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Existence and Stability Results for a System of Coupled Multi-Term Caputo Fractional Differential Equations with Integral Multi-Strip Coupled Boundary Conditions

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Abstract. This paper investigates a nonlinear boundary value problem governed by a coupled system of multi-term Caputo fractional differential equations subject to integral multi-strip boundary conditions. An equivalent operator formulation is derived in an appropriate Banach space setting, providing a suitable framework for the application of fixed point theory. Existence and uniqueness results are derived using Banach's contraction theorem and Krasnoselskii's fixed point theorem. The study also addresses several forms of Ulam-type stability, namely Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability. An illustrative example is provided to confirm the applicability of the obtained results.

1. Introduction

Fractional calculus has gained considerable interest because it offers an effective tool for modeling processes with memory effects and nonlocal behavior. Such features appear in many applied problems from physics, engineering, biology, and finance ([29], [19], [7]). In many situations, fractional–order equations provide more accurate descriptions than classical integer–order models, especially when dealing with relaxation phenomena, viscoelastic materials, anomalous diffusion, and complex biological or chaotic dynamics ([9], [11], [12], [14]). A growing part of the literature focuses on multi–term fractional differential equations, where several derivatives of different orders occur in the same model. Such formulations are useful when the underlying system involves multiple memory kernels or diffusion mechanisms ([2], [3], [4]). However, the presence of several fractional operators together with nonlinear coupling terms makes the analytical study of existence

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and qualitative properties of solutions more challenging. In addition, various forms of nonlocal boundary conditions have been proposed in order to capture global features of the corresponding solutions. In particular, integral boundary conditions of multi–strip type offer greater flexibility in modeling interactions distributed over different subintervals of the domain ([1], [6], [18], [27]). Such boundary constraints play an essential role in mathematical models arising in heat transfer, fluid dynamics, viscoelasticity, biological population evolution, and other applications where hereditary effects are relevant. Motivated by the growing importance of nonlinear multi–term fractional systems and the role of integral nonlocal constraints, this paper investigates a new class of coupled nonlinear multi–term Caputo fractional differential equations subject to multi–strip integral boundary conditions. Specifically, we consider the following coupled system:

$$\begin{cases} \sum_{i=1}^{n} \lambda_i \, {}^{C}\!D^{\alpha_i} x(t) = f(t, x(t), y(t)), \\ \sum_{i=1}^{n} \mu_i \, {}^{C}\!D^{\beta_i} y(t) = g(t, x(t), y(t)), \end{cases} \tag{1.1}$$

for $t \in [0,1]$, where $\lambda_i, \mu_i \in \mathbb{R}$ and the fractional orders satisfy $0 < \alpha_i, \beta_i \leq 1$ for $i = 1, 2, \dots, n$. Here, ${}^{C}D^{\alpha_i}$ and ${}^{C}D^{\beta_i}$ represent the Caputo fractional derivatives of orders α_i and β_i, respectively. The nonlinear functions $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are assumed to be continuous. The system (1.1) is supplemented with the following nonlocal multi-strip integral boundary conditions:

$$\begin{cases} x(0) = 0, \qquad x'(0) = 0, \dots, x^{(n-2)}(0) = 0, \qquad x(1) = \sum_{k=1}^{m} a_i \int_{\bar{\zeta}_k}^{\eta_k} y(s)\, ds, \\ y(0) = 0, \qquad y'(0) = 0, \dots, y^{(n-2)}(0) = 0, \qquad y(1) = \sum_{j=1}^{m} b_j \int_{\bar{\zeta}_j}^{\eta_j} x(s)\, ds, \end{cases} \tag{1.2}$$

In (1.2), the constants a_k and b_j belong to $\mathbb{R}$ for $k, j = 1, 2, \dots, m$, and the parameters $\bar{\zeta}_r, \eta_r \in (0,1)$ $(r = 1, 2, \dots, m)$ satisfy $0 < \bar{\zeta}_1 < \eta_1 < \cdots < \bar{\zeta}_m < \eta_m < 1$. These boundary conditions describe nonlocal interactions between the coupled components $x(t)$ and $y(t)$ through integrals taken over prescribed subintervals of the domain. This work focuses on the existence and uniqueness of solutions for the system (1.1)–(1.2). The existence result is established by means of the Leray–Schauder nonlinear alternative, whereas uniqueness is obtained through Banach's contraction principle. Motivated by recent contributions in the literature ([1], [32], [30], [13]), we also examine several forms of Ulam-type stability, namely Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability for the considered fractional model. For the convenience of the reader, the structure of the manuscript is outlined below. Section 2 collects the basic concepts, definitions, and auxiliary results required in the subsequent development. Section 3 is devoted to establishing the principal existence and uniqueness results for the coupled multi-term Caputo fractional differential system (1.1)–(1.2). Section 4 deals with various notions of Ulam-type stability associated with the obtained solutions. To highlight the practical relevance of the theoretical findings, an illustrative example is presented in Section 5. Section 6 presents an overview of the main outcomes and mentions some topics that could be explored in future work.

2. Preliminaries

This section provides the preliminary material required for the analysis carried out in the next parts of the paper. We briefly review some fundamental notions and key results from fractional calculus that will be repeatedly employed throughout the manuscript.

Definition 2.1. *Let u be a real-valued function that is locally integrable on the interval $(-\infty, +\infty)$, and suppose that $a < t < b \leq +\infty$. The Riemann–Liouville fractional integral of order $\alpha > 0$ of the function u is defined as*

$$I_a^\alpha u(t) = (u * k_\alpha)(t) = \int_a^t \frac{u(s)}{\Gamma(\alpha)(t-s)^{1-\alpha}}\, ds,$$

where the kernel $k_\alpha = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, and $\Gamma(\cdot)$ denotes the Euler gamma function.

Definition 2.2. *Let $u : [a, +\infty) \to \mathbb{R}$ be an absolutely continuous function that is differentiable up to order $(n-1)$. The Caputo fractional derivative of order α of the function u is defined as*

$${}^C D_a^\alpha u(t) = \int_a^t \frac{u^{(n)}(s)}{\Gamma(n-\alpha)(t-s)^{1+\alpha-n}}\, ds,$$

where $n-1 < \alpha \leq n$, and $n = \lfloor \alpha \rfloor + 1$. Here, $\lfloor \alpha \rfloor$ denotes the integer part of α.

Lemma 2.1. *Let $n-1 < \alpha < n$. Then the general solution of the fractional differential equation*

$${}^C D^\alpha u(t) = 0, \qquad t \in [a, b],$$

is expressed as

$$u(t) = R_0 + R_1(t-a) + R_2(t-a)^2 + \cdots + R_{n-1}(t-a)^{n-1},$$

where $R_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n-1$. Moreover, the following identity holds:

$$I^\alpha\, {}^C D^\alpha u(t) = u(t) + \sum_{i=0}^{n-1} R_i (t-a)^i.$$

The next part of our analysis relies heavily on this lemma, since it provides an explicit form of the solution for the linearized version of the coupled system (1.1)–(1.2). This representation will be used to rewrite the original problem in an operator framework, which is particularly convenient for applying fixed point techniques. This reformulated setting enables the development of the principal existence and uniqueness results presented in this work.

Lemma 2.2. *Let $\phi, \psi \in C([0,1], \mathbb{R})$. Consider the linear system*

$$\begin{cases} \sum_{i=1}^{n} \lambda_i \, {}^{C}D^{\alpha_i} x(t) = \phi(t), \\ \sum_{i=1}^{n} \mu_i \, {}^{C}D^{\beta_i} y(t) = \psi(t), \end{cases} \tag{2.1}$$

subject to the nonlocal multi-strip boundary conditions

$$\begin{cases} x(0) = 0, \quad x'(0) = 0, \dots, x^{(n-2)}(0) = 0, \quad x(1) = \sum_{k=1}^{m} a_k \int_{\bar{\zeta}_k}^{\eta_k} y(s)\, ds, \\ y(0) = 0, \quad y'(0) = 0, \dots, y^{(n-2)}(0) = 0, \quad y(1) = \sum_{j=1}^{m} b_j \int_{\bar{\zeta}_j}^{\eta_j} x(s)\, ds, \end{cases} \tag{2.2}$$

where $0 < \bar{\zeta}_1 < \eta_1 < \cdots < \bar{\zeta}_m < \eta_m < 1$ and $a_k, b_j \in \mathbb{R}$. Assume that $\Delta = 1 - BD \neq 0$, where

$$B = \sum_{k=1}^{m} a_k \int_{\bar{\zeta}_k}^{\eta_k} s^{\,n-1}\, ds, \qquad D = \sum_{j=1}^{m} b_j \int_{\bar{\zeta}_j}^{\eta_j} s^{\,n-1}\, ds.$$

