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Four Different Analytical Approaches for the Exact Solution of a Second-Order Scalar Differential Equation with Reflection

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Abstract. This paper proposes different analytical approaches for solving a class of second-order scalar differential equations involving reflection of the argument. The first approach transforms the given model to a coupled system for which the series solution is obtained in an exact form. The second approach is based on the symmetry decomposition method which derives an equivalent system of two separate equations. Such two separate equations are then solved exactly utilizing well-known standard methods. The third approach converts the problem into a 4th-order ordinary differential equation, allowing the derivation of the same exact solution. The fourth approach applies a direct series method with easier computations of the series coefficients in comparison to the second approach. The obtained solution is expressed in terms of entire functions allowing the appearance of periodicity under a specific constraint of the involved parameters. The results provide a unified procedure for treating reflection-type scalar differential equations.

1. Introduction

Equations with proportional delay, often written in the form $(y'(t) = F(t, y(t), y(qt)))$ with $(0 < q < 1)$, have attracted considerable attention due to their scaling structure and their ability to describe processes with memory distributed over geometrically shrinking time intervals. This class includes pantograph-type equations, which have been extensively studied with respect to existence, uniqueness, stability, and asymptotic behavior of solutions. The mathematical challenges stem from the coupling between local dynamics and recursively delayed arguments, which frequently leads to nonstandard qualitative properties compared with constant-delay models [1–3].

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Reflective delays are particularly relevant in neural systems exhibiting bidirectional propagation or symmetric structural organization. In certain nerve fibers, signals may travel in opposite directions due to reflections at boundaries, synaptic interfaces, or branching structures. Differential equations involving reflection arguments, such as terms depending on $(y(-t))$, can therefore represent reversible dynamics, echo effects, and feedback interactions between opposite propagation paths.

Such formulations have been used to analyze oscillatory neural responses, synchronization, and stability of impulse transmission in symmetric neural circuits [4,5]. In this work, we consider the second-order delay differential equation with reflection given by

$$y''(t) = ay(t) + by(-t), \tag{1}$$

subject to the initial conditions (ICs)

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{2}$$

where $a, b \in \mathbb{R}$. Equations with proportional delay are of practical applications as addressed by many researchers [6–9]. These equations include the pantograph equations [10–12] and a special case known as the Ambartsumian equation [13–15]. Several approaches have been suggested to analyze such types of equations including numerical methods [16–19], analytical techniques [20–22], and the Laplace transform (LT) [23]. Additionally, various variable versions have been solved in Refs. [24] and [25]. This paper extends the work [26] by considering a second-order derivative instead of the first order one. It has been shown in Ref. [26] that the solution admits hyperbolic and trigonometric forms using the standard series method. In addition, they have obtained polynomial solutions under several restrictions of the parameters a and b. In contrast with the previous work [26], this paper introduces four different approaches to solve Eqs. (1) and (2). The suggested techniques include the combined transformation-series expansion, the symmetry-decomposition method, the reduction of order approach, and the direct series method. It will be shown that the four approaches lead to the same exact solution. The solution is to be obtained exactly in terms hyperbolic functions. The hyperbolic-form solution is then transformed to a periodic-form solution under a specific relation. The present analysis shows an advantage of symmetry-decomposition method over the other three approaches to treat the 2nd-order scalar model (2) with reflection-arguments.

2. Transformed System & Power Series Solution

Define two auxiliary functions

$$u(t) = y(t), \quad v(t) = y(-t). \tag{3}$$

Differentiating $v(t)$ yields

$$v'(t) = -y'(-t), \quad v''(t) = y''(-t). \tag{4}$$

Eq. (1) leads to the coupled system

$$\begin{cases} u''(t) = au(t) + bv(t),\ u(0) = \alpha,\ u'(0) = \beta, \\ v''(t) = bu(t) + av(t),\ v(0) = \alpha,\ v'(0) = -\beta. \end{cases} \tag{5}$$

Assume a power series solution in the form

$$u(t) = \sum_{n=0}^{\infty} c_n t^n, \quad v(t) = \sum_{n=0}^{\infty} d_n t^n. \tag{6}$$

Substituting into the system yields the recurrence relations

$$c_{n+2} = \frac{ac_n + bd_n}{(n+2)(n+1)}, \quad d_{n+2} = \frac{ad_n + bc_n}{(n+2)(n+1)}. \tag{7}$$

This recurrence scheme holds for all $n \geq 0$. The ICs give

$$c_0 = \alpha, \quad c_1 = \beta, \quad d_0 = \alpha, \quad d_1 = -\beta. \tag{8}$$

