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Well-posedness and Generalized Stability of Caputo Fractional Dynamic Equations on Time Scales with Applications to Epidemiological Models

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Abstract. In this work, we extend the theory of fractional calculus to hybrid systems that combine continuous and discrete dynamics by establishing the existence, uniqueness, and generalized Ulam–Hyers–Rassias stability of solutions to Caputo fractional delta-differential equations on time scales. Using the Schauder fixed-point theorem, we derive sufficient conditions for well-posedness and stability, building on core concepts from fractional calculus and time-scale theory. To demonstrate the applicability and practicality of the proposed framework, we apply the theoretical results to epidemiological modelling. In particular, SIS and SIR models are employed to illustrate how the approach captures memory effects and hybrid dynamics inherent in real-world systems. Thus, the framework not only advances the theoretical foundations of fractional dynamic equations but also highlights their practical relevance.

1. Introduction

Fractional calculus has rapidly emerged as a cornerstone of contemporary mathematical modeling, owing to its ability to extend the classical notions of differentiation and integration to arbitrary non-integer orders [2, 8, 19, 26]. This generalization, often described as "differentiation

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and integration of non-integer order," provides a powerful framework for describing memory effects, hereditary dynamics, and non-local interactions inherent in a wide variety of natural and engineered systems, including physics, biology, engineering, economics, and finance. Unlike integer-order calculus, which models local instantaneous rates of change, fractional calculus intrinsically incorporates the influence of past states, making it indispensable for analyzing processes with long-range dependence or fractal-like dynamics [18].

Theoretical foundations of the field have been rigorously developed in pioneering works such as [3, 10], while applications continue to expand into diverse areas such as viscoelastic materials, anomalous diffusion, epidemiology, and control theory. Several definitions of fractional operators exist, including the Grünwald–Letnikov, Riemann–Liouville, Caputo, Caputo–Fabrizio, and Atangana–Baleanu derivatives. Among these, the Caputo derivative remains the most widely used in applications due to its natural compatibility with classical initial conditions, ensuring that fractional-order models retain meaningful physical interpretations and are accessible for applied scientists.

A central aspect of the theory of fractional differential equations (FDEs) is their qualitative analysis, which parallels the rigorous framework of ordinary differential equations. The notions of existence, uniqueness, stability, and boundedness of solutions form the foundation of well-posedness. In particular, existence and uniqueness ensure that models yield mathematically meaningful and physically interpretable outcomes, while stability guarantees robustness under perturbations. Classical results [3,20,22,23] have employed fixed-point theorems, such as Banach's contraction principle and Schauder's theorem, to establish sufficient conditions in this direction. However, while FDEs are highly effective in modeling systems with memory, they are naturally formulated on continuous domains, which limits their applicability to hybrid dynamical systems that exhibit both continuous evolution and discrete changes.

To overcome this limitation, fractional calculus has been integrated with the theory of time scales, first introduced by Hilger [4]. Time-scale theory unifies continuous and discrete analysis within a single framework by considering dynamics on an arbitrary closed subset of the real line, $\mathbb{T} \subseteq \mathbb{R}$ [6, 7]. When $\mathbb{T} = \mathbb{R}$, one recovers differential equations, and when $\mathbb{T} = \mathbb{Z}$, one recovers difference equations. This unified approach enables the analysis of hybrid systems where discrete sampling, impulsive effects, or switching phenomena occur alongside continuous dynamics. Applications of time-scale theory span digital control, population dynamics, neural networks, and epidemiological models [9].

Recent advances [11–13] have begun extending fractional operators to the time-scale setting, leading to the study of fractional delta-dynamic equations. Despite these developments, the literature on qualitative properties of solutions, particularly existence, uniqueness, and stability in the generalized Ulam–Hyers–Rassias sense remains sparse. This gap motivates the present work. Specifically, we investigate Caputo fractional delta-differential equations on arbitrary time scales and establish sufficient conditions for existence, uniqueness, and generalized Ulam–Hyers–Rassias stability of solutions. Our approach is based on the Schauder fixed-point theorem, which extends beyond Banach's contraction principle and provides a versatile framework for handling nonlinear operators that are continuous and compact but not necessarily contractive. The generalized Ulam–Hyers–Rassias stability framework, in turn, refines classical stability concepts by allowing perturbations to depend on continuous bounding functions rather than fixed constants [16,17,21, 24].

The contributions of this paper are twofold. First, we provide new theoretical results existence, uniqueness and Ulam–Hyers–Rassias stability. Then, we demonstrate the practical significance of our results through applications to epidemiological models, including the fractional SIS and SIR systems, which naturally involve both continuous infection dynamics and discrete intervention or observation schedules. By deriving rigorous conditions for existence, uniqueness, and generalized Ulam-Hyers-Rassias stability, we establish a solid foundation for further investigations into stability analysis, numerical approximation, and control design of fractional-order systems on time scales. Ultimately, this work strengthens the mathematical theory while opening avenues for practical applications to complex, multi-scale systems where discrete and continuous processes coexist.

The remainder of this work is organized as follows. Section 2 introduces the preliminary definitions that serve as the foundation for the discussions in Section 3, which outlines the methodology and presents established lemmas and preliminary theorem employed in proving the main results. Section 4 is devoted to the derivation of the existence, uniqueness, and generalized Ulam–Hyers–Rassias stability of solutions. Finally, Section 5 illustrates the applicability of our results through epidemiological models.

2. Preliminaries

Definition 2.1. *[15] A function $\mathcal{G} : \mathbb{T} \to \mathbb{R}$ is said to be right dense (r-d) continuous if it is continuous at all right-dense points of $\mathbb{T}$, and its left-sided limits exist and are finite at all left-dense points of $\mathbb{T}$. We shall denote right dense continuity by C_{rd}.*

Definition 2.2. *[5] [Time Scale] A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$:*

(i) *Forward jump operator:* $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$
(ii) *Backward jump operator:* $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$
(iii) *Graininess function:* $\mu(t) := \sigma(t) - t$

Definition 2.3. *Equicontinuity on Time Scales A family $\mathcal{F} \subset C_{rd}(\mathbb{T}, X)$ is equicontinuous at $t_0 \in \mathbb{T}^{\kappa}$ if:*

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } t \in (t_0 - \delta, t_0 + \delta) \cap \mathbb{T} \implies d(f(t), f(t_0)) < \epsilon,\ \forall f \in \mathcal{F}$$

It is uniformly equicontinuous if δ is independent of t_0.

Definition 2.4 (Delta-Equicontinuity). *For $\mathcal{F} \subset C^{\lambda}_{rd}(\mathbb{T}, X)$, where $\lambda \in (0,1)$, we say $\mathcal{F}$ is delta-equicontinuous at right-dense t_0 if:*

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } t \in (t_0 - \delta, t_0 + \delta) \cap \mathbb{T} \implies \|f^{\Delta}(t) - f^{\Delta}(t_0)\| < \epsilon,\ \forall f \in \mathcal{F}$$

Definition 2.5 (Uniform Boundedness on Time Scales). *A set $S \subset C_{rd}(\mathbb{T}, X)$ is uniformly bounded if:*

$$\exists M > 0 \text{ such that } \sup_{t \in \mathbb{T}} \|f(t)\| \leq M,\ \forall f \in S$$

For delta-derivatives: $\sup_{t \in \mathbb{T}^{\kappa}} \|f^{\Delta}(t)\| \leq M$.

