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A Novel Concept of Bipolar Fuzzy Sets in *Bd*-Algebras

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Abstract. In this paper, we introduce the concept of bipolar fuzzy *Bd*-subalgebras (BFBd-Subs) within the framework of *Bd*-algebras and examine their fundamental properties. We also provide a thorough analysis of the characterizations of BFBd-Subs, focusing on their structural behavior as revealed by level subsets. Additionally, the study investigates the algebraic properties of BFBd-Subs in relation to homomorphisms of *Bd*-algebras, identifying the conditions under which these structures are preserved.

1. Introduction

The concept of fuzzy sets, proposed by Zadeh [19] in 1965, has become one of the most significant mathematical tools to cope with uncertainty and imprecision in real-world problems. In contrast to the classical set theory, where an element has crisp membership values of 0 or 1, the fuzzy set theory has a continuum of membership degrees between 0 and 1, which makes it more flexible and realistic for gradual membership/non-membership transitions. Since that time, fuzzy set theory has been widely employed in many scientific fields such as control, decision-making, and algebraic structures. Many academics have studied fuzzy subalgebras, fuzzy ideals, and associated structures in various algebraic systems, such as BCK/BCI-algebras, B-algebras, and d-algebras [3,6,7,15].

Thereafter, the concept of bipolar fuzzy sets (BFSs) [20] is an important generalization of fuzzy sets. The BFSs provide a single framework to model both the positive and negative sides of information. A BFS is defined by assigning a positive membership degree in the interval $[0,1]$ that shows how much an element has a certain attribute. Additionally, it assigns a negative membership degree in the range of $[-1,0]$, which indicates how much the element satisfies a

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counter-property. In an algebraic setting, BFSs have been used to analyze subalgebras and ideals in *BCK*/*BCI*-algebras and related structures and obtained rich theoretical conclusions, see [4,16].

The concept of *Bd*-algebras was presented by Bantaojai et al. [1] as a new class of algebraic structures which satisfy certain axiomatic characteristics of *B*-algebras [14] and *d*-algebras [13]. These algebras have been intensively studied in recent years because of their fascinating structural qualities and their relations with other algebraic structures. Since then, several important notions have been studied in the framework of *Bd*-algebras such as fuzzy *Bd*-subalgebras [12], fuzzy *Bd*-ideals [11], prime fuzzy *Bd*-ideals [2], strongest fuzzy dot *Bd*-subalgebras [8], hybrid subalgebras [18], and derivations [10]. These works have together enhanced our understanding of *Bd*-algebras from classical and fuzzy-theoretic perspectives and have set a solid basis for ongoing study of more general algebraic structures in this setting.

Motivated by the theoretical richness of BFSs and the growing body of research on *Bd*-algebras, the present paper introduces a novel concept of BFBd-Subs within the framework of *Bd*-algebras. We systematically investigate the fundamental properties of these structures and provide a thorough characterization of BFBd-Subs through level subsets, demonstrating that each level subset of a BFBd-Sub is itself a *Bd*-subalgebra, and conversely. Furthermore, we examine the behavior of BFBd-Subs with respect to homomorphisms of *Bd*-algebras, establishing conditions under which the bipolar fuzzy subalgebra property is preserved under homomorphic mappings and their pullbacks.

2. Preliminaries

This section reviews some fundamental definitions used throughout this article.

Definition 2.1. *[20] Let X be a nonempty set. A bipolar-valued fuzzy set μ in X is defined as an object of the form:*

$$\mu = \left\{\left(x, \mu^+(x), \mu^-(x)\right) \mid x \in X\right\}$$

where $\mu^+ : X \to [0,1]$ and $\mu^- : X \to [-1,0]$ are mappings. The positive membership degree $\mu^+(x)$ characterizes the satisfaction degree of an element x to the property associated with μ, while the negative membership degree $\mu^-(x)$ represents the satisfaction degree of x to an implicit counter-property. To simplify the notation, we represent the bipolar-valued fuzzy set $\mu = \{(x, \mu^+(x), \mu^-(x)) \mid x \in X\}$ as $\mu = (\mu^+, \mu^-)$, and replace the concept of bipolar-valued fuzzy sets with the concept of bipolar fuzzy sets (BFSs).

Definition 2.2. *[5] Let $\mu = (\mu^+, \mu^-)$ and $\lambda = (\lambda^+, \lambda^-)$ be two BFSs on a nonempty set X.*

(i) *The complement of μ, denoted by $\overline{\mu}$, is defined as:*

$$\overline{\mu} = \left\{\left(x, \overline{\mu^+}(x), \overline{\mu^-}(x)\right) \mid x \in X\right\}$$

where $\overline{\mu^+}(x) = 1 - \mu^+(x)$ and $\overline{\mu^-}(x) = -1 - \mu^-(x)$ for all $x \in X$.

