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Existence Results to Tripled System of Fractional Langevin Equations via Measure of Noncompactness Technique

Ahmed Salem[1,*], **Lamya Almaghamsi**[2]

[1]*Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia*

[2]*Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 21589, Saudi Arabia*

[*]*Corresponding author:* lalmaghamsi@uj.edu.sa, ahmedsalem74@hotmail.com

Abstract. In this study, boundary-constrained tripled systems of fractional Langevin equations with some specified operators and generalized boundary conditions are taken into account. The uniqueness of the solution for the system under consideration are discovered using fractional calculus and the Banach fixed point theorem. We examine the assumptions and conditions that require solutions utilizing the tripled fixed point and the continuity coefficient by introducing noncompactness measures. To demonstrate the outcomes, we create a control problem examples to verify this correctness.

1. Introduction

A differential equation is a mathematical equation that relates some function of one or more variables with its derivatives, differential equations play a vital role in many fields including engineering, physics, economics and biology. Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behavior of complex systems.

Fractional differential equations are often the most effective way to explain the behavior of dynamical systems because standard differential equations may not be able to adequately capture these dynamics. The field of nonlinear fractional differential equations is very important, mainly because it is used as models to describe complex physical phenomena in various fields of engineering and sciences, especially in fluid mechanics, solid-state physics, plasma physics,

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optical fibers and chemical physics. Recently, fractional differential equations have received great attention due to their applications in many important applied fields such as population dynamics, heat conduction in materials with memory, seepage flow in porous media, autonomous mobile robots, fluid dynamics, traffic models, electro magnetic, aeronautics, economics, diffusion theory and so on, see for instance [1–5].

Differential equations can be classified as single, coupled, tripled, multi, or infinite systems based on the number of dependant variables. Each kind of system has several uses and is useful for mathematically simulating a variety of phenomena. Systems, both single and linked, have grown in prominence in a variety of practical applications.

These days, high frequency gadgets' performance is dependent on more than just their geometric dimensions. The surrounding circuitry has a big impact on it. For the extraction and creation of model parameters, this necessitates more (time-consuming) circuit design cycles. Furthermore, for a trustworthy assessment of the circuit operation, high frequency components of a circuit must be modeled with extreme precision. Therefore, it is advised to immediately combine circuit modeling with a device simulation of the high frequency part's components for complex circuits containing high frequency devices. This leads to coupled systems of partial differential equations of the elliptic and parabolic types as well as differential equations.

In other application domains, coupled systems of differential and Partial differential equations are becoming increasingly significant. For such coupled systems, theoretical research is still in its early stages. However, they have already demonstrated that an effective and successful coupled simulation depends on the presence and utilization of specific structures. Coupled systems of fractional-order differential equations equipped with a variety of boundary conditions received considerable attention [6–10]. Such systems find their applications in numerous real world phenomena such as ecological models, disease models, synchronization of chaotic systems, anomalous diffusion, etc. [11, 12].

In continuation, tripled system of fractional differential equations is considered as a generalization of fractional coupled system and constitute of three related fractional differential equations with three boundary, initial conditions or mix of them. However, tripled systems of differential equations are not often taken into account. Gene regulatory networks, three-species food chains, epidemiology, hormone dynamics in endocrine systems, microbial community dynamics, three-stage life cycles, and more are examples of applications of tripled systems of differential equations.

Matar et al. [13] introduced a tripled system of three related fractional differential equations with cyclic permutation boundary conditions. They studied existence and uniqueness results of solution through applying the Banach and Krasnoselskii fixed point theorems. Afshari et al. [14] studied existence and nonexistence of positive solutions to a tripled system of Riemann-Liouville fractional differential equations. In [15], Etemad et al. investigated the existence of tripled fractional differential system via using Banach contraction and tripled fixed point associated with measures of noncompactness due to Kuratowski and Hausdorff.

On the other hand, several problems in physics and engineering can be described by Langevin equation (see [16–22]). The generalization of the Langevin equations is acquired naturally by replacing the ordinary derivative with a fractional-order derivative that yields the popular fractional Langevin equations. According to the classical approach started by Langevin and known as the Einstein-Ornstein-Uhlenbeck theory of Brownian motion, normal diffusion and Brownian motion are associated with Langevin equation. More specifically, the classical Langevin equation addresses the dynamics of a Brownian particle through Newtona's law by incorporating the effect of the Stokes fluid friction and that of thermal fluctuations in the vicinity of the particle into a random force with suitably assigned properties. These properties are derived from the requirement that the particle velocity asymptotically attains a stationary Maxwellian distribution. Over the period of the diffusing particle, the random force arising from molecular collisions undergoes such a rapid fluctuations that it is approximated well by a white noise.

In this paper, we establish sufficient conditions for existence and uniqueness solution for tripled system of fractional Langevin equations in the form:

$$\begin{cases} {}^cD_{0+}^{r_1}({}^cD_{0+}^{s_1}+l_1)\omega_1(t)=\Psi_1(t,\omega_1(\rho_1 t),\omega_2(\rho_2 t),\omega_3(\rho_3 t)),\\ {}^cD_{0+}^{r_2}({}^cD_{0+}^{s_2}+l_2)\omega_2(t)=\Psi_2(t,\omega_1(\rho_1 t),\omega_2(\rho_2 t),\omega_3(\rho_3 t)),\\ {}^cD_{0+}^{r_3}({}^cD_{0+}^{s_3}+l_3)\omega_3(t)=\Psi_3(t,\omega_1(\rho_1 t),\omega_2(\rho_2 t),\omega_3(\rho_3 t)) \end{cases} \tag{1.1}$$