Definition 2.6 (Closedness in Time-Scale Function Spaces). *A set $S \subset C_{rd}(\mathbb{T}, X)$ is closed if for every sequence $\{f_n\} \subset S$ converging uniformly to f on $\mathbb{T}$, then $f \in S$.*

Definition 2.7 (Convexity on Time Scales). *A set $S \subset C_{rd}(\mathbb{T}, X)$ is convex if:*

$$\forall f, g \in S,\ \forall \lambda \in [0,1],\ (\lambda f + (1-\lambda) g)(t) \in S,\ \forall t \in \mathbb{T}$$

3. Methodology and Results

Theorem 3.1 (Time-Scale Arzelà-Ascoli). *Let $\mathbb{T}$ be a time scale and (X, d) a complete metric space. A family $\mathcal{F} \subset C_{rd}(\mathbb{T}, X)$ is relatively compact in the topology of uniform convergence if and only if:*

- (M1) *$\mathcal{F}$ is uniformly bounded: $\exists M > 0$ such that $\sup_{t \in \mathbb{T}} d(f(t), f_0) \leq M$ for some $f_0 \in X$ and all $f \in \mathcal{F}$*
- (M2) *$\mathcal{F}$ is uniformly equicontinuous at all right-dense points of $\mathbb{T}$*
- (M3) *For each left-dense $t_0 \in \mathbb{T}$, the left limits $\lim_{t \to t_0^-} f(t)$ exist uniformly in $\mathcal{F}$*

Proof. We begin by first proving the necessary conditions.
Assume $\mathcal{F}$ is relatively compact in $C_{rd}(\mathbb{T}, X)$. Then:

(i) Uniform boundedness: Since $\mathcal{F}$ is relatively compact, its closure $\overline{\mathcal{F}}$ is compact. The evaluation maps $e_t : f \mapsto f(t)$ are continuous, so $\{f(t) : f \in \overline{\mathcal{F}}\}$ is compact in X for each t. By compactness, there exists finite ϵ-nets, implying uniform boundedness. In fact fixing $f_0 \in X$, the function $f \mapsto \sup_{t \in \mathbb{T}} d(f(t), f_0)$ is continuous on $\overline{\mathcal{F}}$, hence attains its maximum M. Thus $\mathcal{F}$ is uniformly bounded by M.

(ii) Uniform equicontinuity: Given $\epsilon > 0$, let $\{f_1, \ldots, f_n\}$ be an $\epsilon/3$-net for $\overline{\mathcal{F}}$. Each f_i is uniformly continuous on $\mathbb{T}$ (as $\mathbb{T}$ is closed and $f_i \in C_{rd}(\mathbb{T}, X)$).

For right-dense t_0, choose $\delta_i > 0$ such that for all $t \in (t_0 - \delta_i, t_0 + \delta_i) \cap \mathbb{T}$:

$$d(f_i(t), f_i(t_0)) < \epsilon/3$$

Take $\delta = \min \delta_i$. For any $f \in \mathcal{F}$, choose f_k with $\|f - f_k\|_\infty < \epsilon/3$. Then:

$$\begin{aligned} d(f(t), f(t_0)) &\leq d(f(t), f_k(t)) + d(f_k(t), f_k(t_0)) + d(f_k(t_0), f(t_0)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

proving uniform equicontinuity.

(iii) Uniform left-limits: For left-dense t_0, the uniform convergence of $\{f_n\} \subset \mathcal{F}$ implies the existence of uniform left-limits. Suppose not, then there exists $\epsilon > 0$ and sequences $\{f_n\} \subset \mathcal{F}$, $\{t_n\}, \{s_n\} \to t_0^-$ with:

$$d(f_n(t_n), f_n(s_n)) \geq \epsilon$$

But by relative compactness, some subsequence $\{f_{n_k}\}$ converges uniformly, leading to a contradiction when $k \to \infty$.

Now, we shall prove the sufficient condition.

Assume (1)-(3) of the theorem hold. We show $\mathcal{F}$ is totally bounded and complete.

Fix $\epsilon > 0$.

(i) For each right-dense $t_0 \in \mathbb{T}$, choose $\delta(t_0) > 0$ from uniform equicontinuity. The intervals $\{(t_0 - \delta(t_0), t_0 + \delta(t_0))\}$ cover $\mathbb{T}$. By compactness of $\mathbb{T}$ (as closed subset of $\mathbb{R}$), extract finite subcover for right-dense points.

(ii) For isolated points $\{t_1, \dots, t_m\}$, include them directly in the partition.

(iii) For left-dense points $\{\tau_1, \dots, \tau_k\}$, use condition (M3) to choose $\gamma_j > 0$ such that for all $t \in (\tau_j - \gamma_j, \tau_j) \cap \mathbb{T}$:

$$d(f(t), f(\tau_j^-)) < \epsilon/4 \quad \forall f \in \mathcal{F}$$

This yields a finite partition $\mathbb{T} = \bigcup_{i=1}^{N} P_i$ where each P_i is either:

(i) A single isolated point.

(ii) An interval of length $< \delta$ containing a right-dense point.

(iii) A left-neighborhood of a left-dense point.

Also, select representative points $t_i \in P_i$, then by uniform boundedness, $\{f(t_i) : f \in \mathcal{F}\}$ is relatively compact in X^N. Choose finite $\epsilon/2$-net $\{y^1, \dots, y^M\} \subset X^N$ and define piecewise constant approximations $g_j(t) = y_i^j$ for $t \in P_i$. So that, for any $f \in \mathcal{F}$, choose g_j with $d(f(t_i), y_i^j) < \epsilon/2$ for all i. Then:

(i) For isolated points $t = t_i$: $d(f(t), g_j(t)) = d(f(t_i), y_i^j) < \epsilon/2 < \epsilon$

(ii) For t in right-dense interval P_i with representative t_i:

$$\begin{aligned} d(f(t), g_j(t)) &\leq d(f(t), f(t_i)) + d(f(t_i), y_i^j) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \quad \text{(by equicontinuity)} \end{aligned}$$

(iii) For t near left-dense τ_j:

$$\begin{aligned} d(f(t), g_j(t)) &\leq d(f(t), f(\tau_j^-)) + d(f(\tau_j^-), y_k^j) + d(y_k^j, g_j(t)) \\ &< \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon \end{aligned}$$

Thus $\{g_1, \dots, g_M\}$ forms an ϵ-net for $\mathcal{F}$.

Since X is complete and conditions (M1)-(M3) are preserved under uniform limits, $\overline{\mathcal{F}}$ is complete. Combined with total boundedness, $\mathcal{F}$ is relatively compact. □

Lemma 3.1. *[1] Suppose X is a Banach space and $S \subset X$ is a closed, bounded and convex subset. Let $\mathcal{T} : S \to S$ be completely continuous. Then there exists a fixed point of $\mathcal{T}$ in S.*

Consider the following fractional order dynamic equation on time scales

$$\begin{cases} {}^{C}\Delta^{\lambda}x(t) = f(t,x) \\ x(t_0) = x_0, \quad t_0 \geq 0, \end{cases} \tag{3.1}$$

such that $f \in C_{rd}([0,T] \subset \mathbb{T} \times \mathbb{R}, \mathbb{R})$, $f(t,0) = 0$ and $\alpha \in (0,1)$. Some very important results for (3.1) were established in [25].

Lemma 3.2. *[25] Let $f(t,x)$ be C_{rd}, then integral equivalence of problem* (3.1) *is given as*

$$x(t) = x_0 + \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s,x(s))\, \Delta s, \tag{3.2}$$

such that every solution of (3.2) is also the solution of problem (3.1).

Lemma 3.3. *[3] Assume that S is a nonempty closed subset of a Banach space X, and let $p_n \geq 0$, with $\sum_{n=0}^{\infty} p_n$ converging for all $n \in \mathbb{N}$. Let $\mathcal{B} : S \to S$ be a mapping satisfying*

$$\|\mathcal{B}^n x_1 - \mathcal{B}^n x_2\| \leq p_n \|x_1 - x_2\|, \quad x_1, x_2 \in S.$$

Then, there exists a unique fixed point $x^* \in S$ of $\mathcal{B}$.