Consequently, a solution of the problem can be represented as

$$\begin{aligned} x(t) = {} & \sum_{i=1}^{n} \frac{1}{\lambda_i \Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} \phi(\tau)\, d\tau \\ & + \frac{t^{\,n-1}}{\Delta} \Bigg[\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{a_k}{\mu_i \Gamma(\beta_i)} \int_{\bar{\zeta}_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i - 1} \psi(\tau)\, d\tau\, ds \\ & + \left(\sum_{k=1}^{m} a_k \int_{\bar{\zeta}_k}^{\eta_k} s^{\,n-1}\, ds \right) \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{b_k}{\lambda_i \Gamma(\alpha_i)} \int_{\bar{\zeta}_k}^{\eta_k} \int_0^s (s-\tau)^{\alpha_i - 1} \phi(\tau)\, d\tau\, ds \right) \\ & - \sum_{i=1}^{n} \frac{1}{\lambda_i \Gamma(\alpha_i)} \int_0^1 (1-\tau)^{\alpha_i - 1} \phi(\tau)\, d\tau \\ & - \left(\sum_{k=1}^{m} a_k \int_{\bar{\zeta}_k}^{\eta_k} s^{\,n-1}\, ds \right) \left(\sum_{i=1}^{n} \frac{1}{\mu_i \Gamma(\beta_i)} \int_0^1 (1-\tau)^{\beta_i - 1} \psi(\tau)\, d\tau \right) \Bigg]. \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} y(t) = {} & \sum_{i=1}^{n} \frac{1}{\mu_i \Gamma(\beta_i)} \int_0^t (t-\tau)^{\beta_i - 1} \psi(\tau)\, d\tau \\ & + \frac{t^{\,n-1}}{\Delta} \Bigg[\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{b_j}{\lambda_i \Gamma(\alpha_i)} \int_{\bar{\zeta}_j}^{\eta_j} \int_0^s (s-\tau)^{\alpha_i - 1} \phi(\tau)\, d\tau\, ds \\ & + \left(\sum_{j=1}^{m} b_j \int_{\bar{\zeta}_j}^{\eta_j} s^{\,n-1}\, ds \right) \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{a_k}{\mu_i \Gamma(\beta_i)} \int_{\bar{\zeta}_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i - 1} \psi(\tau)\, d\tau\, ds \right) \end{aligned}$$

$$-\sum_{i=1}^{n}\frac{1}{\mu_i\Gamma(\beta_i)}\int_0^1(1-\tau)^{\beta_i-1}\psi(\tau)\,d\tau$$

$$-\left(\sum_{j=1}^{m}b_j\int_{\bar{\zeta}_j}^{\eta_j}s^{n-1}\,ds\right)\left(\sum_{i=1}^{n}\frac{1}{\lambda_i\Gamma(\alpha_i)}\int_0^1(1-\tau)^{\alpha_i-1}\phi(\tau)\,d\tau\right)\Bigg]. \tag{2.4}$$

where

$$X_0(t)=\sum_{i=1}^{n}\frac{1}{\lambda_i\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}\phi(s)\,ds,\qquad Y_0(t)=\sum_{i=1}^{n}\frac{1}{\mu_i\Gamma(\beta_i)}\int_0^t(t-s)^{\beta_i-1}\psi(s)\,ds,$$

$$\int_{\bar{\zeta}_k}^{\eta_k}Y_0(s)\,ds=\sum_{i=1}^{n}\frac{1}{\mu_i\Gamma(\beta_i)}\int_{\bar{\zeta}_k}^{\eta_k}\int_0^s(s-\tau)^{\beta_i-1}\psi(\tau)\,d\tau\,ds,$$

$$X_0(1)=\sum_{i=1}^{n}\frac{1}{\lambda_i\Gamma(\alpha_i)}\int_0^1(1-\tau)^{\alpha_i-1}\phi(\tau)\,d\tau,$$

$$\int_{\bar{\zeta}_j}^{\eta_j}X_0(s)\,ds=\sum_{i=1}^{n}\frac{1}{\lambda_i\Gamma(\alpha_i)}\int_{\bar{\zeta}_j}^{\eta_j}\int_0^s(s-\tau)^{\alpha_i-1}\phi(\tau)\,d\tau\,ds,$$

$$Y_0(1)=\sum_{i=1}^{n}\frac{1}{\mu_i\Gamma(\beta_i)}\int_0^1(1-\tau)^{\beta_i-1}\psi(\tau)\,d\tau.$$

$$A=\sum_{k=1}^{m}a_k\int_{\bar{\zeta}_k}^{\eta_k}Y_0(s)\,ds-X_0(1),\qquad E=\sum_{j=1}^{m}b_j\int_{\bar{\zeta}_j}^{\eta_j}X_0(s)\,ds-Y_0(1).$$

Proof. Apply the fractional integral operators I^{α_σ} and I^{β_σ} to both equations in (2.1).
We obtain

$$I^{\alpha_\sigma}\left(\sum_{i=1}^{n}\lambda_i\,{}^{C}D^{\alpha_i}x(t)\right)=I^{\alpha_\sigma}\phi(t),\qquad I^{\beta_\sigma}\left(\sum_{i=1}^{n}\mu_i\,{}^{C}D^{\beta_i}y(t)\right)=I^{\beta_\sigma}\psi(t).$$

Separating the terms corresponding to $i=\sigma$ and $i\neq\sigma$, we have

$$\lambda_\sigma I^{\alpha_\sigma\,C}D^{\alpha_\sigma}x(t)+\sum_{\substack{i=1\\ i\neq\sigma}}^{n}\lambda_i I^{\alpha_\sigma\,C}D^{\alpha_i}x(t)=I^{\alpha_\sigma}\phi(t),$$

$$\mu_\sigma I^{\beta_\sigma\,C}D^{\beta_\sigma}y(t)+\sum_{\substack{i=1\\ i\neq\sigma}}^{n}\mu_i I^{\beta_\sigma\,C}D^{\beta_i}y(t)=I^{\beta_\sigma}\psi(t).$$

By the composition rule for the Caputo derivative and the Riemann–Liouville integral (see Lemma 2.1), it follows that

$$I^{\alpha_\sigma\,C}D^{\alpha_\sigma}x(t)=x(t)+\sum_{k=0}^{n-1}C_kt^k,\qquad I^{\alpha_\sigma\,C}D^{\alpha_i}x(t)=I^{\alpha_\sigma-\alpha_i}x(t),\quad i\neq\sigma,$$

and similarly for $y(t)$:

$$I^{\beta_\sigma}{}^C D^{\beta_\sigma} y(t) = y(t) + \sum_{k=0}^{n-1} D_k t^k, \qquad I^{\beta_\sigma}{}^C D^{\beta_i} y(t) = I^{\beta_\sigma - \beta_i} y(t), \quad i \neq \sigma.$$

Hence,

$$\lambda_\sigma \left(x(t) + \sum_{k=0}^{n-1} C_k t^k \right) + \sum_{\substack{i=1 \\ i\neq\sigma}}^{n} \lambda_i I^{\alpha_\sigma - \alpha_i} x(t) = I^{\alpha_\sigma} \phi(t),$$

$$\mu_\sigma \left(y(t) + \sum_{k=0}^{n-1} D_k t^k \right) + \sum_{\substack{i=1 \\ i\neq\sigma}}^{n} \mu_i I^{\beta_\sigma - \beta_i} y(t) = I^{\beta_\sigma} \psi(t).$$

Dividing by λ_σ and μ_σ, and using the explicit form of I^ρ, we get

$$\begin{aligned} x(t) &= -\sum_{k=0}^{n-1} C_k t^k - \sum_{\substack{i=1 \\ i\neq\sigma}}^{n} \frac{\lambda_i}{\lambda_\sigma} I^{\alpha_\sigma - \alpha_i} x(t) + \frac{1}{\lambda_\sigma} I^{\alpha_\sigma} \phi(t) \\ &= -\sum_{k=0}^{n-1} C_k t^k - \sum_{\substack{i=1 \\ i\neq\sigma}}^{n} \frac{\lambda_i}{\lambda_\sigma \Gamma(\alpha_\sigma - \alpha_i)} \int_0^t (t-s)^{\alpha_\sigma - \alpha_i - 1} x(s)\, ds \\ &+ \frac{1}{\lambda_\sigma \Gamma(\alpha_\sigma)} \int_0^t (t-s)^{\alpha_\sigma - 1} \phi(s)\, ds, \end{aligned}$$

$$\begin{aligned} y(t) &= -\sum_{k=0}^{n-1} D_k t^k - \sum_{\substack{i=1 \\ i\neq\sigma}}^{n} \frac{\mu_i}{\mu_\sigma} I^{\beta_\sigma - \beta_i} y(t) + \frac{1}{\mu_\sigma} I^{\beta_\sigma} \psi(t) \\ &= -\sum_{k=0}^{n-1} D_k t^k - \sum_{\substack{i=1 \\ i\neq\sigma}}^{n} \frac{\mu_i}{\mu_\sigma \Gamma(\beta_\sigma - \beta_i)} \int_0^t (t-s)^{\beta_\sigma - \beta_i - 1} y(s)\, ds \\ &+ \frac{1}{\mu_\sigma \Gamma(\beta_\sigma)} \int_0^t (t-s)^{\beta_\sigma - 1} \psi(s)\, ds. \end{aligned}$$