for all $t\in J:=[0,1]$. Subject to the following tripled boundary conditions

$$\begin{cases} \omega_1(0)=0,\ \omega_1(1)=\alpha_1\omega_1(\eta_1),\ {}^cD_{0+}^{s_1}\omega_1(0)=\beta_1{}^cD_{0+}^{s_1}\omega_1(\xi_1),\\ \omega_2(0)=0,\ \omega_2(1)=\alpha_2\omega_2(\eta_2),\ {}^cD_{0+}^{s_2}\omega_2(0)=\beta_2{}^cD_{0+}^{s_2}\omega_2(\xi_2),\\ \omega_3(0)=0,\ \omega_3(1)=\alpha_3\omega_3(\eta_3),\ {}^cD_{0+}^{s_3}\omega_3(0)=\beta_3{}^cD_{0+}^{s_3}\omega_3(\xi_3) \end{cases} \tag{1.2}$$

where ${}^cD_{0+}^{r_i}$ and ${}^cD_{0+}^{s_i}$ denote the Caputo fractional derivative of order $1<r_i\le 2$ and $0<s_i\le 1$, respectively, $\Psi_i:[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $0<\rho_i\le 1$, $l_i\in\mathbb{R}$ are the dissipative parameters, $\alpha_i,\ \beta_i\in\mathbb{R}$ and $0<\eta_i,\ \xi_i<1$ for $i=1,2,3$.

It worth mentioning that, when it comes to explaining physical processes like the branching process, brain networks, and polymer dissociation, infinite systems of differential equations play a crucial role. Investigating infinite systems of ODEs is commonly instructed by solving PDEs numerically. Another example relates to the use of semidiscretization to solve a number of difficult parabolic PDE problems.

One particularly interesting area of nonlinear functional analysis is the measures of noncompactness theory. As generalizations of compact operators, it allows us to select a very significant class of operators. In terms of a metric of noncompactness, those operators fulfill the Darbo contractions. Fixed point theory makes extensive use of noncompactness measurements. By using measures of noncompactness approaches, the Darbo fixed point theorem is a useful tool for examining the existence and uniqueness of solutions to differential and integral equations [23, 24]. It worth mentioning that many authors investigate the existence of solutions for infinite systems of differential and integral equations using the technique of the measure of noncompactness [25–30].

Our study is based on the use of the Banach fixed point to investigate the uniqueness of solution. Furthermore, it is based on use of the Darbo fixed point theorems in conjunction with the measure of noncompactness approach in a Banach space.

2. Basic Definitions and Lemmas

This part aims to present some fundamentals fixed point theorems and essential preliminaries related to the fractional calculus, especially in regard to the Caputo fractional-order operator.

Definition 2.1 ([31,32])**.** *The fractional integral of order $\sigma > 0$ with the lower limit a for a function ϕ can be defined as*

$$I^{\sigma}_{a^+}\psi(t) = \frac{1}{\Gamma(\sigma)}\int_a^t \frac{\psi(s)}{(t-s)^{1-\alpha}}ds, \qquad t > a$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2.2 ([31,32])**.** *The Caputo derivative of order σ with the lower limit a for a function ψ can be written as*

$${}^cD^{\sigma}_a\psi(t) = \frac{1}{\Gamma(n-\sigma)}\int_a^t \frac{\psi^{(n)}(s)}{(t-s)^{\sigma+1-n}}ds = I^{n-\sigma}_a\psi^{(n)}(t), \quad t > a,\ n-1 < \sigma < n,\ n \in \mathbb{N}.$$

It is not intractable to exhibit the following semi-group properties.

Lemma 2.1. *Let $r-1 < \sigma \le r$, $r \in \mathbb{N}$ and $\delta > -1$. Then,*

$$\begin{aligned}
&I^{\sigma}_{a^+}(t-a)^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\delta+\sigma+1)}(t-a)^{\delta+\sigma},\\
&{}^cD^{\sigma}_{a^+}(t-a)^{\delta} = \begin{cases} \dfrac{\Gamma(\delta+1)}{\Gamma(\delta-\sigma+1)}(t-a)^{\delta-\sigma}, & \delta \notin \{0,1,\cdots,r-1,\sigma-1\},\\ 0, & \delta \in \{0,1,\cdots,r-1,\sigma-1\},\end{cases}\\
&{}^cD^{\sigma}_{a^+}I^{\sigma}_{a^+}\psi(t) = \psi(t),\\
&I^{\sigma}_{a^+}\,{}^cD^{\sigma}_{a^+}\psi(t) = \psi(t) - \frac{(t-a)^{\sigma-1}}{\Gamma(\sigma)}I^{1-\sigma}_{a^+}\psi(a).
\end{aligned}$$

Theorem 2.1 (Banach fixed point theorem)**.** *There is a unique fixed point for every contraction mapping on a complete metric space.*

Let $\mathbf{B}(x,r)$ be a closed ball in Banach space $\mathbb{E}$ centered at x with radius r and in case of if $x = 0$, then we denote $\mathbf{B}_r$ instead of $\mathbf{B}(0,r)$. Let X be a nonempty subset of $\mathbb{E}$ such that $\overline{X}$, and $ConvX$ be closure, and convex closure of X, respectively. Throughout this work, the symbol $\mathfrak{M}_{\mathbb{E}}$ represents the family of the nonempty and bounded subsets of $\mathbb{E}$ while $\mathfrak{N}_{\mathbb{E}}$ denotes the subfamily of all relatively compact subsets of $\mathfrak{M}_{\mathbb{E}}$.