4. Main Results

Our aim in this section is to prove the existence, uniqueness and generalized Ulam-Hyers-Rassias stability of the solution of the fractional differential equations (3.1) on time scales.

4.1. **Existence of Caputo-type Fractional Dynamic System.** To begin, the following axioms will be useful in the sequel;

(A1): Suppose that $f(t,x) : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, the f satisfies the Lipschitz condition:

$$|f(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2|, \quad \text{where } L > 0.$$

(A2): Suppose that $f(t,x)$ has a weak singularity with respect to t, then there exists a constant $0 < \alpha \leq 1$ such that

$$(\mathcal{G}x)(t) = t^{\alpha} f(t,x)$$

is a C_{rd} bounded operator defined on $[0,T] \times [0,T]$, for $T > 0$, $[0,T] \subset \mathbb{T}$.

Next, we prove the existence, uniqueness and generalized Ulam-Hyers-Rassias Stability theorems using the above axioms.

Theorem 4.1. *Suppose that axioms (A_1)-(A_2) hold. Then at least one solution of the problem (3.1) exists such that $x \in C_{rd}[0,k]$, for $k \in (0,T]$.*

Proof. We define the set,

$$\mathcal{H} = \{x \in C_{rd}[0,T] : \|x - x_0\|_{C_{rd}[0,T]} = \sup_{t\in[0,T]} |x - x_0| \le r\}$$

where $r > 0$.
Since $\mathcal{G}$ is bounded, then there exists a positive constant $M > 0$ such that

$$\sup\{|(\mathcal{G}x)(t)| : t \in [0,T], x \in \mathcal{H}\} \le M.$$

Again, suppose

$$\Omega_k = \{x : x \in C_{rd}[0,k], \sup_{t\in[0,T]} |x - x_0| \le r\},$$

where $k = \min\left\{\left(\frac{r\Gamma(\lambda+1-\alpha)}{M\Gamma(1-\alpha)}\right)^{\frac{1}{\lambda-\alpha}}, T\right\}$, $\lambda > \alpha$.
It is clear that $\Omega_r \subseteq C_{rd}([0,k])$ is a bounded, closed, nonempty and convex subset.
Moreover, $k \le T$, and we can define the integral equivalence of the problem (3.1) as follows;

$$(Jx)(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\lambda-1} f(s, x(s))\Delta s, \quad t \in [0,k]. \tag{4.1}$$

From (4.1), we have

$$|Jx - x_0| \le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\lambda-1}s^{-\alpha}\Delta s \le \frac{M\Gamma(1-\alpha)}{\Gamma(\lambda+1-\alpha)}t^{\lambda-\alpha} \le r,$$

for any $x \in C_{rd}([0,k])$, implying that $J\Omega_k \subset \Omega_k$.
We can now show C_{rd} of the operator J. To do so, let $x_n, x \in \Omega_k$ such that $\|x_n - x\|_{C([0,k])} \to 0$ as $n \to \infty$.
Recall that the operator $\mathcal{G}$ is C_{rd} and so, we have that

$$\|\mathcal{G}x_n - \mathcal{G}x\|_{C([0,k])} \to 0 \quad \text{as } n \to \infty.$$

Now

$$\begin{aligned}
\|Jx_n - Jx\| &= \left|\frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1} f(s, x_n(s))\Delta s - \frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1} f(s, x_k(s))\Delta s\right| \\
&\le \frac{1}{\Gamma(\lambda)}\left|\int_0^t (t-s)^{\lambda-1}\left[f(s, x_n(s)) - f(s, x_k(s))\right]\Delta s\right| \\
&\le \frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}\left|f(s, x_n(s)) - f(s, x_k(s))\right|\Delta s \\
&\le \frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}S^{-\alpha}|\mathcal{G}x_n(s) - \mathcal{G}x(s)|\Delta s \\
&\le \frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}S^{-\alpha}\|\mathcal{G}x_n(s) - \mathcal{G}x(s)\|_{[0,k]}\Delta s \\
&\le \frac{\Gamma(1-\lambda)}{\Gamma(\lambda+1-\alpha)}t^{\lambda-\alpha}\|\mathcal{G}x_n(s) - \mathcal{G}x(s)\|_{[0,t]}
\end{aligned}$$

Then

$$\|Jx_n(s) - Jx(s)\|_{[0,t]} \to 0 \quad \text{as } n \to \infty$$

which shows that J is C_{rd}. We now prove the C_{rd} of the $J\Omega_k$. To do this, let $x \in \Omega_k$, and $t_1, t_2 \in [0,k]$, with $t_1 \le t_2$.

Given $\epsilon > 0$,

$$\frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} S^{-\alpha} \Delta s = \frac{\Gamma(1-\alpha)}{\Gamma(\lambda+1-\alpha)} t^{\lambda-\alpha} \to 0 \text{ as } t \to 0^+,$$

where $0 \le \alpha < 1$; then there exist $\beta > 0$, such that

$$\frac{2M}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} s^{-\alpha} \Delta s < \epsilon, \quad t \in [0,k],$$

holds.

Suppose $t_1, t_2 \in [0,\beta] \subset \mathbb{T}$, then

$$\begin{aligned}
&\Big|\frac{1}{\Gamma(\lambda)} \int_0^{t_1} (t_1-s)^{\lambda-1} f(s,x(s)) \Delta s - \frac{1}{\Gamma(\lambda)} \int_0^{t_2} (t_2-s)^{\lambda-1} f(s,x(s)) \Delta s\Big| \\
&\quad \le \frac{M}{\Gamma(\lambda)} \int_0^{t_1} (t_1-s)^{\lambda-1} s^{-\alpha} \Delta s + \frac{M}{\Gamma(\lambda)} \int_0^{t_2} (t_2-s)^{\lambda-1} s^{-\alpha} \Delta s \\
&\quad < \epsilon.
\end{aligned} \tag{4.2}$$

Again, suppose $\frac{\beta}{2} \le t_1 \le t_2 \le k$,

$$\begin{aligned}
|(Jx)(t_1) - (Jx)(t_2)| &= \left|\frac{1}{\Gamma(\lambda)} \int_0^{t_1} (t_1-s)^{\lambda-1} f(s,x(s)) \Delta s - \frac{1}{\Gamma(\lambda)} \int_0^{t_2} (t_2-s)^{\lambda-1} f(s,x(s)) \Delta s\right| \\
&\le \left|\frac{1}{\Gamma(\lambda)} \int_0^{t_1} [(t_1-s)^{\lambda-1} - (t_2-s)^{\lambda-1}] f(s,x(s)) \Delta s\right| \\
&\quad + \left|\frac{1}{\Gamma(\lambda)} \int_{t_1}^{t_2} (t_2-s)^{\lambda-1} f(s,x(s)) \Delta s\right|.
\end{aligned} \tag{4.3}$$