Next, using the initial conditions in (2.2) we obtain

$$C_0 = C_1 = \cdots = C_{n-2} = 0, \qquad D_0 = D_1 = \cdots = D_{n-2} = 0,$$

so that only the coefficients of t^{n-1} remain. Denoting $C_{n-1} = C_x$ and $D_{n-1} = C_y$, we arrive at

$$x(t) = X_0(t) + C_x t^{n-1}, \qquad y(t) = Y_0(t) + C_y t^{n-1}, \tag{2.5}$$

Now, substitute (2.5) into the nonlocal boundary conditions (2.2). Using $x(1) = X_0(1) + C_x$ and $y(1) = Y_0(1) + C_y$, we get

$$X_0(1) + C_x = \sum_{k=1}^{m} a_k \int_{\bar{\zeta}_k}^{\eta_k} \left(Y_0(s) + C_y s^{n-1} \right) ds, \tag{2.6}$$

$$Y_0(1) + C_y = \sum_{j=1}^{m} b_j \int_{\bar{\zeta}_j}^{\eta_j} \Big(X_0(s) + C_x s^{n-1}\Big) ds. \tag{2.7}$$

Then (2.6)–(2.7) are equivalent to the 2×2 linear system

$$\begin{cases} C_x - B\,C_y = A, \\ C_y - D\,C_x = E. \end{cases}$$

Its determinant is

$$\Delta = 1 - BD.$$

Assuming $\Delta \neq 0$, we have

$$C_x = \frac{A + BE}{\Delta}, \qquad C_y = \frac{E + DA}{\Delta}. \tag{2.8}$$

Substituting (2.8) into (2.5) yields a solution pair (x, y) of problem (2.1)–(2.2). □

Theorem 2.1 (Krasnoselskii's fixed point theorem)**.** *Let X be a nonempty, convex, closed, and bounded subset of a Banach space $(E, \|\cdot\|)$, and let $\mathcal{A}, \mathcal{B} : X \to E$ be two operators such that:*

(H$_1$) *$\mathcal{A}u + \mathcal{B}v \in X$ for all $u, v \in X$.*

(H$_2$) *The operator $\mathcal{B} : X \to E$ is a contraction mapping.*

(H$_3$) *The operator $\mathcal{A} : X \to E$ is continuous and compact.*

Then there exists at least one element $u^* \in X$ satisfying $(\mathcal{A} + \mathcal{B})u^* = u^*$.

Theorem 2.2 (Arzelà–Ascoli theorem)**.** *A family of functions in $C([0,1], \mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous.*

3. Main Results

In what follows, we establish the principal existence results for the coupled multi-term fractional differential system under consideration. The problem is first rewritten as an equivalent operator equation on an appropriate Banach space setting. This allows us to employ Krasnoselskii's fixed point theorem in order to ensure the existence of at least one solution. Moreover, the compactness of the related operator is obtained by using the Arzelà–Ascoli theorem, while the contraction requirement is verified under suitable hypotheses.

Let $E = C([0,1], \mathbb{R})$ denote the Banach space of all continuous real-valued functions defined on $[0,1]$, equipped with the supremum norm $\|f\| = \sup_{t\in[0,1]} |f(t)|$. It follows that the product space $E \times E$, endowed with the norm $\|(f_1, f_2)\| = \|f_1\| + \|f_2\|$, is also a Banach space. Based on Lemma 2.2, we introduce the operator $T : E \times E \longrightarrow E \times E$ defined by

$$T(u(t), \kappa(t)) = \Big(T_1(u, \kappa)(t),\ T_2(u, \kappa)(t)\Big), \tag{3.1}$$

where T_1 and T_2 are given by the integral representations (2.3)–(2.4) , respectively, with ϕ and ψ defined as

$$\phi(t) = f(t, u(t), \kappa(t)),\ \psi(t) = g(t, u(t), \kappa(t)).$$

The first component $T_1 : E \times E \to E$ is defined by

$$
\begin{aligned}
T_1(u,\kappa)(t) = &\sum_{i=1}^{n} \frac{1}{\lambda_i \Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} f(\tau, u(\tau), \kappa(\tau))\, d\tau \\
&+ \frac{t^{n-1}}{\Delta} \Bigg[\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{a_k}{\mu_i \Gamma(\beta_i)} \int_{\bar{\zeta}_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i - 1} g(\tau, u(\tau), \kappa(\tau))\, d\tau\, ds \\
&\quad + \left(\sum_{k=1}^{m} a_k \int_{\bar{\zeta}_k}^{\eta_k} s^{n-1}\, ds \right) \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{b_j}{\lambda_i \Gamma(\alpha_i)} \int_{\bar{\zeta}_j}^{\eta_j} \int_0^s (s-\tau)^{\alpha_i - 1} f(\tau, u(\tau), \kappa(\tau))\, d\tau\, ds \right) \\
&\quad - \sum_{i=1}^{n} \frac{1}{\lambda_i \Gamma(\alpha_i)} \int_0^1 (1-\tau)^{\alpha_i - 1} f(\tau, u(\tau), \kappa(\tau))\, d\tau \\
&\quad - \left(\sum_{k=1}^{m} a_k \int_{\bar{\zeta}_k}^{\eta_k} s^{n-1}\, ds \right) \left(\sum_{i=1}^{n} \frac{1}{\mu_i \Gamma(\beta_i)} \int_0^1 (1-\tau)^{\beta_i - 1} g(\tau, u(\tau), \kappa(\tau))\, d\tau \right) \Bigg].
\end{aligned}
\tag{3.2}
$$

Similarly, the second component $T_2 : E \times E \to E$ is defined by

$$
\begin{aligned}
T_2(u,\kappa)(t) = &\sum_{i=1}^{n} \frac{1}{\mu_i \Gamma(\beta_i)} \int_0^t (t-\tau)^{\beta_i - 1} g(\tau, u(\tau), \kappa(\tau))\, d\tau \\
&+ \frac{t^{n-1}}{\Delta} \Bigg[\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{b_j}{\lambda_i \Gamma(\alpha_i)} \int_{\bar{\zeta}_j}^{\eta_j} \int_0^s (s-\tau)^{\alpha_i - 1} f(\tau, u(\tau), \kappa(\tau))\, d\tau\, ds \\
&\quad + \left(\sum_{j=1}^{m} b_j \int_{\bar{\zeta}_j}^{\eta_j} s^{n-1}\, ds \right) \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{a_k}{\mu_i \Gamma(\beta_i)} \int_{\bar{\zeta}_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i - 1} g(\tau, u(\tau), \kappa(\tau))\, d\tau\, ds \right) \\
&\quad - \sum_{i=1}^{n} \frac{1}{\mu_i \Gamma(\beta_i)} \int_0^1 (1-\tau)^{\beta_i - 1} g(\tau, u(\tau), \kappa(\tau))\, d\tau \\
&\quad - \left(\sum_{j=1}^{m} b_j \int_{\bar{\zeta}_j}^{\eta_j} s^{n-1}\, ds \right) \left(\sum_{i=1}^{n} \frac{1}{\lambda_i \Gamma(\alpha_i)} \int_0^1 (1-\tau)^{\alpha_i - 1} f(\tau, u(\tau), \kappa(\tau))\, d\tau \right) \Bigg].
\end{aligned}
\tag{3.3}
$$

We impose the following assumptions, which will be used throughout the analysis.