(ii) *The intersection of μ and λ, denoted by $\mu \cap \lambda$, is defined as:*

$$\mu \cap \lambda = \left\{\left(x, \left(\mu^+ \cap \lambda^+\right)(x), \left(\mu^- \cup \lambda^-\right)(x)\right) \mid x \in X\right\}$$

where $(\mu^+ \cap \lambda^+)(x) = \min\{\mu^+(x), \lambda^+(x)\}$ *and* $(\mu^- \cup \lambda^-)(x) = \max\{\mu^-(x), \lambda^-(x)\}$ *for all* $x \in X$*.*

(iii) *The union of* μ *and* λ*, denoted by* $\mu \cup \lambda$*, is defined as:*

$$\mu \cup \lambda = \left\{\left(x, \left(\mu^+ \cup \lambda^+\right)(x), (\mu^- \cap \lambda^-)(x)\right) \mid x \in X\right\}$$

where $(\mu^+ \cup \lambda^+)(x) = \max\{\mu^+(x), \lambda^+(x)\}$ *and* $(\mu^- \cap \lambda^-)(x) = \min\{\mu^-(x), \lambda^-(x)\}$ *for all* $x \in X$*.*

(iv) μ *contained in* λ*, denoted by* $\mu \subseteq \lambda$*, is defined as:*

$$\mu \subseteq \lambda \text{ if and only if } \mu^+(x) \leq \lambda^+(x) \text{ and } \mu^-(x) \geq \lambda^-(x) \text{ for all } x \in X.$$

Definition 2.3. *[17] Let* X *be a nonempty set and* $\mu = (\mu^+, \mu^-)$ *be a BFS on* X*. For any* $(t, s) \in [0, 1] \times [-1, 0]$*, we have:*

(i) *the set* $P_L(\mu^+, t) = \{x \in X \mid \mu^+(x) \leq t\}$ *is called the positive lower t-cut of* μ*;*
(ii) *the set* $P_U(\mu^+, t) = \{x \in X \mid \mu^+(x) \geq t\}$ *is called the positive upper t-cut of* μ*;*
(iii) *the set* $N_L(\mu^-, s) = \{x \in X \mid \mu^-(x) \leq s\}$ *is called the negative lower s-cut of* μ*;*
(iv) *the set* $N_U(\mu^-, s) = \{x \in X \mid \mu^-(x) \geq s\}$ *is called the negative upper s-cut of* μ*;*
(v) *the set* $P^*_L(\mu^+, t) = \{x \in X \mid \mu^+(x) < t\}$ *is called the strong positive lower t-cut of* μ*;*
(vi) *the set* $P^*_U(\mu^+, t) = \{x \in X \mid \mu^+(x) > t\}$ *is called the strong positive upper t-cut of* μ*;*
(vii) *the set* $N^*_L(\mu^-, s) = \{x \in X \mid \mu^-(x) < s\}$ *is called the strong negative lower s-cut of* μ*;*
(viii) *the set* $N^*_U(\mu^-, s) = \{x \in X \mid \mu^-(x) > s\}$ *is called the strong negative upper s-cut of* μ*;*

Definition 2.4. *[1] An algebra* $(X, *, 0)$ *is referred to as a Bd-algebra if it satisfies the following conditions for all* $x, y \in X$*: (i)* $x * 0 = x$ *and (ii)* $x * y = 0$ *and* $y * x = 0$ *implies* $x = y$*.*

For convenience, we shall simplify the notation of a *Bd*-algebra $(X, *, 0)$ by denoting it with the symbol X.

Definition 2.5. *[1] Let* X *be a Bd-algebra. A nonempty subset* A *of* X *is defined as a Bd-subalgebra (Bd-Sub) of* X *provided that* $0 \in A$ *and* $x * y \in A$ *for any* $x, y \in A$*.*

3. Bipolar fuzzy *Bd*-subalgebras

In this section, we present the concept of bipolar fuzzy *Bd*-subalgebras within the framework of *Bd*-algebras and explore specific properties associated with them.

Definition 3.1. *A BFS* $\mu = (\mu^+, \mu^-)$ *on a Bd-algebra* X *is called a bipolar fuzzy Bd-subalgebra (BFBd-Sub) of* X *if it satisfies the following conditions:*

(i) $\mu^+(0) \geq \mu^+(x)$ *for all* $x \in X$*.*
(ii) $\mu^+(x * y) \geq \min\{\mu^+(x), \mu^+(y)\}$ *for all* $x, y \in X$*.*
(iii) $\mu^-(0) \leq \mu^-(x)$ *for all* $x \in X$*.*
(iv) $\mu^-(x * y) \leq \max\{\mu^-(x), \mu^-(y)\}$ *for all* $x, y \in X$*.*

Example 3.1. *Let* $X = \{0, a, b\}$ *be a set with a binary operation* $*$ *defined on* X *by the following table:*

$*$	0	a	b
0	0	0	0
a	a	a	a
b	b	b	b

Then $(X, *, 0)$ is a Bd-algebra, see [10]. Now, we define a BFS $\mu = (\mu^+, \mu^-)$ on X as follows:

X	0	a	b
μ^+	0.8	0.6	0.3

X	0	a	b
μ^-	-0.9	-0.7	-0.1

It is straightforward to indicate that $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of X.

Proposition 3.1. *Let $\mu = (\mu^+, \mu^-)$ and $\lambda = (\lambda^+, \lambda^-)$ be BFBd-Subs of X. Then $\mu \cap \lambda$ is a BFBd-Sub of X.*

