Int. J. Anal. Appl. (2026), 24:224

International Journal of Analysis and Applications

A Novel Study on Certain Covering Properties of Quad Topological Spaces

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Abstract. In this paper, we defined Q-topological spaces and Q-open sets, discussing and proving most of their neighboring properties with illustrative examples. We also studied the covering properties of Q-topological spaces based on Q-open sets, such as:Q-compact spaces, Q-lindelöf spaces, Q-countably compact spaces, S-Q-compact spaces, S-Q-lindelöf spaces, S-Q-countably compact spaces. We proved several new theories and facts, demonstrating the main relationship between these spaces and supporting each case with examples.

1. General Introduction

In 2011, Palaniammal [9], studied fuzzy q-b-open sets in fuzzy quad topological spaces along with their several properties and characterizations. Also, studied fuzzy q-open sets, fuzzy q-gb-open sets, fuzzy q-b-continuous mapping and fuzzy q-b-separation and obtain some of their basic features.

In 2013, Mukundan [8], defined a new fundamental concept of quad topological spaces and identified new types of closed sets such as q-closed, q-b-closed and q-bt-closed sets.

Tapi et al., (2014), [13], discussed the separation axioms in quad topological spaces and studied some of their properties and characteristics.

Tapi and Sharma (2015), [16], studied and discussed various properties of q-open and q- closed sets and revealed a q-continuous function in quad topological spaces.

In 2017, Chandrasekar et al., [1], identified a new type of topological structure and named it the nano quad star topology resulting from two nano bi-topological systems.

Received: Jun. 4, 2026.

2020 *Mathematics Subject Classification.* 54D10, 54A05, 54A40, 54E55, 03E72, 54F45.

Key words and phrases. Q-topological spaces; Q-open set; Q-closure of a set; Q-interior of a set; Q-compact spaces; Q-Lindelöf spaces; Q-countably compact spaces; S-Q-compact spaces; S-Q-Lindelöf spaces; S-Q-countably compact spaces.

https://doi.org/10.28924/2291-8639-24-2026-224
ISSN: 2291-8639

Khattak et al., (2017), [3], they focus on this study to look for soft regular separation axioms in soft quad topological spaces. They talk over and focus their concentration on soft regular separation axioms in soft quad topological spaces with respect to ordinary elements and soft points. Moreover they study the inherited features at different angles with respect to ordinary points and soft elements. Some of their central traits in soft quad topological spaces are also brought under analysis.

In 2016, Tapi et al., [14], introduced two new types of open sets, namely quad-semi open and quad pre-open sets in quad topological spaces along with their several properties and characterizations. As application to quad open sets, tri semi open and tri pre- open sets, the authors introduced quad continuous, quad semi continuous, and quad pre continuous functions to obtain some of their basic features.

In 2017, Lavanya & Amsaveni [6], in this study, the initiation of quad fuzzy topological spaces is given and $Q\tau T_{\alpha^m}$ compactness is investigated. Also some of their characterizations and features of $Q\tau T_{\alpha^m}$ com $Q\tau T_{\alpha^m}$ compactness spaces are examined.

Sharma et al. 2018, [11], introduced two new types of fuzzy open sets namely fuzzy q-semi-open sets and fuzzy q-pre-open sets in fuzzy q-topological spaces and also defined the fuzzy continuity namely fuzzy q-continuity, fuzzy q-semi-continuity and fuzzy q-pre-continuity in fuzzy q-topological spaces.

Sharma et al., 2018, [12], the main objective of this paper is to study fuzzy open collections and fuzzy tri b-open sets in fuzzy tri topological domains along with their several properties and descriptions. They study fuzzy tri continuous, fuzzy tri distinction, fuzzy tri b-continuous, fuzzy tri-b separation and obtain few of their basic features.

In 2018, Jamal et al., [2], the central objective of this article is to highlight soft separations axioms in soft quad topological spaces. Moreover to discuss the connection among them through two routes.

In the study of Khattak et al., 2018, [4], the main objective of this paper is to discuss and publish soft β-separation axioms in soft quad topological spaces. The focus is on soft β-separation axioms in soft quad topological spaces with respect to the concepts of ordinary points and soft points. Furthermore, the properties inherited from various aspects of soft points and ordinary points are studied.

Khodair et al., 2021, [5], defined a new types of quad topological spaces called a nano quad topological spaces. The concepts of nano open sets was also presented, and their various properties were studied.

In 2022, Mehmood et al., [7], explores the study of the notions of some new concepts and results. Three definitions are given namely, neutrosophic soft quad semi-open, neutrosophic soft quad pre-open and neutrosophic soft quad b^*-open sets in neutrosophic soft quad-topological spaces. Among these one of the interesting neutrosophic soft quad generalized open set known as neutrosophic soft quad pre-open set is defined, then on the basis of this definition some fundamentals are obtained and derived. These are including, neutrosophic soft quad interior, neutrosophic soft quad boundary, neutrosophic soft quad exterior and neutrosophic soft quad closure.

Rashid and Musa's, 2025, [10], study addressed the key properties of maximal *iDg#*-open sets in the field of topology. The study focused on a set of important theoretical findings, including a proof of the decomposition theory, which demonstrates how these sets can be divided into less significant subsets while preserving their fundamental properties.

Also, in 2025, Tapi & Sharma, [15], introduced new type of open sets, namely T_α-open sets in quad topological spaces along with their several properties.

Throughout this research, we will use the following symbols to represent their respective topological concepts:

Symbol	**Topological Concept**
$\mathbb{R}$	The set of all real number.
$\mathbb{Z}$	The set of all integer numbers.
$\mathbb{N}$	The set of all natural numbers.
$\mathbb{Q}$	The set of all rational numbers.
I_{rr}	The set of all irrational numbers.
$\sigma_{l.r}$	The left ray topology on $\mathbb{R}$.
$\sigma_{r.r}$	The right ray topology on $\mathbb{R}$.
σ_u	The usual topology on $\mathbb{R}$.
σ_{cof}	The co-finite topology.
σ_{coc}	The co-countable topology.
σ_{dis}	The discrete topology.
σ_{ind}	The indiscrete topology.
σ_p	The include point topology on $\mathbb{R}$.
$\sigma_{Exc(p)}$	The exclude point topology on $\mathbb{R}$.
Q- open set	Quad open set
Q-$cl(A)$ or $Q-(\overline{A})$	Quad closure of A
Q-$int(A)$ or Q-(A°)	Quad interior of A
Q-compact	Quad compact
S-Q-compact	Semi quad compact
Q-Lindel$\ddot{o}$f	Quad Lindel$\ddot{o}$f
S-Q-Lindel$\ddot{o}$f	Semi quad Lindel$\ddot{o}$f
Q-countably compact	Quad countably compact
S-Q-countably compact	Semi quad countably compact

2. Foundations of Quad Topological Spaces

2.1. **Basic Definitions and Properties of a quad Topological Spaces.** We discussed the definition of Q-topological spaces and Q-open sets with many illustrative examples. Also, studied most of their properties.

Definition 2.1. *(Mukundan,2013) Given a non-empty set D and* $\tau_Q = \left(\sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}}\right)$ *be an ordered quad of topologies on D (4-tuples of topologies) on D. Then,* $\left(D, \sigma_Q\right)$ *is called a quad topological space.*

Remark 2.1. *We use the symbol Q-topological space to denote the quad topological space* $\left(D, \tau_Q\right)$*.*

Definition 2.2. *A subset E of Q-topological space D is called a quad-open set if* $E \in \left(\sigma_{D_{(1)}} \cup \sigma_{D_{(2)}} \cup \sigma_{D_{(3)}} \cup \sigma_{D_{(4)}}\right)$ *and it is denoted by (Q-open set).*

Definition 2.3. *The complement of a quad-open set in any* $\tau_{D_{(i)}}$ *where* $(i = 1, 2, 3, 4)$ *is called a quad-closed set and it is denoted by (Q-closed set).*

Example 2.1. *Let* $D = \mathbb{R}$ *and* $\sigma_{D_{(1)}} = \sigma_{dis}$, $\sigma_{D_{(2)}} = \sigma_u$, $\sigma_{D_{(3)}} = \sigma_{(l.r)}$ *and* $\sigma_{D_{(4)}} = \sigma_{(r.r)}$*. Then,* $\left(D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}}\right)$ *is a Q-topological space.*

$(2, 3)$ *is a Q-open set, while its complement* $(-\infty, 2] \cup [3, \infty)$ *is a Q-closed set.*

$\{5\}$ *is a Q-open and Q-closed set, hence it is a Q-clopen set, since* $R - \{5\} = (-\infty, 5) \cup (5, \infty)$ *is a Q-open set in* $\left(\mathbb{R}, \sigma_{D_{(1)}}\right) = (\mathbb{R}, \sigma_{dis})$*.*

Definition 2.4. *Given a non-empty set D and* $\sigma_T = \left(\sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}\right)$ *be an ordered triple of topologies on D (3-tuples of topologies) on D. Then,* (D, σ_T) *is called a tri-topological space and denoted by* $(T.T.S)$ *or T-topological space.*

Remark 2.2. *Any tri-topological space* $\left(D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}\right)$ *is a Q-topological space when we repeated only one of the topologies* $\sigma_{D_{(i)}}$*, where* $i \in \{1, 2, 3\}$*.*

Example 2.2. *Given the tri-topological space* $\left(\mathbb{Q}, \sigma_{cof}, \sigma_5, \sigma_7\right)$*. Repeat* σ_5 *once, we have* $\left(\mathbb{Q}, \sigma_{cof}, \sigma_5, \sigma_7, \sigma_5\right)$ *is a quad topological space.*