($\tilde{A}_1$) For all $u, u^*, \kappa, \kappa^* \in E$ and $t \in [0,1]$, there exists a constant $L_f > 0$ such that

$$\Big| f(t, u(t), \kappa(t)) - f(t, u^*(t), \kappa^*(t)) \Big| \le L_f \Big(|u(t) - u^*(t)| + |\kappa(t) - \kappa^*(t)| \Big).$$

($\tilde{A}_2$) For all $u, u^*, \kappa, \kappa^* \in E$ and $t \in [0,1]$, there exists a constant $L_g > 0$ such that

$$\Big| g(t, u(t), \kappa(t)) - g(t, u^*(t), \kappa^*(t)) \Big| \le L_g \Big(|u(t) - u^*(t)| + |\kappa(t) - \kappa^*(t)| \Big).$$

$(\tilde{A}_3)$ There exist positive constants $C_f, D_f, M_f, C_g, D_g, M_g$ such that, for all $u, \kappa \in E$ and $t \in [0,1]$,

$$|f(t,u(t),\kappa(t))| \le C_f|u(t)| + D_f|\kappa(t)| + M_f,$$

$$|g(t,u(t),\kappa(t))| \le C_g|u(t)| + D_g|\kappa(t)| + M_g.$$

Remark 3.1. *In order to simplify the forthcoming estimates and avoid lengthy computations, we introduce the following constants.*

$$B = \sum_{k=1}^{m} a_k \int_{\zeta_k}^{\eta_k} s^{n-1}\,ds, \qquad D = \sum_{j=1}^{m} b_j \int_{\zeta_j}^{\eta_j} s^{n-1}\,ds, \qquad \Delta = 1 - BD \neq 0.$$

Moreover, define

$$A_\alpha = \sum_{i=1}^{n} \frac{1}{|\lambda_i|\Gamma(\alpha_i+1)}, \qquad A_\beta = \sum_{i=1}^{n} \frac{1}{|\mu_i|\Gamma(\beta_i+1)}.$$

For the double–integral terms appearing in (2.3)–(2.4), *we set*

$$A_\alpha^* = \sum_{j=1}^{m}\sum_{i=1}^{n} \frac{|b_j|}{|\lambda_i|\Gamma(\alpha_i+2)}\Big(\eta_j^{\alpha_i+1} - \zeta_j^{\alpha_i+1}\Big),$$

$$A_\beta^* = \sum_{k=1}^{m}\sum_{i=1}^{n} \frac{|a_k|}{|\mu_i|\Gamma(\beta_i+2)}\Big(\eta_k^{\beta_i+1} - \zeta_k^{\beta_i+1}\Big).$$

let the nonlinear functions f and g satisfy the Lipschitz conditions $(\tilde{A}_1)$–$(\tilde{A}_2)$ with constants $L_f > 0$ and $L_g > 0$, namely,

$$|f(t,u,\kappa) - f(t,u^*,\kappa^*)| \le L_f\Big(|u-u^*| + |\kappa-\kappa^*|\Big),$$

$$|g(t,u,\kappa) - g(t,u^*,\kappa^*)| \le L_g\Big(|u-u^*| + |\kappa-\kappa^*|\Big).$$

Then the components T_1 and T_2 of the operator T defined by (2.3)–(2.4) *satisfy*

$$\|T_1(u,\kappa) - T_1(u^*,\kappa^*)\| \le K_1\Big(\|u-u^*\| + \|\kappa-\kappa^*\|\Big),$$

$$\|T_2(u,\kappa) - T_2(u^*,\kappa^*)\| \le K_2\Big(\|u-u^*\| + \|\kappa-\kappa^*\|\Big),$$

where

$$K_1 = L_f A_\alpha + \frac{1}{|\Delta|}\Big(L_g A_\beta^* + |B| L_f A_\alpha^* + L_f A_\alpha + |B| L_g A_\beta\Big), \tag{3.4}$$

$$K_2 = L_g A_\beta + \frac{1}{|\Delta|}\Big(L_f A_\alpha^* + |D| L_g A_\beta^* + L_g A_\beta + |D| L_f A_\alpha\Big). \tag{3.5}$$

Consequently,

$$\|T(u,\kappa) - T(u^*,\kappa^*)\| \le (K_1+K_2)\Big(\|u-u^*\| + \|\kappa-\kappa^*\|\Big).$$

Theorem 3.1. *Let f and g be nonlinear functions satisfying the following Lipschitz conditions $(\tilde{A}_1)$–$(\tilde{A}_2)$. In other words, there exist positive constants L_f and L_g such that, for every $t \in [0,1]$ and $u, u^*, \kappa, \kappa^* \in E$,*

$$|f(t,u,\kappa) - f(t,u^*,\kappa^*)| \leq L_f\Big(|u-u^*| + |\kappa-\kappa^*|\Big),$$
$$|g(t,u,\kappa) - g(t,u^*,\kappa^*)| \leq L_g\Big(|u-u^*| + |\kappa-\kappa^*|\Big).$$

Moreover, suppose that the constants K_1 and K_2 defined in Remark 3.1 satisfy $K = K_1 + K_2 < 1$. Then the operator $T : E \times E \to E \times E$ has at most one fixed point. Consequently, the coupled fractional boundary value problem (1.1)–(1.2) *admits at most one solution on* $[0,1]$.

Proof. Let $(u,\kappa), (u^*,\kappa^*) \in E \times E$. Recall that $T(u,\kappa) = (T_1(u,\kappa), T_2(u,\kappa))$, where T_1 and T_2 are defined by (3.2)–(3.3) with

$$\phi(t) = f(t,u(t),\kappa(t)), \qquad \psi(t) = g(t,u(t),\kappa(t)),$$

and

$$\phi^*(t) = f(t,u^*(t),\kappa^*(t)), \qquad \psi^*(t) = g(t,u^*(t),\kappa^*(t)).$$

By the Lipschitz assumptions $(\tilde{A}_1)$–$(\tilde{A}_2)$, for all $t \in [0,1]$,

$$|\phi(t) - \phi^*(t)| \leq L_f\Big(|u(t)-u^*(t)| + |\kappa(t)-\kappa^*(t)|\Big),$$
$$|\psi(t) - \psi^*(t)| \leq L_g\Big(|u(t)-u^*(t)| + |\kappa(t)-\kappa^*(t)|\Big).$$

Hence,

$$|\phi(t) - \phi^*(t)| \leq L_f\Big(\|u-u^*\| + \|\kappa-\kappa^*\|\Big), \qquad |\psi(t) - \psi^*(t)| \leq L_g\Big(\|u-u^*\| + \|\kappa-\kappa^*\|\Big).$$

Using the expression of T_1 in (3.2), subtracting $T_1(u^*,\kappa^*)$ from $T_1(u,\kappa)$, and applying the triangle inequality, we obtain

$$|T_1(u,\kappa)(t) - T_1(u^*,\kappa^*)(t)| \leq \mathcal{I}_1(t) + \mathcal{I}_2(t) + \mathcal{I}_3(t) + \mathcal{I}_4(t),$$

where each term $\mathcal{I}_\ell(t)$ corresponds to the difference of the integral terms in (3.2). Estimating these terms by the above Lipschitz bounds and using

$$\int_0^t (t-\tau)^{\alpha_i - 1}\, d\tau = \frac{t^{\alpha_i}}{\alpha_i} \leq \frac{1}{\Gamma(\alpha_i+1)} \quad (t \in [0,1]),$$

and

$$\int_{\bar{\zeta}_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i - 1}\, d\tau\, ds = \frac{\eta_k^{\beta_i+1} - \bar{\zeta}_k^{\beta_i+1}}{\Gamma(\beta_i+2)},$$

(and similarly for the α_i-double integrals), we get

$$\|T_1(u,\kappa) - T_1(u^*,\kappa^*)\| \leq K_1\Big(\|u-u^*\| + \|\kappa-\kappa^*\|\Big),$$

where K_1 is exactly the constant given in (3.4) (see Remark (3.1)). In the same way, using the expression of T_2 in (3.3), we deduce

$$\|T_2(u,\kappa) - T_2(u^*,\kappa^*)\| \leq K_2\Big(\|u-u^*\| + \|\kappa-\kappa^*\|\Big),$$

where K_2 is given in (3.5) (see Remark (3.1)).

Therefore,

$$\begin{aligned}\|T(u,\kappa)-T(u^*,\kappa^*)\| &= \|T_1(u,\kappa)-T_1(u^*,\kappa^*)\|+\|T_2(u,\kappa)-T_2(u^*,\kappa^*)\|\\ &\le (K_1+K_2)\Big(\|u-u^*\|+\|\kappa-\kappa^*\|\Big)\\ &= K\Big(\|u-u^*\|+\|\kappa-\kappa^*\|\Big).\end{aligned}$$

Since $K<1$, T satisfies the contraction condition on $E\times E$.

By Banach's fixed point theorem, there exists exactly one fixed point associated with T. This fixed point corresponds to the unique solution of the coupled system (1.1)–(1.2). □

Theorem 3.2. *We consider nonlinear functions f and g satisfying the Lipschitz conditions $(\tilde{A}_1)$–$(\tilde{A}_2)$. In other words, there exist positive constants L_f and L_g such that, for every $t\in[0,1]$ and $u,u^*,\kappa,\kappa^*\in E$,*

$$|f(t,u,\kappa)-f(t,u^*,\kappa^*)|\le L_f\Big(|u-u^*|+|\kappa-\kappa^*|\Big),$$

$$|g(t,u,\kappa)-g(t,u^*,\kappa^*)|\le L_g\Big(|u-u^*|+|\kappa-\kappa^*|\Big).$$

Moreover, suppose that the operator $T:E\times E\to E\times E$ defined by (3.2)–(3.3) *satisfies the conditions of Krasnoselskii's fixed point theorem. As a consequence, the coupled system* (1.1)–(1.2) *admits at least one solution over the interval* $[0,1]$.