Proof. Let $x, y \in X$. Then we have

$$\left(\mu^+ \cap \lambda^+\right)(0) = \min\left\{\mu^+(0), \lambda^+(0)\right\} \geq \min\left\{\mu^+(x), \lambda^+(x)\right\} = \left(\mu^+ \cap \lambda^+\right)(x),$$

$$\begin{aligned}\left(\mu^+ \cap \lambda^+\right)(x * y) &= \min\left\{\mu^+(x * y), \lambda^+(x * y)\right\} \\ &\geq \min\left\{\min\left\{\mu^+(x), \mu^+(y)\right\}, \min\left\{\lambda^+(x), \lambda^+(y)\right\}\right\} \\ &= \min\left\{\min\left\{\mu^+(x), \lambda^+(x)\right\}, \min\left\{\mu^+(y), \lambda^+(y)\right\}\right\} \\ &= \min\left\{\left(\mu^+ \cap \lambda^+\right)(x), \left(\mu^+ \cap \lambda^+\right)(y)\right\},\end{aligned}$$

$$\left(\mu^- \cup \lambda^-\right)(0) = \max\left\{\mu^-(0), \lambda^-(0)\right\} \leq \max\left\{\mu^-(x), \lambda^-(x)\right\} = \left(\mu^- \cup \lambda^-\right)(x),$$

and

$$\begin{aligned}\left(\mu^- \cup \lambda^-\right)(x * y) &= \max\left\{\mu^-(x * y), \lambda^-(x * y)\right\} \\ &\leq \max\left\{\max\left\{\mu^-(x), \mu^-(y)\right\}, \max\left\{\lambda^-(x), \lambda^-(y)\right\}\right\} \\ &= \max\left\{\max\left\{\mu^-(x), \lambda^-(x)\right\}, \max\left\{\mu^-(y), \lambda^-(y)\right\}\right\} \\ &= \max\left\{\left(\mu^- \cup \lambda^-\right)(x), \left(\mu^- \cup \lambda^-\right)(y)\right\}.\end{aligned}$$

Hence, $\mu \cap \lambda$ is a BFBd-Sub of X. □

Example 3.2. *Let $X = \{0, a, b, c\}$ be a set equipped with a binary operation $*$ on X, defined by the following table:*

$*$	0	a	b	c
0	0	0	c	0
a	a	0	b	c
b	b	b	b	c
c	c	b	b	c

It can be shown that $(X, *, 0)$ is a Bd-algebra, see [1]. Next, we define the BFSs $\mu = (\mu^+, \mu^-)$ and $\lambda = (\lambda^+, \lambda^-)$ on X as follows:

X	0	a	b	c
μ^+	0.7	0.7	0.4	0.4

X	0	a	b	c
μ^-	-0.5	-0.5	-0.2	-0.2

X	0	a	b	c
λ^+	0.6	0.3	0.6	0.3

X	0	a	b	c
λ^-	-0.9	-0.2	-0.9	-0.2

It can be seen that $\mu = (\mu^+, \mu^-)$ and $\lambda = (\lambda^+, \lambda^-)$ are BFBd-Subs of X. Next, we consider

$$(\mu^+ \cup \lambda^+)(c * a) = \max\{\mu^+(c * a), \lambda^+(c * a)\} = \max\{0.4, 0.3\} = 0.4$$

and

$$\begin{aligned}\min\left\{\left(\mu^+ \cup \lambda^+\right)(c), \left(\mu^+ \cup \lambda^+\right)(a)\right\} &= \min\left\{\max\left\{\mu^+(c), \lambda^+(c)\right\}, \max\left\{\mu^+(a), \lambda^+(a)\right\}\right\} \\ &= \min\{\max\{0.4, 0.6\}, \max\{0.7, 0.3\}\} = 0.6.\end{aligned}$$

It is evident that $(\mu^+ \cup \lambda^+)(c * a) \ngeq \min\{(\mu^+ \cup \lambda^+)(c), (\mu^+ \cup \lambda^+)(a)\}$. In conclusion, $\mu \cup \lambda$ is not a BFBd-Sub of X.

Let $\mu = (\mu^+, \mu^-)$ and $\lambda = (\lambda^+, \lambda^-)$ be BFSs of X such that $\mu \subseteq \lambda$. We observe that $\mu \cup \lambda = \lambda$. Based on the conditions stated above, the following proposition holds true.

Proposition 3.2. *Let $\mu = (\mu^+, \mu^-)$ and $\lambda = (\lambda^+, \lambda^-)$ be BFBd-Subs of a Bd-algebra X such that $\mu \subseteq \lambda$ or $\lambda \subseteq \mu$. Then $\mu \cup \lambda$ is a BFBd-Sub of X.*

4. Characterizations of bipolar fuzzy *Bd*-subalgebras

This section analyzes the characterizations of BFBd-Subs within *Bd*-algebras based on certain characteristics. Moreover, we particularly characterize BFBd-Subs of *Bd*-algebras considering their level subsets.

Theorem 4.1. *Let $\mu = (\mu^+, \mu^-)$ be a BFS of a Bd-algebra X. Then $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of X if and only if $\overline{\mu} = (\overline{\mu^-}, \overline{\mu^+})$ is a BFBd-Sub of X.*

Proof. Assume that $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of X. Let $x, y \in X$. Then we have

$$\overline{\mu^-}(0) = -1 - \mu^-(0) \geq -1 - \mu^-(x) = \overline{\mu^-}(x),$$

$$\begin{aligned}\overline{\mu^-}(x * y) &= -1 - \mu^-(x * y) \\ &\geq -1 - \max\{\mu^-(x), \mu^-(y)\} \\ &= \min\{-1 - \mu^-(x), -1 - \mu^-(y)\} \\ &= \min\left\{\overline{\mu^-}(x), \overline{\mu^-}(y)\right\},\end{aligned}$$

$$\overline{\mu^+}(0) = 1 - \mu^+(0) \leq 1 - \mu^+(x) = \overline{\mu^+}(x),$$

and

$$\overline{\mu^+}(x * y) = 1 - \mu^+(x * y)$$

$$\begin{aligned} &\leq 1-\min\left\{\mu^{+}(x),\mu^{+}(y)\right\} \\ &= \max\left\{1-\mu^{+}(x),1-\mu^{+}(y)\right\} \\ &= \max\left\{\overline{\mu^{+}}(x),\overline{\mu^{+}}(y)\right\}. \end{aligned}$$

Thus, $\overline{\mu}=(\overline{\mu^{-}},\overline{\mu^{+}})$ is a BFBd-Sub of X.