Remark 2.3. *Every* $\sigma_{D_{(i)}}$*-open set is a Q-open set for* $i = 1, 2, 3,$ *and* 4*.*

Remark 2.4. *The union of Q-open sets is not valid to be a Q-open set in all general cases, as we shall see in the next example:*

Example 2.3. *Let* $D = \{a, b, c\}$, $\sigma_{D_{(1)}} = \{D, \varnothing\}$, $\sigma_{D_{(2)}} = \{\varnothing, D, \{a\}\}$, $\sigma_{D_{(3)}} = \{\varnothing, D, \{a\}, \{a, b\}\}$ *and* $(\sigma_{D_{(4)}} = \{\varnothing, D, \{c\}\}$*. Then, the Q-open sets are:* $\varnothing, D, \{a\}, \{a, b\}$ *and* $\{c\}$*.*

The Q-closed sets are: $\varnothing, D, \{b, c\}, \{c\}, \{a, b\}$ *. Now,* $\{a\} \cup \{c\} = \{a, c\}$ *is not Q-open set, since* $\{a, c\} \notin \tau_{D_{(i)}}$ *for* $i = 1, 2, 3$ *and* 4*.*

Remark 2.5. *The intersection of Q-closed sets is not valid to be a Q-closed set in all general cases. In the previous Example2.3, the sets* $\{a, b\}$ *and* $\{b, c\}$ *are two Q-closed sets, but* $\{a, b\} \cap \{b, c\} = \{b\}$ *is not a Q-closed sets.*

Definition 2.5. *Let* $\left(D,\sigma_Q\right) = (D,\ \sigma_{D_{(1)}}\ ,\sigma_{D_{(2)}}\ ,\sigma_{D_{(3)}}\ ,\sigma_{D_{(4)}})$ *and* $\left(D,\sigma_Q\right) = (D,\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}})$ *be two Q-topological spaces then :*

1. *The intersection of* $\left(D,\tau_Q\right)$ *and* $\left(D,\sigma_Q\right)$ *is denoted by*
$$\left(D,\tau_Q\cap\sigma_Q\right) = \left(D,\tau_{D_{(1)}}\cap\sigma_{D_{(1)}},\tau_{D_{(2)}}\cap\sigma_{D_{(2)}},\tau_{D_{(3)}}\cap\sigma_{D_{(3)}},\tau_{D_{(4)}}\cap\sigma_{D_{(4)}}\right).$$
2. *The union of* $\left(D,\tau_Q\right)$ *and* $\left(D,\sigma_Q\right)$ *is denoted by*
$$\left(D,\tau_Q\cup\sigma_Q\right) = \left(D,\tau_{D_{(1)}}\cup\sigma_{D_{(1)}},\tau_{D_{(2)}}\cup\sigma_{D_{(2)}},\tau_{D_{(3)}}\cup\sigma_{D_{(3)}},\tau_{D_{(4)}}\cup\sigma_{D_{(4)}}\right).$$

Remark 2.6. *The intersection of any two Q-topological spaces is a Q-topological space.*

Remark 2.7. *The union of any two Q-topological spaces is not a Q-topological space.*

Example 2.4. *Consider the Q-topological spaces*
$$\left(D,\sigma_Q\right) = (\mathbb{R},\sigma_{ind},\sigma_u,\sigma_{l.r},\sigma_4\ )\ \text{ and }\ \left(D,\tau_Q\right) = (\mathbb{R},\tau_{ind},\tau_u,\tau_{r.r},\tau_4\ )$$
then, $\tau_Q\cap\sigma_Q$ *is a Q-topology on* $\mathbb{R}$ *but* $\tau_Q\cup\sigma_Q$ *is not a Q-topology on* $\mathbb{R}$*, since* $(\sigma_{l.r}\cup\sigma_{r.r})$ *is not a topology on* $\mathbb{R}$.

2.2. **Quad Closure of Sets.** In this part, we introduced and discussed the concept of Q-closure of sets and studied its properties, giving many results and several illustrative examples of this concept.

Definition 2.6. *Let* $\left(D,\sigma_Q\right) = \left(D,\ \sigma_{D_{(1)}}\ ,\sigma_{D_{(2)}}\ ,\sigma_{D_{(3)}}\ ,\sigma_{D_{(4)}}\right)$ *be a Q- topological space and* $A\subseteq D$ *then the Q-closure of A is the result of the intersection of the smallest closed sets that containing A in each* $\sigma_{D_{(i)}}$*, for* $i=1\ ,\ 2\ ,\ 3\ ,\ 4$ *and denoted by* $Q-cl(A)$ *or* $Q-\left(\overline{A}\right)$.

Remark 2.8. *We conclude from the previous main definition that , if* $A\subseteq D$ *then the* $Q-cl(A) = \left(\sigma_{D_{(1)}}-cl(A)\right)\cap\left(\sigma_{D_{(2)}}-cl(A)\right)\cap\left(\sigma_{D_{(3)}}-cl(A)\right)\cap\left(\sigma_{D_{(4)}}cl(A)\right) = \bigcap_{i=1}^{4}\left(\sigma_{D_{(i)}}-cl(A)\right)$.

Example 2.5. *Given* $D=\{a,b,c\}$, $\tau_{D_{(1)}}=\{\varnothing,D,\{a\}\}$, $\tau_{D_{(2)}}=\{\varnothing,D,\{b\}\}$, $\tau_{D_{(3)}}=\{\varnothing,D,\{c\}\ ,\{b,c\}\}$ *and* $\tau_{D_{(4)}}=\{\varnothing,D,\{a,b\}\ ,\{c\}\}$. *If* $E=\{\ c\}$ *, then the* $Q-cl(E)=\{b,c\}\cap\{\ a,c\}\cap D\cap\{c\}=\{c\}$.

Remark 2.9. *It can be noted that the Q-closure of the set is not always a Q-closed, as we see in the following example.*

Example 2.6. *Consider the Q-topological space* $(\mathbb{R},\sigma_{ind},\sigma_5,\sigma_{l.r},\sigma_{r.r})$ *and* $A=(\ 1,7]$ *then*
$$Q-cl(A)= \mathbb{R}\cap\mathbb{R}\cap[\ 1,\infty)\cap(\ -\infty,7] = [\ 1,7]$$
is not Q-closed set.

Theorem 2.1. *If A is a subset of a Q-topological space* $(D,\ \sigma_{D_{(1)}}\ ,\sigma_{D_{(2)}}\ ,\sigma_{D_{(3)}}\ ,\sigma_{D_{(4)}})$ *and A is closed set at least in one* $\sigma_{D_{(i)}}$, $(i\ =\ 1\ ,\ 2\ ,\ 3\ ,\ 4)$ *then* $Q-cl(A)=A$.

Proof: Since A is closed in some $\sigma_{D_{(i)}}$, $(i = 1 , 2 , 3 , 4)$ then $\sigma_{D_{(i)}} - cl(A) = A,$ so

$$Q - cl(A) = \bigcap_{i=1}^{4}\left(\sigma_{D_{(i)}} - cl(A)\right) = A.$$

Theorem 2.2. *Let $\left(D, \sigma_Q\right)$ be a Q-topological space, $d \in D$ and $A \subseteq D$, then $d \in (Q - cl(A))$ if and only if for each Q-open set H with $d \in H$ we have $H \cap A \neq \varnothing$.*

Proof: Let $d \in (Q - cl(A))$ and suppose that there is a Q-open set H with $d \in H$ and $H \cap A = \varnothing$. Now, $A \subseteq (D - H)$, so $(Q - cl(A)) \subseteq (Q - cl(D - H)) = (D - H)$. Hence $d \in (Q - cl(A)) \subseteq (D - H)$, so $d \in (D - H)$, which is a contradiction. Thus, $H \cap A \neq \varnothing$ for each Q-open set H with $d \in H$.

Conversely, suppose that $H \cap A \neq \varnothing$ for each Q-open set H with $d \in H$. But $d \notin (Q - cl(A))$, so $d \in D - (Q - cl(A)) = D - \bigcap_{i=1}^{4}\left(\sigma_{D_{(i)}} - cl(A)\right) = \bigcup_{i=1}^{4}\left(D - \sigma_{D_{(i)}} - cl(A)\right)$. Therefore $d \in \left(D - \left(\sigma_{D_{(i)}} - cl(A)\right)\right)$ for some $(i = 1 , 2 , 3 , 4)$.

But $D - \left(\sigma_{D_{(i)}} - cl(A)\right)$ is a Q-open set then $A \cap \left(D - \left(\sigma_{D_{(i)}} - cl(A)\right)\right) \neq \varnothing$, which is a contradiction, because $A \cap \left(D - \left(\sigma_{D_{(i)}} - cl(A)\right)\right) = \varnothing$.

Some properties of Q-Closure of Sets

Let $\left(D, \sigma_Q\right)$ be a Q-topological space and A , $B \subseteq D$ then the following properties are holds:

1) $A \subseteq Q - cl(A)$.
2) If $A \subseteq B$ then $Q - cl(A) \subseteq Q - cl(B)$.

Proof: Since $A \subseteq B$ then, $\left(\sigma_{D_{(i)}} - cl(A)\right) \subseteq \left(\sigma_{D_{(i)}} - cl(B)\right)$, for all $i = 1 ,2 , 3 , 4$. So,

$$\bigcap_{i=1}^{4}\left(\sigma_{D_{(i)}} - cl(A)\right) \subseteq \bigcap_{i=1}^{4}\left(\sigma_{D_{(i)}} - cl(A)\right).$$

Hence, $Q - cl(A) \subseteq Q - cl(B)$.