From (4.3), we have

$$\begin{aligned}
&\Big|\frac{1}{\Gamma(\lambda)} \int_0^{t_1} [(t_1-s)^{\lambda-1} - (t_2-s)^{\lambda-1}] f(s,x(s)) \Delta s\Big| \\
&\quad \le \frac{M}{\Gamma(\lambda)} \int_0^{t_1} \Big|[(t_1-s)^{\lambda-1} - (ct_2-s)^{\lambda-1}] s^{-\alpha}\Big| \Delta s \\
&\quad \le \frac{M}{\Gamma(\lambda)} \int_0^{\frac{\beta}{2}} \Big|[(t_1-s)^{\lambda-1} - (t_2-s)^{\lambda-1}]\Big| s^{-\alpha} \Delta s \\
&\quad + \frac{M(\frac{\beta}{2})^{-\alpha}}{\Gamma(\lambda)} \int_{\frac{\beta}{2}}^{t_1} \Big|[(t_1-s)^{\lambda-1} - (t_2-s)^{\lambda-1}]\Big| \Delta s \\
&\quad \le \frac{2M}{\Gamma(\lambda)} \int_0^{\frac{\alpha_1}{2}} (\frac{\sigma}{2} - s)^{\lambda-1} s^{-\alpha} \Delta s \\
&\quad + \frac{N(\frac{\beta}{2})^{-\alpha}}{\Gamma(\lambda)} \Big[(t_2-t_1)^{\lambda} + (t_1 - \frac{\beta}{2})^{\lambda} - (t_2 - \frac{\beta}{2})^{\lambda}\Big]
\end{aligned}$$

$$\leq \epsilon + \frac{N(\frac{\beta}{2})^{-\alpha}}{\Gamma(\lambda)}\left[(t_2-t_1)^{\lambda} + (t_1-\frac{\beta}{2})^{\lambda} - (t_2-\frac{\beta}{2})^{\lambda}\right].$$

Again, from (4.3), we have

$$\begin{aligned}\left|\frac{1}{\Gamma(\lambda)}\int_{t_1}^{t_2}(t_2-s)^{\lambda-1}f(s,x(s))\Delta s\right| &\leq \frac{N(\frac{\alpha_1}{2})^{-\alpha}}{\Gamma(\lambda)}\int_{t_1}^{t_2}(t_2-s)^{\lambda-1}\Delta s \\ &\leq \frac{N(B_2)^{-\alpha}}{\Gamma(\lambda+1)}(t_2-t_1)^{\lambda}. \qquad (4.4)\end{aligned}$$

Clearly, from the fact $\frac{\Gamma(\lambda)}{\Gamma(\lambda+1-\alpha)}t^{\lambda-\alpha} \to 0$ as $t \to 0^+$, then there exist $\beta > 0$, such that $\frac{\beta}{2} \leq t_1 \leq t_2 \leq k$ and $|t_1 - t_2| < \beta$, then

$$|(Jx)(t_1) - (Jx)(t_2)| < 2\epsilon. \qquad (4.5)$$

From (4.2) and (4.5), we have that $\{(Jx)(t) : x \in \Omega_k\}$ is equicontinuous on $\mathbb{T}$. For the fact that $J\Omega_k \subset \Omega_k$, then $\{(Jx)(t) : x \in \Omega_k\}$ is bounded and J is pre-compact. The map J is C_{rd}. By Theorem 1 and Lemma 1, problem (3.1) has a local solution. □

4.2. **Uniqueness of solution of Caputo-type Fractional Dynamic System.**

Theorem 4.2. *Suppose that the axioms (A1) and (A2) are satisfied. Then the problem (3.1) has a unique solution for $x \in C_{rd}([0,k]$, where $k \in [0,T] \subset \mathbb{T}$.*

Proof. From Lemma 2, (3.1) and (3.2) are equivalent; so our aim here is to show that (3.2) has exactly one solution. First, we construct the following set;

$$\Omega_k = \{x : x \in C_{rd}[0,k], \sup_{t\in[0,T]} |x - x_0| \leq \varpi\},$$

which is closed and non-empty subset of the Banach space X.

Adopting the following operator,

$$(Jx)(t) = x_0 + \frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1} f(s,x(s))\Delta s, \quad t \in [0,k].$$

Using the following fixed-point problem

$$x(t) = (Jx)(t).$$

So that if we prove that the fixed point equation has a unique fixed point, then we can infer that (3.2) has a unique fixed solution. We begin by proving that

$$\begin{aligned}\|(Jx)(t) - x_0\| &\leq \frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}|f(s,x(s))|\Delta s \\ &\leq \frac{\|f\|_{C[0,k]}\Gamma(1-\lambda)}{\Gamma(1-\alpha)}\frac{\varpi\Gamma(\lambda+1-\alpha)}{\|f\|_{C[0,k]}\Gamma(1-\lambda)} = \varpi,\end{aligned}$$

for any $x \in \Omega_k$. There exists $Jx \in \Omega_k$ if $x \in \Omega_k$.

Next, for $0 \le t_1 \le t_2 \le k$,

$$\begin{aligned}
|(Jx)(t_1) - (Jx)(t_2)| &= \bigg| \frac{1}{\Gamma(\lambda)} \int_0^{t_1} (t_1 - s)^{\lambda-1} f(s, x(s)) \Delta s \\
&- \frac{1}{\Gamma(\lambda)} \int_0^{t_2} (t_2 - s)^{\lambda-1} f(s, x(s)) \Delta s \bigg| \\
&\le \frac{1}{\Gamma(\lambda)} \int_0^{t_1} |(t_1 - s)^{\lambda-1} - (t_2 - s)^{\lambda-1}| |f(s, x(s))| \Delta s \\
&+ \frac{1}{\Gamma(\lambda)} \int_{t_1}^{t_2} (t_2 - s)^{\lambda-1} |f(s, x(s))| \Delta s \\
&\le \frac{\|f\|_{C[0,k]}}{\Gamma(\lambda)} \int_0^{t_1} [(t_1 - s)^{\lambda-1} - (t_2 - s)^{\lambda-1}] \Delta s \\
&+ \frac{\|f\|_{C[0,k]}}{\Gamma(\lambda)} \int_{t_1}^{t_2} (t_2 - s)^{\lambda-1} \Delta s \\
&= \frac{\|f\|_{C[0,k]}}{\Gamma(1+\lambda)} \left[t_1^{\lambda} - t_1^{\lambda} + (t_2 - t_1)^{\lambda} \right],
\end{aligned}$$

which shows that Jx is C_{rd}.
However,

$$\|J^n x - J^n x^*\|_{C[0,t]} \le \frac{(L\delta^{\lambda})^n}{\Gamma(1+n\lambda)} \|x - x^*\|_{C_{rd}[0,t]}, \tag{4.6}$$

for every $n \in \mathbb{N}$ and $t \in [0,k] \subset \mathbb{T}$.
Suppose $n = 0$, then (4.6) is true.
By induction, (4.6) also holds for $n - 1$ such that

$$\begin{aligned}
\|J^n x - J^n x^*\|_{C_{rd}[0,t]} &= \|J(J^{n-1}x) -_{rd} J(J^{n-1}x^*)\|_{C_{rd}[0,t]} \\
&\le \frac{1}{\Gamma(\lambda)} \max_{0 \in [\delta, t]} \left| \int_0^{\delta} (\delta - s)^{\lambda-1} [f(s, J^{n-1}x(s)) - f(s, J^{n-1}x^*(s))] \Delta s \right|.
\end{aligned}$$

By Lipschitz condition, (4.6) suffices. Now,

$$\sum_{n=0}^{\infty} \frac{(L\delta^{\lambda})^n}{\Gamma(1+n\lambda)} = E_{\lambda}(L\delta^{\lambda}).$$

where $E_{\lambda}(\cdot)$ represents the Mittag-Leffler function. Therefore, by applying Lemma 3, the proof is complete. □

4.3. **Stability.** We consider the generalized Ulam-Hyers-Rassias stability of the Caputo-type fractional dynamic equation 3.1. Given $r > 0$ and a continuous function $\mathfrak{Q} : [0, T] \subset T \times \mathbb{R} \to \mathbb{R}$. Considering the following inequalities

$$\|^{C}\Delta^{\lambda} x(t) - f(t, x)\| \le r \qquad t \in [0, T] \tag{4.7}$$

$$\|^{C}\Delta^{\lambda} x(t) - f(t, x)\| \le r\mathfrak{Q}(t) \qquad t \in [0, T] \tag{4.8}$$

$$\|{}^{C}\Delta^{\lambda x(t)} - f(t,x)\| \leq \mathfrak{Q}(t) \qquad t \in [0,T]. \tag{4.9}$$