Definition 2.3 ([33]). *A mapping $\chi : \mathfrak{M}_{\mathbb{E}} \to [0,\infty)$ is called a noncompactness measure in $\mathbb{E}$ if the all conditions below hold:*

- **(Ω_1):** $\ker \chi = \{X \in \mathfrak{M}_{\mathbb{E}} : \chi(X) = 0\} \neq \phi$, $\ker \chi \subset \mathfrak{N}_{\mathbb{E}}$,
- **(Ω_2):** $Y \subset X$, *then* $\chi(Y) \leq \chi(X)$,
- **(Ω_3):** $\chi(Y) = \chi(\overline{Y}) = \chi(ConvY)$,
- **(Ω_4):** $\chi(\lambda_1 Y + \lambda_2 X) \leq \lambda_1 \chi(Y) + \lambda_2 \chi(X)$, $\lambda_1 + \lambda_2 = 1$,
- **(Ω_5):** *If* (Y_n) *is a sequence of closed subsets of* $\mathfrak{M}_{\mathbb{E}}$ *with* $Y_{n+1} \subset Y_n$ $(n \geq 1)$ *and* $\lim\limits_{n\to\infty} \chi(Y_n) = 0$, *then* $\bigcap\limits_{n=1}^{\infty} Y_n \neq \phi$.

Definition 2.4 ([34]). *Let Y be a bounded nonempty subset of Banach space $\mathbb{E}$. The function $f \in Y$ is said to be modulus of continuous function, denoted by $\varpi(f,\epsilon)$, if $\forall f \in Y$ and $\forall \epsilon > 0$, we have*

$$\varpi(f,\epsilon) = \sup\{|f(t) - f(s)| : t,s \in J,\ |t-s| \leq \epsilon\}.$$

In addition,

$$\varpi(Y,\epsilon) = \sup\{\varpi(f,\epsilon) : f \in Y\},$$

and

$$\varpi_0(Y) = \lim_{\epsilon\to 0} \varpi(Y,\epsilon).$$

where it satisfies that $\varpi_0(Y) = 2\chi(Y)$.

One of the most well-known applications for existence issues is the Schauder fixed point theorem, which concentrates on compact operators.

Theorem 2.2 (Schauder fixed point theorem). *In a Banach space $\mathbb{E}$, let $\Omega \neq \phi$ be a convex closed set satisfying the boundedness property. For each continuous compact mapping $\mathfrak{F}\colon \Omega \to \Omega$, there is a $x \in \Omega$ such that $x = \mathfrak{F}x$.*

Definition 2.5 ([35]). *Let $\mathbb{E}(J)$ be Banach algebra. A noncompactness measure χ in $\mathbb{E}(J)$ satisfies condition (m) if the condition below holds*

$$\chi(fg) \leq \|f\|\chi(g) + \|g\|\chi(f)$$

for all $f,g \in \mathfrak{M}_{C(J)}$.

Lemma 2.2 ([36]). *The condition (m) can be held by the noncompactness measure ϖ_0 on $\mathbb{E}(J)$.*

Consider the following class $\mathcal{S}$ of all functions $\psi : (0,\infty) \to (b,\infty)$ such that the following condition holds:

$$\forall\, (x_n) \subset (0,\infty)\,,\ \lim_{n\to\infty} \psi(x_n) = b \iff \lim_{n\to\infty} x_n = 0.$$

Now, we introduce Darbo's fixed point theorem that we can rely on it to illustrate the existence of at least one fixed point.

Theorem 2.3 ([37]). *Consider Banach space $\mathbb{E}$ containing a bounded, closed, convex and nonempty subset, say Ω. Let $\mathfrak{F}\colon \Omega \longrightarrow \Omega$ be a continuous mapping. Assume that there is $\theta \in [0,1)$ with χ as a noncompactness measure in $\mathbb{E}$ satisfying the following*

$$\chi(\mathfrak{F}Y) \leq \theta\chi(Y),\ \phi \neq Y \subseteq \Omega.$$

Then, T has a fixed point in Ω.

3. Auxiliary Results

This section presents some auxiliary results that will provide a basis for introducing some results associated with the existence of the solutions to tripled system of fractional Langevin equation.

Lemma 3.1. *Let $i = 1,2,3$ and $\Psi_i \in C([0,1],\mathbb{R})$ such that $\Psi_i(t) = \Psi_i(t,\omega_1(\rho_1 t),\omega_2(\rho_2 t),\omega_3(\rho_3 t))$. Then, the solution of the tripled system* (1.1)-(1.2) *is equivalent to the system of integral equations*

$$\begin{aligned}\omega_i(t) = &\frac{1}{\Gamma(r_i+s_i)}\int_0^t (t-s)^{r_i+s_i-1}\Psi_i(s)ds - \frac{l_i}{\Gamma(s_i)}\int_0^t (t-s)^{s_i-1}\omega_i(s)ds\\ &+ \frac{\Upsilon_{B_i}(t)}{\Delta_i}\Pi_i - \frac{\Upsilon_{A_i}(t)}{\Delta_i}\Lambda_i \end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\Pi_i =&\frac{1}{\Gamma(r_i+s_i)}\left[\alpha_i\int_0^{\eta_i}(\eta_i-s)^{r_i+s_i-1}\Psi_i(s)ds - \int_0^1 (1-s)^{r_i+s_i-1}\Psi_i(s)ds\right]\\ &- \frac{l_i}{\Gamma(s_i)}\left[\alpha_i\int_0^{\eta_i}(\eta_i-s)^{s_i-1}\omega_i(s)ds - \int_0^1(1-s)^{s_i-1}\omega_i(s)ds\right],\\ \Lambda_i =&\frac{\beta_i}{\Gamma(r_i)}\int_0^{\xi_i}(\xi_i-s)^{r_i-1}\Psi_i(s)ds - \frac{l_i\beta_i}{\Gamma(r_i+s_i)}\int_0^{\xi_i}(\xi_i-s)^{r_i+s_i-1}\Psi_i(s)ds\\ &+\frac{l_i^2\beta_i}{\Gamma(s_i)}\int_0^{\xi_i}(\xi_i-s)^{s_i-1}\omega_i(s)ds,\\ \Upsilon_{a_i}(t) =&(a_i(1)-a_i(0)t)t^{s_i},\\ \Delta_i =&A_i(0)B_i(1)-A_i(1)B_i(0) \neq 0\end{aligned}$$

such that, for $n = 0,1$,

$$\begin{aligned}A_i(n) &= 1-\alpha_i\eta_i^{s_i+n},\\ B_i(n) &= (1-\beta_i-n)\xi_i^n\Gamma(s_i+1+n)+l_i\beta_i\xi_i^{s_i+n}.\end{aligned}$$