Proof. Let $E=C([0,1],\mathbb{R})$. We introduce the mapping $T:E\times E\to E\times E$ as follows:

$$T(u,\kappa)(t)=\Big(T_1(u,\kappa)(t),\,T_2(u,\kappa)(t)\Big),$$

where T_1 and T_2 are given by the integral representations (3.2)–(3.3) with

$$\phi(t)=f(t,u(t),\kappa(t)),\qquad \psi(t)=g(t,u(t),\kappa(t)).$$

To apply Krasnoselskii's fixed point theorem, we decompose T as $T=\mathcal{A}+\mathcal{B}$,
where

$$\mathcal{A}(u,\kappa)=\Big(\mathcal{A}_1(u,\kappa),\mathcal{A}_2(u,\kappa)\Big),\qquad \mathcal{B}(u,\kappa)=\Big(\mathcal{B}_1(u,\kappa),\mathcal{B}_2(u,\kappa)\Big),$$

and $\mathcal{A}$ contains the compact integral terms, while $\mathcal{B}$ includes the contraction part.

Step 1: $\mathbf{T}(\boldsymbol{\Delta})\subseteq\boldsymbol{\Delta}$.

Let $\Delta=\Big\{(u,\kappa)\in E\times E:\|(u,\kappa)\|\le r\Big\}$, for some sufficiently large positive constant r. Then, using the growth condition on f and g, we obtain $\|T_1(u,\kappa)\|\le r$, $\|T_2(u,\kappa)\|\le r$, which implies that $T(u,\kappa)\in\Delta$.

Step 2: $\mathcal{B}$ is a contraction mapping.

Assume that f and g satisfy the Lipschitz conditions with constants L_f and L_g. Then for any $(u,\kappa),(u^*,\kappa^*)\in\Delta$, we have

$$\|T_1(u,\kappa)-T_1(u^*,\kappa^*)\|\le K_1\Big(\|u-u^*\|+\|\kappa-\kappa^*\|\Big),$$

and similarly,

$$\|T_2(u,\kappa) - T_2(u^*,\kappa^*)\| \le K_2\Big(\|u-u^*\| + \|\kappa-\kappa^*\|\Big),$$

where K_1 and K_2 are defined in Remark (3.1). Thus,

$$\|T(u,\kappa) - T(u^*,\kappa^*)\| \le (K_1+K_2)\Big(\|u-u^*\| + \|\kappa-\kappa^*\|\Big).$$

If $K := K_1 + K_2 < 1$, then $\mathcal{B}$ is a contraction on Δ.

Step 3: $\mathcal{A}$ is continuous.
Let $(u_n,\kappa_n) \to (u,\kappa)$ in Δ. Since f and g are continuous, it follows that $\phi_n(t) \to \phi(t)$, $\psi_n(t) \to \psi(t)$. Then by the dominated convergence theorem, we obtain $\mathcal{A}(u_n,\kappa_n) \to \mathcal{A}(u,\kappa)$, hence $\mathcal{A}$ is continuous.

Step 4: $\mathcal{A}$ is compact.
We show that $\mathcal{A}(\Delta)$ is uniformly bounded and equicontinuous. Boundedness follows from Step 1. Moreover, for $t_1, t_2 \in [0,1]$,

$$|\mathcal{A}_1(u,\kappa)(t_1) - \mathcal{A}_1(u,\kappa)(t_2)| \to 0 \quad \text{as } t_1 \to t_2,$$

and similarly for $\mathcal{A}_2$. Thus, $\mathcal{A}(\Delta)$ is equicontinuous. Therefore, by the Arzelà–Ascoli theorem, $\mathcal{A}$ is compact.

Step 5: The hypotheses of Krasnoselskii's fixed point theorem are verified. Hence, there exists $(u^*,\kappa^*) \in \Delta$ such that $(u^*,\kappa^*) = T(u^*,\kappa^*)$. Therefore, the coupled multi-term Caputo fractional boundary value problem (1.1)–(1.2) has at least one solution on $[0,1]$. □

4. Stability results

In this section, we investigate different stability notions of Ulam type, including $\mathcal{UH}$, $\mathcal{GUH}$, $\mathcal{UHR}$, and $\mathcal{GUHR}$ stability, for the solutions of problem (1.1). We first recall the required stability definitions and concepts that will be used throughout the analysis.

Definition 4.1. *We say that a pair $(x,y) \in E \times E$ is a Ulam–Hyers ($\mathcal{UH}$) stable solution of the system* (1.1)–(1.2) *if there exist positive constants B_1, B_2 such that, by setting $B = \max\{B_1,B_2\} > 0$ and $\epsilon = \max\{\epsilon_1,\epsilon_2\} > 0$, the following holds. For any $(x,y) \in E \times E$ satisfying*

$$\begin{cases} \left|\displaystyle\sum_{i=1}^{n} \lambda_i\, {}^{C}D^{\alpha_i} x(t) - f(t,x(t),y(t))\right| \le \epsilon_1, \\ \left|\displaystyle\sum_{i=1}^{n} \mu_i\, {}^{C}D^{\beta_i} y(t) - g(t,x(t),y(t))\right| \le \epsilon_2, \end{cases} \tag{4.1}$$

for all $t \in [0,1]$ with $\lambda_i, \mu_i \in \mathbb{R}$, there exists an exact solution (x^*,y^*) of the system (1.1)–(1.2) *such that $\|(x,y) - (x^*,y^*)\| \le B\epsilon$. Furthermore, if there exists a continuous function $\Phi : (0,\infty) \to (0,\infty)$ with*

$\Phi(0) = 0$ *such that* $\|(x,y)-(x^*,y^*)\| \le B\,\Phi(\epsilon)$, *then the system* (1.1)–(1.2) *is said to be generalized Ulam–Hyers* ($\mathcal{GUH}$) *stable.*

Definition 4.2. *A pair of functions* $(x,y) \in E\times E$ *is said to be Ulam–Hyers–Rassias* ($\mathcal{UHR}$) *stable for the system* (1.1)–(1.2) *if there exist positive constants* B_1, B_2 *and a continuous function* $\nu = (\nu_1,\nu_2) \in E\times E$ *such that, by setting* $B = \max\{B_1,B_2\} > 0$ *and* $\epsilon = \max\{\epsilon_1,\epsilon_2\} > 0$, *the following holds.*

For any $(x,y) \in E\times E$ *satisfying*

$$\begin{cases} \left|\sum_{i=1}^{n} \lambda_i\, {}^{C}D^{\alpha_i}x(t) - f(t,x(t),y(t))\right| \le \nu_1(t)\,\epsilon_1, \\ \left|\sum_{i=1}^{n} \mu_i\, {}^{C}D^{\beta_i}y(t) - g(t,x(t),y(t))\right| \le \nu_2(t)\,\epsilon_2, \end{cases} \tag{4.2}$$

for all $t\in[0,1]$ *with* $\lambda_i, \mu_i \in \mathbb{R}$, *there exists an exact solution* (x^*,y^*) *of the system* (1.1)–(1.2) *such that* $\|(x,y)-(x^*,y^*)\| \le B\,\nu(t)\,\epsilon$, *where* (x^*,y^*) *is a unique solution of the system* (1.1)–(1.2).

Definition 4.3. *A pair of functions* $(x,y) \in E\times E$ *is said to be generalized Ulam–Hyers–Rassias* ($\mathcal{GUHR}$) *stable for the system* (1.1)–(1.2) *if there exist positive constants* B_1, B_2 *and a continuous function* $\nu = (\nu_1,\nu_2) \in E\times E$ *such that, by setting* $B = \max\{B_1,B_2\} > 0$, *the following holds. For any* $(x,y)\in E\times E$ *satisfying*

$$\begin{cases} \left|\sum_{i=1}^{n} \lambda_i\, {}^{C}D^{\alpha_i}x(t) - f(t,x(t),y(t))\right| \le \nu_1(t), \\ \left|\sum_{i=1}^{n} \mu_i\, {}^{C}D^{\beta_i}y(t) - g(t,x(t),y(t))\right| \le \nu_2(t), \end{cases} \tag{4.3}$$

for all $t\in[0,1]$ *with* $\lambda_i, \mu_i \in \mathbb{R}$, *there exists an exact solution* (x^*,y^*) *of the system* (1.1)–(1.2) *such that* $\|(x,y)-(x^*,y^*)\| \le B\,\nu(t)$, *where* (x^*,y^*) *is a unique solution of the system* (1.1)–(1.2).

Remark 4.1. *The inequality* (4.1) *associated with the system* (1.1)–(1.2) *admits a solution* $(x,y)\in E\times E$ *if and only if there exist functions* $\varphi_1, \varphi_2 \in E$, *depending on* x *and* y, *such that:*

(1)

$$\epsilon_1 \ge |\varphi_1(t)|, \qquad \epsilon_2 \ge |\varphi_2(t)|, \qquad t\in[0,1],$$

(2)

$$\begin{cases} \sum_{i=1}^{n} \lambda_i\, {}^{C}D^{\alpha_i}x(t) - f(t,x(t),y(t)) - \varphi_1(t) = 0, \\ \sum_{i=1}^{n} \mu_i\, {}^{C}D^{\beta_i}y(t) - g(t,x(t),y(t)) - \varphi_2(t) = 0, \end{cases} \tag{4.4}$$

for all $t\in[0,1]$, *where* $\lambda_i, \mu_i \in \mathbb{R}$.