Definition 4.1. *Equation 3.1 is Ulam-Hyers stable if there exists a positive real number ϕ_f such that for each $r > 0$ and for each solution $x^* \in C_{rd}[0,T]$, of the inequality 4.7, there exists a unique solution $x \in C_{rd}[0,T]$ of equation 3.1 such that*

$$\|x^*(t) - x(t)\| \leq \phi_f r. \tag{4.10}$$

Definition 4.2. *Equation 3.1 is generalized Ulam-Hyers stable if there exists a function $\psi_f \in C_{rd}[0,T]$, $\psi_f(0) = 0$ such that for each $x^* \in C_{rd}[0,T]$, of the inequality 4.7, there exists a unique solution $x \in C_{rd}[0,T]$ of equation 3.1 such that*

$$\|x^*(t) - x(t)\| \leq \psi_f r. \tag{4.11}$$

Definition 4.3. *Equation 3.1 is Ulam-Hyers-Rassias stable with respect to $\mathfrak{Q}$ if there exists a positive real number $\phi_{f,\mathfrak{Q}}$ such that for each $r > 0$ and for each solution $x^* \in C_{rd}[0,T]$, of the inequality 4.8, there exists a unique solution $x \in C_{rd}[0,T]$ of equation 3.1 such that*

$$\|x^*(t) - x(t)\| \leq \phi_{f,\mathfrak{Q}} r \mathfrak{Q}(t). \tag{4.12}$$

Definition 4.4. *Equation 3.1 is generalized Ulam-Hyers-Rassias stable with respect to $\mathfrak{Q}$ if there exists a positive real number $\phi_{f,\mathfrak{Q}}$ such that for each solution $x^* \in C_{rd}[0,T]$, of the inequality 4.9, there exists a unique solution $x \in C_{rd}[0,T]$ of equation 3.1 such that*

$$\|x^*(t) - x(t)\| \leq \phi_{f,\mathfrak{Q}} \mathfrak{Q}(t). \tag{4.13}$$

Remark 4.1. *We said that the function $x^* \in C_{rd}[0,T]$ is a solution of the inequality 4.7 if and only if it satisfied the following*

(i) $|p(t)| \leq r$
(ii) ${}^{C}\Delta^{\lambda} x(t) = f(t,x) + p(t)$.

where the function $p \in C_{rd}[0,T]$.

Inequalities 4.8 and 4.9 have similar remarks

Remark 4.2. *Let $0 < \lambda < 1$, if $x^* \in C_{rd}[0,T]$ is a solution of the inequality 4.7, then x^* is a solution of the following integral inequality*

$$|x^*(t) - x^*(0) - \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s, x^*(s))\, \Delta s| \leq \frac{rT^\lambda}{\Gamma(\lambda+1)}.$$

From remark 4.1, we have

$${}^{C}\Delta^{\lambda} x(t) = f(t,x) + p(t).$$

Then

$$x^*(t) = x^*(0) + \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s, x^*(s))\, \Delta s + \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} p(s)\, \Delta s.$$

From this we have that

$$|x^*(t) - x^*(0) - \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s, x^*(s))\, \Delta s| = |\frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} p(s)\, \Delta s|$$

$$\leq \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} |p(s)|\, \Delta s = \frac{rT^\lambda}{\Gamma(\lambda+1)}.$$

The solutions of inequalities 4.8 and 4.9 have similar remarks.

Theorem 4.3. *Assume $(A_1) - (A_2)$ hold. Suppose there exist $\beta_{\mathfrak{Q}} > 0$ such that*

$$\frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} \mathfrak{Q}(s)\, \Delta s \leq \beta \mathfrak{Q}(t),$$

for each $t \in [0, T]$ where $\mathfrak{Q} \in C_{rd}[0, T] \times \mathbb{R}$ is a non-decreasing function. Then equation 3.1 is said to be generalized Ulam-Hyers-Rassias stable.

Proof. Let $x* \in C_{rd}[0, T] \times \mathbb{R}$ be a solution of the inequality 4.9. Let $x \in C_{rd}[0, T] \times \mathbb{R}$ be a unique solution of fractional dynamic equation 3.1.
By the differential inequality 4.9, we have

$$|x^*(t) - x^*(0) - \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s, x^*(s))\, \Delta s| = |\frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} p(s)\, \Delta s|$$

$$\leq \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} s{-}\alpha |\mathfrak{Q}(s)|\, \Delta s = \frac{T^\lambda}{\Gamma(\lambda+1)} \mathfrak{Q}(t) = \beta \mathfrak{Q}(t)$$

$$||x^*(t) - x(t)| = |x^*(0) + \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s, x^*(s))\, \Delta s + \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} p(s)\, \Delta s|$$

$$-|x^*(0) - \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s, x(s))\, \Delta s|$$

$$\leq |\frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} p(s)\, \Delta s| + |\frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} [f(s, x^*(s)) - f(s, x(s))]\, \Delta s|$$

$$\leq \beta \mathfrak{Q}(t) + |\frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} |x^*(s)) - x(s)|\, \Delta s\Big|$$

$$||x - x^*||_{C[0,t]} \leq \beta \mathfrak{Q}(t) + \frac{T^\lambda}{\Gamma(\lambda+1)} ||x - x^*||_{C_{rd}[0,t]}$$

$$||x - x^*||_{C[0,t]} \leq \frac{1}{1 - \frac{T^\lambda}{\Gamma(\lambda+1)}} \beta \mathfrak{Q}(t).$$

Therefore equation 3.1 is said to be generalized Ulam-Hyers-Rassias stable. □

5. Application

Fractional SIS model

Consider the Caputo fractional dynamic system (SIS model) of order $\lambda \in (0,1]$:

$$^{C}\Delta^{\lambda} I(t) = f(t, I(t)), \qquad t \in [0,T] \subseteq \mathbb{T}, \tag{5.1}$$

with initial condition $I(0) = I_0 \geq 0$.

Where
N represents the total population which is constant, $I(t)$ is the infected population,

$$f(t,I) = \beta(t)\,\frac{(N-I)I}{N} - \gamma\, I, \tag{5.2}$$

and $\beta(t)$ is a continuous transmission rate and $\gamma > 0$ is the recovery rate. The susceptible population is given by $S(t) = N - I(t)$ (no births/deaths), and the infection transmission is proportional to SI/N, and per-capita recovery γ.
The fractional derivative order $\lambda < 1$ models memory in infection dynamics.

Taking the partial derivative of (5.2) with respect to I, we obtain:

$$\frac{\partial f}{\partial I}(t,I) = \beta(t)\frac{N-2I}{N} - \gamma.$$

Let $I \in [0,N]$ and assume $\beta(t)$ is continuous and bounded on $[0,T] \subseteq \mathbb{T}$ with $\beta(t) \leq \beta_{\max}$ for $t \in [0,T] \subseteq \mathbb{T}$, then

$$\left|\frac{\partial f}{\partial I}(t,I)\right| \leq \beta_{\max} + \gamma := L.$$

For any $I_1, I_2 \in [0,N]$,

$$|f(t,I_1) - f(t,I_2)| \leq L\,|I_1 - I_2|.$$

with Lipschitz constant $L = \beta_{\max} + \gamma > 0$. Implying that axiom $(A1)$ is satisfied.

From axiom $(A2)$, we deduce that if f is rd-continuous on $[0,T] \times [0,N]$, then $t^{\alpha} f(t,x)$ is continuous and bounded (because t^{α} is bounded on $[0,T]$). So we can simply pick any $\alpha \in (0,1]$, like $\alpha = \frac{1}{2}$, and axiom (A2) holds for (5.1).

