\}$ of $P_n^\star(\mathfrak{O})$ there exists a finite subfamily $\{O_m : m \in \Gamma_0\}$ such that $P_n^\star(\mathfrak{O}) \setminus \sqcup_{m \in \Gamma_0} O_m \notin \mathfrak{G}$. Such a subfamily is said to be proximate subcover modulo $\mathfrak{G}$ ($\mathfrak{G}$ proximate subcover).*

The evidence of the subsequent theorem is evident from the discourse in section 2.

Theorem 4.1. *For a space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$, the following are equivalent:*
(1) $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is qhc;
(2) $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is $(P_n^\star(\mathfrak{O}))$qhc;
(3) $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is $\mathfrak{G}_f$qhc, where $\mathfrak{G}_f$ the grill of all finite subsets of $P_n^\star(\mathfrak{O})$;
(4) $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is $\mathfrak{G}_n$qhc, where $\mathfrak{G}_n$ the grill of all nowhere dense subsets in a ShTs.

A family $\mathcal{L}$ of subsets of $P_n^\star(\mathfrak{O})$ possesses a finite-intersection property modulo a grill $\mathfrak{G}$ on $P_n^\star(\mathfrak{O})$ (denoted as $\mathfrak{G}$*FIP*) if the intersection of any finite subfamily of $\mathcal{L}$ is not a member of $\mathfrak{G}$.

Theorem 4.2. *For a space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$, and a grill $\mathfrak{G}$ on $P_n^\star(\mathfrak{O})$, the following are equivalent:*
(1) $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is $\mathfrak{G}$qhc;
(2) For each family $\mathcal{L}$ of closed sets having empty intersection, there is a finite subfamily $\{\updownarrow_1, \updownarrow_2, \updownarrow_3, ..., \updownarrow_n\}$ such that $\sqcap_{i=1}^n int(\updownarrow_i) \notin \mathfrak{G}$;
(3) For each family $\mathcal{L}$ of closed sets such that $\{int(\updownarrow) : \updownarrow \in \mathcal{L}\}$ has $(\mathfrak{G})$FIP, one has $\sqcap\{\updownarrow : \updownarrow \in \mathcal{L}\} \in \mathfrak{G}$;
(4) There is a finite $(\mathfrak{G})$ proximate subcover for each regular open cover;
(5) For any family $\mathcal{L}$ of non-empty regular closed sets with an empty intersection, there exists a finite subfamily $\{\updownarrow_1, \updownarrow_2, \updownarrow_3, ..., \updownarrow_n\}$ such that $\sqcap_{i=1}^n int(\updownarrow_i) \in \mathfrak{G}$;
(6) For each collection $\mathcal{L}$ of non empty regular closed sets such that $\{int(\updownarrow) : \updownarrow \in \mathcal{L}\}$ has $(\mathfrak{G})$FIP, one has $\sqcap\{\updownarrow : \updownarrow \in \mathcal{L}\} \in \mathfrak{G}$;
(7) For each open filter base $\mathbb{B}$ on $P_n^\star(\mathfrak{O})$-$\mathfrak{G}$, $\sqcap\{cl(B) : B \in \mathbb{B}\} \in \mathfrak{G}$.

5. Compactness of Hausdorff Superhyper Topological Spaces via Grill

We shall delineate the relationship between $\mathfrak{G}$ and α-quasi h-closed.

Definition 5.1. *A ShT_s $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is said to be α-quasi h-closed (α qhc) if for every α-open cover O of $P_n^\star(\mathfrak{O})$, there is a finite sub-collection O_0 of O such that $P_n^\star(\mathfrak{O}) = \sqcup\{cl_{ShT}(w : w \in O_0\}$. A Hausdorff α qhc space is called an α h-closed Space (α qhcS).*

Proposition 5.1. *Have $\mathfrak{G}$ denote a grill on a ShT_s $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$, such that $\mathfrak{T}_{ShT} \setminus \phi \sqsubseteq \mathfrak{G}$. If $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact, it is designated as a α qhc.*

Proof. Consider the set $\{O_m : m \in \Gamma\}$ be a fundamental $\mathfrak{T}^{\mathfrak{G}}_{ShT}\alpha$-open cover of $P^\star_n(\mathfrak{O})$. If $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact, it follows that there is a finite subset Γ_0 of Γ which is $P^\star_n(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{G}$. Consequently $int(P^\star_n(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m) = \phi$. Otherwise, $int(P^\star_n(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m) \in \mathfrak{T}^{\mathfrak{G}}_{ShT}\backslash\phi \sqsubseteq \mathfrak{G}$ which implies $P^\star_n(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m \in \mathfrak{G}$, a contradiction.
Consequently, $P^\star_n(\mathfrak{O})\backslash cl(P^\star_n(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m)^c = \phi$. Hence, $P^\star_n(\mathfrak{O}) = \sqcup_{m\in\Gamma_0} cl(O_m)$ and $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α *qhcS*. □