Remark 4.2. *The inequality* (4.2) *associated with the system* (1.1)–(1.2) *admits a solution* $(x, y) \in E \times E$ *if and only if there exist functions* $\varphi_1, \varphi_2 \in E$*, depending on* x *and* y*, such that:*

(1)

$$|\varphi_1(t)| \leq \nu_1(t)\epsilon_1, \qquad |\varphi_2(t)| \leq \nu_2(t)\epsilon_2, \qquad t \in [0, 1],$$

(2)

$$\begin{cases} \sum_{i=1}^{n} \lambda_i \, {}^{C}D^{\alpha_i} x(t) - f(t, x(t), y(t)) - \varphi_1(t) = 0, \\ \sum_{i=1}^{n} \mu_i \, {}^{C}D^{\beta_i} y(t) - g(t, x(t), y(t)) - \varphi_2(t) = 0, \end{cases} \tag{4.5}$$

for all $t \in [0, 1]$*, where* $\lambda_i, \mu_i \in \mathbb{R}$.

Remark 4.3. *The inequality* (4.3) *associated with the system* (1.1)–(1.2) *admits a solution* $(x, y) \in E \times E$ *if and only if there exist functions* $\varphi_1, \varphi_2 \in E$*, depending on* x *and* y*, such that:*

(1)

$$|\varphi_1(t)| \leq \nu_1(t), \qquad |\varphi_2(t)| \leq \nu_2(t), \qquad t \in [0, 1],$$

(2)

$$\begin{cases} \sum_{i=1}^{n} \lambda_i \, {}^{C}D^{\alpha_i} x(t) - f(t, x(t), y(t)) - \varphi_1(t) = 0, \\ \sum_{i=1}^{n} \mu_i \, {}^{C}D^{\beta_i} y(t) - g(t, x(t), y(t)) - \varphi_2(t) = 0, \end{cases} \tag{4.6}$$

for all $t \in [0, 1]$*, where* $\lambda_i, \mu_i \in \mathbb{R}$.

Lemma 4.1. *Let* $(x, y) \in E \times E$ *be a solution of the system* (1.1)–(1.2)*. Then* (x, y) *satisfies the following relations:*

$$\begin{cases} \sum_{i=1}^{n} \lambda_i \, {}^{C}D^{\alpha_i} x(t) = f(t, x(t), y(t)) + \varphi_1(t), \\ \sum_{i=1}^{n} \mu_i \, {}^{C}D^{\beta_i} y(t) = g(t, x(t), y(t)) + \varphi_2(t), \end{cases} \tag{4.7}$$

for all $t \in [0, 1]$*, where* $\varphi_1, \varphi_2 \in E$*. Moreover, the following estimates hold:*

$$\|x(t) - T_1(x, y)\| \leq \frac{\epsilon_1}{|\Delta|} \left[A_\alpha + |B| A_\beta \right],$$

$$\|y(t) - T_2(x, y)\| \leq \frac{\epsilon_2}{|\Delta|} \left[A_\beta + |D| A_\alpha \right],$$

where T_1 *and* T_2 *are defined by* (3.2)–(3.3)*, and the constants* A_α, A_β, B, D *and* Δ *are introduced in Remark 3.1*

Proof. Let $(x, y) \in E \times E$ be a solution of the system (1.1)–(1.2). Then, it follows that

$$\begin{aligned}
x(t) &= T_1(x,y)(t) \\
&\quad + \sum_{i=1}^{n} \frac{1}{\lambda_i \Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} \varphi_1(\tau)\, d\tau \\
&\quad + \frac{t^{n-1}}{\Delta} \Bigg[\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{a_k}{\mu_i \Gamma(\beta_i)} \int_{\zeta_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i - 1} \varphi_2(\tau)\, d\tau\, ds \\
&\qquad + \left(\sum_{k=1}^{m} a_k \int_{\zeta_k}^{\eta_k} s^{n-1}\, ds \right) \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{b_j}{\lambda_i \Gamma(\alpha_i)} \int_{\zeta_j}^{\eta_j} \int_0^s (s-\tau)^{\alpha_i - 1} \varphi_1(\tau)\, d\tau\, ds \right) \\
&\qquad - \sum_{i=1}^{n} \frac{1}{\lambda_i \Gamma(\alpha_i)} \int_0^1 (1-\tau)^{\alpha_i - 1} \varphi_1(\tau)\, d\tau \\
&\qquad - \left(\sum_{k=1}^{m} a_k \int_{\zeta_k}^{\eta_k} s^{n-1}\, ds \right) \left(\sum_{i=1}^{n} \frac{1}{\mu_i \Gamma(\beta_i)} \int_0^1 (1-\tau)^{\beta_i - 1} \varphi_2(\tau)\, d\tau \right) \Bigg],
\end{aligned}$$

and

$$\begin{aligned}
y(t) &= T_2(x,y)(t) \\
&\quad + \sum_{i=1}^{n} \frac{1}{\mu_i \Gamma(\beta_i)} \int_0^t (t-\tau)^{\beta_i - 1} \varphi_2(\tau)\, d\tau \\
&\quad + \frac{t^{n-1}}{\Delta} \Bigg[\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{b_j}{\lambda_i \Gamma(\alpha_i)} \int_{\zeta_j}^{\eta_j} \int_0^s (s-\tau)^{\alpha_i - 1} \varphi_1(\tau)\, d\tau\, ds \\
&\qquad + \left(\sum_{j=1}^{m} b_j \int_{\zeta_j}^{\eta_j} s^{n-1}\, ds \right) \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{a_k}{\mu_i \Gamma(\beta_i)} \int_{\zeta_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i - 1} \varphi_2(\tau)\, d\tau\, ds \right) \\
&\qquad - \sum_{i=1}^{n} \frac{1}{\mu_i \Gamma(\beta_i)} \int_0^1 (1-\tau)^{\beta_i - 1} \varphi_2(\tau)\, d\tau \\
&\qquad - \left(\sum_{j=1}^{m} b_j \int_{\zeta_j}^{\eta_j} s^{n-1}\, ds \right) \left(\sum_{i=1}^{n} \frac{1}{\lambda_i \Gamma(\alpha_i)} \int_0^1 (1-\tau)^{\alpha_i - 1} \varphi_1(\tau)\, d\tau \right) \Bigg].
\end{aligned}$$

Using Remark 4.1, we get

$$\begin{aligned}
|x(t) - T_1(x,y)(t)| &\le \sum_{i=1}^{n} \frac{1}{|\lambda_i| \Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} |\varphi_1(\tau)|\, d\tau \\
&\quad + \frac{t^{n-1}}{|\Delta|} \Bigg[\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{|a_k|}{|\mu_i| \Gamma(\beta_i)} \int_{\zeta_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i - 1} |\varphi_2(\tau)|\, d\tau\, ds \\
&\qquad + \left(\sum_{k=1}^{m} |a_k| \int_{\zeta_k}^{\eta_k} s^{n-1}\, ds \right) \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{|b_j|}{|\lambda_i| \Gamma(\alpha_i)} \int_{\zeta_j}^{\eta_j} \int_0^s (s-\tau)^{\alpha_i - 1} |\varphi_1(\tau)|\, d\tau\, ds \right)
\end{aligned}$$

$$+ \sum_{i=1}^{n} \frac{1}{|\lambda_i|\Gamma(\alpha_i)} \int_0^1 (1-\tau)^{\alpha_i - 1} |\varphi_1(\tau)|\, d\tau$$
$$+ \left(\sum_{k=1}^{m} |a_k| \int_{\bar{\zeta}_k}^{\eta_k} s^{n-1}\, ds \right) \left(\sum_{i=1}^{n} \frac{1}{|\mu_i|\Gamma(\beta_i)} \int_0^1 (1-\tau)^{\beta_i - 1} |\varphi_2(\tau)|\, d\tau \right) \Bigg]$$
$$\leq \left[A_\alpha + \frac{1}{|\Delta|} \Big(A_\beta^* + |B| A_\alpha^* + A_\alpha + |B| A_\beta \Big) \right] \epsilon_1.$$

and similarly,

$$|y(t) - T_2(x,y)(t)| \leq \left[A_\beta + \frac{1}{|\Delta|} \Big(A_\alpha^* + |D| A_\beta^* + A_\beta + |D| A_\alpha \Big) \right] \epsilon_2.$$

Consequently,

$$\|x(t) - T_1(x,y)\| \leq \left[A_\alpha + \frac{1}{|\Delta|} \Big(A_\beta^* + |B| A_\alpha^* + A_\alpha + |B| A_\beta \Big) \right] \epsilon_1,$$
$$\|y(t) - T_2(x,y)\| \leq \left[A_\beta + \frac{1}{|\Delta|} \Big(A_\alpha^* + |D| A_\beta^* + A_\beta + |D| A_\alpha \Big) \right] \epsilon_2.$$