Conversely, assume that $\overline{\mu}=(\overline{\mu^{-}},\overline{\mu^{+}})$ is a BFBd-Sub of X. We consider

$$1-\mu^{+}(0)=\overline{\mu^{+}}(0)\leq\overline{\mu^{+}}(x)=1-\mu^{+}(x),$$

$$\begin{aligned} 1-\mu^{+}(x*y) &= \overline{\mu^{+}}(x*y) \\ &\leq \max\left\{\overline{\mu^{+}}(x),\overline{\mu^{+}}(y)\right\} \\ &= \max\left\{1-\mu^{+}(x),1-\mu^{+}(y)\right\} \\ &= 1-\min\left\{\mu^{+}(x),\mu^{+}(y)\right\}, \end{aligned}$$

$$-1-\mu^{-}(0)=\overline{\mu^{-}}(0)\geq\overline{\mu^{-}}(x)=-1-\mu^{-}(x),$$

and

$$\begin{aligned} -1-\mu^{-}(x*y) &= \overline{\mu^{-}}(x*y) \\ &\geq \min\left\{\overline{\mu^{-}}(x),\overline{\mu^{-}}(y)\right\} \\ &= \min\left\{-1-\mu^{-}(x),-1-\mu^{-}(y)\right\} \\ &= -1-\max\left\{\mu^{-}(x),\mu^{-}(y)\right\}. \end{aligned}$$

It follows that $\mu^{+}(0)\geq\mu^{+}(x)$, $\mu^{+}(x*y)\geq\min\{\mu^{+}(x),\mu^{+}(y)\}$, $\mu^{-}(0)\leq\mu^{-}(x)$, and $\mu^{-}(x*y)\leq\max\{\mu^{-}(x),\mu^{-}(y)\}$. Therefore, $\mu=(\mu^{+},\mu^{-})$ is a BFBd-Sub of X. □

The following theorem is derived directly from Theorem 4.1.

Theorem 4.2. *Let $\mu=(\mu^{+},\mu^{-})$ be a BFS of a Bd-algebra X. Then $\mu=(\mu^{+},\mu^{-})$ is a BFBd-Sub of X if and only if the $\square\mu=(\mu^{+},\overline{\mu^{+}})$ and $\triangle\mu=(\overline{\mu^{-}},\mu^{-})$ are BFBd-Subs of X.*

Theorem 4.3. *Let X be a Bd-algebra and $\mu=(\mu^{+},\mu^{-})$ be a BFS of X. Then $\mu=(\mu^{+},\mu^{-})$ is a BFBd-Sub of X if and only if for every $(t,s)\in[0,1]\times[-1,0]$, $P_U(\mu^{+},t)$ and $N_L(\mu^{-},s)$ are Bd-Subs of X when they are nonempty.*

Proof. Assume that $\mu=(\mu^{+},\mu^{-})$ is a BFBd-Sub of X. Let $(t,s)\in[0,1]\times[-1,0]$ be such that $P_U(\mu^{+},t)$ and $N_L(\mu^{-},s)$ are nonempty. Let $x,y\in P_U(\mu^{+},t)$ and $a,b\in N_L(\mu^{-},s)$. Then $\mu^{+}(x)\geq t$, $\mu^{+}(y)\geq t$, $\mu^{-}(a)\leq s$, and $\mu^{-}(b)\leq s$. By assumption, we have $\mu^{+}(0)\geq\mu^{+}(x)\geq t$ and $\mu^{-}(0)\leq\mu^{-}(a)\leq s$. Also, $0\in P_U(\mu^{+},t)$ and $0\in N_L(\mu^{-},s)$. Next, we consider $\mu^{+}(x*y)\geq\min\{\mu^{+}(x),\mu^{+}(y)\}\geq t$ and $\mu^{-}(a*b)\leq\max\{\mu^{-}(a),\mu^{-}(b)\}\leq s$. We obtain that $x*y\in P_U(\mu^{+},t)$ and $a*b\in N_L(\mu^{-},s)$. Therefore, $P_U(\mu^{+},t)$ and $N_L(\mu^{-},s)$ are Bd-Subs of X.

Conversely, assume that $P_U(\mu^+, t) \neq \emptyset$ and $N_L(\mu^-, s) \neq \emptyset$ are Bd-Subs of X for all $(t,s) \in [0,1] \times [-1,0]$. Let $x, y \in X$. Choose $\mu^+(x) = t_0$ and $\mu^-(x) = s_0$ for some $(t_0, s_0) \in [0,1] \times [-1,0]$. Then $x \in P_U(\mu^+, t_0)$ and $x \in N_L(\mu^-, s_0)$; that is, $P_U(\mu^+, t_0) \neq \emptyset$ and $N_L(\mu^-, s_0) \neq \emptyset$. By the hypothesis, we have $P_U(\mu^+, t_0)$ and $N_L(\mu^-, s_0)$ are Bd-Subs of X. So, $0 \in P_U(\mu^+, t_0)$ and $0 \in N_L(\mu^-, s_0)$. It turns out that $\mu^+(0) \geq t_0 = \mu^+(x)$ and $\mu^-(0) \leq s_0 = \mu^-(x)$. Now, we letting $t_1 = \min\{\mu^+(x), \mu^+(y)\}$ and $s_1 = \max\{\mu^-(x), \mu^-(y)\}$ for some $(t_1, s_1) \in [0,1] \times [-1,0]$. It follows that $\mu^+(x) \geq t_1$, $\mu^+(y) \geq t_1$, $\mu^-(x) \leq s_1$, and $\mu^-(y) \leq s_1$. This implies that $x, y \in P_U(\mu^+, t_1)$ and $x, y \in N_L(\mu^-, s_1)$. By the given assumption, we have $x * y \in P_U(\mu^+, t_1)$ and $x * y \in N_L(\mu^-, s_1)$. We get that $\mu^+(x * y) \geq t_1 = \min\{\mu^+(x), \mu^+(y)\}$ and $\mu^-(x * y) \leq s_1 = \max\{\mu^-(x), \mu^-(y)\}$. Consequently, $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of X. □