Proposition 5.2. *Suppose that $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is an α qhcS, where $\mathfrak{T}_{ShT}$ is a discrete ShT on $P^\star_n(\mathfrak{O})$. It follows that $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is $\alpha\mathfrak{G}\delta$-compact, where $\mathfrak{G}\delta$ a grill defined as $\mathfrak{G}\delta = \{\mathcal{A} \sqsubseteq P^\star_n(\mathfrak{O}) : int(cl(\mathcal{A})) \neq \phi\}$.*

Proof. Suppose that $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is an α *qhcS*, and $O = \{O_m : m \in \Gamma\}$ be a fundamental $\mathfrak{T}^{\mathfrak{G}}_{ShT}\alpha$-open cover of $P^\star_n(\mathfrak{O})$. It follows that exists a finite sub-collection $O_0 = \{O_m : m \in \Gamma_0\}$ of O such that $P^\star_n(\mathfrak{O}) = \sqcup\{cl(O_m) : O_m \in O_0\}$. Then $P^\star_n(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{G}\delta$. Otherwise, $P^\star_n(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m \in \mathfrak{G}\delta$, and hence $P^\star_n(\mathfrak{O})\backslash clint(\sqcup_{m\in\Gamma_0} O_m) \neq \phi$. Thus $P^\star_n(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} cl(O_m) \neq \phi$, a contradiction. Therefore $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is $\alpha\mathfrak{G}\delta$ compact. □

Definition 5.2. *Consider $\mathfrak{G}$ represent a grill on a ShT_s $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$. The space $P^\star_n(\mathfrak{O})$ is classified as α-$\mathfrak{G}$ regular if, for any α-closed set $\mathcal{L}$ in $P^\star_n(\mathfrak{O})$ where $l \notin \mathcal{L}$, that exist disjoint α-open sets O and $\mathcal{M}$ such that $l \in O$ and $\mathcal{L}\backslash\mathcal{M} \notin \mathfrak{G}$.*

Proposition 5.3. *Let $\mathfrak{G}$ denote a grill on ShT Hausdorff Space $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$. Then, $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ regular if it is α-$\mathfrak{G}$ compact.*

Proof. If $\mathcal{L}$ be a α-closed subset of $P^\star_n(\mathfrak{O})$ such that $l \notin \mathcal{L}$. Because $P^\star_n(\mathfrak{O})$ is *ShT* Hausdorff Space, there are two separate open sets O_l and Q_y such that $l \in O_l$ and $y \in Q_y$ for each $y \in \mathcal{L}$. Therefore, $\{Q_y : y \in \mathcal{L}\} \sqcup \{P^\star_n(\mathfrak{O})\backslash\mathcal{L}\}$ represents a α-open cover of $P^\star_n(\mathfrak{O})$. Given that $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact, then there exists finitely many points $y_1, y_2, ..., y_n \in \mathcal{L}$ such that $P^\star_n(\mathfrak{O})\backslash[(\sqcup^n_{i=1} Q_{y_i}) \sqcup (P^\star_n(\mathfrak{O})\backslash\mathcal{L})] \notin \mathfrak{G}$. Let $\mathcal{F} = P^\star_n(\mathfrak{O})\backslash \sqcup^n_{i=1} clQ_{y_i}$ and $\mathcal{A} = \sqcup^n_{i=1} Q_{y_i}$. Then $\mathcal{F}$ and $\mathcal{A}$ are disjoint non-empty α-open sets in $P^\star_n(\mathfrak{O})$ such that $l \in \mathcal{F}$ (since $l \notin clQ_{y_i} \forall i = 1, 2, ..., n$) and $\mathcal{L}\backslash\mathcal{A} = \mathcal{L} \sqcup [P^\star_n(\mathfrak{O})\backslash \sqcup^n_{i=1} Q_{y_i}] = P^\star_n(\mathfrak{O})\backslash[(\sqcup^n_{i=1} Q_{y_i}) \sqcup (P^\star_n(\mathfrak{O})\backslash\mathcal{L})] \notin \mathfrak{G}$. So, $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ regular. If $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact, it is both α-hc and α-$\mathfrak{G}$ regular. □

Corollary 5.1. *If $\mathfrak{G}$ denote a a grill on the ShT Hausdorff Space $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ such that $\mathfrak{T}_{ShT}\backslash\phi \sqsubseteq \mathfrak{G}$. If $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}$ compact, it is both α-hc and α-$\mathfrak{G}$ regular.*