3) $Q - cl(A \cap B) \subseteq Q - cl(A) \cap Q - cl(B)$.

Remark 2.10. *$Q - cl(A) \cap Q - cl(B) \nsubseteq Q - cl(A \cap B)$, as we will see in the following example 2.7.*

Example 2.7. *Given the Q-topological space $(\mathbb{R}, \tau_u, \tau_{l.r}, \tau_{ind}, \tau_{r.r})$ and $A = \mathbb{Q}$, $B = Irr$ then $Q - cl(A \cap B) = \mathbb{Q} - cl(\varnothing) = \varnothing$. While, $Q - cl(A) = \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} = \mathbb{R}$ and $Q - cl(B) = \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} = \mathbb{R}$. So, $(Q - cl(A) \cap Q - cl(B)) = \mathbb{R} \nsubseteq Q - cl(A \cap B) = \varnothing$.*

4) $Q - cl\,(Q - cl(A)) = Q - cl(A)$.

Proof: $\supseteq$) Since $A \subseteq Q - cl(A)$, then by (2), $Q - cl(A) \subseteq Q - cl\,(Q - cl(A))$.

$\subseteq$) Since $Q - cl(A) \subseteq Q - cl(A) = \bigcap_{i=1}^{4}\left(\sigma_{D_{(i)}} - cl(A)\right)$, then $Q - cl(A) \subseteq \sigma_{D_{(i)}} - cl(A)$ for all $i = 1 , 2 , 3 , 4$. So $\sigma_{D_{(i)}} - cl\,(Q - cl(A)) \subseteq \sigma_{D_{(i)}} - cl(A)$ for all $i = 1 , 2 , 3 , 4$. Therefore,

$$\bigcap_{i=1}^{4}\left(\sigma_{D_{(i)}} - cl\,(Q - cl(A))\right) \subseteq \bigcap_{i=1}^{4}\left(\sigma_{D_{(i)}} - cl(A)\right).$$

Hence, $Q-cl\,(Q-cl(A)\,)\subseteq Q-cl(A)$.

5) $Q-cl(A\cup B)=Q-cl(A)\cup Q-cl(B)$.

Proof: Since $A\subseteq Q-cl(A)$ and $B\subseteq Q-cl(B)$, then $A\cup B\subseteq[Q-cl(A)\cup Q-cl(B)]$.

So, $Q-cl(A\cup B)\subseteq Q-cl\,(Q-cl(A)\cup Q-cl(B))=Q-cl(A)\cup Q-cl(B)$.

Conversely, Since $A\subseteq A\cup B$ then $Q-cl(A)\subseteq Q-cl(A\cup B)$, similarly, $B\subseteq A\cup B$ then $Q-cl(B)\subseteq Q-cl(A\cup B)$. So, $Q-cl(A)\cup Q-cl(B)\subseteq Q-cl(A\cup B)$.

Definition 2.7. *Let $\left(D,\sigma_Q\right)=\left(D,\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}}\right)$ be a Q- topological space and $A\subseteq D$ then A is called a Q- dense set in D if $Q-cl(A)=D$.*

Example 2.8. *Let D= $\mathbb{R}$ and consider the Q-topological space $\left(\mathbb{R},\sigma_{ind},\sigma_{l.r},\sigma_u,\sigma_{cof}\right)$. Then A= $\mathbb{Q}$ is a Q-dense set in $\mathbb{R}$, but $A=(2\,,\,\infty)$ is not a Q-dense set in $\mathbb{R}$.*

2.3. **Quad Interior of Sets.** In this part, we introduced and discussed the concept of Q-interior of sets and studied its properties, giving many results and several illustrative examples of this concept.

Definition 2.8. *Let $(\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}})$ be a Q-topological space and $A\subseteq D$ then the quad interior of A in each σ_{D_i}, for $i=1,2,3,$ and 4, and denoted by $Q-int(A)$ or $Q-(A^O)$ that is meaning $Q-int(A)=\bigcup_{i=1}^{4}\sigma_{D_i}-int(A)$. Where $\sigma_{D_i}-int(A)$ is the interior of A in (D,σ_{D_i}).*

Example 2.9. *Consider the Q-topological space $(\mathbb{R},\sigma_{ind},\sigma_4,\sigma_{dis},\sigma_{l.r})$ and $A=[\,2,3)$ then: the $Q-int(A)=\varnothing\cup\varnothing\cup[\,2,3)\,\cup\varnothing=[\,2,3)$.*

Example 2.10. *Given the Q-topological space $(\mathbb{R},\sigma_2,\sigma_3,\sigma_4,\sigma_5)$ and $A=(1,4)$ then*

$$Q-int(A)=(1,4)\cup(1,4)\cup\varnothing\cup\varnothing=(1,4).$$

Some Properties of Q-Interior of Sets:

1) Let (D,σ_Q) be a Q-topological space and $A,B\subseteq D$ then: $Q-int(A)\subseteq A$

Proof: Since $\sigma_{D_i}-int(A)\subseteq A$ for all $i=1,2,3,4$, then $\bigcup_{i=1}^{4}\sigma_{D_i}-int(A)\subseteq A$ so $Q-int(A)\subseteq A$.

2) $Q-int(A)$ is not a Q-open set every cases.

Example 2.11. *Let $D=\{1,2,3,4\}$ and $\sigma_{D_{(1)}}=\{\varnothing,D,\{1\}\}$, $\sigma_{D_{(2)}}=\{\varnothing,D,\{2\}\}$, $\sigma_{D_{(3)}}=\{\varnothing,D,\{3\}\}$, $\sigma_{D_{(4)}}=\{\varnothing,D,\{4\}\}$. If $A=\{1,2\}$ then $Q-int(A)=\{1\}\cup\{2\}\cup\varnothing\cup\varnothing=\{1,2\}$ but $\{1,2\}$ is not Q-open set.*

3) If $A\subseteq B$, then $Q-int(A)\subseteq Q-int(B)$.

Proof: Since $A\subseteq B$, then $\sigma_{D_{(i)}}-int(A)\subseteq\sigma_{D_{(i)}}-int(B)$ for all $i=1,2,3$ and 4. Hence, $\bigcup_{i=1}^{4}\sigma_{D_{(i)}}-\ int(A)\subseteq\bigcup_{i=1}^{4}\sigma_{D_{(i)}}-\ int(B)$ Therefore, $Q-int(A)\subseteq Q-int(B)$.

4) $Q-int(A)=Q-int\,(Q-int(A))$.

Proof: $\supseteq$: since $Q-int(A)\subseteq A$ then, $Q-int\,(Q-int(A))\subseteq Q-int(A)$

$\subseteq$: since $\sigma_{D_{(i)}}-int(A)\subseteq\bigcup_{i=1}^{4}\sigma_{D_{(i)}}-int(A)=Q-int(A)$ for $i=1,2,3,4$ then $\tau_{D_{(i)}}-int(A)=\sigma_{D_{(i)}}-int\left(\sigma_{D_{(i)}}-int(A)\right)\subseteq\sigma_{D_{(i)}}-int\,(Q-int(A))$ so, $\bigcup_{i=1}^{4}\sigma_{D_{(i)}}-int(A)\subseteq\bigcup_{i=1}^{4}\sigma_{D_{(i)}}-int\,(Q-int(A))$. Hence $Q-int(A)\subseteq Q-int\,(Q-int(A))$.

5) $Q-int(A\cap B)=Q-int(A)\cap Q-int(B)$.

Proof: Since $A\cap B\subseteq A$ then $Q-int(A\cap B)\subseteq Q-int(A)$ similarly $Q-int(A\cap B)$ $\subseteq$: $Q-int(B)$. Therefore$Q-int(A\cap B)\subseteq Q-int(A)\cap Q-int(B)$ conversely.

6) $Q-int(A)\cup Q-int(B)\subseteq Q-int(A\cup B)$.

Proof: Since $A\subseteq A\cup B$ then $Q-int(A)\subseteq Q-int(A\cup B)$ in the same way $Q-int(B)\subseteq Q-int(A\cup B)$, hence the result.

Remark 2.11. *The opposite side of property* (6)*is not always true.*

Example 2.12. *Consider the Q-topological space* $(\mathbb{R},\sigma_{ind},\sigma_u,\sigma_{l.r},\sigma_{r.r})$ *and* $A=[\,0,1)$, $B=[\,1,3)$. *Now,* $Q-int(A)=(0,1)$ *and* $Q-int(B)=(1,3)$ *and* $Q-int(A\cup B)=(0,3)\nsubseteq Q-int(A)\cup Q-int(B)$.

3. Some Covering Properties of Quad Topological Spaces

3.1. ***Q*-Compact Spaces.** In this section, *Q*-open sets will be used to define some types of *Q*-open covers such as: $\sigma_{(1234)}$-open covers and $\sigma_{(1-4)}$-open cover in order to use them in defining *Q*-compact spaces and *S*-*Q*-compact spaces. The characteristics of these spaces will be discussed and many facts about these concepts and the relationship between them and other topological spaces will be concluding.