Proof. Operating by $I_{0+}^{r_i}$ gives

$$({}^cD_{0+}^{s_i}+l_i)\omega_i(t) = a_i + b_i t + \frac{1}{\Gamma(r_i)}\int_0^t (t-s)^{r_i-1}\Psi_i(s)ds. \tag{3.2}$$

Hence,

$$\omega_i(t) = \frac{a_i t^{s_i}}{\Gamma(s_i+1)} + \frac{b_i t^{s_i+1}}{\Gamma(s_i+2)} + \frac{1}{\Gamma(r_i+s_i)}\int_0^t (t-s)^{r_i+s_i-1}\Psi_i(s)ds$$

$$-\frac{l_i}{\Gamma(s_i)}\int_0^t (t-s)^{s_i-1}\omega_i(s)ds + c_i. \tag{3.3}$$

where a_i, b_i and $c_i,\ i=1,2,3$ are arbitrary constants. Substituting the two boundary conditions of (1.2), $\omega_i(0)=0$ and $\omega_i(1)=\alpha_i\omega_i(\eta_i)$, into the previous equation give $c_i=0$ and

$$\frac{a_i}{\Gamma(s_i+1)}A_i(0)+\frac{b_i}{\Gamma(s_i+2)}A_i(1)=\Pi_i. \tag{3.4}$$

From (3.2), we get ${}^cD^{s_i}_{0+}\omega_i(0)=a_i$ and

$${}^cD^{s_i}_{0+}\omega_i(\xi_i)=a_i+b_i\xi_i+\frac{1}{\Gamma(r_i)}\int_0^{\xi_i}(\xi_i-s)^{r_i-1}\Psi_i(s)ds-l_i\omega_i(\xi_i).$$

Substituting the value of $\omega_i(\xi_i)$ from (3.3) into the previous equation and using the boundary condition of (1.2), ${}^cD^{s_i}_{0+}\omega_i(0)=\beta_i{}^cD^{s_i}_{0+}\omega_i(\xi_i)$, we get

$$\frac{a_i}{\Gamma(s_i+1)}B_i(0)+\frac{b_i}{\Gamma(s_i+2)}B_i(1)=\Lambda_i. \tag{3.5}$$

Solving the equations (3.4) and (3.5) implies

$$\frac{a_i}{\Gamma(s_i+1)}=\frac{B_i(1)\Pi_i-A_i(1)\Lambda_i}{\Delta_i},\qquad \frac{b_i}{\Gamma(s_i+2)}=\frac{A_i(0)\Lambda_i-B_i(0)\Pi_i}{\Delta_i}$$

which is the desired result (3.1) after substituting into (3.3).

Conversely, it is easy to see that the result (3.1) satisfies all conditions of (1.2). Operating by ${}^cD^{s_i}_{0+}$ on (3.1) with using Lemma 2.1 gives

$$\begin{aligned}{}^cD^{s_i}_{0+}\omega_i(t)&=\frac{1}{\Gamma(r_i)}\int_0^t(t-s)^{r_i-1}\Psi_i(s)ds-l_i\omega_i(t)\\&\quad+\frac{B_i(1)\Gamma(s+1)-B_i(0)\Gamma(s+2)t}{\Delta_i}\Pi_i-\frac{A_i(1)\Gamma(s+1)-A_i(0)\Gamma(s+2)t}{\Delta_i}\Lambda_i.\end{aligned}$$

Again, operating by ${}^cD^{r_i}_{0+}$ on the previous equation, with noting that ${}^cD^{r_i}_{0+}t^n=0,\ n=0,1,\ 1<r_i\le 2$ according Lemma 2.1, gives equations (1.1) which means that the result in (3.1) is the unique representation of solution to the boundary value problem (1.1) and (1.2). □

Lemma 3.2. *Let the functions Υ_{a_i} be defined in Lemma 3.1, $0\le t_1<t_2\le 1$ and there exists $\epsilon>$ such that $t_2-t_1<\epsilon$. Then, the functions Υ_{a_i} satisfy the following property:*

$$|\Upsilon_{a_i}(t_2)-\Upsilon_{a_i}(t_1)|\le|(a_i(1)|\epsilon^{s_i}+|a_i(0)|(s_i+1)\epsilon.$$

Proof. It is easy to see that $x^a-y^a\le(x-y)^a$ for all $0\le y<x\le b$ and $0\le a\le 1$. If $a>1$, we can use the mean value theorem to deduce that $x^a-y^a\le at^{a-1}(x-y)$ such that $y<t<x<b$ and so $x^a-y^a\le ab^{a-1}(x-y)$. To prove our result, let $0\le t_1<t_2\le 1$ and there exists $\epsilon>0$ such that $t_2-t_1<\epsilon$. Then, with the previous facts, we get

$$\begin{aligned}|\Upsilon_{a_i}(t_2)-\Upsilon_{a_i}(t_1)|&=|(a_i(1)-a_i(0)t_2)t_2^{s_i}-(a_i(1)-a_i(0)t_1)t_1^{s_i}|\\&\le|(a_i(1)|(t_2^{s_i}-t_1^{s_i})+|a_i(0)|(t_2^{s_i+1}-t_1^{s_i+1})\end{aligned}$$

$$\leq |(a_i(1)|(t_2-t_1)^{s_i} + |a_i(0)|(s_i+1)t^{s_i}(t_2-t_1), \qquad t \in (t_1,t_2)$$
$$\leq |(a_i(1)|\epsilon^{s_i} + |a_i(0)|(s_i+1)\epsilon.$$