Proof. Proposition 5.2 and 5.5 have direct consequence □

Theorem 5.1. *Assume O is a α-open subbase for a ShT Hausdorff Space $(P^\star_n(\mathfrak{O}), \mathfrak{T}_{ShT})$. If $P^\star_n(\mathfrak{O})$ has an open $\underline{\mathfrak{G}}$-cover then there is a $\underline{\mathfrak{G}}$-cover of $P^\star_n(\mathfrak{O})$, comprising elements of O.*

Proof. If C represent the collection of all open $\underline{\mathfrak{G}}$-covers of $P^\star_n(\mathfrak{O})$. Clear that C is partially order set. Then by hypothesis $C \neq \phi$, let $\{P_r\}$ be a linearly ordered subset of C. Then $\sqcup_r P_r$ is a covering of

$P_n^\star(\mathfrak{O})$. We assert that it constitutes an open $\underline{\mathfrak{G}}$-cover of $P_n^\star(\mathfrak{O})$. For, if not, there is $\mathrm{L}_1, \mathrm{L}_2, ..., \mathrm{L}_\mathrm{n} \in \sqcup_r P_r$ such that $P_n^\star(\mathfrak{O}) \backslash \sqcup_{i=1}^n \mathrm{L}_\mathrm{i} \notin \mathfrak{G}$. Now, there exist a $P_{r_0} \in C$ such that $\mathrm{L}_1, \mathrm{L}_2, ..., \mathrm{L}_\mathrm{n} \in P_{r_0}$. Thus $P_{r_0} \notin C$, a contradiction. Thus, C includes a greatest element P. If C is an open set and $C \notin P$, then there exist finitely many $\mathrm{L}_1, \mathrm{L}_2, ..., \mathrm{L}_\mathrm{n} \in P$ such that $P_n^\star(\mathfrak{O}) \backslash (C \sqcup \mathrm{L}_1 \sqcup \mathrm{L}_2 \sqcup ... \sqcup \mathrm{L}_\mathrm{n}) \notin \mathfrak{G}$. The assemblage of open sets excluded from P forms a filter. To finalize the proof, it is adequate to demonstrate that $O \sqcap P$ constitutes a $\underline{\mathfrak{G}}$-cover of $P_n^\star(\mathfrak{O})$. Let $f \in P_n^\star(\mathfrak{O})$, and since P constitutes open cover of $P_n^\star(\mathfrak{O})$, there exists an open set $\mathcal{F} \in P$ such that $f \in \mathcal{F}$. Given that O is α-open subbase for $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$, there exist $\int_1, \int_2, ..., \int_n \in O$ such that $f \in \int_1 \sqcap \int_2 \sqcap ... \sqcap \int_n \sqsubseteq \mathcal{F}$. It follows that there exists an $\int_i$ for some $i = 1, 2, ..., n$, such that $\int_i \in P$. For otherwise, if $\int_i \notin P$ for all $i = 1, 2, ..., n$, then $\sqcap_{i=1}^n \int_i \in \mathfrak{T}_{ShT} \backslash P$. Thus $\mathcal{F} \in \mathfrak{T}_{ShT} \backslash P$ and $\mathcal{F} \notin P$, a contradiction. Consequently, $l \in \int_i \in O \sqcap P$, hence $O \sqcap P$ constitutes a $\underline{\mathfrak{G}}$-cover of $P_n^\star(\mathfrak{O})$. □

6. α-$\mathfrak{G}$ Compact Sets with Relation to a Superhyper Topological Space

Definition 6.1. *Take $\mathfrak{G}$ define a grill on the ShT$_s$ $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$. A subset $\mathcal{A}$ of the space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is defined to be α-$\mathfrak{G}$ compact relative to $P_n^\star(\mathfrak{O})$ if, for every cover $\{O_m : m \in \Gamma\}$ of $\mathcal{A}$ by α-open sets of $P_n^\star(\mathfrak{O})$, there exists a finite subset Γ_0 of Γ such that $\mathcal{A} \backslash \sqcup_{m \in \Gamma_0} O_m \notin \mathfrak{G}$.*

Theorem 6.1. *The subsequent statements are identical for a subset $\mathcal{A}$ of $P_n^\star(\mathfrak{O})$.:*
(i) $\mathcal{A}$ is α-$\mathfrak{G}$ compact relative to $P_n^\star(\mathfrak{O})$, where the grill $\mathfrak{G} = P^\star(P_n^\star(\mathfrak{O}))$;
(ii) $\mathcal{A}$ is α compact relative to $P_n^\star(\mathfrak{O})$.