Definition 3.1. *Let* $D=\left(D,\ \sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}}\right)$ *be a Q-topological space, a cover* $\widetilde{Q}$ *of D is called* $\sigma_{(1-4)}$*-open cover if* $\widetilde{Q}\subseteq\sigma_{D_{(1)}}\cup\sigma_{D_{(2)}}\cup\sigma_{D_{(3)}}\cup\sigma_{D_{(4)}}$. *If* $\widetilde{Q}$ *has at least one element from each* $\sigma_{D_{(i)}}$ *it is called* $\sigma_{(1234)}$*-open cover.*

Definition 3.2. *A Q-topological space* $D=(D,\sigma_Q)$ *is called S-Q-compact, if every* $\sigma_{(1-4)}$*-open cover of D has a finite sub-cover.*

Definition 3.3. *A Q-topological space* $D=(D,\sigma_Q)$ *is called a Q-compact, if every* $\sigma_{(1234)}$*-open cover of D has a finite sub-cover.*

Definition 3.4. *A Q-topological space* $D=(D,\sigma_Q)$ *is called compact if* $(D,\sigma_{D_{(i)}})$ *is compact for all* $i=1\ ,\ 2\ ,\ 3\ ,\ 4$.

Remark 3.1. *Every* $\sigma_{(1234)}$*-open cover is a* $\sigma_{(1-4)}$*-open cover.*

Remark 3.2. *Every $\sigma_{D_{(i)}}$-open cover is a $\sigma_{(1-4)}$-open cover.*

Example 3.1. *The Q-topological space $\left(\mathbb{R},\sigma_{ind},\sigma_{cof},\sigma_{r.r},\sigma_{l.r}\right)$ is a Q-compact but is not S-Q-compact.*

Since any $\sigma_{(1234)}$-open cover must contains $\mathbb{R}$ as a mandatory element taken from σ_{ind} then $\{\mathbb{R}\}$ is a finite sub-cover of any $\sigma_{(1234)}$-open cover of $\mathbb{R}$. But $\left(\mathbb{R},\sigma_{ind},\sigma_{cof},\sigma_{r.r},\sigma_{l.r}\right)$ is not S-Q- compact, since $\widetilde{Q}=\{(-\infty,a) : a\in Q\}$ is a $\sigma_{(1-4)}$-open cover of $\mathbb{R}$ which has no finite sub-cover.

Example 3.2. *The Q-topological space $\left(\mathbb{R},\sigma_{dis},\sigma_{cof},\sigma_{coc},\sigma_{u}\right)$ is a Q-compact but is not $S-Q$-compact. Let $\widetilde{Q}=\{Q_\alpha : \alpha\in\Delta\}$ be a $\sigma_{(1234)}$-open cover of the space $\mathbb{R}$, then there exist at least one member of $\widetilde{Q}$ such $U\in\sigma_{cof}$. Since $u\in\sigma_{cof}$ then $(\mathbb{R}-U)=$ finite set say; $(\mathbb{R}-U)=\{r_1,r_2,r_3,\ldots,r_n\}$. Now; for each $r_i\in(\mathbb{R}-U)$ chose $Q_i\in\widetilde{Q}$ such that $r_i\in Q_i$, then $\widetilde{\acute{Q}}=U\cup\{Q_1,Q_2,\ldots,Q_n\}$ is a finite sub-cover of $\widetilde{Q}$ for $\mathbb{R}$.*

Therefore, $\left(\mathbb{R},\sigma_{dis},\sigma_{cof},\sigma_{coc},\sigma_{u}\right)$ is Q-compact. But $\left(\mathbb{R},\sigma_{dis},\sigma_{cof},\sigma_{coc},\sigma_{u}\right)$ is not S-Q-compact, since $\widetilde{Q}=\{\{q\} : q\in\mathbb{R}\}\subseteq\tau_{dis}$ is a $\sigma_{(1-4)}$-open cover which has no finite sub-cover.

Example 3.3. *The Q-topological space $\left(\mathbb{R},\sigma_{ind},\sigma_{cof},\sigma_{ind},\sigma_{cof}\right)$ is S-Q-compact, Q-compact and compact space.*

Example 3.4. *Let $D=\mathbb{R}$ and $\mathcal{B}_1=\{D,\{d\} : d\in D-\{0\}\}$, $\mathcal{B}_2=\{D,\{d\} : d\in D-\{1\}\}$, $\mathcal{B}_3=\{D,\{d\} : d\in D-\{2\}\}$, $\mathcal{B}_4=\{D,\{d\} : d\in D-\{3\}\}$.*

Let $\sigma_1,\sigma_2,\sigma_3$ and σ_4 be the topologies induced by $\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3$ and $\mathcal{B}_4$ respectively. Now $(\mathbb{R},\sigma_i)$ is compact for all $i=1$, 2 , 3 , 4, since any σ_i -open cover must contain $\mathbb{R}$, so $\{\mathbb{R}\}$ is a finite sub-cover for any σ_i-open cover of $\mathbb{R}$. Therefore, $(\mathbb{R},\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ is compact. Now, $(\mathbb{R},\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ is not Q-compact and not S-Q-compact, since the $\widetilde{Q}=\{\{d\} : d\in\{\mathbb{R}\}\}$ is a $\sigma_{(1234)}$-open cover which is also a $\sigma_{(1-4)}$-open cover has no finite sub-cover.

Definition 3.5. *Let $\left(D,\sigma_Q\right)=(\,D,\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}})$ be a Q-topological space then the least upper bound topology or the supremum topology of $\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}}$ and $\sigma_{D_{(4)}}$ is the smallest topology that contains all the topologies$\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}}$ and $\sigma_{D_{(4)}}$ and denoted by σ_{sub}.*

Example 3.5. *Consider the Q-topological space $\left(\mathbb{R},\sigma_{ind},\sigma_{cof},\sigma_{coc},\sigma_{u}\right)$ then the suprem topology of σ_{ind}, σ_{cof}, σ_{coc} and σ_u is σ_u.*

Theorem 3.1. *Let $\left(D,\sigma_Q\right)=(\,D,\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}})$ be a Q-topological space. If (D,σ_{sub}) is compact then $\left(D,\sigma_Q\right)$ is S-Q-compact.*

Proof: Let $\widetilde{Q}=\{Q_\alpha : \alpha\in\Delta\}$ be any $\sigma_{(1-4)}$-open cover of $\left(D,\sigma_Q\right)$ then $\widetilde{Q}\subseteq\bigcup_{i=1}^{4}\sigma_{D_{(i)}}\subseteq\sigma_{sub}$. Hence $\widetilde{Q}$ is an open cover of a compact space (D,σ_{sub}), so $\widetilde{Q}$ has a finite sub-cover. Therefore $\left(D,\sigma_Q\right)$ is $S-Q$-compact.

Remark 3.3. *If we say that a Q-topological space $\left(D,\sigma_Q\right)$ has a particular topological property P then we mean that all $\sigma_{D_{(1)}}$, $\sigma_{D_{(2)}}$, $\sigma_{D_{(3)}}$ and $\sigma_{D_{(4)}}$ have the property P.*
For instance, $\left(D,\sigma_Q\right)$ is said to be T_0-space if$\left(D,\sigma_{D_{(1)}}\right)$, $\left(D,\sigma_{D_{(2)}}\right)$, $\left(D,\sigma_{D_{(3)}}\right)$and $(D,\sigma_{D_{(4)}})$ are T_0-space.

Example 3.6. *The Q-topological space* $(\mathbb{R}, \sigma_{dis}, \sigma_{l.r}, \sigma_{r.r}, \sigma_u)$ *is* T_0*-space.*

Theorem 3.2. *If* D *is a finite set then let* $\left(D, \sigma_Q\right) = (\ D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}})$ *is Q-compact , S-Q-compact and compact.*

Proof: Suppose that D is a finite set say $D = \{d_1, d_2, d_3, \ldots, d_n\}$. Let $\widetilde{Q} = \{Q_\alpha : \alpha \in \Delta\}$ be any $\sigma_{(1-4)}-$open cover of D then for all $d_i \in D$ choose $Q_{\alpha_i} \in \widetilde{Q}$ such that $d_i \in Q_{\alpha_i}$, so

$$\overset{\frown}{\widetilde{Q}} = \{Q_{\alpha_1}, Q_{\alpha_2}, Q_{\alpha_3}, \ldots, Q_{\alpha_n}\}$$

is a finite sub-cover of $\widetilde{Q}$ for D thus $\left(D, \sigma_Q\right)$ is S-Q-compact. Since any $\sigma_{(1234)}$ or any $\sigma_{D_{(i)}}-$open cover is a $\sigma_{(1-4)}$-open cover we have $\left(D, \sigma_Q\right)$ is Q-compact and compact.