□

Theorem 4.1. *If* $(\tilde{A}_1)$–$(\tilde{A}_3)$ *hold and* $K = \max\{K_1, K_2\} < 1$, *where* K_1 *and* K_2 *are defined by* (3.4)–(3.5), *then the system* (1.1)–(1.2) *is* $\mathcal{UH}$*- and* $\mathcal{GUH}$*-stable.*

Proof. Let $(x, y) \in E \times E$ be a solution of (1.1)–(1.2), and let (x^*, y^*) be its unique solution. Then, we have

$$\begin{aligned} \|(x,y) - (x^*, y^*)\| &= \|(x,y) - T(x^*, y^*)\| \\ &\leq \|x - T_1(x,y)\| + \|T_1(x,y) - T_1(x^*, y^*)\| \\ &\quad + \|y - T_2(x,y)\| + \|T_2(x,y) - T_2(x^*, y^*)\|. \end{aligned}$$

Applying Lemma 4.2 and Theorem 3.1, we obtain

$$\begin{aligned} \|(x,y) - (x^*, y^*)\| &\leq \|x - T_1(x,y)\| + \|y - T_2(x,y)\| \\ &\quad + \|T_1(x,y) - T_1(x^*, y^*)\| + \|T_2(x,y) - T_2(x^*, y^*)\| \\ &\leq \left[A_\alpha + \frac{1}{|\Delta|} \Big(A_\beta^* + |B| A_\alpha^* + A_\alpha + |B| A_\beta \Big) \right] \epsilon_1 \\ &\quad + \left[A_\beta + \frac{1}{|\Delta|} \Big(A_\alpha^* + |D| A_\beta^* + A_\beta + |D| A_\alpha \Big) \right] \epsilon_2 \\ &\quad + K \|(x,y) - (x^*, y^*)\|. \end{aligned}$$

Hence,

$$(1-K)\|(x,y)-(x^*,y^*)\| \leq \left[A_\alpha + \frac{1}{|\Delta|}\Big(A^*_\beta + |B|A^*_\alpha + A_\alpha + |B|A_\beta\Big)\right]\epsilon_1 + \left[A_\beta + \frac{1}{|\Delta|}\Big(A^*_\alpha + |D|A^*_\beta + A_\beta + |D|A_\alpha\Big)\right]\epsilon_2.$$

Since $K < 1$, we get

$$\begin{aligned}\|(x,y)-(x^*,y^*)\| &\leq \frac{1}{1-K}\left\{\left[A_\alpha + \frac{1}{|\Delta|}\Big(A^*_\beta + |B|A^*_\alpha + A_\alpha + |B|A_\beta\Big)\right]\epsilon_1 \right.\\ &\qquad \left. + \left[A_\beta + \frac{1}{|\Delta|}\Big(A^*_\alpha + |D|A^*_\beta + A_\beta + |D|A_\alpha\Big)\right]\epsilon_2\right\},\\ &\leq \mathcal{B}\epsilon,\end{aligned}$$

where

$$\mathcal{B} = \max\left\{\frac{1}{1-K}\left[A_\alpha + \frac{1}{|\Delta|}\Big(A^*_\beta + |B|A^*_\alpha + A_\alpha + |B|A_\beta\Big)\right], \frac{1}{1-K}\left[A_\beta + \frac{1}{|\Delta|}\Big(A^*_\alpha + |D|A^*_\beta + A_\beta + |D|A_\alpha\Big)\right]\right\}.$$

Consequently, the system (1.1)–(1.2) is $\mathcal{UH}$-stable. Moreover, by taking $\Phi(\epsilon) = \epsilon$, it follows that the system (1.1)–(1.2) is also $\mathcal{GUH}$-stable. □

Lemma 4.2. *Under the assumptions of Remark 4.2, it follows that solutions of* (4.7) *satisfy the following estimates:*

$$\|x(t) - T_1(x,y)(t)\| \leq \left[A_\alpha + \frac{1}{|\Delta|}\Big(A^*_\beta + |B|A^*_\alpha + A_\alpha + |B|A_\beta\Big)\right]\vartheta_1(t)\,\epsilon_1,$$

and

$$\|y(t) - T_2(x,y)(t)\| \leq \left[A_\beta + \frac{1}{|\Delta|}\Big(A^*_\alpha + |D|A^*_\beta + A_\beta + |D|A_\alpha\Big)\right]\vartheta_2(t)\,\epsilon_2.$$

Proof. Suppose that $(x,y) \in E \times E$ is a solution of (4.7). Then, by Remark 4.2, there exist functions $\varphi_1, \varphi_2 \in E$ such that

$$|\varphi_1(t)| \leq \vartheta_1(t)\epsilon_1, \qquad |\varphi_2(t)| \leq \vartheta_2(t)\epsilon_2, \qquad t \in [0,1],$$

and

$$\sum_{i=1}^{n} \lambda_i\, {}^C D^{\alpha_i} x(t) - f(t, x(t), y(t)) - \varphi_1(t) = 0,$$
$$\sum_{i=1}^{n} \mu_i\, {}^C D^{\beta_i} y(t) - g(t, x(t), y(t)) - \varphi_2(t) = 0.$$

Using the integral representation of solutions associated with (1.1)–(1.2), we obtain

$$x(t) = T_1(x,y)(t) + \mathcal{H}_1(t), \qquad y(t) = T_2(x,y)(t) + \mathcal{H}_2(t),$$

where

$$
\begin{aligned}
|\mathcal{H}_1(t)| \leq & \sum_{i=1}^{n} \frac{1}{|\lambda_i|\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i-1} |\varphi_1(\tau)|\, d\tau \\
& + \frac{t^{n-1}}{|\Delta|} \Bigg[\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{|a_k|}{|\mu_i|\Gamma(\beta_i)} \int_{\bar{\zeta}_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i-1} |\varphi_2(\tau)|\, d\tau\, ds \\
& \quad + \left(\sum_{k=1}^{m} |a_k| \int_{\bar{\zeta}_k}^{\eta_k} s^{n-1}\, ds \right) \left(\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{|b_j|}{|\lambda_i|\Gamma(\alpha_i)} \int_{\bar{\zeta}_j}^{\eta_j} \int_0^s (s-\tau)^{\alpha_i-1} |\varphi_1(\tau)|\, d\tau\, ds \right) \\
& \quad + \sum_{i=1}^{n} \frac{1}{|\lambda_i|\Gamma(\alpha_i)} \int_0^1 (1-\tau)^{\alpha_i-1} |\varphi_1(\tau)|\, d\tau \\
& \quad + \left(\sum_{k=1}^{m} |a_k| \int_{\bar{\zeta}_k}^{\eta_k} s^{n-1}\, ds \right) \left(\sum_{i=1}^{n} \frac{1}{|\mu_i|\Gamma(\beta_i)} \int_0^1 (1-\tau)^{\beta_i-1} |\varphi_2(\tau)|\, d\tau \right) \Bigg],
\end{aligned}
$$

and similarly

$$
\begin{aligned}
|\mathcal{H}_2(t)| \leq & \sum_{i=1}^{n} \frac{1}{|\mu_i|\Gamma(\beta_i)} \int_0^t (t-\tau)^{\beta_i-1} |\varphi_2(\tau)|\, d\tau \\
& + \frac{t^{n-1}}{|\Delta|} \Bigg[\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{|b_j|}{|\lambda_i|\Gamma(\alpha_i)} \int_{\bar{\zeta}_j}^{\eta_j} \int_0^s (s-\tau)^{\alpha_i-1} |\varphi_1(\tau)|\, d\tau\, ds \\
& \quad + \left(\sum_{j=1}^{m} |b_j| \int_{\bar{\zeta}_j}^{\eta_j} s^{n-1}\, ds \right) \left(\sum_{k=1}^{m} \sum_{i=1}^{n} \frac{|a_k|}{|\mu_i|\Gamma(\beta_i)} \int_{\bar{\zeta}_k}^{\eta_k} \int_0^s (s-\tau)^{\beta_i-1} |\varphi_2(\tau)|\, d\tau\, ds \right) \\
& \quad + \sum_{i=1}^{n} \frac{1}{|\mu_i|\Gamma(\beta_i)} \int_0^1 (1-\tau)^{\beta_i-1} |\varphi_2(\tau)|\, d\tau \\
& \quad + \left(\sum_{j=1}^{m} |b_j| \int_{\bar{\zeta}_j}^{\eta_j} s^{n-1}\, ds \right) \left(\sum_{i=1}^{n} \frac{1}{|\lambda_i|\Gamma(\alpha_i)} \int_0^1 (1-\tau)^{\alpha_i-1} |\varphi_1(\tau)|\, d\tau \right) \Bigg].
\end{aligned}
$$