Proof. From (i) to (ii). Let $\{O_m : m \in \Gamma\}$ denote a cover of $\mathcal{A}$ via α-open sets of $P_n^\star(\mathfrak{O})$. Because $\mathcal{A}$ is α-$\mathfrak{G}$ compact with respect to $P_n^\star(\mathfrak{O})$, there exists a finite subset Γ_0 of Γ such that $\mathcal{A} \backslash \sqcup_{m \in \Gamma_0} O_m \notin \mathfrak{G}$. Given that $\mathfrak{G} = P^\star(P_n^\star(\mathfrak{O}))$, it follows that $\mathcal{A} \backslash \sqcup_{m \in \Gamma_0} O_m = \phi$. Consequently, $\mathcal{A} = \sqcup_{m \in \Gamma_0} O_m$, indicating that $\mathcal{A}$ is α compact with respect to $P_n^\star(\mathfrak{O})$.
From (ii) to (i). Let $\{O_m : m \in \Gamma\}$ be a cover of $\mathcal{A}$ by α-open sets of $P_n^\star(\mathfrak{O})$. Since $\mathcal{A}$ is α compact with respect to $P_n^\star(\mathfrak{O})$, then there is a finite subcover $\{O_m : m \in \Gamma_0\}$ of $\mathcal{A}$. Consequently, $\mathcal{A} \backslash \sqcup_{m \in \Gamma_0} O_m = \phi \notin \mathfrak{G}$; hence, $\mathcal{A}$ is α-$\mathfrak{G}$ compact with respect to $P_n^\star(\mathfrak{O})$. □

Proposition 6.1. *Let $\mathfrak{G}$ denote a grill on the ShT$_s$ $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$, If $\mathcal{A}_m, m = 1, 2$ are α-$\mathfrak{G}$ compact subsets with respect to $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$, then $\mathcal{A}_1 \sqcup \mathcal{A}_2$ is α-$\mathfrak{G}$ compact with respect to $P_n^\star(\mathfrak{O})$.*

Proof. Consider $\{O_m : m \in \Gamma\}$ as a cover of $\mathcal{A}_1 \sqcup \mathcal{A}_2$ by α-open sets of $P_n^\star(\mathfrak{O})$. Consequently, it constitutes a α-open cover of $\mathcal{A}_m$ where, $m = 1, 2$. Given that $\mathcal{A}_m$ is α-$\mathfrak{G}$ compact concerning $P_n^\star(\mathfrak{O})$, There is a finite subset Γ_1 of Γ such that $\mathcal{A}_1 \backslash \sqcup_{m \in \Gamma_1} O_m \notin \mathfrak{G}$, as well as exists another finite subset Γ_2 of Γ such that $\mathcal{A}_2 \backslash \sqcup_{m \in \Gamma_2} O_m \notin \mathfrak{G}$. Given that $(\mathcal{A}_1 \backslash \sqcup_{m \in \Gamma_1} O_m) \sqcup \mathcal{A}_2 \backslash (\sqcup_{m \in \Gamma_2} O_m) \sqsupseteq (\mathcal{A}_1 \sqcup \mathcal{A}_2) \backslash \sqcup_{m \in \Gamma_1 \sqcup \Gamma_2} O_m \notin \mathfrak{G}$. Therefore, $\mathcal{A}_1 \sqcup \mathcal{A}_2$ is α-$\mathfrak{G}$ compact with respect to $P_n^\star(\mathfrak{O})$ since there is a finite subset $\Gamma_1 \sqcup \Gamma_2$ of Γ such that $(\mathcal{A}_1 \sqcup \mathcal{A}_2) \backslash \sqcup_{m \in \Gamma_1 \sqcup \Gamma_2} O_m \notin \mathfrak{G}$. □

Theorem 6.2. *Let $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ denote a ShT$_s$ with grill $\mathfrak{G}$ on $P_n^\star(\mathfrak{O})$. $\mathcal{A}$ is α-$\mathfrak{G}$ compact with respect to $P_n^\star(\mathfrak{O})$, then $(\mathcal{A}, \mathfrak{T}_{ShT}/\mathcal{A})$ is $\mathfrak{G}/\mathcal{A}$-compact.*

Proof. Let $\{(O_m \sqcap \mathcal{A}) : m \in \Gamma\}$ be $\mathfrak{T}_{ShT}/\mathcal{A}$-open cover of $\mathcal{A}$, where $O_m \in \mathfrak{T}_{ShT}$ for every $m \in \Gamma$. The collection $\{O_m : m \in \Gamma\}$ constitutes a cover of $\mathcal{A}$ by α-open subsets of $P_n^\star(\mathfrak{O})$. Given that $\mathcal{A}$ is α-$\mathfrak{G}$ compact concerning $P_n^\star(\mathfrak{O})$, it follows that there exists a finite subset Γ_0 of Γ such that $\mathcal{A}\backslash \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{G}$. Consequently, $\mathcal{A} \sqcap [\mathcal{A}\backslash \sqcup_{m\in\Gamma_0} O_m] \notin \mathcal{A} \sqcap \mathfrak{G}$. Since $\mathcal{A}\backslash \sqcup_{m\in\Gamma_0} (O_m \sqcap \mathcal{A}) = \mathcal{A}\backslash \sqcup_{m\in\Gamma_0} O_m = \mathcal{A} \sqcap [\mathcal{A}\backslash \sqcup_{m\in\Gamma_0} O_m] \notin \mathcal{A} \sqcap \mathfrak{G}$, Here exists a finite subset Γ_0 of Γ that contains $\mathcal{A}\backslash \sqcup_{m\in\Gamma_0} (O_m \sqcap \mathcal{A}) \notin \mathfrak{G}/\mathcal{A}$. Consequently, $(\mathcal{A}, \mathfrak{T}_{ShT}/\mathcal{A})$ is $\mathfrak{G}/\mathcal{A}$-compact. □