The first few components are evaluated as

$$\begin{cases} c_0 &= \alpha,\ c_1 = \beta,\ c_2 = \frac{(a+b)\alpha}{2!}, \\ c_3 &= \frac{(a-b)\beta}{3!},\ c_4 = \frac{(a+b)^2\alpha}{4!},\ c_5 = \frac{(a-b)^2\beta}{5!}, \\ c_6 &= \frac{(a+b)^3\alpha}{6!},\ c_7 = \frac{(a-b)^3\beta}{7!},\ c_8 = \frac{(a+b)^4\alpha}{8!}, \end{cases} \tag{9}$$

and

$$\begin{cases} d_0 &= \alpha,\ d_1 = -\beta,\ d_2 = \frac{(a+b)\alpha}{2!}, \\ d_3 &= -\frac{(a-b)\beta}{3!},\ d_4 = \frac{(a+b)^2\alpha}{4!},\ d_5 = -\frac{(a-b)^2\beta}{5!}, \\ d_6 &= \frac{(a+b)^3\alpha}{6!},\ d_7 = -\frac{(a-b)^3\beta}{7!},\ d_8 = \frac{(a+b)^4\alpha}{8!}. \end{cases} \tag{10}$$

Accordingly, we have

$$c_{2n} = \alpha\frac{(a+b)^n}{(2n)!}, \quad c_{2n+1} = \beta\frac{(a-b)^n}{(2n+1)!}, \tag{11}$$

and

$$d_{2n} = \alpha\frac{(a+b)^n}{(2n)!}, \quad d_{2n+1} = -\beta\frac{(a-b)^n}{(2n+1)!}. \tag{12}$$

One can find

$$\begin{aligned} u(t) &= \sum_{n=0}^{\infty} c_{2n} t^{2n} + \sum_{n=0}^{\infty} c_{2n+1} t^{2n+1}, \\ &= \alpha\sum_{n=0}^{\infty}\frac{(a+b)^n}{(2n)!} + \beta\sum_{n=0}^{\infty}\frac{(a-b)^n}{(2n+1)!}, \\ &= \alpha\sum_{n=0}^{\infty}\frac{(\sqrt{a+b})^{2n}}{(2n)!} + \frac{\beta}{\sqrt{a-b}}\sum_{n=0}^{\infty}\frac{(\sqrt{a-b})^{2n+1}}{(2n+1)!},\ a > b \\ &= \alpha\cosh\left(\sqrt{a+b}\ t\right) + \frac{\beta}{\sqrt{a-b}}\sinh\left(\sqrt{a-b}\ t\right). \end{aligned} \tag{13}$$

Similarly, one can obtain $v(t)$ as

$$v(t) = \alpha \cosh\Big(\sqrt{a+b}\ t\Big) - \frac{\beta}{\sqrt{a-b}} \sinh\Big(\sqrt{a-b}\ t\Big),\ a > b. \tag{14}$$

Since $y(t) = u(t)$, then the solution is expressed in terms of hyperbolic and trigonometric functions as

$$y(t) = \begin{cases} \alpha \cosh\Big(\sqrt{a+b}\ t\Big) + \frac{\beta}{\sqrt{a-b}} \sinh\Big(\sqrt{a-b}\ t\Big),\ a > b, \\ \alpha \cos\Big(\sqrt{a+b}\ t\Big) + \frac{\beta}{\sqrt{b-a}} \sin\Big(\sqrt{b-a}\ t\Big),\ b > a. \end{cases} \tag{15}$$

Remark 1. *In Eq. (15), the solution is given separately for $a > b$ and $b > a$. However, the solution of borderline case $a = b$ can also be obtained via calculating the limit of Eq. (15) as $b \to a$. This leads to*

$$y(t) = \begin{cases} \alpha \cosh\Big(\sqrt{2a}\ t\Big) + \beta t, & a > 0, \\ \alpha \cos\Big(\sqrt{-2a}\ t\Big) + \beta t, & a < 0, \end{cases}$$

as solutions of the second-order ODE $y''(t) = a\,[y(t) + y(-t)]$.

3. Symmetry Decomposition Method

This section shows that the symmetry decomposition method can be implemented to derive the same obtained exact solution provided in the previous section. Let us start with defining the even and odd components:

$$y_e(t) = \frac{y(t) + y(-t)}{2}, \quad y_o(t) = \frac{y(t) - y(-t)}{2}. \tag{16}$$

Differentiating once implies

$$y_e'(t) = \frac{y'(t) - y'(-t)}{2}, \quad y_o'(t) = \frac{y'(t) + y'(-t)}{2}. \tag{17}$$

Differentiating again gives

$$y_e''(t) = \frac{y''(t) + y''(-t)}{2}, \quad y_o''(t) = \frac{y''(t) - y''(-t)}{2}. \tag{18}$$