This ends the proof. □

4. Uniqueness via Banach Contraction

In this section, we explain the existence criteria for our system. Before we start our main work, we need the following hypothesis. Let us introduce the space $\mathbb{Z} = \{z(t) \colon z(t) \in C([0,1],\mathbb{R})\}$ endowed with the norm $\|z\| = \sup_{t\in[0,1]} |z(t)|$. Obviously, $(\mathbb{Z}, \|\cdot\|)$ is a Banach space. Then the product space $\mathbb{E} = (\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}, \|(z_1,z_2,z_3)\|)$ is also a Banach space equipped with the norm

$$\|(z_1,z_2,z_2)\| = \|z_1\| + \|z_2\| + \|z_3\|.$$

From Lemma 3.1, we define the operator $\Phi : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ by

$$\Phi(\omega_1,\omega_2,\omega_3)(t) = \begin{pmatrix} \Phi_1(\omega_1,\omega_2,\omega_3)(t) \\ \Phi_2(\omega_1,\omega_2,\omega_3)(t) \\ \Phi_3(\omega_1,\omega_2,\omega_3)(t) \end{pmatrix},$$

where

$$\Phi_i(\omega_1,\omega_2,\omega_3)(t) = \frac{1}{\Gamma(r_i+s_i)}\int_0^t (t-s)^{r_i+s_i-1}\Psi_i(s)ds - \frac{l_i}{\Gamma(s_i)}\int_0^t (t-s)^{s_i-1}\omega_i(s)ds + \frac{\Upsilon_{B_i}(t)}{\Delta_i}\Pi_i - \frac{\Upsilon_{A_i}(t)}{\Delta_i}\Lambda_i \tag{4.1}$$

For convenience, we set

$$\begin{aligned}
\delta_i(s) &= \tfrac{\alpha_i\eta_i^s+1}{\Gamma(s+1)}, & \tau_i(s) &= \tfrac{\beta_i\xi_i^s}{\Gamma(s+1)}, \\
\Upsilon_{A_i} &= \sup_{t\in J}|\Upsilon_{A_i}(t)|, & \Upsilon_{B_i} &= \sup_{t\in J}|\Upsilon_{B_i}(t)|, \\
\Upsilon_A &= \max_i \Upsilon_{A_i}, & \Upsilon_B &= \max_i \Upsilon_{B_i}, & \Delta &= \min_i |\Delta_i|.
\end{aligned} \tag{4.2}$$

To prove our main results in this section, we need the following assumption:

($\mathbf{H}_1$): Let $i = 1,1,3$ and $\Psi_i : [0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the Lipschitz conditions

$$|\Psi_i(t,u_1,u_2,u_3)) - \Psi_i(t,v_1,v_2,v_3)| \leq C_i(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|).$$

for all $t\in[0,1]$ and $u_i,\ v_i \in \mathbb{R}$, where C_i are positive constants.

($\mathbf{H}_2$): There exists a positive constant $\mathfrak{R} < 1$ where

$$\mathfrak{R} = \sum_{i=1}^{3}\frac{C_i}{\Gamma(r_i+s_i+1)} + \frac{\ell}{\Gamma(s+1)} + \frac{\Upsilon_B}{\Delta}\left(\ell\delta + \sum_{i=1}^{3}C_i\delta_i(s_i+r_i)\right) + \frac{\Upsilon_A}{\Delta}\left(\ell^2\tau + \sum_{i=1}^{3}\left[\tau_i(r_i) + |l_i|\tau_i(r_i+s_i)\right]C_i\right).$$

where $\ell = \max_i |l_i|$, $\tau = \max_i |\tau_i(s_i)|$, $\delta = \max_i |\delta_i(s_i)|$ and $\Gamma(s+1) = \min_i \Gamma(s_i+1)$.

Theorem 4.1. *Suppose that the assumptions* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *are satisfied. Then, the tripled system* (1.1) *and* (1.2) *has a unique solution.*

Proof. Define $\sup\limits_{t\in[0,1]} |\Psi_i(t,0,0,0)| \leq \aleph_i$ and $\kappa > 0$ such that

$$\kappa > \frac{\hbar}{1-\mathfrak{R}}$$

where

$$\hbar = \sum_{i=1}^{3} \frac{\aleph_i}{\Gamma(r_i+s_i+1)} + \frac{\Upsilon_B}{\Delta}\sum_{i=1}^{3} \aleph_i \delta_i(s_i+r_i) + \frac{\Upsilon_A}{\Delta}\sum_{i=1}^{3} \left[\tau_i(r_i) + |l_i|\tau_i(r_i+s_i)\right]\aleph_i.$$

Let the closed ball $\mathbf{B}_\kappa$ with radius κ be derived as

$$\mathbf{B}_\kappa = \{\omega(t) = (\omega_1(\rho_1 t), \omega_2(\rho_2 t), \omega_3(\rho_3 t)) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} : \|(\omega_1,\omega_2,\omega_3)\| \leq \kappa\}.$$