The reverse of Theorem 5.4 is false, as demonstrated in Example 5.5.

Example 6.1. *Let $\mathfrak{O} = \{1, 2, 3\}$. The power set excluding the empty set is $P(\mathfrak{O})\backslash\phi$. The first Power Set $P_1^\star$ of $\mathfrak{O}$ is:*
$P_1^\star(\mathfrak{O}) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ Then, the ShT $\mathfrak{T}_{ShT}$ on $p_1^\star(\mathfrak{O})$ is
$\mathfrak{T}_{ShT} = \{\phi, p_1^\star(\mathfrak{O}), \{\{1\}\}\}$. The set $O = \{\{1\}, \{2\}\}$ is α-open set in $\mathfrak{T}_{ShT}$. Let $\mathcal{A} = \{\{2\}, \{3\}\}$, and $\mathfrak{T}_{ShT}/\mathcal{A} = \{\{\phi\}, \mathcal{A}\}$. Obviously, $O \sqcap \mathcal{A} = \{\{2\}\}$ not open in $\mathfrak{T}_{ShT}/\mathcal{A}$.

7. Countably α-$\mathfrak{G}$ Compact Superhyper Topological Spaces

Definition 7.1. *Consider $\mathfrak{G}$ represent a grill on the c $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$. A space is termed countably α-$\mathfrak{G}$ compact if, for every countable α-open cover $\{O_m : m \in \Gamma\}$ of $P_n^\star(\mathfrak{O})$, there is a finite subset Γ_0 of Γ such that $P_n^\star(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{G}$, where Γ denotes the set that includes positive numbers.*

Proposition 7.1. *Consider $\mathfrak{G}$ denote a grill on the Consider $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$. If the Space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is classified as countably α-$\mathfrak{G}$ compact, then any countable collection $\{\mathcal{L}_m : m \in \Gamma\}$ of α-closed sets of $P_n^\star(\mathfrak{O})$ such that $\sqcap\{\mathcal{L}_m : m \in \Gamma\}$ is empty, there is a finite subset Γ_0 of Γ such that $\sqcap\{\mathcal{L}_m : m \in \Gamma_0\}$ does not belong to $\mathfrak{G}$.*

Proof. Let $\{\mathcal{L}_m : m \in \Gamma\}$ denote a countable collection of α-closed sets within $P_n^\star(\mathfrak{O})$ such that $\sqcap\{\mathcal{L}_m : m \in \Gamma\} = \emptyset$. The collection $\{P_n^\star(\mathfrak{O})\backslash\mathcal{L}_m : m \in \Gamma\}$ constitutes a countable α-open cover of $P_n^\star(\mathfrak{O})$. Consequently, there is a finite subset Γ_0 of Γ such that $P_n^\star(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} (P_n^\star(\mathfrak{O})\backslash\mathcal{L}_m) \notin \mathfrak{G}$. Consequently, $\sqcap_{m\in\Gamma_0}[P_n^\star(\mathfrak{O})\backslash(P_n^\star(\mathfrak{O})\backslash\mathcal{L}_m)] \notin \mathfrak{G}$. □

Proposition 7.2. *If the Space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is classified as countably α-$\mathfrak{G}_1$ compact, $\mathfrak{G}_1$, and $\mathfrak{G}_2$ are two grills on $P_n^\star(\mathfrak{O})$ such that $\mathfrak{G}_1 \sqsupseteq \mathfrak{G}_2$, then $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}_2$ compact.*

Proof. Let $\{O_m : m \in \Gamma\}$ denote a countable α-open cover of $P_n^\star(\mathfrak{O})$. Because $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is a countably α-$\mathfrak{G}_1$ compact, There exists a finite subset Γ_0 of Γ such that $P_n^\star(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{G}_1$. Since $\mathfrak{G}_1 \sqsupseteq \mathfrak{G}_2$, thus $P_n^\star(\mathfrak{O})\backslash \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{G}_2$. Thus $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is α-$\mathfrak{G}_2$ compact. □

Theorem 7.1. *If $\mathfrak{G} = P^\star(P_n^\star(\mathfrak{O}))$ represents a grill on the ShT$_s$ $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$, the subsequent statements are equivalent*
(i) The Space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is countably α compact;
(ii) The Space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is countably α-$\mathfrak{G}$ compact.