Theorem 3.3. *A Q-topological space* $\left(D, \sigma_Q\right)$ *is S-Q-compact if and only if it is compact and Q-compact.*

proof: Suppose that the space $\left(D, \sigma_Q\right)$ is S-Q-compact then any $\sigma_{D_{(i)}}$-open cover for $i = 1\ ,\ 2\ ,\ 3\ ,\ 4$ or any $\sigma_{(1234)}$-open cover is a $\sigma_{(1-4)}$-open cover, so it has a finite sub-cover. Thus, $\left(D, \sigma_Q\right)$ is compact and Q-compact. Conversely, suppose that $\left(D, \sigma_Q\right)$ is compact and Q-compact. let $\widetilde{Q} = \{Q_\alpha : \alpha \in \Delta\}$ be any $\sigma_{(1-4)}$-open cover of the space $\left(D, \sigma_Q\right)$. If $\widetilde{Q}$ is a $\sigma_{(1234)}-$open cover, it has a finite sub-cover.
If $\widetilde{Q}$ is a $\sigma_{D_{(i)}}$-open cover for some $i = 1\ ,\ 2\ ,\ 3\ ,\ 4$ then it has a finite sub-cover because $(D, \sigma_{D_{(i)}})$, is compact. If $\widetilde{Q}$ is not $\sigma_{(1234)}$-open cover and not $\sigma_{D_{(i)}}$-open cover then we have two cases:

Case (1): $\widetilde{Q} \subseteq \sigma_{D_{(j)}} \cup \sigma_{D_{(k)}}$ for some $j, k \in \{1,2,3,4\}$ say $\widetilde{Q} = \{U_\alpha : \alpha \in \Delta\} \cup \left\{V_\beta : \beta \in \Delta\right\}$ where $U_\alpha \in \sigma_{D_{(j)}}$ and $V_\beta \in \sigma_{D_{(k)}}$ $\forall\ \alpha, \beta \in \Delta$ since $(D,\ \sigma_{D_{(j)}})$ and $(D,\ \sigma_{D_{(k)}})$ are compact space there exist a finite subset Δ_1, Δ_2 of Δ such that $\overset{\frown}{\widetilde{Q}} = \{u_\alpha : \alpha \in \Delta_1\} \cup \left\{v_\beta : \beta \in \Delta_2\right\}$ is a finite sub-cover of $\widetilde{Q}$ for Δ.

Case (2): $\widetilde{Q} \subseteq \sigma_{D_{(j)}} \cup \sigma_{D_{(k)}} \cup \sigma_{D_{(m)}}$ for some $j, k, m \in \{1,2,3,4\}$ say, $\widetilde{Q} = \{U_\alpha : \alpha \in \Delta\} \cup \left\{v_\beta : \beta \in \Delta\right\} \cup \{w_\delta : \delta \in \Delta\}$ where $u_\alpha \in \sigma_{D_{(j)}}$, $v_\beta \in \sigma_{D_{(k)}}$, $w_\delta \in \sigma_{D_{(m)}}$ for all $\alpha, \beta, \delta \in \Delta$ then there exist a finite subset Δ_1, Δ_2 and Δ_3 of Δ such that $\widetilde{Q} = \{u_\alpha : \alpha \in \Delta_1\} \cup \left\{v_\beta : \beta \in \Delta_2\right\} \cup \{w_\delta : \delta \in \Delta_3\}$ is a finite sub-cover of $\widetilde{Q}$ for D Therefore, $\left(D, \sigma_Q\right)$ is S-Q-compact.

Example 3.7. *The Q-topological space* $\left(\mathbb{R}, \sigma_5, \sigma_6, \sigma_7, \sigma_{cof}\right)$ *is Q-compact but not compact so the space is not S-Q-compact.*

Remark 3.4. *A topological space* (D, σ_D) *is called hereditary compact if every subspace of* D *is compact.*

Definition 3.6. *A Q-topological space* $(\ D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}})$ *is hereditary compact if each* $(D, \sigma_{D_{(i)}})$ *is hereditary compact for all* $i = 1\ ,\ 2\ ,\ 3\ ,\ 4$.

Theorem 3.4. *If* $(\ D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}})$ *is a hereditary compact space then it is S-Q-compact.*

proof: Let $\widetilde{Q} = \{u_\alpha : \alpha \in \Delta\} \cup \left\{v_\beta : \beta \in \Delta\right\} \cup \{w_\delta : \delta \in \Delta\} \cup \{H_\xi : \xi \in \Delta\}$ be a $\sigma_{(1-4)}$-open cover of the space D, where $u_\alpha \in \sigma_{D_{(1)}}$, $v_\beta \in \sigma_{D_{(2)}}, w_\delta \in \tau_{D_{(3)}}$ and $H_\xi \in \sigma_{D_{(4)}}$ for each $\alpha, \beta, \delta, \xi \in \Delta$. Since $U_1 = \bigcup \{u_\alpha : \alpha \in \Delta\}$ is $\sigma_{D_{(1)}}$-compact there exist a finite set $\Delta_1 \subseteq \Delta$ such that $U_1 = \bigcup \{u_\alpha : \alpha \in \Delta_1\}$.

Since $U_2 = \bigcup\left\{v_\beta : \beta \in \Delta\right\}$ is $\sigma_{D_{(2)}}-$ compact there exist a finite set $\Delta_2 \subseteq \Delta$ such that $U_2 = \bigcup\left\{v_\beta : \beta \in \Delta_2\right\}$.

Since $U_3 = \bigcup\{w_\delta : \delta \in \Delta\}$ is $\sigma_{D_{(3)}}$-compact there exist a finite set $\Delta_3 \subseteq \Delta$ such that $U_3 = \bigcup\{w_\delta : \delta \in \Delta_3\}$.

Since $U_4 = \bigcup\{H_\xi : \xi \in \Delta\}$ is $\sigma_{D_{(4)}}$-compact there exist a finite set $\Delta_4 \subseteq \Delta$ such that $U_4 = \cup\{H_\xi : \xi \in \Delta_4\}$.

Thus $\hat{\widetilde{Q}} = \{u_\alpha : \alpha \in \Delta_1\} \cup \left\{v_\beta : \beta \in \Delta_2\right\} \cup \{w_\delta : \delta \in \Delta_3\} \cup \{H_\xi : \xi \in \Delta_4\}$ is a finite sub-cover of $\widetilde{Q}$ for D therefore $\left(D, \sigma_Q\right)$ is *S*-*Q*-compact.

Theorem 3.5. *Every $\sigma_{D_{(i)}}$-closed proper subset of a S-Q-compact space is $\sigma_{D_{(i)}}$-compact for $(i \neq j,\ i, j = 1, 2, 3, 4)$.*

Proof: Let K be a non-empty $\sigma_{D_{(i)}}$-closed proper subset of $\left(D, \sigma_Q\right)$. Let $\widetilde{U} = \left\{U_j : j \in \Delta\right\}$ be a $\sigma_{D_{(i)}}$-open cover of K. For each $d \in (D - K)$, then there exist a $\sigma_{D_{(i)}}$-open set $H_{(d)}$ such that $d \in H_{(d)} \subseteq (D - K)$. Since $\emptyset \neq K \neq D$, then $\left\{U_j : j \in \Delta\right\} \cup \left\{H_{(d)} : d \in D - K\right\}$ is a $\sigma_{(1-4)}$-open cover of a *S*-*Q*-compact space $\left(D, \sigma_Q\right)$, so there exist a finite subset $\Delta_1 \subseteq \Delta$ and a finite set $\{d_1, d_2, \ldots, d_n\} \subseteq D - K$ such that $\left\{U_j : j \in \Delta_1\right\} \cup \left\{H_{(d_1)}, H_{(d_2)}, \ldots, H_{(d_n)}\right\}$ is a finite sub-cover of the space D. So $\widetilde{V} = \left\{U_j : j \in \Delta_1\right\}$ is a finite sub-cover of $\widetilde{U}$ for K. Therefore, K is a $\sigma_{D_{(i)}}$-compact.

Example 3.8. *Consider $\left(\mathbb{R}, \sigma_{ind}, \sigma_{cof}, \sigma_{cof}, \sigma_{ind}\right)$ and $K = \{1\ ,\ 2\ ,\ 3\ , \ldots, n\}$, $n \in \mathbb{N}$ then K is a closed set in $\left(\mathbb{R},\ \sigma_{cof}\right)$, so K is compact in $(\mathbb{R}, \sigma_{ind})$, and $\left(\mathbb{R}, \sigma_{cof}\right)$.*

Definition 3.7. *Let $\left(D, \tau_Q\right) = (\ D, \tau_{D_{(1)}}, \tau_{D_{(2)}}, \tau_{D_{(3)}}, \tau_{D_{(4)}})$ and $\left(E, \sigma_Q\right) = \left(E\ ,\ \sigma_{E_{(1)}},\ \sigma_{E_{(2)}},\ \sigma_{E_{(3)}}, \sigma_{E_{(4)}}\right)$ be a Q-topological spaces, then $f : \left(D, \tau_Q\right) \longrightarrow \left(E, \sigma_Q\right)$ is Q-continuous (open , closed , homeomorphism; respectively) if $f : (D, \tau_{D_i}) \longrightarrow (E, \sigma_{E_i})$ is continuous (open , closed , homeomorphism; respectively).*

Theorem 3.6. *Every Q-continuous image of a Q-compact space is a Q-compact.*

Proof: Let $f : (D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}}) \longrightarrow (E, \sigma_{E_{(1)}}, \sigma_{E_{(2)}}, \sigma_{E_{(3)}}, \sigma_{E_{(4)}})$ be a *Q*-continuous function and $\left(D, \sigma_Q\right)$ is a *Q*-compact. Given $\widetilde{E} = \left\{U_j : j \in \Delta\right\} \cup \left\{V_\beta : \beta \in \Delta\right\} \cup \{W_\delta : \delta \in \Delta\} \cup \{H_\xi : \xi \in \Delta\}$ be a $\sigma_{(1234)}$-open cover of E where $U_\alpha \in \tau_{E_{(1)}}$, $\forall \alpha \in \Delta$, $V_\beta \in \tau_{E_{(2)}}, \forall \beta \in \Delta$, $W_\delta \in \tau_{E_{(3)}}, \forall \delta \in \Delta$ and $H_\xi \in \sigma_{D_{(4)}}, \forall \xi \in \Delta$. Let $\widetilde{D} = \left\{f^{-1}(U_\alpha) : \alpha \in \Delta\right\} \cup \left\{f^{-1}\left(V_\beta\right) : \beta \in \Delta\right\} \cup \left\{f^{-1}(W_\delta) : \delta \in \Delta\right\} \cup \left\{f^{-1}(H_\xi) : \xi \in \Delta\right\}$ then $f^{-1}(U_\alpha) \in \sigma_{D_{(1)}}$, $\forall \alpha \in \Delta$, $f^{-1}\left(V_\beta\right) \in \sigma_{D_{(2)}}$, $\forall \beta \in \Delta$, $f^{-1}(W_\delta) \in \tau_{D_{(3)}}$, $\forall \delta \in \Delta$, $f^{-1}(H_\xi) \in \sigma_{D_{(4)}}$, $\forall \xi \in \Delta$, because f is *Q*-continuous. Now $\widetilde{D}$ is a $\sigma_{(1234)}$-open cover of D so there exist a finite subset $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 of Δ such that $\widetilde{D^*} = \left\{f^{-1}(U_\alpha) : \alpha \in \Delta_1\right\} \cup \left\{f^{-1}\left(V_\beta\right) : \beta \in \Delta_2\right\} \cup \left\{f^{-1}(W_\delta) : \delta \in \Delta_3\right\} \cup \left\{f^{-1}(H_\xi) : \xi \in \Delta_4\right\}$ is a finite sub-cover of $\widetilde{D}$ for D.