We show that $\Phi\mathbf{B}_\kappa \subset \mathbf{B}_\kappa$. By the assumption $(\mathbf{H}_1)$, we have

$$\begin{aligned}
|\Psi_i(t,\omega(t)) &= |\Psi_i(t,\omega_1(\rho_1 t),\omega_2(\rho_2 t),\omega_3(\rho_3 t))| \\
&\leq |\Psi_i(t,\omega_1(\rho_1 t),\omega_2(\rho_2 t),\omega_3(\rho_3 t)) - \Psi_i(t,0,0,0)| + |\Psi_i(t,0,0,0)| \\
&\leq C_i(|\omega_1(\rho_1 t)| + |\omega_2(\rho_2 t)|) + |\omega_3(\rho_3 t)| + \aleph_i \\
&\leq C_i(\|\omega_1\| + \|\omega_2\| + \|\omega_3\|) + \aleph_i = C_i\kappa + \aleph_i
\end{aligned}$$

which leads to

$$\begin{aligned}
|\Pi_i| &\leq (C_i\kappa + \aleph_i)\delta_i(s_i+r_i) + |l_i||\omega_i(\rho_i t)|\delta_i(s_i), \\
|\Lambda_i| &\leq (C_i\kappa + \aleph_i)[\tau_i(r_i) + |l_i|\tau_i(s_i+r_i)] + l_i^2|\omega_i(\rho_i t)|\tau_i(s_i).
\end{aligned}$$

Hence,

$$|(\Phi_i\omega)(t)| \leq \frac{C_i\kappa + \aleph_i}{\Gamma(r_i+s_i+1)} + \frac{|l_i||\omega_i(\rho_i t)|}{\Gamma(s_i+1)} + \frac{\Upsilon_{B_i}}{|\Delta_i|}|\Pi_i| + \frac{\Upsilon_{A_i}}{|\Delta_i|}|\Lambda_i|.$$

Consequently,

$$\begin{aligned}
\|\Phi\omega\| &= \sum_{i=1}^{3} \sup_{t\in[0,1]} |(\Phi_i\omega)(t)| \\
&\leq \kappa\sum_{i=1}^{3} \frac{C_i}{\Gamma(r_i+s_i+1)} + \sum_{i=1}^{3}\frac{\aleph_i}{\Gamma(r_i+s_i+1)} + \frac{\ell}{\Gamma(s+1)}\sum_{i=1}^{3}\sup_{t\in[0,1]}|\omega_i(\rho_i t)| \\
&\quad + \frac{\Upsilon_B}{\Delta}\sum_{i=1}^{3}|\Pi_i| + \frac{\Upsilon_A}{\Delta}\sum_{i=1}^{3}|\Lambda_i| = \mathfrak{R}\kappa + \hbar \leq \kappa
\end{aligned}$$

which implies that $\Phi\mathbf{B}_\kappa \subset \mathbf{B}_\kappa$.

The next step is to find a sufficient condition for contraction. To do this, let $\omega(t) = (\omega_1(\rho_1 t), \omega_2(\rho_2 t), \omega_3(\rho_3 t))$, $\omega'(t) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ for all $t \in J = [0,1]$. Then,

$$|(\Pi_i\omega)(t) - (\Pi_i\omega')(t)| \leq \frac{1}{\Gamma(r_i+s_i)}\left[|\alpha_i|\int_0^{\eta_i}(\eta_i - s)^{r_i+s_i-1}|\Psi_i(s,\omega(s)) - \Psi_i(s,\omega(s))|ds\right.$$

$$
\begin{aligned}
&+ \int_0^1 (1-s)^{r_i+s_i-1} |\Psi_i(s,\omega(s)) - \Psi_i(s,\omega(s))| ds \Bigg] \\
&+ \frac{|l_i|}{\Gamma(s_i)} \left[|\alpha_i| \int_0^{\eta_i} (\eta_i - s)^{s_i-1} ds + \int_0^1 (1-s)^{s_i-1} ds \right] \|\omega_i - \omega_i'\| \\
&\le \frac{|\alpha_i|\eta_i^{s_i+r_i} + 1}{\Gamma(r_i+s_i+1)} C_i \sum_{j=1}^{3} \|\omega_j - \omega_j'\| + \frac{|l_i|\left(|\alpha_i|\eta_i^{s_i}+1\right)}{\Gamma(s_i+1)} \|\omega_i - \omega_i'\| \\
&= C_i \delta_i(r_i+s_i) \|\omega - \omega'\| + |l_i| \delta_i(s_i) \|\omega_i - \omega_i'\|.
\end{aligned}
$$

Similarly,

$$
|\Lambda_i(\omega)(\rho t) - \Lambda_i(\omega')(\rho t)| \le \left[C_i \tau_i(r_i) + |l_i| C_i \tau_i(r_i+s_i) \right] \|\omega - \omega'\| + l_i^2 \tau_i(s_i) \|\omega_i - \omega_i'\|.
$$

These conclude that

$$
\begin{aligned}
\|\Phi_i(\omega) - \Phi_i(\omega')\| \le{}& \frac{C_i}{\Gamma(r_i+s_i+1)} \|\omega - \omega'\| + \frac{|l_i|}{\Gamma(s_i+1)} \|\omega_i - \omega_i'\| \\
&+ \frac{\Upsilon_{B_i}|}{|\Delta_i|} \|\Pi_i(\omega) - \Pi_i(\omega')\| + \frac{\Upsilon_{A_i}}{|\Delta_i|} \|\Lambda_i(\omega) - \Lambda_i(\omega')\|
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\Phi(\omega) - \Phi(\omega')\| \le{}& \sum_{i=1}^{3} \Bigg(\frac{C_i}{\Gamma(r_i+s_i+1)} + \frac{\Upsilon_{B_i}}{|\Delta_i|} C_i \delta_i(r_i+s_i) \\
&+ \frac{\Upsilon_{A_i}}{|\Delta_i|} \left[C_i \tau_i(r_i) + |l_i| C_i \tau_i(r_i+s_i) \right] \Bigg) \|\omega - \omega'\| \\
&+ \left(\frac{\ell}{\Gamma(s+1)} + \frac{\Upsilon_B}{\Delta} \ell \delta + \frac{\Upsilon_A}{\Delta} \ell^2 \tau \right) \sum_{i=1}^{3} \|\omega_i - \omega_i'\| \\
={}& \Re \|\omega - \omega'\|.
\end{aligned}
$$