Theorem 4.4. *Let $\mu = (\mu^+, \mu^-)$ be a BFS of a Bd-algebra X. Then $\mu = (\mu^+, \mu^-)$ is a BFBd-sub of X if and only if for every $(t,s) \in [0,1] \times [-1,0]$, $P^*_U(\mu^+, t)$ and $N^*_L(\mu^-, s)$ are Bd-Subs of X when they are nonempty.*

Proof. Assume that $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of X. Let $(t,s) \in [0,1] \times [-1,0]$ be such that $P^*_U(\mu^+, t) \neq \emptyset$ and $N^*_L(\mu^-, s) \neq \emptyset$. Let $x, y \in P^*_U(\mu^+, t)$ and $a, b \in N^*_L(\mu^-, s)$. Then $\mu^+(x) > t$, $\mu^+(y) > t$, $\mu^-(a) < s$, and $\mu^-(b) < s$. So, $\mu^+(0) \geq \mu^+(x) > t$ and $\mu^-(0) \leq \mu^-(a) < s$. Thus, $0 \in P^*_U(\mu^+, t)$ and $0 \in N^*_L(\mu^-, s)$. Next by assumption, we have $\mu^+(x * y) \geq \min\{\mu^+(x), \mu^+(y)\} > t$ and $\mu^-(a * b) \leq \max\{\mu^-(a), \mu^-(b)\} < s$. Hence, $x * y \in P^*_U(\mu^+, t)$ and $a * b \in N^*_L(\mu^-, s)$. Therefore, $P^*_U(\mu^+, t)$ and $N^*_L(\mu^-, s)$ are Bd-Subs of X.

Conversely, assume that $P^*_U(\mu^+, t) \neq \emptyset$ and $N^*_L(\mu^-, s) \neq \emptyset$ are Bd-Subs of X for all $(t,s) \in [0,1] \times [-1,0]$. Suppose that $t_0 = \mu^+(0) < \mu^+(a)$ for some $a \in X$ and $t_0 \in [0,1]$. Then $a \in P^*_U(\mu^+, t_0)$. That is, $P^*_U(\mu^+, t_0) \neq \emptyset$. By assume that, we have $P^*_U(\mu^+, t_0)$ is a Bd-Sub of X. So, $0 \in P^*_U(\mu^+, t_0)$, it follows $\mu^+(0) > t_0 = \mu^+(0)$. This is a contradiction. Thus, $\mu^+(0) \geq \mu^+(x)$ for all $x \in X$. Similarly, we can prove that $\mu^-(0) \leq \mu^-(x)$ for all $x \in X$. Now, suppose that $\mu^+(a * b) < \min\{\mu^+(a), \mu^+(b)\}$ for some $a, b \in X$. Let $\mu^+(a * b) = t_1$ for some $t_1 \in [0,1]$. We have that $\mu^+(a) > t_1$ and $\mu^+(b) > t_1$, it implies $a, b \in P^*_U(\mu^+, t_1)$. By the hypothesis, we have $P^*_U(\mu^+, t_1)$ is a Bd-Sub of X. Thus, $a * b \in P^*_U(\mu^+, t_1)$. This means that $\mu^+(a * b) > t_1 = \mu^+(a * b)$, which is a contradiction. Hence, $\mu^+(x * y) \geq \min\{\mu^+(x), \mu^+(y)\}$ for all $x, y \in X$. Similarly, we can show that $\mu^-(x * y) \leq \max\{\mu^-(x), \mu^-(y)\}$ for all $x, y \in X$. Consequently, $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of X. □

Theorem 4.5. *Let X be a Bd-algebra and $\overline{\mu} = \left(\overline{\mu^+}, \overline{\mu^-}\right)$ be a BFS of X. Then $\overline{\mu} = \left(\overline{\mu^+}, \overline{\mu^-}\right)$ is a BFBd-Sub of X if and only if for every $(t,s) \in [0,1] \times [-1,0]$, $P_L(\mu^+, t)$ and $N_U(\mu^-, s)$ are Bd-Subs of X when they are nonempty.*

Proof. Assume that $\overline{\mu} = \left(\overline{\mu^+}, \overline{\mu^-}\right)$ is a BFBd-Sub of X. Let $x, y \in P_L(\mu^+, t)$. Then $\mu^+(x) \leq t$ and $\mu^+(y) \leq t$. Thus, $1 - \mu^+(0) = \overline{\mu^+}(0) \geq \overline{\mu^+}(x) = 1 - \mu^+(x)$, and then $\mu^+(0) \leq \mu^+(x) \leq t$. So, $0 \in P_L(\mu^+, t)$. Since $\overline{\mu^+}(x * y) \geq \min\{\overline{\mu^+}(x), \overline{\mu^+}(y)\}$, we have $1 - \mu^+(x * y) \geq \min\{1 - \mu^+(x), 1 - \mu^+(y)\} = 1 - \max\{\mu^+(x), \mu^+(y)\}$. Also, $\mu^+(x * y) \leq \max\{\mu^+(x), \mu^+(y)\} \leq t$. So, $x * y \in P_L(\mu^+, t)$. Hence, $P_L(\mu^+, t)$ is a Bd-Sub of X. Now, let $a, b \in N_U(\mu^-, s)$. Then $\mu^-(a) \geq s$ and $\mu^-(b) \geq s$.