Proof. (i) ⇒ (ii)

Let $\{O_m : m \in \Gamma\}$ denote countable α-open cover of $P_n^\star(\mathfrak{O})$. By (i), there is a finite subcover $\{O_m : m \in \Gamma_0\}$ of $P_n^\star(\mathfrak{O})$. Then $P_n^\star(\mathfrak{O})\setminus \sqcup_{m\in\Gamma_0} O_m = \phi \notin \mathcal{G}$. Thus $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is countably α-$\mathfrak{G}$ compact.

(ii) ⇒ (i) Let $\{O_m : m \in \Gamma\}$ denote countable α-open cover of $P_n^\star(\mathfrak{O})$. By (ii), there is a finite subset Γ_0 of Γ such that $P_n^\star(\mathfrak{O})\setminus \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{G}$. Since $\mathfrak{G} = P^\star(P_n^\star(\mathfrak{O}))$, then $P_n^\star(\mathfrak{O})\setminus \sqcup_{m\in\Gamma_0} O_m = \phi$. This indicates that $\{O_m : m \in \Gamma_0\}$ constitutes a finite subcover of $P_n^\star(\mathfrak{O})$. Thus, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is countably α compact. □

Proposition 7.3. *If the Space $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is classified as countably α compact, then $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is countably α-$\mathfrak{G}$ compact, such that $\mathfrak{G}$ is a grill on $P_n^\star(\mathfrak{O})$.*

Proof. It is apparent. □

Definition 7.2. *A ShT_s $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is designated as α $Lindelöf$ if every α-open cover of $P_n^\star(\mathfrak{O})$ possesses a countable sub-cover.*

Theorem 7.2. *If $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathcal{G})$ is designated as countably α-$\mathfrak{G}$ compact and α $Lindelöf$ space, then $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathfrak{G})$ is α-$\mathfrak{G}$ compact, where $\mathfrak{G}$ denotes a grill on $P_n^\star(\mathfrak{O})$.*

Proof. Suppose $\{O_m : m \in \Gamma\}$ be α-open cover of $P_n^\star(\mathfrak{O})$. Since $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is *α $Lindelöf$*, there is a countably subset Γ_1 of Γ such that $P_n^\star(\mathfrak{O}) = \sqcup_{m\in\Gamma_1} O_m$. However, $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathfrak{G})$ is countably α-$\mathfrak{G}$ compact, therefore is a finite subset Γ_0 of Γ_1 such that $P_n^\star(\mathfrak{O})\setminus \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{G}$. Thus $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathfrak{G})$ is α-$\mathfrak{G}$ compact. □

Theorem 7.3. *Take $\mathfrak{f} : (P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathcal{G}) \to (P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT})$ be an α-irresolute surjective mapping. If $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathcal{G})$ is countably α-$\mathfrak{G}$ compact, then $(P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT}, \mathfrak{f}(\mathfrak{G}))$ is countably α-$\mathfrak{f}(\mathfrak{G})$ compact.*

Proof. If $\{\mathcal{W}_n : n \in \Gamma\}$ represent a countable α-open cover of $P_n^\star(\mathfrak{y})$. Given that $\mathfrak{f}$ is α-irresolute, the collection $\{\mathfrak{f}^{-1}(\mathcal{W}_n) : n \in \Gamma\}$ constitutes a countable α-open cover of $P_n^\star(\mathfrak{O})$. Consequently, there is a finite subset Γ_0 of Γ such that $P_n^\star(\mathfrak{O})\setminus \sqcup_{n\in\Gamma_0} \mathfrak{f}^{-1}(\mathcal{W}_n) \notin \mathfrak{G}$. Given that $\mathfrak{f}$ is surjective, it follows that $P_n^\star(\mathfrak{y})\setminus \sqcup_{n\in\Gamma_0} \mathcal{W}_n = \mathfrak{f}[P_n^\star(\mathfrak{O})\setminus \sqcup_{n\in\Gamma_0} \mathfrak{f}^{-1}(\mathcal{W}_n)] \notin \mathfrak{f}(\mathfrak{G})$. Consequently, $(P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT}, \mathfrak{f}(\mathfrak{G}))$ is countably α-$\mathfrak{f}(\mathfrak{G})$ compact. □

Theorem 7.4. *Let $\mathfrak{f} : (P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathfrak{G}) \to (P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT})$ be an α-continuous surjective function. If $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathfrak{G})$ is countably α-$\mathfrak{G}$ compact, then $(P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT}, \mathfrak{f}(\mathcal{G}))$ is countably α-$\mathfrak{f}(\mathfrak{G})$ compact.*