Since axioms (A1) and (A2) hold on $[0,T] \subseteq \mathbb{T}$ for system (5.1), then there exists at least one function $I \in C_{rd}[0,k]$ (for any $k \in (0,T]$) that satisfies (5.1) which is the solution function.

Also, by Lemma 2, any solution of (5.1), will also satisfy

$$I(t) = I_0 + \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s, I(s))\, \Delta s. \tag{5.3}$$

Next, we show that this solution is unique. Suppose I_1 and I_2 are two solutions with initial conditions $I_1(0) = I_2(0) = I_0$, then

$$I_1(t) - I_2(t) \quad = \quad \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} \Big[f(s, I_1(s)) - f(s, I_2(s))\Big]\, \Delta s$$

$$\leq \quad \frac{L}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} |e(s)| \, \Delta s.$$

where $|f(t, I(t))| \leq e(t)$, $|e(t)| \leq 0 \cdot E_\lambda(Lt^\lambda) = 0$ for $t \in [0, T] \subseteq \mathbb{T}$, $e(0) = 0$ and E_λ is the one-parameter Mittag–Leffler function. Therefore, $e(t) \equiv 0$ and the solution $I(t)$ of system (5.1) is unique.

To show that the solution is also generalized Ulam–Hyers–Rassias stable, let $y(t)$ be an approximate solution satisfying

$$\left| {}^C\Delta^\lambda y(t) - f(t, y(t)) \right| \leq Q(t),$$

for a non-decreasing $\mathfrak{Q} = Q(t) \in C_{rd}[0, T]$. If $x(t)$ is the exact solution, then from (5.3), we compute $d(t) = y(t) - x(t)$ as,

$$\begin{aligned} d(t) =& \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} \Big[f(s, y(s)) - f(s, x(s)) \Big] \, \Delta s \\ &+ \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} r(s) \, \Delta s, \\ |d(t)| \leq & \frac{L}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} |d(s)| \, ds \ + \ \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} Q(s) \, \Delta s. \end{aligned}$$

with $|r(s)| \leq Q(s)$.

By hypothesis of Theorem (4.3), there is $\beta_{\mathfrak{Q}} > 0$ so that

$$\frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} Q(s) \, ds \leq \beta_{\mathfrak{Q}} \, Q(t).$$

Thus,

$$|d(t)| \leq \frac{L}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} |d(s)| \, \Delta s + \beta_{\mathfrak{Q}} \, Q(t).$$

Applying the well-known result that if $u(t) \leq a(t) + b * u(t)$ then $u(t) \leq a(t) E_\lambda(bt^\lambda))$ we obtain.

$$|d(t)| \leq \beta_{\mathfrak{Q}} \, Q(t) \, E_\lambda(Lt^\lambda) \quad \text{for } t \in [0, T] \subseteq \mathbb{T}.$$

Since Q is non-decreasing, $E_\lambda(Lt^\lambda)$ is bounded on $[0, T] \subseteq \mathbb{T}$. Thus there is a constant $C > 0$ (take $C = \beta_{\mathfrak{Q}} \max_{t \in [0,T]} E_\lambda(Lt^\lambda)$) such that

$$|y(t) - x(t)| \leq C \, Q(t),$$

verifying the generalized Ulam–Hyers–Rassias stability of (5.1).

We can obtain the numerical simulation using simple integrator (Volterra discretization) method and the result is shown below:

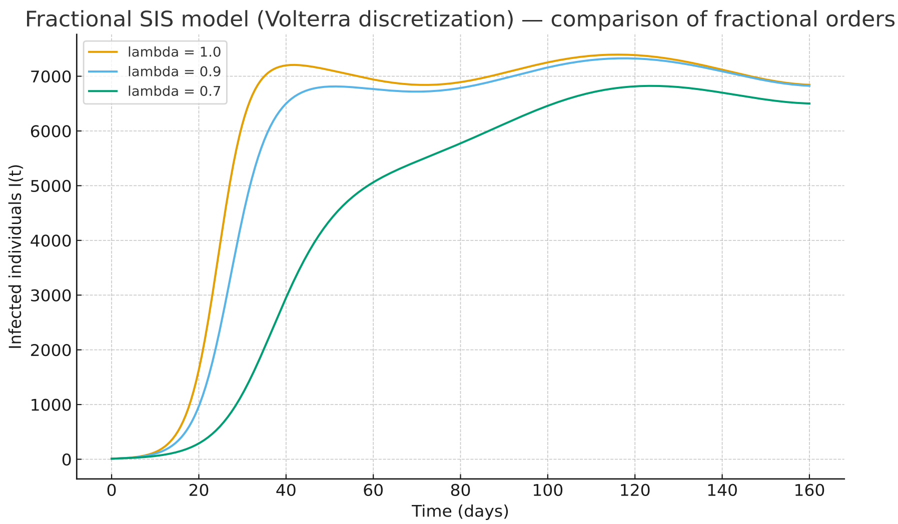

Figure 1. Numerical simulation of the fractional SIS epidemic model using the Volterra discretization scheme. The trajectories of infected individuals $I(t)$ are shown for three different fractional orders, $\lambda = 1.0, 0.9,)$ *and* $(0.7$, with parameters $N = 10{,}000$, $\beta(t) = 0.35, (1 + 0.1 \sin(2\pi t/90))$, $\gamma = 0.1$, and $I(0) = 10$. The integer-order case $(\lambda = 1)$ exhibits the fastest growth and highest epidemic peak, while fractional orders $(\lambda < 1)$ slow the infection dynamics, lower and delay the peak, and extend the transient phase, illustrating the memory effect inherent in fractional-order models.

The figure 1 above overlays trajectories of the infected population $I(t)$ for $\lambda = 1.0, , 0.9, 0.7$ using the Volterra discretization.

The $\lambda = 1$ curve (integer order ODE case) rises fastest and reaches the highest early peak. That matches the intuition: without memory, the rate of change depends only on the present state and responds immediately to the high transmission term $\beta(t)$, SI/N. As λ decreases to 0.9 and 0.7 the epidemic grows more slowly and the peak is delayed and lower. The curve for $\lambda = 0.7$ is the slowest to rise and attains the smallest peak over the plotted window. Also, the lower the order, the longer the transient approach toward a quasi-stationary infected level. This indicates that memory (nonlocality) spreads the influence of past infections across time, new infections are damped by the history of the system.

Realistically, a fractional derivative of order $\lambda < 1$ encodes temporal memory or non-exponential waiting times for transitions. Epidemiologically, this may represent behaviors or biological processes that cause the infection process at time t to depend on the whole past (like persistent cautious behavior, immune waning on long timescales, or non-Markovian contact patterns). The numerical results show that such memory tends to slow the rise of infections and flatten/delay the epidemic peak compared to the classical $(\lambda = 1)$ model.

Furthermore, memory reduces the instantaneous rate of increase, so peak prevalence decreases; however the infected population can remain elevated for longer (longer tail/transient), which is important for control decisions, short but sharp epidemics (high λ) versus drawn-out moderate epidemics (low λ) require different intervention timing. If the process has strong memory, interventions applied early may have delayed observed effects (because past infection history continues to influence dynamics), so evaluation windows must account for that lag. Conversely, for classical dynamics ($\lambda \approx 1$), interventions can show quicker results.