Let $\widetilde{E^*} = \left\{U_j : j \in \Delta_1\right\} \cup \left\{V_\beta : \beta \in \Delta_2\right\} \cup \{W_\delta : \in \Delta_3\} \cup \{H_\xi : \xi \in \Delta_4\}$ then $\widetilde{E^*}$ is a finite sub cover of $\widetilde{E}$ for E. Therefore, $\left(E, \sigma_Q\right)$ is *Q*-compact space.

Theorem 3.7. *Every Q-continuous image of a S-Q-compact space is a S-Q-compact.*

Proof: The prove is similar as we show in theorem 3.6.

3.2. Q **Lindel$\ddot{o}$f Space.** In this section, Q-open sets will be used to define some types of Q-open covers such as: $\sigma_{(1234)}$-open covers and $\sigma_{(1-4)}$open cover in order to use them in defining Q-lindel$\ddot{o}$f spaces and S-Q-lindel$\ddot{o}$f spaces. The characteristics of these spaces will be discussed and many facts about these concepts and the relationship between them and other Q-topological spaces will be concluding.

Definition 3.8. *A quad topological space $D = (D, \sigma_Q)$ is called S-Q-lindelöf, if every $\sigma_{(1234)}$-open cover of D has a countable sub-cover.*

Definition 3.9. *A quad topological space $D = (D, \sigma_Q)$ is called S-Q-lindelöf, if every $\sigma_{(1-4)}$-open cover of D has a countable sub-cover.*

Definition 3.10. *A quad topological space $D = (D, \sigma_Q)$ is called lindelöf, if $(D, \sigma_{D_{(i)}})$ is lindelöf for all $i = 1, 2, 3, 4$.*

Example 3.9. *The Q-topological space $(\mathbb{R}, \sigma_{ind}, \sigma_{coc}, \sigma_{r.r}, \sigma_{l.r})$ is a Q-lindelöf, S-Q-lindelöf and lindelöf space. Since any $\sigma_{(1234)}$-open cover must contains $\mathbb{R}$ as a mandatory element take from τ_{ind}, then $\{\mathbb{R}\}$ is a countable sub-cover of any $\sigma_{(1234)}$-open cover of $\mathbb{R}$.*

Since each of the space $(\mathbb{R}, \sigma_{ind})$, $(\mathbb{R}, \sigma_{r.r})$, $(\mathbb{R}, \sigma_{l.r})$ and $(\mathbb{R}, \tau_{coc})$ are lindelöf space the $(\mathbb{R}, \sigma_{ind}, \sigma_{coc}, \sigma_{r.r}, \sigma_{l.r})$ is S-Q-lindelöf. Now; to show that $(\mathbb{R}, \sigma_{ind}, \sigma_{coc}, \sigma_{l.r})$ is S-Q-lindelöf, let $\widetilde{Q} = \{Q_\alpha : \alpha \in \Delta\}$ be any $\sigma_{(1-4)}$-open cover of $\mathbb{R}$. If $Q_\alpha \in \sigma_{ind}$ for some $\alpha \in \Delta$ then $Q_\alpha \in \mathbb{R}$ so $\{\mathbb{R}\}$ is countable sub-cover of any $\sigma_{(1-4)}$-open cover of $\mathbb{R}$. If $Q_\alpha \in \sigma_{coc}$ for some $\alpha \in \Delta$, then $(D - Q_\alpha)$ is a countable set, say $(D - Q_\alpha) = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. So for each $\alpha_i \in \Delta$, choose $Q_{\alpha_i} \in Q$ such that $\alpha_i \in Q_{\alpha_i}$, hence $\overset{\frown}{Q} = \{D - Q_\alpha\} \cup \{Q_{\alpha_1}, Q_{\alpha_2}, Q_{\alpha_3}, \ldots\}$ is a countable sub-cover of $\widetilde{Q}$ for $\mathbb{R}$.

Thus $(\mathbb{R}, \sigma_{ind}, \sigma_{coc}, \sigma_{r.r}, \sigma_{l.r})$ is S-Q-lindelöf. If $\widetilde{Q} \subseteq \tau_{l.r} \cup \tau_{r.r}$ then $\widetilde{Q}$ has a countable sub-cover of the form $\{(-\infty, a), (b, \infty) : a > 0, b < 0\}$, thus $(\mathbb{R}, \sigma_{ind}, \sigma_{coc}, \sigma_{r.r}, \sigma_{l.r})$ is a S-Q-lindelöf.

Example 3.10. $(\mathbb{R}, \sigma_5, \sigma_{coc}, \sigma_u, \sigma_{dis})$ *is Q-lindelöf not lindelöf and S-Q-lindelöf.*

Let $\widetilde{Q} = \{Q_\alpha : \alpha \in \Delta\}$ be any $\sigma_{(1234)}$-open cover of $\mathbb{R}$ then exist at least one $Q_\alpha \in \sigma_{coc}$. So $\mathbb{R}-Q_\alpha$ is a countable set say $(\mathbb{R}-Q_\alpha) = \{r_1, r_2, r_3, \ldots\}$. Now for each $r_i \in \mathbb{R}-Q_\alpha$, choose $Q_{\alpha_i} \in \widetilde{Q}$ such that $r_i \in Q_{\alpha_i}$. Take $\overset{\frown}{Q} = \{\mathbb{R}-Q_\alpha\} \cup \{Q_{\alpha_1}, Q_{\alpha_2}, Q_{\alpha_3}, \ldots\}$, then $\overset{\frown}{Q}$ is a countable sub-cover of $\widetilde{Q}$ for $\mathbb{R}$. Therefore, $(\mathbb{R}, \sigma_5, \sigma_{coc}, \sigma_u, \sigma_{dis})$ is Q-lindelöf. Now $(\mathbb{R}, \sigma_5, \sigma_{coc}, \sigma_u, \sigma_{dis})$ is not lindelöf and not S-Q-lindelöf since $\widetilde{Q} = \{\{d\} : d \in \mathbb{R}\}$ is a $\sigma_{(1-4)}$-open cover which is also σ_{dis}-open cover which has no countable sub-cover.

Example 3.11. *Let $D = \mathbb{R}$ and $\mathcal{B}_1 = \{\emptyset, D, \{d\} : d \in \mathbb{R} - \{1\}\}$, $\mathcal{B}_2 = \{\emptyset, D, \{d\} : d \in \mathbb{R} - \{2\}\}$, $\mathcal{B}_3 = \{\emptyset, D, \{d\} : d \in \mathbb{R} - \{3\}\}$, $\mathcal{B}_4 = \{\emptyset, D, \{d\} : d \in \mathbb{R} - \{4\}\}$, let $\sigma_1, \sigma_2, \sigma_3$, and σ_4 be the topologies induced by the boys $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$and $\mathcal{B}_4$ respectively, then $(\mathbb{R}, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is a lindelöf, not Q-lindelöf and not S-Q-lindelöf.*

Theorem 3.8. *Every Q(resp. $S - Q$)-compact space is Q(resp. $S - Q$)$-$lindelöf.*

Proof: Since every finite sub-cover is a countable sub-cover the result is directly indeed.

Theorem 3.9. *If D is a countable set then $(D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}})$ is Q-lindelöf, S-Q-lindelöf and lindelöf.*

Proof: Suppose that D is a countable set say $D = \{d_1, d_2, d_3, \ldots\}$. Let $\widetilde{Q} = \{Q_\alpha : \alpha \in \Delta\}$ be any $\sigma_{(1-4)}$-open cover of D then for all $d_i \in D$ choose $Q_{\alpha_i} \in \widetilde{Q}$ such that $d_i \in Q_{\alpha_i}$, so $\acute{\widetilde{Q}} = \{Q_{\alpha_1}, Q_{\alpha_2}, Q_{\alpha_3}, \ldots\}$ is a countable sub-cover of $\widetilde{Q}$ for D. Thus $\left(D, \sigma_Q\right)$ is S-Q-lindelöf. Since any $\sigma_{(1234)}$ or any $\sigma_{D_{(i)}}$-open cover is a $\sigma_{(1-4)}$-open cover we have $\left(D, \sigma_Q\right)$ is Q-lindelöf and lindelöf.

Theorem 3.10. *Every compact Q-topological space is a Q-lindelöf.*

Proof: Since every compact space is lindelöf, then any compact Q-topological space is lindelöf.