Proof. Consider $\{\mathcal{W}_n : n \in \Gamma\}$ represent a countable open cover of $P_n^\star(\mathfrak{y})$. Consequently, $\{P_n^\star(\mathfrak{y})\setminus\mathcal{W}_n : n \in \Gamma\}$ constitutes a countable closed subset of $P_n^\star(\mathfrak{y})$. Given that $\mathfrak{f}$ is α-continuous, then $\{\mathfrak{f}^{-1}(P_n^\star(\mathfrak{y})\setminus\mathcal{W}_n) : n \in \Gamma\}$ constitutes a countable α-closed subsets of $P_n^\star(\mathfrak{O})$. Then $\{P_n^\star(\mathfrak{O})\setminus\mathfrak{f}^{-1}(P_n^\star(\mathfrak{y})\setminus\mathcal{W}_n) : n \in \Gamma\}$ is a countable α-open subsets of $P_n^\star(\mathfrak{O})$. Since $\{P_n^\star(\mathfrak{O})\setminus\mathfrak{f}^{-1}(P_n^\star(\mathfrak{y})\setminus\mathcal{W}_n) = \mathfrak{f}^{-1}(\mathcal{W}_n)$, then $\{\mathfrak{f}^{-1}(\mathcal{W}_n : n \in \Gamma\}$ be a countable α-open cover of $P_n^\star(\mathfrak{O})$. Since $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathfrak{G})$ is countably α-$\mathfrak{G}$ compact, there exists a finite subset Γ_0 of Γ such that $P_n^\star(\mathfrak{O})\setminus \sqcup_{n\in\Gamma_0} \mathfrak{f}^{-1}(\mathcal{W}_n) \notin \mathfrak{G}$. Consequently, $\mathfrak{f}(P_n^\star(\mathfrak{O})\setminus \sqcup_{n\in\Gamma_0} \mathfrak{f}^{-1}(\mathcal{W}_n)) \notin \mathfrak{f}(\mathfrak{G})$, which implies $P_n^\star(\mathfrak{y})\setminus \sqcup_{n\in\Gamma_0} \mathcal{W}_n \notin \mathfrak{f}(\mathfrak{G})$, and therefore $(P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT}, \mathfrak{f}(\mathfrak{G}))$ is countably α-$\mathfrak{f}(\mathfrak{G})$ compact. □

Theorem 7.5. *If $\mathfrak{f} : (P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}) \to (P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT}, \mathfrak{G})$ is pre-α-open bijection and $(P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT}, \mathfrak{G})$ is countably α-$\mathfrak{G}$ compact, then $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is countably $\alpha\mathfrak{f}^{-1}(\mathfrak{G})$ compact.*

Proof. If $\mathfrak{f}$ is pre-α-open, then $\mathfrak{f}(\mathfrak{U})$ is α-open in $(P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT}, \mathfrak{G})$ for every α-open set $\mathfrak{U}$ in $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$. Given that $\mathfrak{f}$ is a bijection, it follows that the inverse function $\mathfrak{f}^{-1} : (P_n^\star(\mathfrak{y}), \mathfrak{R}_{ShT}, \mathfrak{G}) \to (P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ exists and is a α-irresolute surjection. Consequently, the proof is derived from the preceding theorem. □

Theorem 7.6. *If $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathfrak{G})$ has countably α-$\mathfrak{G}$ compact and $\mathfrak{G} \sqsupseteq \mathfrak{T}_{ShT}$, then $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is feebly compact.*

Proof. Let $\{O_m : m \in \Gamma\}$ denote a countable open cover of $P_n^\star(\mathfrak{O})$, thereby constituting it is α-open cover of $P_n^\star(\mathfrak{O})$. Since $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT}, \mathfrak{G})$ is countably α-$\mathfrak{G}$ compact, then there exists a finite number Γ_0 of Γ that $P_n^\star(\mathfrak{O})\setminus \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{G}$. Given that $\mathfrak{T}_{ShT} \sqsubseteq \mathfrak{G}$, then $P_n^\star(\mathfrak{O})\setminus \sqcup_{m\in\Gamma_0} O_m \notin \mathfrak{T}_{ShT}$. Thus $int(P_n^\star(\mathfrak{O})\setminus \sqcup_{m\in\Gamma_0} O_m) = \phi$, then $P_n^\star(\mathfrak{O})\setminus cl(\sqcup_{m\in\Gamma_0} O_m) = \phi$. It follows that $P_n^\star(\mathfrak{O}) = cl\{\sqcup O_m : m \in \Gamma_0\}$ and hence $(P_n^\star(\mathfrak{O}), \mathfrak{T}_{ShT})$ is feebly compact.