Then Eq. (1) splits into the following two separate equations

$$\begin{cases} y_e''(t) = (a+b) y_e(t), \\ y_o''(t) = (a-b) y_o(t), \end{cases} \tag{19}$$

under the ICs

$$\begin{cases} y_e(0) = \alpha,\ y_e'(0) = 0, \\ y_o(0) = 0,\ y_o'(0) = \beta. \end{cases} \tag{20}$$

Thus, the problem reduces to two independent second-order ODEs, yielding exact solutions:

$$\begin{cases} y_e(t) = C_1 \cosh(\sqrt{a+b}\, t) + C_2 \sinh(\sqrt{a+b}\, t), \\ y_o(t) = C_3 \cosh(\sqrt{a-b}\, t) + C_4 \sinh(\sqrt{a-b}\, t). \end{cases} \tag{21}$$

Applying the ICs (20) on Eqs. (21) implies $C_1 = \alpha, C_2 = 0, C_3 = 0,$ and $C_4 = \beta/\sqrt{a-b}$. Accordingly, we have

$$\begin{cases} y_e(t) = \alpha\cosh(\sqrt{a+b}\,t), \\ y_o(t) = \frac{\beta}{\sqrt{a-b}}\sinh(\sqrt{a-b}\,t), \quad a > b. \end{cases} \tag{22}$$

Therefore

$$y(t) = y_e(t) + y_o(t) = \alpha\cosh(\sqrt{a+b}\,t) + \frac{\beta}{\sqrt{a-b}}\sinh(\sqrt{a-b}\,t), \quad a > b. \tag{23}$$

Remark 2. *When $a = b$, the second-order ODE becomes $y''(t) = a\,[y(t) + y(-t)]$. The even/odd decomposition then gives $y_o''(t) = 0$, leading to a linear solution $y_o(t) = \beta t$ for the odd part. The even part $y_e''(t) = 2ay_e(t)$ with the solution $y_e(t) = \alpha\cosh(\sqrt{2a}\,t)$ for $a > 0$ or $y_e(t) = \alpha\cos(\sqrt{-2a}\,t)$ for $a < 0$. Adding the odd part solution to the even part solution gives the same results highlighted in remark 1.*

4. Reduction to a 4th-Order IVP

Theorem 1. *The SDE (1) is equivalent to the 4th-order IVP:*

$$\begin{cases} y^{(4)}(t) - 2ay^{(2)}(t) + (a^2 - b^2)y(t) = 0, \\ y(0) = \alpha,\ y^{(1)}(0) = \beta,\ y^{(2)}(0) = \alpha(a+b),\ y^{(3)}(0) = \beta(a-b). \end{cases} \tag{24}$$

Proof. Firstly, we rewrite Eq. (1) as

$$y^{(2)}(t) = ay(t) + by(-t), y(0) = \alpha,\ y^{(1)}(0) = \beta. \tag{25}$$

Differentiating once we obtain

$$y^{(3)}(t) = ay^{(1)}(t) - by^{(1)}(-t). \tag{26}$$

Differentiating once again, then

$$y^{(4)}(t) = ay^{(2)}(t) + by^{(2)}(-t). \tag{27}$$

From Eq. (25), we have

$$y^{(2)}(-t) = ay(-t) + by(t). \tag{28}$$

Substituting Eq. (25) and Eq. (28) into Eq. (27) yields

$$y^{(4)}(t) - 2ay^{(2)}(t) + (a^2 - b^2)y(t) = 0. \tag{29}$$

At $t = 0$, Eq. (25) and Eq. (26) give

$$y^{(2)}(0) = (a+b)y(0) = \alpha(a+b), \tag{30}$$

and

$$y^{(3)}(0) = (a-b)y^{(1)}(0) = \beta(a-b), \tag{31}$$

respectively, this completes the proof. □

Remark 3. *It may be important to refer to that the conditions $y^{(2)}(0) = \alpha(a+b)$ and $y^{(3)}(0) = \beta(a-b)$ are always consistent with the original second-order problem (1-2).*

Theorem 2. *The exact solution of the 4th-order IVP (24) is given by Eq. (23).*

Proof. From the 4th-order ODE (24), we have the characteristic equation

$$\mu^4 - 2a\mu^2 + (a^2 - b^2) = 0. \tag{32}$$

Solving this equation implies four roots, given as

$$\mu = \pm\sqrt{a+b},\ \pm\sqrt{a-b}. \tag{33}$$

The general solution is

$$y(t) = A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t} + A_3 e^{\mu_3 t} + A_4 e^{\mu_4 t}. \tag{34}$$