Since $-1-\mu^-(0) = \overline{\mu^-}(0) \leq \overline{\mu^-}(a) = -1-\mu^-(a)$, we have $\mu^-(0) \geq \mu^-(a) \geq s$. It follows that $0 \in N_U(\mu^-, s)$. By assumption, we get $\overline{\mu^-}(a*b) \leq \max\{\overline{\mu^-}(a), \overline{\mu^-}(b)\}$. That is, $-1-\mu^-(a*b) \leq \max\{-1-\mu^-(a), -1-\mu^-(b)\} = -1-\min\{\mu^-(a), \mu^-(b)\}$. So, $\mu^-(a*b) \geq \min\{\mu^-(a), \mu^-(b)\} \geq s$. This means that $a*b \in N_U(\mu^-, s)$. Therefore, $N_U(\mu^-, s)$ is a Bd-Sub of X.

Conversely, let $x, y \in X$. Choose $t_0 = \mu^+(x)$ for some $t_0 \in [0,1]$. Then $x \in P_L(\mu^+, t_0)$. By the given assumption, we have $P_L(\mu^+, t_0)$ is a Bd-Sub of X. This implies that $0 \in P_L(\mu^+, t_0)$; that is, $\mu^+(0) \leq t_0 = \mu^+(x)$. Also, $\overline{\mu^+}(0) = 1-\mu^+(0) \geq 1-\mu^+(x) = \overline{\mu^+}(x)$ Now, letting $t_1 = \max\{\mu^+(x), \mu^+(y)\}$ for some $t_1 \in [0,1]$. It turns out that $\mu^+(x) \leq t_1$ and $\mu^+(y) \leq t_1$. Then $x, y \in P_L(\mu^+, t_1)$, it implies $P_L(\mu^+, t_1) \neq \emptyset$. By the hypothesis, we have $x*y \in P_L(\mu^+, t_1)$. So, $\mu^+(x*y) \leq t_1 = \max\{\mu^+(x), \mu^+(y)\}$. We obtain that

$$\begin{aligned} \overline{\mu^+}(x*y) &= 1-\mu^+(x*y) \\ &\geq 1-\max\{\mu^+(x), \mu^+(y)\} \\ &= \min\{1-\mu^+(x), 1-\mu^+(y)\} \\ &= \min\{\overline{\mu^+}(x), \overline{\mu^+}(y)\}. \end{aligned}$$

Next, let $\mu^-(x) = s_0$ for some $s_0 \in [-1,0]$. Then $x \in N_U(\mu^-, s_0)$. By assumption, we have $0 \in N_U(\mu^-, s_0)$, and so $\mu^-(0) \geq s_0 = \mu^-(x)$. Thus, $\overline{\mu^-}(0) = -1-\mu^-(0) \leq -1-\mu^-(x) = \overline{\mu^-}(x)$. Let $\min\{\mu^-(x), \mu^-(y)\} = s_1$ for some $s_1 \in [-1,0]$. Then $\mu^-(x) \geq s_1$ and $\mu^-(y) \geq s_1$. This means that $x, y \in N_U(\mu^-, s_1)$. By the given assumption, we have $x*y \in N_U(\mu^-, s_1)$. So, $\mu^-(x*y) \geq s_1 = \min\{\mu^-(x), \mu^-(y)\}$. We have that

$$\begin{aligned} \overline{\mu^-}(x*y) &= -1-\mu^-(x*y) \\ &\leq -1-\min\{\mu^-(x), \mu^-(y)\} \\ &= \max\{-1-\mu^-(x), -1-\mu^-(y)\} \\ &= \max\{\overline{\mu^-}(x), \overline{\mu^-}(y)\}. \end{aligned}$$

Consequently, $\overline{\mu} = \left(\overline{\mu^+}, \overline{\mu^-}\right)$ is a BFBd-Sub of X. □

Theorem 4.6. *Let $\overline{\mu} = \left(\overline{\mu^+}, \overline{\mu^-}\right)$ be a BFS of a Bd-algebra X. Then $\overline{\mu} = \left(\overline{\mu^+}, \overline{\mu^-}\right)$ is a BFBd-sub of X if and only if for every $(t,s) \in [0,1] \times [-1,0]$, $P^*_L(\mu^+, t)$ and $N^*_U(\mu^-, s)$ are Bd-Subs of X when they are nonempty.*

Proof. Let $(t,s) \in [0,1] \times [-1,0]$ be such that $P^*_L(\mu^+, t) \neq \emptyset$ and $N^*_U(\mu^-, s) \neq \emptyset$. Let $x, y \in P^*_L(\mu^+, t)$. Then $\mu^+(x) < t$ and $\mu^+(y) < t$. Since $1-\mu^+(0) = \overline{\mu^+}(0) \geq \overline{\mu^+}(x) = 1-\mu^+(x)$, we have $\mu^+(0) \leq \mu^+(x) < t$. This implies that $0 \in P^*_L(\mu^+, t)$. We have that

$$\begin{aligned} 1-\mu^+(x*y) &= \overline{\mu^+}(x*y) \\ &\geq \min\{\overline{\mu^+}(x), \overline{\mu^+}(y)\} \\ &= \min\{1-\mu^+(x), 1-\mu^+(y)\} \\ &= 1-\max\{\mu^+(x), \mu^+(y)\}. \end{aligned}$$