This concludes that Φ is a contraction operator with a constant $\Re < 1$. Therefore, from Banach fixed point theorem, the operator Φ has a unique fixed point, which is the unique solution of system (1.1), (1.2). This completes the proof. □

Example 4.1. *Consider the following tripled system of nonlinear fractional Langevin equation*

$$
\begin{cases}
{}^cD_{0+}^{1.3}({}^cD_{0+}^{0.7} + 0.01)\omega_1(t) = \frac{e^{-t}}{(3-t)^6-4} \cos(\omega_1(0.4t)) - \frac{1}{60} \arctan(\omega_2(0.5t)) + \frac{(1+t)^2 \omega_3^2(0.6t)}{15(5-t)^2(1+\omega_3^2(0.6t))}, \\[2ex]
{}^cD_{0+}^{1.5}({}^cD_{0+}^{0.5} + 0.01)\omega_2(t) = \frac{e^{-t}}{(4-t)^4-1} \cos(\omega_2(0.5t)) - \frac{1}{80} \arctan(\omega_3(0.6t)) + \frac{t^2 \omega_1^2(0.4t)}{20(3-t)^2)(1+\omega_1^2(0.4t)}, \\[2ex]
{}^cD_{0+}^{1.7}({}^cD_{0+}^{0.3} + 0.01)\omega_3(t) = \frac{e^{-t}}{4(5+t)^2} \cos(\omega_3(0.6t)) - \frac{1}{100} \arctan(\omega_1(0.4t)) + \frac{(2+t)^2 \omega_2^2(0.5t)}{25(7-t)^2(1+\omega_2^2(0.5t))}
\end{cases}
$$

subject to the following coupled boundary conditions

$$\begin{cases} \omega_1(0) = 0, & \omega_1(1) = 5\omega_1(0.2), \qquad {}^cD^{0.7}_{0+}\omega_1(0) = 4{}^cD^{0.7}_{0+}\omega_1(0.2) \\ \\ \omega_2(0) = 0, & \omega_2(1) = 4\omega_2(0.5), \qquad {}^cD^{0.5}_{0+}\omega_2(0) = 6{}^cD^{0.5}_{0+}\omega_2(0.4) \\ \\ \omega_3(0) = 0, & \omega_3(1) = 3\omega_2(0.8), \qquad {}^cD^{0.3}_{0+}\omega_2(0) = 8{}^cD^{0.3}_{0+}\omega_2(0.6) \end{cases}$$

Here, we take

$$\begin{array}{llllllll} r_1 = 1.3, & s_1 = 0.7, & l_1 = \frac{1}{100}, & \rho_1 = 0.2, & \eta_1 = 0.2, & \xi_1 = 0.2, & \alpha_1 = 5, & \beta_1 = 4, \\ r_2 = 1.5, & s_2 = 0.5, & l_2 = \frac{1}{100}, & \rho_2 = 0.5, & \eta_2 = 0.5, & \xi_2 = 0.4, & \alpha_2 = 4, & \beta_2 = 6, \\ r_3 = 1.7, & s_3 = 0.3, & l_3 = \frac{1}{100}, & \rho_3 = 0.8, & \eta_3 = 0.8, & \xi_3 = 0.6, & \alpha_3 = 3, & \beta_3 = 8, \end{array}$$

and

$$\begin{aligned} \Psi_1(t,\omega(\rho t)) &= \frac{e^{-t}}{(3-t)^6-4}\cos(\omega_1(0.4t)) - \frac{1}{60}\arctan(\omega_2(0.5t)) + \frac{(1+t)^2\omega_3^2(0.6t)}{15(5-t)^2(1+\omega_3^2(0.6t))}, \\ \Psi_2(t,\omega(\rho t)) &= \frac{e^{-t}}{(4-t)^4-1}\cos(\omega_2(0.5t)) - \frac{1}{80}\arctan(\omega_3(0.6t)) + \frac{t^2\omega_1^2(0.4t)}{20(3-t)^2(1+\omega_1^2(0.4t))}, \\ \Psi_3(t,\omega(\rho t)) &= \frac{e^{-t}}{4(5+t)^2}\cos(\omega_3(0.6t)) - \frac{1}{100}\arctan(\omega_1(0.4t)) + \frac{(2+t)^2\omega_2^2(0.5t)}{25(7-t)^2(1+\omega_2^2(0.5t))}. \end{aligned}$$

According to the mean value theorem, we know that $|\cos x - \cos y| \le |x-y|$ and $|\arctan x - \arctan y| \le |x-y|$. Also, we have

$$\begin{aligned} \left| f(t)\frac{u^2}{1+u^2} - f(t)\frac{v^2}{1+v^2} \right| &= |f(t)|\left|\frac{u^2-v^2}{(1+u^2)(1+v^2)}\right| \\ &= |f(t)|\frac{|u+v|}{(1+u^2)(1+v^2)}|u-v| \\ &= |f(t)|\frac{\sqrt{(1+u^2)(1+v^2)-(1-uv)^2}}{(1+u^2)(1+v^2)}|u-v| \\ &\le |f(t)|\frac{\sqrt{(1+u^2)(1+v^2)}}{(1+u^2)(1+v^2)}|u-v| \le |f(t)||u-v|. \end{aligned}$$