□

8. Conclusion

This work introduces a systematic method for building and evaluating SuperHyper Topological spaces with the idea of grill. We examined the extensions of compactness using the concept of the grill within Superhyper Topology. The Superhyper Topology augments the topology obtained from the nth-Powerset. We established the relationship between the grill notion and α-quasi H-closed, along with some significant characterizations. Also, introduced the concept of α-$\mathfrak{G}$ compact relative to SuperHyper Topological spaces. Finally, The notion of countably α-$\mathfrak{G}$ compact SuperHyper Topological was examined.

In future work, we plan to further extend this framework by introducing the ideal concept with SuperHyper Topological spaces, and comparison of the primary characteristics with the grill.

Acknowledgment: The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/ 2026/01/45067).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Z. Pawlak, Rough Set Theory and Its Applications to Data Analysis, Cybern. Syst. 29 (1998), 661–688. https://doi.org/10.1080/019697298125470.

[2] J. Aluja, A.M. Gil Lafuente, Towards an Advanced Modelling of Complex Economic Phenomena, Springer, 2012. https://doi.org/10.1007/978-3-642-24812-2.

[3] Y. Zhao, Y. Yao, F. Luo, Data Analysis Based on Discernibility and Indiscernibility, Inf. Sci. 177 (2007), 4959–4976. https://doi.org/10.1016/j.ins.2007.06.031.

[4] F. Smarandache, Foundation of Revolutionary Topologies: An Overview, Examples, Trend Analysis, Research Issues, Challenges, and Future Directions, Neutrosophic Syst. Appl. 13 (2023), 45–66. https://doi.org/10.61356/j.nswa.2024.125.

[5] G. Choquet, Sur les Notions de Filtre et de Grille, C. R. Hebd. Séances Acad. Sci. 224 (1947), 171–173.

[6] B. Roy, M.N. Mukherjee, On a Typical Topology Induced by a Grill, Soochow J. Math. 33 (2007), 771–786.

[7] A. Al-Omari, T. Noiri, Decompositions of Continuity via Grills, Jordan J. Math. Stat. 4 (2025), 33–46.

[8] E. Hatir, S. Jafari, On Some New Classes of Sets and a New Decomposition of Continuity via Grills, J. Adv. Math. Stud. 3 (2010), 33–41.

[9] C. Janaki, I. Arockiarani, Γ-Open Sets and Decomposition of Continuity via Grills, Int. J. Math. Arch. 2 (2011), 1087–1093.

[10] A. Karthika, I. Arockiarani, Generalization of Compactness Using Grills, Int. J. Math. Arch. 4 (2013), 284–288.

[11] N. Karthikeyan, N. Rajesh, Faint Continuous via Topological Grills, Int. J. Math. Arch. 6 (2015), 24–28.

[12] D. Mandal, M.N. Mukherjee, On a Class of Sets via Grill: A Decomposition of Continuity, An. Univ. Ovidius Constanta Ser. Mat. 20 (2012), 307–316. https://doi.org/10.2478/v10309-012-0020-9.

[13] A.A. Nasef, A.A. Azzam, Some Topological Operators via Grills, J. Linear Topol. Algebra 5 (2016), 199–204.

[14] A.A. Nasef, A.A. Azzam, On Class of Function via Grill, J. New Theory 17 (2017), 18–25. https://dergipark.org.tr/en/pub/jnt/article/381394.

[15] A.A. Azzam, Grill Nano Topological Spaces with Grill Nano Generalized Closed Sets, J. Egypt. Math. Soc. 25 (2017), 164–166. https://doi.org/10.1016/j.joems.2016.10.005.

[16] A.A. Azzam, S.S. Hussein, H.S. Osman, Compactness of Topological Spaces with Grills, Ital. J. Pure Appl. Math. 44 (2020), 198–207.

[17] B. Roy, M.N. Mukherjee, On a Type of Compactness via Grills, Mat. Vesn. 59 (2007), 113–120. https://eudml.org/doc/130222.

[18] T. Fujita, Superhypergraph Neural Networks and Plithogenic Graph Neural Networks: Theoretical Foundations, preprint, arXiv:2412.01176, 2024. https://arxiv.org/abs/2412.01176.

[19] F. Smarandache, Introduction to SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra, J. Algebr. Hyperstruct. Log. Algebr. 3 (2022), 17–24. https://doi.org/10.52547/hatef.jahla.3.2.2.

[20] O. Njåstad, On Some Classes of Nearly Open Sets, Pac. J. Math. 15 (1965), 961–970. https://doi.org/10.2140/pjm.1965.15.961.

[21] S.N. Maheshwari, S.S. Thakur, On α-Irresolute Mappings, Tamkang J. Math. 11 (1980), 209–214.