Thus, $\mu^+(x*y) \leq \max\{\mu^+(x), \mu^+(y)\} < t$. So, $x*y \in P_L^*(\mu^+, t)$. Hence, $P_L^*(\mu^+, t)$ is a Bd-Sub of X. Now, let $x, y \in N_U^*(\mu^-, s)$. Then $\mu^-(x) > s$ and $\mu^-(y) > s$. We have that $-1 - \mu^-(0) = \overline{\mu^-}(0) \leq \overline{\mu^-}(x) = -1 - \mu^-(x)$, and so $\mu^-(0) \geq \mu^-(x) > s$. That is, $0 \in N_U^*(\mu^-, s)$. Next, we consider

$$\begin{aligned}
-1 - \mu^-(x*y) &= \overline{\mu^-}(x*y) \\
&\leq \max\{\overline{\mu^-}(x), \overline{\mu^-}(y)\} \\
&= \max\{-1 - \mu^-(x), -1 - \mu^-(y)\} \\
&= -1 - \min\{\mu^-(x), \mu^-(y)\}.
\end{aligned}$$

This means that $\mu^-(x*y) \geq \min\{\mu^-(x), \mu^-(y)\} > s$. It turns out that $x*y \in N_U^*(\mu^-, s)$. Therefore, $N_U^*(\mu^-, s)$ is a Bd-Sub of X.

Conversely, let $x, y \in X$. Suppose that $\overline{\mu^+}(0) < \overline{\mu^+}(x)$ such that $\mu^+(0) = t_0$ for some $t_0 \in [0, 1]$. Then $1 - \mu^+(0) < 1 - \mu^+(x)$. It implies that $t_0 = \mu^+(0) > \mu^+(x)$. So, $x \in P_L^*(\mu^+, t_0)$. By the hypothesis, we have $0 \in P_L^*(\mu^+, t_0)$. It follows that $\mu^+(0) < t_0 = \mu^+(0)$, which is a contradiction. Thus, $\overline{\mu^+}(0) \geq \overline{\mu^+}(x)$. Next, suppose that there exists $t_1 \in [0, 1]$ such that $\mu^+(x*y) = t_1$ and $\overline{\mu^+}(x*y) < \min\{\overline{\mu^+}(x), \overline{\mu^+}(y)\}$. Then

$$1 - \mu^+(x*y) < \min\{1 - \mu^+(x), 1 - \mu^+(y)\} = 1 - \max\{\mu^+(x), \mu^+(y)\}.$$

That is, $\mu^+(x*y) > \max\{\mu^+(x), \mu^+(y)\}$, and so $\mu^+(x) < \mu^+(x*y) = t_1$ and $\mu^+(y) < \mu^+(x*y) = t_1$. It means that $x, y \in P_L^*(\mu^+, t_1)$. By the given assumption, we have $x*y \in P_L^*(\mu^+, t_1)$. Thus, $\mu^+(x*y) < t_1 = \mu^+(x*y)$. This is a contradiction. Hence, $\overline{\mu^+}(x*y) \geq \min\{\overline{\mu^+}(x), \overline{\mu^+}(y)\}$. Similarly, we can prove that $\overline{\mu^-}(0) \leq \overline{\mu^-}(x)$ and $\overline{\mu^-}(x*y) \leq \max\{\overline{\mu^-}(x), \overline{\mu^-}(y)\}$. Consequently, $\overline{\mu} = \left(\overline{\mu^+}, \overline{\mu^-}\right)$ is a BFBd-Sub of X. □

5. Homomorphism of bipolar fuzzy *Bd*-subalgebras

This section discusses the relationships of BFBd-Subs under a homomorphism of *Bd*-algebras. Let $f : X \to Y$ be a mapping of nonempty sets X and Y, and let $\mu = (\mu^+, \mu^-)$ be a BFS of Y. The BFS $\mu_f = \left(\mu_f^+, \mu_f^-\right)$ of X by under f is defined by $\mu_f^+(x) = \mu^+(f(x))$ and $\mu_f^-(x) = \mu^-(f(x))$ for all $x \in X$. Let $(X, *, 0)$ and $(Y, \circ, 0')$ be *Bd*-algebras. The mapping $f : X \to Y$ is called a *homomorphism* (see, [8]) if $f(0) = 0'$ and $f(x*y) = f(x) \circ f(y)$ for all $x, y \in X$.

Theorem 5.1. *Let $f : X \to Y$ be a homomorphism of Bd-algebras $(X, *, 0)$ and $(Y, \circ, 0')$. If $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of Y, then $\mu_f = \left(\mu_f^+, \mu_f^-\right)$ is a BFBd-Sub of X.*

Proof. Assume that $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of Y. Let $x, y \in X$. Then

$$\mu_f^+(0) = \mu^+(f(0)) = \mu^+(0') \geq \mu^+(f(x)) = \mu_f^+(x) \text{ and}$$
$$\mu_f^-(0) = \mu^-(f(0)) = \mu^-(0') \leq \mu^-(f(x)) = \mu_f^-(x).$$

Moreover,

$$\mu_f^+(x*y) = \mu^+(f(x*y))$$

$$
\begin{aligned}
&= \mu^+(f(x) \circ f(y)) \\
&\geq \min\{\mu^+(f(x)), \mu^+(f(y))\} \\
&= \min\{\mu_f^+(x), \mu_f^+(y)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_f^-(x * y) &= \mu^-(f(x * y)) \\
&= \mu^-(f(x) \circ f(y)) \\
&\leq \max\{\mu^-(f(x)), \mu^-(f(y))\} \\
&= \max\{\mu_f^-(x), \mu_f^-(y)\}.
\end{aligned}
$$

Hence, $\mu_f = \left(\mu_f^+, \mu_f^-\right)$ is a BFBd-Sub of X. $\square$

However, the converse of Theorem 5.1 is not always true in general, as shown in the following example.