Remark 3.5. *A Q-topological space $(D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}})$ is hereditary $lindelöf$ if each $\left(D\ ,\ \sigma_{D_{(i)}}\right)$ is a hereditary lindelöf for all $i = 1\ ,\ 2\ ,\ 3\ ,\ 4$.*

Theorem 3.11. *If $(D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}})$ is a hereditary lindelöf space then it is S-Q- lindelöf.*

Proof: Let $\widetilde{Q} = \{Q_\alpha : \alpha \in \Delta\} \cup \left\{V_\beta : \beta \in \Delta\right\} \cup \{W_\delta : \delta \in \Delta\} \cup \{H_\xi : \xi \in \Delta\}$ be a $\sigma_{(1-4)}$-open cover of the space D, where $U_\alpha \in \sigma_{D_{(1)}}$, $V_\beta \in \sigma_{D_{(2)}}, W_\delta \in \sigma_{D_{(3)}}$ and $H_\xi \in \sigma_{D_{(4)}}$ for each $\alpha, \beta, \delta, \xi \in \Delta$. Since $U_1 = \bigcup \{U_\alpha : \alpha \in \Delta\}$ is $\sigma_{D_{(1)}} - \mathit{lindelöf}$ there exist a countable set $\Delta_1 \subseteq \Delta$ such that $U_1 = \bigcup \{U_\alpha : \alpha \in \Delta_1\}$. Since $U_2 = \bigcup \left\{V_\beta : \beta \in \Delta\right\}$ is $\sigma_{D_{(2)}} - \mathit{lindelöf}$ there exist a countable set $\Delta_2 \subseteq \Delta$ such that $U_2 = \bigcup \left\{V_\beta : \beta \in \Delta_2\right\}$. Since $U_3 = \cup \{W_\delta : \delta \in \Delta\}$ is $\sigma_{D_{(3)}} - \mathit{lindelöf}$ there exist a countable set $\Delta_3 \subseteq \Delta$ such that $U_3 = \bigcup \{W_\delta : \delta \in \Delta_3\}$. Since $U_4 = \bigcup \{H_\xi : \xi \in \Delta\}$ is $\sigma_{D_{(4)}} - \mathit{lindelöf}$ there exist a countable set $\Delta_4 \subseteq \Delta$ such that $U_4 = \bigcup \{H_\xi : \xi \in \Delta_4\}$.Thus $\acute{\widetilde{Q}} = \{U_\alpha : \alpha \in \Delta_1\} \cup \left\{V_\beta : \beta \in \Delta_2\right\} \cup \{W_\delta : \delta \in \Delta_3\} \cup \{H_\xi : \xi \in \Delta_4\}$ is a countable sub-cover of $\widetilde{Q}$ for D. Therefore, $\left(D, \sigma_Q\right)$ is S-Q-lindelöf.

Theorem 3.12. *Every $\sigma_{D_{(i)}}$-closed proper subset of a S-Q-lindelöf space is a $\sigma_{D_{(i)}} - \mathit{lindelöf}$ for $(i \neq j,\ i, j = 1\ ,\ 2\ ,\ 3\ ,\ 4)$.*

Proof: Let K be a non-empty $\sigma_{D_{(i)}}$-closed proper subset of $\left(D, \sigma_Q\right)$. Let $\widetilde{U} = \left\{U_j : j \in \Delta\right\}$ be a $\sigma_{D_{(i)}}$-open cover of K. For each $d \in (D - K)$ then there a $\sigma_{D_{(i)}}$-open set $H_{(d)}$ such that $d \in H_{(d)} \subseteq (D - K)$. Since $\varnothing \neq K \neq D$ then $\left\{U_j : j \in \Delta\right\} \cup \left\{H_{(d)} : d \in D - K\right\}$ is a $\sigma_{(1-4)}$-open cover of the space D. So $\widetilde{U} = \left\{U_j : j \in \Delta\right\}$ is a countable sub-cover of $\widetilde{U}$ for K. Therefore , K is $\sigma_{D_{(i)}}$-lindelöf.

Remark 3.6. *(Willard , 1972) Every second countable space is lindelöf.*

Theorem 3.13. *Every second countable Q-topological space is S-Q-lindelöf.*

Proof: Since every second countable space is lindelöf then every second countable Q-topological space is a hereditary lindelöf and so it is a S-Q-lindelöf.

Definition 3.11. *A family $\mathcal{F}$ of subset of a Q-topological space $\left(D, \sigma_Q\right)$ is $\sigma_{(1-4)}$-closed if every member of $\mathcal{F}$ is contain in $\bigcup_{i=1}^{4} \sigma_{D_{(i)}}$.*

Definition 3.12. *A family $\mathcal{F}$ of subset of a Q-topological space $\left(D, \sigma_Q\right)$ is $\sigma_{(1234)}$-closed if every member of $\mathcal{F}$ is contains at least one member of each $\sigma_{D_{(i)}}$ for $i = 1\ ,\ 2\ ,\ 3\ ,\ 4$.*

Definition 3.13. *(The countable intersection property), (Willard,1972) the family $\{D_\alpha : \alpha \in \Delta\}$ has the countable intersection property if every countable non-empty sub-family $D_{\alpha_1}, D_{\alpha_2}, D_{\alpha_3}, \dots$ of $\{D_\alpha : \alpha \in \Delta\}$ has the property $\bigcap_{i=1} D_{\alpha_i} \neq \varnothing$.*

Theorem 3.14. *A Q-topological space $\left(D, \sigma_Q\right)$ is S-Q-lindelöf if and only if every $\sigma_{(1-4)}$-closed family with the countable intersection property has non-empty intersection.*

Proof: Suppose that $\left(D, \sigma_Q\right)$ is S-Q-lindelöf and $\{D_\alpha : \alpha \in \Delta\}$ be a family of $\sigma_{(1-4)}$-closed sets which having the countable intersection property. Assume that $\bigcap_{\alpha \in \Delta} D_\alpha = \varnothing$. Then $\{D - D_\alpha : \alpha \in \Delta\}$ is a $\tau_{(1-4)}$-open sets and $\bigcup_{\alpha \in \Delta} (D - D_\alpha) = D - \bigcap_{\alpha \in \Delta} D_\alpha = D - \varnothing = D$. So $\{D - D_\alpha : \alpha \in \Delta\}$ is a $\tau_{(1-4)}$-open cover of a S-Q-lindelöf space D, so it has a countable sub-cover $\{(D - D_{\alpha_1}), (D - D_{\alpha_2}), (D - D_{\alpha_3}), \dots\}$ consequently, $\bigcup_{i=1} (D - D_{\alpha_i}) = D$ so $D - \bigcap_{i=1} D_{\alpha_i} = D$ which is meaning $\bigcap_{i=1} D_{\alpha_i} = \varnothing$. which is contradiction the fact that $\{D - D_\alpha : \alpha \in \Delta\}$ has the countable intersection property. Conversely, suppose that the $\sigma_{(1-4)}$-closed family $\{D_\alpha : \alpha \in \Delta\}$ having the countable intersection property has a non-empty intersection. let $\widetilde{Q} = \{Q_\alpha : \alpha \in \Delta\}$ be any $\sigma_{(1-4)}-$open cover of D. Then $\bigcup_{\alpha = \Delta} (Q_\alpha) = D$. Now $\{D - Q_\alpha : \alpha \in \Delta\}$ is a family of $\sigma_{(1-4)} - closed$ sets in D and $\bigcap_{\alpha \in \Delta} (D - Q_\alpha) = D - \bigcup_{\alpha = \Delta} (Q_\alpha) = D - D = \varnothing$. Since $\bigcap_{\alpha \in \Delta} (D - Q_\alpha) = \varnothing$, the family $\{D - Q_\alpha : \alpha \in \Delta\}$ dosńt have the countable intersection property, so there exist a countable sub-family $\{(D - Q_{\alpha_1}), (D - Q_{\alpha_2}), (D - Q_{\alpha_3}), \dots\}$ of $\{D - Q_\alpha : \alpha \in \Delta\}$ such that $\bigcap_{i=1} D - Q_{\alpha_i} = \varnothing \Rightarrow D - \bigcup_{i=1} (Q_{\alpha_i}) = \varnothing$ and so $\bigcup_{i=1} (Q_{\alpha_i}) = D$, hence $\{Q_{\alpha_1}, Q_{\alpha_2}, Q_{\alpha_3}, \dots\}$ is a countable sub-cover of $\widetilde{Q}$ for D. Thus $\left(D, \sigma_Q\right)$ is S-Q-lindelöf.