Fractional SIR model

Consider the Caputo fractional SIR system of order $0 < \lambda < 1$:

$$\begin{aligned} {}^{C}\Delta^{\lambda} S(t) &= -\beta(t)\frac{S(t), I(t)}{N}, \\ {}^{C}\Delta^{\lambda} I(t) &= \beta(t)\frac{S(t), I(t)}{N} - \gamma, I(t), \\ {}^{C}\Delta^{\lambda} R(t) &= \gamma, I(t), \end{aligned} \tag{5.4}$$

$t \in [0, T]$, with initial data

$$S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad R(0) = R_0 \geq 0, \quad \text{where} \quad S_0 + I_0 + R_0 = N.$$

Also, $\mathbf{x}(t) = (S(t), I(t), R(t))^{\top} \in \mathbb{R}^3$ and we define the vector field

$$\mathbf{F}(t, \mathbf{x}) = \begin{pmatrix} -\beta(t)\dfrac{S, I}{N} \\ \beta(t)\dfrac{S, I}{N} - \gamma I \\ \gamma I \end{pmatrix}, \qquad \mathbf{x} = (S, I, R).$$

Assuming $\beta(\cdot) \in C_{rd}([0, T])$ such that $\beta_{\max} := \sup_{t \in [0,T]} \beta(t) < \infty$ and $\gamma > 0$ and $N > 0$ are constants.

Then from Lemma 2, the system (5.4) is equivalent to the Volterra Delta integral system

$$\mathbf{x}(t) = \mathbf{x}_0 + \frac{1}{\Gamma(\lambda)} \int_0^t (t - s)^{\lambda - 1} \mathbf{F}(s, \mathbf{x}(s)) \Delta s, \qquad \mathbf{x}_0 = (S_0, I_0, R_0). \tag{5.5}$$

Computing the Jacobian $J(x) = D_{\mathbf{x}}\mathbf{F}(t, \mathbf{x}) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}$ we obtain:

$F_1(S, I, R; t) = -\beta(t)\dfrac{SI}{N}$.

$$\frac{\partial F_1}{\partial S} = -\beta(t)\frac{I}{N}, \qquad \frac{\partial F_1}{\partial I} = -\beta(t)\frac{S}{N}, \qquad \frac{\partial F_1}{\partial R} = 0.$$

$F_2(S, I, R; t) = \beta(t)\dfrac{SI}{N} - \gamma I$.

$$\frac{\partial F_2}{\partial S} = \beta(t)\frac{I}{N}, \qquad \frac{\partial F_2}{\partial I} = \beta(t)\frac{S}{N} - \gamma, \qquad \frac{\partial F_2}{\partial R} = 0.$$

$F_3(S, I, R; t) = \gamma I$.

$$\frac{\partial F_3}{\partial S} = 0, \qquad \frac{\partial F_3}{\partial I} = \gamma, \qquad \frac{\partial F_3}{\partial R} = 0.$$

Thus, the Jacobian becomes

$$J(t,\mathbf{x}) = D_{\mathbf{x}}\mathbf{F}(t,\mathbf{x}) = \begin{pmatrix} -, \dfrac{\beta(t)I}{N} & -, \dfrac{\beta(t)S}{N} & 0 \\ \dfrac{\beta(t)I}{N} & \dfrac{\beta(t)S}{N} - \gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}. \tag{5.6}$$

Let

$$\mathcal{D} := \{\mathbf{x} = (S,I,R) \in \mathbb{R}^3 : \ 0 \le S,I,R \le N, \ S+I+R = N\}.$$

This implies that; $0 \le S \le N, \ 0 \le I \le N$. Using the uniform bound $\beta(t) \le \beta_{\max}$, we can compute the absolute row-sums (infinity norm bound) of (5.6) as follow:
For the first row:

$$|J_{11}| + |J_{12}| + |J_{13}| = \frac{\beta(t)I}{N} + \frac{\beta(t)S}{N} + 0 = \beta(t)\frac{S+I}{N} \le \beta(t)\frac{N}{N} \le \beta_{\max}.$$

For the second row:

$$|J_{21}| + |J_{22}| + |J_{23}| = \frac{\beta(t)I}{N} + \Big|\frac{\beta(t)S}{N} - \gamma\Big| + 0 \le \frac{\beta(t)I}{N} + \frac{\beta(t)S}{N} + \gamma = \beta(t)\frac{S+I}{N} + \gamma \le \beta_{\max} + \gamma.$$

For the third row:

$$|J_{31}| + |J_{32}| + |J_{33}| = 0 + \gamma + 0 = \gamma.$$

Clearly, we deduce that

$$\|J(t,\mathbf{x})\|_\infty \ = \ \max_i \sum_j |J_{ij}| \ \le \ \max\{\beta_{\max}, \ \beta_{\max} + \gamma, \ \gamma\} \ = \ \beta_{\max} + \gamma.$$

By the mean-value theorem for vector-valued maps we have that,

$$\mathbf{F}(t,\mathbf{x}) - \mathbf{F}(t,\mathbf{y}) = \Big(\int_0^1 J\big(t, \mathbf{y} + \theta(\mathbf{x}-\mathbf{y})\big)\,\Delta\theta\Big)(\mathbf{x}-\mathbf{y}),$$

so taking the sup (∞) norm,

$$\|\mathbf{F}(t,\mathbf{x}) - \mathbf{F}(t,\mathbf{y})\|_\infty \le (\beta_{\max} + \gamma)\,\|\mathbf{x}-\mathbf{y}\|_\infty.$$

So that the vector Lipschitz constant is given by

$$L_v := \beta_{\max} + \gamma.$$

Now, we prove local existence and uniqueness.

Define the operator $\mathcal{T}$ on $X = C([0,k];\mathbb{R}^3)$ equipped with the sup(∞) norm $\|x\|_\infty = \sup_{t\in[0,k]} \|x(t)\|_\infty$ to be:

$$(\mathcal{T}x)(t) := \mathbf{x}_0 + \frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}\mathbf{F}(s,x(s))\,\Delta s.$$

Let M_F be a bound on $\|\mathbf{F}(t,\mathbf{x})\|_\infty$ for $(t,\mathbf{x}) \in [0,k] \times \mathcal{D}_R$ where $\mathcal{D}_R$ is a ball of radius R. We define a simple global bound on $\mathbf{F}$ on $\mathcal{D}$ as:

$$\|\mathbf{F}(t,\mathbf{x})\|_\infty \le N(\beta_{\max} + \gamma) \quad \text{for } \mathbf{x} \in \mathcal{D},$$

since $\frac{SI}{N} \le N$ and $I \le N$, so that $M_F := N(\beta_{\max} + \gamma)$.

Choose $R > 0$ such that the closed ball $B_R = \{x \in X : \|x - \mathbf{x}_0\|_\infty \le R\}$ is contained in functions taking values in a region where $\mathbf{F}$ is bounded by M_F. For any $x \in B_R$,

$$\|(\mathcal{T}x)(t) - \mathbf{x}_0\|_\infty \le \frac{1}{\Gamma(\lambda)} \sup_{t\in[0,k]} \int_0^t (t-s)^{\lambda-1} \|\mathbf{F}(s,x(s))\|_\infty \Delta s \le \frac{M_F}{\Gamma(\lambda+1)} k^\lambda.$$

$\implies \|\mathcal{T}x - \mathbf{x}_0\|_\infty \le \dfrac{M_F k^\lambda}{\Gamma(\lambda+1)}$.

Choose k small enough that

$$\frac{M_F k^\lambda}{\Gamma(\lambda+1)} \le R,$$

then $\mathcal{T}(B_R) \subset B_R$.