Hence

$$y(t) = A_1 e^{\mu_1 t} + A_2 e^{-\mu_1 t} + A_3 e^{\mu_2 t} + A_4 e^{-\mu_2 t}, \tag{35}$$

where

$$\mu_1 = \sqrt{a+b}, \quad \mu_2 = \sqrt{a-b}. \tag{36}$$

Applying the ICs given in (24), we obtain

$$\begin{cases} A_1 + A_2 + A_3 + A_4 = \alpha, \\ \mu_1(A_1 - A_2) + \mu_2(A_3 - A_4) = \beta, \\ \mu_1^2(A_1 + A_2) + \mu_2^2(A_3 + A_4) = \alpha(a+b), \\ \mu_1^3(A_1 - A_2) + \mu_2^3(A_3 - A_4) = \beta(a-b). \end{cases} \tag{37}$$

Solving this system yields

$$A_1 = A_2 = \frac{\alpha}{2}, \quad A_4 = -A_3 = -\frac{\beta}{2\mu_2}. \tag{38}$$

Substituting these values into Eq. (34) leads to

$$y(t) = \frac{\alpha}{2}\left(e^{\mu_1 t} + e^{-\mu_1 t}\right) + \frac{\beta}{2\mu_2}\left(e^{\mu_2 t} - e^{-\mu_2 t}\right), \tag{39}$$

which can be re-expressed in terms of the hyperbolic function as

$$y(t) = \alpha\cosh(\mu_1 t) + \frac{\beta}{\mu_2}\sinh(\mu_2 t), \tag{40}$$

where μ_1 and μ_2 are already given in Eqs. (33). This implies the same exact solution obtained in Eq. (23) and hence finalizes the proof. $\square$

5. Direct Power Series Method

Instead of transforming the model to a coupled system as made in section 2, one can assume a direct series expansion as

$$y(t) = \sum_{n=0}^{\infty} h_n t^n. \tag{41}$$

On inserting (41) into (1), we derive

$$h_{n+2} = \frac{a + b(-1)^n}{(n+2)(n+1)} h_n, \quad n \geq 0. \tag{42}$$

The given ICs require $h_0 = \alpha$ and $h_1 = \beta$. Accordingly, we have the recurrence scheme:

$$\begin{cases} h_0 = \alpha, \quad h_1 = \beta, \\ h_{n+2} = \frac{a+b(-1)^n}{(n+2)(n+1)} h_n, \quad n \geq 0. \end{cases} \tag{43}$$

Using this recurrence scheme we derive the first eleven components of the coefficients h_n as follows

$$\begin{cases} h_0 = \alpha, \\ h_1 = \beta, \\ h_2 = \frac{a+b}{2!}\,\alpha, \\ h_3 = \frac{a-b}{3!}\,\beta, \\ h_4 = \frac{(a+b)^2}{4!}\,\alpha, \\ h_5 = \frac{(a-b)^2}{5!}\,\beta, \\ h_6 = \frac{(a+b)^3}{6!}\,\alpha, \\ h_7 = \frac{(a-b)^3}{7!}\,\beta, \\ h_8 = \frac{(a+b)^4}{8!}\,\alpha, \\ h_9 = \frac{(a-b)^4}{9!}\,\beta, \\ h_{10} = \frac{(a+b)^5}{10!}\,\alpha. \end{cases} \tag{44}$$

It is clear that the even-indexed coefficients depend only on α, while the odd-indexed coefficients depend only on β, reflecting the symmetry induced by the reflection term $y(-t)$. By separating even and odd indices, we derive explicit formulas for the coefficients as

$$h_{2n} = \alpha \frac{(a+b)^n}{(2n)!}, \quad h_{2n+1} = \beta \frac{(a-b)^n}{(2n+1)!}. \tag{45}$$

Therefore, the power series solution can be written in the compact form

$$y(t) = \alpha \sum_{n=0}^{\infty} \frac{(a+b)^n}{(2n)!} t^{2n} + \beta \sum_{n=0}^{\infty} \frac{(a-b)^n}{(2n+1)!} t^{2n+1}, \tag{46}$$

which can be treated to give the same exact solutions given by Eqs. (15). The main observation here is that the compact recurrence relation (43) significantly simplifies computation in comparison with the algorithms introduced in section 2.

6. Conclusions

In this paper, various robust analytical techniques combining transformation-series expansion, symmetry-decomposition, reduction of order, and direct series methods were introduced for solving a second-order scalar equation with argument-reflection. The implemented approaches yield the same exact solution. The obtained solution was successfully expressed in terms of entire functions utilizing hyperbolic functions. Moreover, a periodic solution was derived under a certain restriction of the model's parameters. Comparing the symmetry-decomposition method with the other three approaches revealed its simplicity. Actually, the symmetry-decomposition method converted the give equation to a system of two separate ODEs which were easily solved via well-known standard methods in differential calculus. The obtained results reflect the effectiveness of the symmetry-decomposition method to treat higher-order scalar equations of reflection-arguments types. Furthermore, the direct series method is computationally simpler for coefficient extraction, while the decomposition method is conceptually simpler.

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