Theorem 3.15. *Every Q-continuous image of a Q-lindelöf space is a Q-lindelöf.*

Proof : Let $f : (D, \sigma_{D_{(1)}}, \sigma_{D_{(2)}}, \sigma_{D_{(3)}}, \sigma_{D_{(4)}}) \longrightarrow (E, \sigma_{E_{(1)}}, \sigma_{E_{(2)}}, \sigma_{E_{(3)}}, \sigma_{E_{(4)}})$ be a Q-continuous function and $\left(D, \sigma_Q\right)$ is a Q-lindelöf. Given $\widetilde{E} = \left\{U_j : j \in \Delta\right\} \cup \left\{V_\beta : \beta \in \Delta\right\} \cup \{W_\delta : \delta \in \Delta\} \cup \{H_\xi : \xi \in \Delta\}$

be a $\sigma_{(1234)}$-open cover of E where $U_\alpha \in \sigma_{E_{(1)}}$, $\forall\alpha \in \Delta$ $V_\beta \in \sigma_{E_{(2)}}, \forall\beta \in \Delta$, $W_\delta \in \sigma_{E_{(3)}}, \forall\delta \in \Delta$ and $H_\xi \in \sigma_{D_{(4)}}, \forall\xi \in \Delta$. Let $\widetilde{D} = \left\{f^{-1}(U_\alpha) : \alpha \in \Delta\right\} \cup \left\{f^{-1}\left(V_\beta\right) : \beta \in \Delta\right\} \cup \left\{f^{-1}(W_\delta) : \delta \in \Delta\right\} \cup \left\{f^{-1}(H_\xi) : \xi \in \Delta\right\}$ then $f^{-1}(U_\alpha) \in \sigma_{D_{(1)}}$, $\forall\alpha \in \Delta$, $f^{-1}\left(V_\beta\right) \in \sigma_{D_{(2)}}$, $\forall\beta \in \Delta$,$f^{-1}(W_\delta) \in \sigma_{D_{(3)}}$, $\forall\delta \in \Delta$,$f^{-1}(H_\xi) \in \sigma_{D_{(4)}}$, $\forall\xi \in \Delta$, because f is Q-continuous. Now $\widetilde{D}$ is a $\sigma_{(1234)}$-open cover of D so there exist a countably subset $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 of Δ such that $\widetilde{D^*} = \left\{f^{-1}(U_\alpha) : \alpha \in \Delta_1\right\} \cup \left\{f^{-1}\left(V_\beta\right) : \beta \in \Delta_2\right\} \cup \left\{f^{-1}(W_\delta) : \delta \in \Delta_3\right\} \cup \left\{f^{-1}(H_\xi) : \xi \in \Delta_4\right\}$ is a countable sub-cover of $\widetilde{D}$ for D.

Let $\widetilde{E^*} = \left\{U_j : j \in \Delta_1\right\} \cup \left\{V_\beta : \beta \in \Delta_2\right\} \cup \{W_\delta :\in \Delta_3\} \cup \{H_\xi : \xi \in \Delta_4\}$ then $\widetilde{E^*}$ is a countable sub cover of $\widetilde{E}$ for E. Therefore, $\left(E, \sigma_Q\right)$ is Q-lindelöf space.

Theorem 3.16. *Every Q-continuous image of a S-Q-lindelöf space is a S-Q-lindelöf.*

Proof: The prove is similar as we show in theorem 3.15.

3.3. Q**-Countably Compact Spaces.** In this section, Q-open sets will be used to define some types of Q-open covers such as: $\sigma_{(1234)}$-open covers and $\sigma_{(1-4)}$-open cover in order to use them in defining Q-countably compact spaces and S-Q-countably compact spaces. The characteristics of these spaces will be discussed and many facts about these concepts and the relationship between them and other Q-topological spaces will be concluding.

Definition 3.14. *A quad topological space $D = (D, \sigma_Q)$ is called Q-countably compact if every countably $\sigma_{(1234)}$-open cover of D has a finite sub-cover.*

Definition 3.15. *A quad topological space $D = (D, \sigma_Q)$ is called a S-Q-compact, if every countably $\sigma_{(1-4)}$-open cover of D has a finite sub-cover.*

Definition 3.16. *A quad topological space $D = (D, \sigma_Q)$ is called a countably compact if $(Q, \tau_{\sigma_{(i)}})$ is countably compact for all $i = 1$, 2 , 3 , 4.*

Example 3.12. *The Q-topological space $\left(\mathbb{R}, \sigma_{cof}, \sigma_5, \sigma_4, \sigma_2\right)$ is $\sigma_{(1234)}$-compact, $\sigma_{(1234)}$-lindelöf and $\sigma_{(1234)}$-countably compact. But the Q-space is not compact, not lindelöf and not countably compact. It is also the Q-space is not S-Q-compact, not S-Q-lindelöf and not S-Q-countably compact.*

Example 3.13. *The Q-topological space $\left(\mathbb{Z}, \sigma_{ind}, \sigma_{dis}, \sigma_{cof}, \sigma_{coc}\right)$ is $\sigma_{(1234)}$-compact, $\sigma_{(1234)}$-lindelöf and $\sigma_{(1234)}$-countably compact. In addition the Q-space is not S-Q-compact, S-Q-lindelöf and not S-Q-countably compact. We can show that the Q-space is not compact, lindelöf and not countably compact.*

Example 3.14. *The Q-topological space $\left(\mathbb{R}, \sigma_{dis}, \sigma_{coc}, \sigma_{Exc(5)}, \sigma_5\right)$ is $\sigma_{(1234)}$-lindelöf, not $\sigma_{(1234)}$-compact and not $\sigma_{(1234)}$-countably compact, since the $\sigma_{(1234)}$-open cover $\{\mathbb{R} - \mathbb{N}\} \cup \{\{n\} : n \in \mathbb{N}\}$ of $\mathbb{R}$ is also a $\sigma_{(1234)}$-countably open cover which has no finite sub-cover. It is also the Q-space is not compact, not lindelöf, not countably compact, not S-Q-compact, not S-Q-lindelöf and not S-Q-countably compact. To explain the previous cases. Consider the cover $\{\mathbb{R} - \mathbb{Q}\} \cup \{\{q\} : q \in \mathbb{Q}\}$ is a $\sigma_{(1-4)}$-open cover which is exactly a countably $\sigma_{(1-4)}$-open cover has no finite sub-cover. Thus, the Q-space is not S-Q-compact and not S-Q-countably* *compact. The $\sigma_{(1-4)}$-open cover $\{\{q\} : q \in \mathbb{R}\}$ of $\mathbb{R}$ has no countably sub-cover, so the Q-space is not S-Q-lindelöf. Since $(\mathbb{R},\sigma_5)$ is not compact, not lindelöf and not countably compact, the Q-space is not compact, not lindelöf and not countably compact.*

Theorem 3.17. *Let $\left(D,\sigma_Q\right) = (\ D,\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}})$ be a Q-topological space. If (D,σ_{sub}) is countably compact then $\left(D,\sigma_Q\right)$ is S-Q-countably compact.*

Proof: Let $\widetilde{Q} = \{Q_\alpha : \alpha \in \Delta\}$ be any $\sigma_{(1-4)}$-open cover of $\left(D,\sigma_Q\right)$ then $\widetilde{Q} \subseteq \bigcup_{i=1}^{4} \sigma_{D_{(i)}} \subseteq \sigma_{sub}$. Hence $\widetilde{Q}$ is an open cover of a countably compact space (D,σ_{sub}), so $\widetilde{Q}$ has a finite sub-cover. Therefore $\left(D,\sigma_Q\right)$ is $S-Q$-countably compact.

Theorem 3.18. *If D is a finite set then let $\left(D,\sigma_Q\right) = (\ D,\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}})$ is Q-countably compact, S-Q- countably compact and countably compact.*

Proof: The proof is exactly the same as the proof of theorem 3.2.

Theorem 3.19. *A Q-topological space $\left(D,\sigma_Q\right)$ is S-Q-countably compact if and only if it is countably compact and Q- countably compact.*

proof: The proof is exactly the same as the proof of theorem 3.3.

Example 3.15. *The Q-topological space $\left(\mathbb{R},\sigma_5,\sigma_6,\sigma_7,\sigma_{cof}\right)$ is Q-countably compact but not countably compact so the space is not S-Q-countably compact.*

Remark 3.7. *A topological space (D,σ_D) is called hereditary countably compact if every subspace of D is countably compact.*

Definition 3.17. *A Q-topological space $(\ D,\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}})$ is hereditary countably compact if each $(D,\sigma_{D_{(i)}})$ is hereditary countably compact for all $i = 1\ ,\ 2\ ,\ 3\ ,\ 4$.*

Theorem 3.20. *If $(\ D,\sigma_{D_{(1)}},\sigma_{D_{(2)}},\sigma_{D_{(3)}},\sigma_{D_{(4)}})$ is a hereditary countably compact space then it is S-Q-countably compact.*

proof: The proof is exactly the same as the proof of theorem 3.4.

Theorem 3.21. *Every $\sigma_{D_{(i)}}$-closed proper subset of a S-Q-countably compact space is $\sigma_{D_{(i)}}$-countably compact for $(i \neq j,\ i,j = 1,2,3,4)$.*

Proof: The proof is exactly the same as the proof of theorem 3.5.

Theorem 3.22. *Every Q-continuous image of a Q-countably compact space is a Q-countably compact.*

Proof: The proof is exactly the same as the proof of theorem 3.6.

Theorem 3.23. *Every Q-continuous image of a S-Q-countably compact space is a S-Q-countably compact.*

Proof: The prove is similar as we show in theorem 3.22.

4. Conclusion

Based on the basic concepts known in quad topological spaces, this study reveals and proves new properties and features of these spaces by introducing the concept of Q-open sets. Most of the properties and general facts relating to Q-open sets have been studied.

We discussed the concept of covering properties for quad topological spaces, such as Q-compact, Q-Lindel*ö*f, Q-countably compact spaces, S-Q-compact. S-Q-Lindel*ö*f and S-Q-countably compact spaces. We subsequently arrived at remarkable results, established facts, and new theories that support the efforts and ambitions of researchers in future studies related to Q-topological spaces.

5. Future Work

We recommend generalizing the results of this study to encompass various topological properties of quad, penta, and hexa topological spaces. We suggest prioritizing studies that explore and discuss new topics related to advanced covering properties in quad topological spaces.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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