For $x, y \in B_R$, using the Lipschitz constant L_v we have that,

$$\begin{aligned} \|\mathcal{T}x - \mathcal{T}y\|_\infty &\le \frac{1}{\Gamma(\lambda)} \sup_{t\in[0,k]} \int_0^t (t-s)^{\lambda-1} \|\mathbf{F}(s,x(s)) - \mathbf{F}(s,y(s))\|_\infty \Delta s \\ &\le \frac{L_v}{\Gamma(\lambda)} \left(\sup_{t\in[0,k]} \int_0^t (t-s)^{\lambda-1}\, \Delta s \right) \|x-y\|_\infty. \end{aligned}$$

Since $\int_0^t (t-s)^{\lambda-1} \Delta s = t^\lambda/\lambda$, and $\Gamma(\lambda)\lambda = \Gamma(\lambda+1)$, then

$$\|\mathcal{T}x - \mathcal{T}y\|_\infty \le \frac{L_v\, k^\lambda}{\Gamma(\lambda+1)}\, \|x-y\|_\infty.$$

Let $q := \frac{L_v\, k^\lambda}{\Gamma(\lambda+1)}$. If we choose $k > 0$ so that $q < 1$, then $\mathcal{T}$ is a contraction on B_R. Then by Lemma 3, $\mathcal{T}$ has a unique fixed point $x \in B_R$ (a unique solution) on $[0,k]$ where k satisfies $L_v k^\lambda/\Gamma(\lambda+1) < 1$.

Without loss of generality, this can be further extended to a global interval $[0,T]$. Solving on $[0,k]$ as above; the unique solution $x(t)$ belongs to $C([0,k];\mathbb{R}^3)$ and its values stay in a compact set (bounded by N). In particular the solution is bounded on $[0,k]$.

Now, Taking a new initial data $x(k)$ at time $t = k$ and repeating the contraction argument on the interval $[k,2k]$ using the same constant L_v (since the field is Lipschitz on the compact domain, the same choice of k) gives existence and uniqueness on $[k,2k]$. Next, cover $[0,T]$ by m subintervals of length $\le k$, with m finite. By piecing together the unique local solutions we obtain a unique global solution on the whole interval $[0,T]$.

Clearly, no blowup occurs since the population is bounded; the structure of the SIR model preserves nonnegativity and the total population $S + I + R = N$, so solution remains in the compact physical domain.

Numerically simulating this Caputo fractional SIR model using the Diethelm's predictor–corrector method, we get the following results as shown in figures 2 and 3 below:

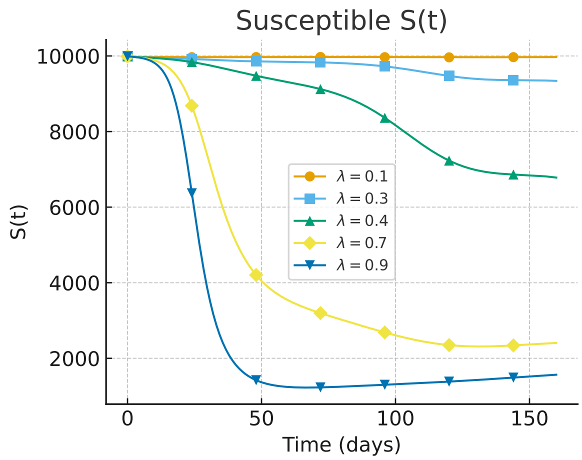

Figure 2. Dynamics of the Susceptible population $S(t)$ in the fractional SIR model for different values of $\lambda = 0.1, 0.3, 0.4, 0.7, 0.9$. The trajectories show how memory effects alter the rate at which susceptible individuals decline over time. Lower λ values preserve more memory, leading to slower depletion of the susceptible pool, while higher λ values accelerate the decline.

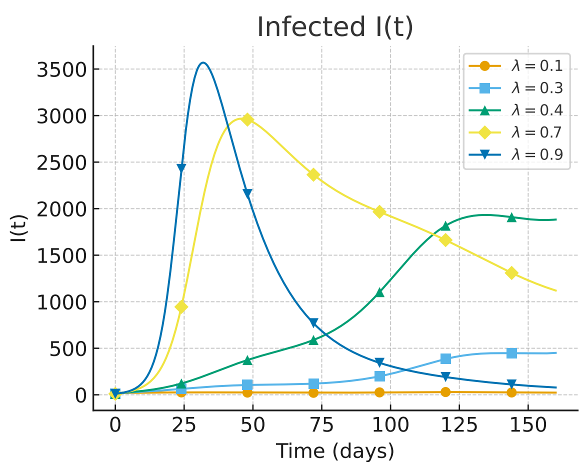

Figure 3. Dynamics of the Infected population $I(t)$ in the fractional SIR model for different values of $\lambda = 0.1, 0.3, 0.4, 0.7, 0.9$. The results show that as λ increases, the infection peaks faster and at higher levels, reflecting reduced memory of past states. Lower λ values delay and suppress the infection peak, highlighting the role of fractional dynamics in controlling outbreak intensity.

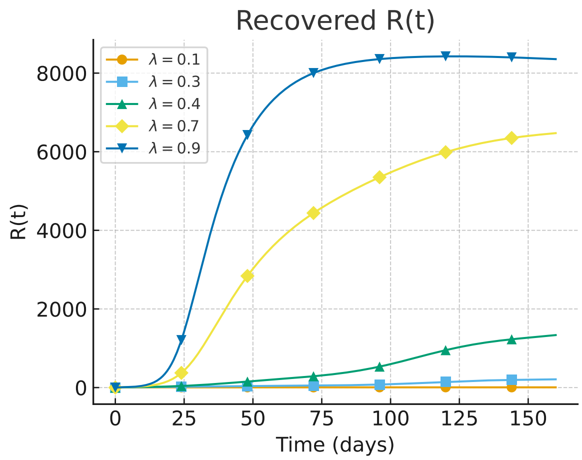

Figure 4. Dynamics of the Recovered population $R(t)$ in the fractional SIR model for different values of $\lambda = 0.1, 0.3, 0.4, 0.7, 0.9$. Higher λ values lead to faster accumulation of recovered individuals, corresponding to quicker epidemic progression. Lower λ values slow down recovery, reflecting the impact of memory and long-term dependence in the fractional framework.

The Caputo fractional SIR system introduces a memory effect controlled by the fractional order λ. Unlike the classical model, where transitions depend only on present states, this model incorporate the system's history, making past interactions relevant to the current dynamics.

In figure 2, the Susceptible population (S), lower order λ imply strong memory effects. This slows down the rate at which susceptible individuals move into the infected class because past states weigh heavily on current dynamics. Conversely, at higher λ, the system behaves closer to the classical case, with susceptibles declining quickly due to weaker memory effects.

In figure 3, the Infected population (I), higher order produce sharper and earlier infection peaks, indicating rapid transmission with minimal delay from memory effects. For lower order, infections spread more slowly, and the peak is flattened. This demonstrates the ability of our fractional dynamics to delay and mitigate outbreaks.

Finally, in figure 4, for the Recovered population (R), increasing λ speeds up recovery, since infections resolve more quickly and accumulate in the recovered class. Lower order delay recovery, reflecting the persistence of memory, which prolongs the epidemic tail.

In summary, the fractional order λ controls the pace and intensity of the epidemic, that is Low order slows epidemic, delayed peak, longer duration while high fractional order means fast epidemic, sharp peak, shorter duration.

6. Conclusion

In this work, we established sufficient conditions for the existence, uniqueness, and generalized Ulam-Hyers–Rassias stability of solutions of Caputo fractional delta-differential equations on time scales using the Schauder fixed-point theorem, thereby extending stability analysis to hybrid systems with both continuous and discrete dynamics. The application to SIS and SIR epidemic models revealed the critical role of the fractional order λ in shaping transmission dynamics: higher orders ($\lambda \to 1$) produce sharper and earlier peaks characteristic of classical epidemic models, while lower orders ($\lambda < 1$) incorporate strong memory effects, slowing infection spread, flattening peaks, and prolonging epidemic tails. This demonstrates how fractional time-scale models can capture the persistence of past interactions and non-local influences, offering a more realistic framework for analyzing and controlling epidemics. Beyond their theoretical contribution, these results highlight how memory-dependent hybrid models can guide intervention strategies by distinguishing between short, sharp outbreaks and prolonged, moderate epidemics, thereby enriching both the mathematical theory and its epidemiological applications.

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