Example 5.1. *Let $X = \{0, u, v\}$ and $Y = \{0', a, b, c\}$ be two sets equipped with binary operations $*$ and $\circ$ respectively, forming Bd-algebras $(X, *, 0)$ and $(Y, \circ, 0')$ as defined by the following tables:*

$*$	0	u	v
0	0	u	v
u	u	0	0
v	v	v	v

$\circ$	$0'$	a	b	c
$0'$	$0'$	$0'$	$0'$	$0'$
a	a	a	a	a
b	b	b	b	b
c	c	c	a	a

Define the mapping $f : X \to Y$ as follows:

$$
f(x) = \begin{cases} 0' & \text{if } x \in \{0, u\} \\ b & \text{otherwise} \end{cases} \quad \text{for all } x \in X.
$$

Then f is a homomorphism between the two Bd-algebras, see [9]. Next, we define a BFS $\mu = (\mu^+, \mu^-)$ on Y with the following membership values:

Y	$0'$	a	b	c
μ^+	0.7	0.2	0.2	0.5

Y	$0'$	a	b	c
μ^-	−0.6	−0.3	−0.3	−0.5

Consider the induced BFS $\mu_f = \left(\mu_f^+, \mu_f^-\right)$ on X under the mapping f, defined by $\mu_f^+(x) = \mu^+(f(x))$ and $\mu_f^-(x) = \mu^-(f(x))$ for all $x \in X$. The membership values for μ_f are calculated as:

X	0	u	v
μ_f^+	0.7	0.7	0.2

X	0	u	v
μ_f^-	−0.6	−0.6	−0.3

It can be verified that $\mu_f = \left(\mu_f^+, \mu_f^-\right)$ satisfies the conditions of BFBd-Sub of X. However, we observe that the BFS $\mu = (\mu^+, \mu^-)$ is not a BFBd-Sub of Y, because $\mu^+(c \circ c) = 0.2 \ngeq 0.5 = \min\{\mu^+(c), \mu^+(c)\}$.

Theorem 5.2. *Let $f : X \to Y$ be a homomorphism and a surjective of Bd-algebras $(X, *, 0)$ and $(Y, \circ, 0')$. If $\mu_f = \left(\mu_f^+, \mu_f^-\right)$ is a BFBd-Sub of X, then $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of Y.*

Proof. Assume that $\mu_f = \left(\mu_f^+, \mu_f^-\right)$ is a BFBd-Sub of X. Let $a, b \in Y$. Then there exist $x, y \in X$ such that $f(x) = a$ and $f(y) = b$. We have that $\mu^+(0') = \mu^+(f(0)) = \mu_f^+(0) \geq \mu_f^+(x) = \mu^+(f(x)) = \mu^+(a)$ and $\mu^-(0') = \mu^-(f(0)) = \mu_f^-(0) \leq \mu_f^-(x) = \mu^-(f(x)) = \mu^-(a)$. In addition,

$$\begin{aligned}
\mu^+(a \circ b) &= \mu^+(f(x) \circ f(y)) \\
&= \mu^+(f(x * y)) \\
&= \mu_f^+(x * y) \\
&\geq \min\{\mu_f^+(x), \mu_f^+(y)\} \\
&= \min\{\mu^+(f(x)), \mu^+(f(y))\} \\
&= \min\{\mu^+(a), \mu^+(b)\}
\end{aligned}$$

and

$$\begin{aligned}
\mu^-(a \circ b) &= \mu^-(f(x) \circ f(y)) \\
&= \mu^-(f(x * y)) \\
&= \mu_f^-(x * y) \\
&\leq \max\{\mu_f^-(x), \mu_f^-(y)\} \\
&= \max\{\mu^-(f(x)), \mu^-(f(y))\} \\
&= \max\{\mu^-(a), \mu^-(b)\}.
\end{aligned}$$

Therefore, $\mu = (\mu^+, \mu^-)$ is a BFBd-Sub of Y □

6. Conclusion

In this study, we have presented the novel notion of BFBd-Subs within the *Bd*-algebras. The research has delineated the fundamental conditions for categorizing a BFS as a Bd-Sub, underpinned by precise definitions and exemplifications. Our characterization analysis indicates that BFBd-Subs can be effectively described by their level subsets. This analysis demonstrates that these level sets are Bd-Subs if and only if the corresponding BFS is a BFBd-Sub. Additionally, we investigated the algebraic properties of these structures in relation to homomorphisms, determining that the characteristic of being a BFBd-Sub is maintained under homomorphic mappings and pullbacks. These findings improve understanding of the connection between bipolar fuzzy set theory and the structural properties of *Bd*-algebras. In further study, we plan to expand this conceptual framework by exploring the notion of bipolar fuzzy *Bd*-ideals within *Bd*-algebras. We will concentrate on examining their essential properties, defining their characterizations, and investigating their connections with the BFBd-Subs developed in this research.

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