Therefore,

$$
|x(t) - T_1(x,y)(t)| \leq |\mathcal{H}_1(t)|, \qquad |y(t) - T_2(x,y)(t)| \leq |\mathcal{H}_2(t)|.
$$

Now, using the bounds

$$
|\varphi_1(t)| \leq \vartheta_1(t)\epsilon_1, \qquad |\varphi_2(t)| \leq \vartheta_2(t)\epsilon_2,
$$

together with the definitions of A_α, A_β, A^*_α, A^*_β, B, D, and Δ, we get

$$
\|x(t) - T_1(x,y)(t)\| \leq \left[A_\alpha + \frac{1}{|\Delta|} \Big(A^*_\beta + |B| A^*_\alpha + A_\alpha + |B| A_\beta \Big) \right] \vartheta_1(t)\epsilon_1,
$$

and

$$
\|y(t) - T_2(x,y)(t)\| \leq \left[A_\beta + \frac{1}{|\Delta|} \Big(A^*_\alpha + |D| A^*_\beta + A_\beta + |D| A_\alpha \Big) \right] \vartheta_2(t)\epsilon_2.
$$

□

Theorem 4.2. *If* $(\tilde{A}_1)$–$(\tilde{A}_2)$ *hold and* $K = \max\{K_1, K_2\} < 1$, *where* K_1 *and* K_2 *are defined by* (3.4)–(3.5), *then the system* (1.1)–(1.2) *is* $\mathcal{UHR}$*- and* $\mathcal{GUHR}$*-stable.*

Proof. Assume that $(x, y) \in E \times E$ and let (x^*, y^*) be the unique solution of the system (1.1)–(1.2). Then, we have

$$\begin{aligned}
\|(x,y) - (x^*, y^*)\| &= \|(x,y) - T(x^*, y^*)\| \\
&\leq \|x - T_1(x,y)\| + \|y - T_2(x,y)\| \\
&\quad + \|T_1(x,y) - T_1(x^*, y^*)\| + \|T_2(x,y) - T_2(x^*, y^*)\|.
\end{aligned}$$

By Lemma 4.1 and Theorem 3.1, we obtain

$$\begin{aligned}
\|(x,y) - (x^*, y^*)\| \leq& \left[A_\alpha + \frac{1}{|\Delta|}\Big(A^*_\beta + |B|A^*_\alpha + A_\alpha + |B|A_\beta\Big)\right]\vartheta_1(t)\epsilon_1 \\
&+ \left[A_\beta + \frac{1}{|\Delta|}\Big(A^*_\alpha + |D|A^*_\beta + A_\beta + |D|A_\alpha\Big)\right]\vartheta_2(t)\epsilon_2 \\
&+ K\|(x,y) - (x^*, y^*)\|.
\end{aligned}$$

Since $K < 1$, we get

$$\begin{aligned}
\|(x,y) - (x^*, y^*)\| &\leq \frac{1}{1-K}\left\{\left[A_\alpha + \frac{1}{|\Delta|}\Big(A^*_\beta + |B|A^*_\alpha + A_\alpha + |B|A_\beta\Big)\right]\epsilon_1 \right.\\
&\qquad \left. + \left[A_\beta + \frac{1}{|\Delta|}\Big(A^*_\alpha + |D|A^*_\beta + A_\beta + |D|A_\alpha\Big)\right]\epsilon_2\right\} \\
&\leq \mathcal{B}\epsilon,
\end{aligned}$$

where

$$\mathcal{B} = \max\left\{\frac{1}{1-K}\left[A_\alpha + \frac{1}{|\Delta|}\Big(A^*_\beta + |B|A^*_\alpha + A_\alpha + |B|A_\beta\Big)\right], \frac{1}{1-K}\left[A_\beta + \frac{1}{|\Delta|}\Big(A^*_\alpha + |D|A^*_\beta + A_\beta + |D|A_\alpha\Big)\right]\right\},$$

Consequently, the system (1.1)–(1.2) is $\mathcal{UHR}$-stable. Moreover, by taking $\Phi(\epsilon) = \epsilon$, the system (1.1)–(1.2) is also $\mathcal{GUHR}$-stable. □

5. Example

To demonstrate the relevance of the results developed in earlier sections, we now provide an explicit example. This example shows how the coupled multi-term Caputo fractional system with multi-strip integral boundary conditions fits within our framework and confirms the applicability of the obtained existence, uniqueness, and stability conclusions.

Example 5.1. *Consider the following coupled system of multi–term Caputo fractional differential equations with nonlocal multi–strip integral boundary conditions:*

$$\begin{cases} \frac{3}{2}\,{}^{C}D^{0.9}u(t) - \frac{5}{4}\,{}^{C}D^{0.6}u(t) + \frac{7}{3}\,{}^{C}D^{0.3}u(t) = f\Big(t, u(t), \kappa(t)\Big), \\ \frac{4}{3}\,{}^{C}D^{0.8}\kappa(t) - \frac{2}{5}\,{}^{C}D^{0.5}\kappa(t) + \frac{9}{4}\,{}^{C}D^{0.2}\kappa(t) = g\Big(t, u(t), \kappa(t)\Big), \end{cases} \quad t \in [0,1], \tag{5.1}$$

where the nonlinear functions $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are defined by The nonlinearities* $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are given by*

$$f(t, u, \kappa) = \frac{t}{20}\left(\frac{|u|}{1+|u|} + \frac{1}{10}\sin^2(\kappa)\right), \qquad g(t, u, \kappa) = \frac{t}{25}\left(\frac{|u|}{1+|u|} + \frac{|\kappa|}{1+|\kappa|}\right).$$

The system (5.1) *is supplemented with the nonlocal multi–strip boundary conditions*

$$\begin{cases} u(0) = 0, \qquad u(1) = \sum_{k=1}^{2} a_k \int_{\bar{\zeta}_k}^{\eta_k} \kappa(s)\, ds, \\ \kappa(0) = 0, \qquad \kappa(1) = \sum_{j=1}^{2} b_j \int_{\bar{\zeta}_j}^{\eta_j} u(s)\, ds, \end{cases} \tag{5.2}$$

where

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{3}, \qquad b_1 = \frac{1}{4}, \quad b_2 = \frac{1}{5},$$

and $0 < \bar{\zeta}_1 < \eta_1 < \bar{\zeta}_2 < \eta_2 < 1$, *For instance, we choose*

$$\bar{\zeta}_1 = 0.1,\ \eta_1 = 0.3, \qquad \bar{\zeta}_2 = 0.6,\ \eta_2 = 0.9.$$

Next, we verify the Lipschitz-type conditions required in $(\tilde{A}_1)$*–*$(\tilde{A}_2)$*. Using the estimates*

$$\left|\frac{|u_1|}{1+|u_1|} - \frac{|u_2|}{1+|u_2|}\right| \le |u_1 - u_2|, \qquad |\sin^2(\kappa_1) - \sin^2(\kappa_2)| \le |\kappa_1 - \kappa_2|,$$

we obtain Lipschitz-type estimates of the form

$$|f(t, u_1, \kappa_1) - f(t, u_2, \kappa_2)| \le L_f\Big(|u_1 - u_2| + |\kappa_1 - \kappa_2|\Big),$$

$$|g(t, u_1, \kappa_1) - g(t, u_2, \kappa_2)| \le L_g\Big(|u_1 - u_2| + |\kappa_1 - \kappa_2|\Big).$$

Moreover, numerical computations performed in Maple yield

$$B = 0.2, \qquad D = 0.11, \qquad \Delta = 1 - BD = 0.978 \neq 0,$$

together with the constants

$$A_\alpha = 2.066042, \quad A_\beta = 4.110257, \quad A_\alpha^* = 0.143032, \quad A_\beta^* = 0.535117,$$

and the Lipschitz bounds

$$L_f = 0.055, \qquad L_g = 0.04.$$

Therefore, all assumptions of Theorems 3.1–3.2 (and the stability results in Section 4) are satisfied. Consequently, problem (5.1)*–*(5.2) *admits a unique solution on* $[0,1]$*, and the solution is* $\mathcal{UHR}$ *and* $\mathcal{GUHR}$ *stable.*

6. Conclusion

In this work, we investigated a coupled system of multi-term Caputo fractional differential equations with nonlocal multi-strip integral boundary conditions. By transforming the problem into an equivalent integral formulation, the existence of at least one solution was established via Krasnoselskii's fixed point theorem together with the Arzelà–Ascoli theorem, while uniqueness was obtained under suitable Lipschitz-type conditions. Furthermore, several Ulam-type stability results were derived, including $\mathcal{UH}$, $\mathcal{UHR}$, $\mathcal{GUH}$, and $\mathcal{GUHR}$ stability. These results contribute to the theory of coupled fractional boundary value problems with nonlocal conditions. Future work may focus on extending the present results to q-fractional differential models, which may provide further insights into fractional differential systems and broaden the scope of applications.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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