These imply that the assumption $(\mathbf{H}_1)$ holds with $C_1 = 1/60$, $C_2 = 1/80$ and $C_3 = 1/100$. With a given data, we find that

$$\begin{array}{llll} A_1(0) = -0.620657, & A_1(1) = 0.675869, & B_1(0) = -2.71295, & B_1(1) = -1.23316, \\ A_2(0) = -1.82843, & A_2(1) = -0.414214, & B_2(0) = -4.39319, & B_2(1) = -3.17524, \\ A_3(0) = -1.80575, & A_3(1) = -1.2446, & B_3(0) = -6.21366, & B_3(1) = -5.55904 \end{array}$$

which lead to

$$\begin{array}{l} \Upsilon_{A_1} = 1.29653, \quad \Upsilon_{A_2} = 2.24264, \quad \Upsilon_{A_3} = 3.05034 \Rightarrow \Upsilon_A = 3.05034, \\ \Upsilon_{B_1} = 3.94611, \quad \Upsilon_{B_2} = 7.56843, \quad \Upsilon_{B_3} = 11.7727 \Rightarrow \Upsilon_B = 11.7727. \end{array}$$

Also, we have

$$\begin{array}{llllll}
\Delta_1 = 2.599, & \delta_1(0.7) = 2.884, & \delta_1(2) = 0.6, & \tau_1(0.7) = 1.427, & \tau_1(1.3) = 0.423, & \tau_1(2) = 0.08, \\
\Delta_2 = 3.986, & \delta_2(0.5) = 4.319, & \delta_2(2) = 1, & \tau_2(0.5) = 4.282, & \tau_2(1.5) = 1.142, & \tau_2(2) = 0.48, \\
\Delta_3 = 2.304, & \delta_3(0.3) = 4.241, & \delta_3(2) = 1.46, & \tau_3(0.3) = 7.647, & \tau_3(1.7) = 2.173, & \tau_3(2) = 1.44
\end{array}$$

which yield that $\Delta = 2.30471$, $\delta = 4.31992$ and $\tau = 7.64742$. These imply that the assumption $(\mathbf{H}_2)$ holds with $\aleph = 0.499331 < 1$. Therefore, all conditions of Theorem 4.1 are satisfied. Hence, the system in our Example 4.1 has a unique solution.

5. Existence via Measures of Noncompactness

In this section, we explain the existence criteria for our system by using measures of noncompactness technique. The following results uses the Schauder Theorem 2.2 to investigate the presence of a tripled fixed point to the operator $\Phi\colon \mathbb{E} = \mathbb{Z}^3 \to \mathbb{Z}^3$ that leads to a solution for the fractional Langevin system (1.1)-(1.2). These theories, which are listed below, are required to determine the outcomes:

($\mathcal{M}_1$): Let $\Psi_i\colon J \times \mathbb{R}^3 \to \mathbb{R}$ be Carathéodory functions, and there exist nondecreasing kth-fractional integrable function $u\colon J \to [0,\infty)$ and nondecreasing continuous function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\psi(s) < s$ for all $s > 0$ equipped with

$$|\Psi_i(t,x) - \Psi_i(t,y)| \le u(t)\psi(\|x-y\|), \quad t \in J$$

for all $x, y \in \mathbb{R}^3$, k is chosen such that $I_0^k u(t) \le \mathfrak{C}_u$. Furthermore, let $\sup\limits_{t\in[0,1]} |\Psi_i(t,0,0,0)| \le \aleph_i$.

($\mathcal{M}_2$): There exists a positive constat $\mathcal{H}$ such that

$$\mathcal{H} = \sum_{i=1}^{3}\left(\frac{1}{\Gamma(r_i+s_i+1)} + \frac{\Upsilon_{B_i}}{|\Delta_i|}\delta_i(r_i+s_i) + \frac{\Upsilon_{A_i}}{|\Delta_i|}\left[\tau_i(r_i) + |l_i|\tau_i(r_i+s_i)\right]\right)\mathfrak{C}_u + \left(\frac{\ell}{\Gamma(s+1)} + \frac{\Upsilon_B}{\Delta}\ell\delta + \frac{\Upsilon_A}{\Delta}\ell^2\tau\right).$$

($\mathcal{M}_3$): For each bounded set $S \in \mathbb{Z}^3$ and for each $t \in [0,1]$, we have

$$\chi\{(\Psi_i(t,\omega(t) : \omega \in S\}) \le \varrho_i(t)\chi(S)$$

where $\varrho_i\colon J \to \mathbb{R}_+$ be so that $(I_0^{r_i+s_i}\varrho_i)(t) \le I_i$.

($\mathcal{M}_4$): There are positive real constants $\mathcal{H}_i;\ i = 1,2,3$ such that

$$\mathcal{H}_i = I_i + \frac{|l_i|}{\Gamma(s_i+1)} + \frac{\Upsilon_{B_i}}{|\Delta_i|}\left[(I_0^{r_i+s_i}\varrho_i)(\eta_i) + (I_0^{r_i+s_i}\varrho_i)(1) + |l_i|\delta_i(s_i)\right] + \frac{\Upsilon_{A_i}|}{|\Delta_i|}\left[|\beta_i|(I_0^{r_i}\varrho_i)(\xi_i) + |l_i||\beta_i|(I_0^{r_i+s_i}\varrho_i)(\xi_i) + l_i^2\tau_i(s_i)\right]$$

and satisfying $\mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 < 1$.

Lemma 5.1 ([38]). *Regard $\Psi\colon I\times\mathbb{R}^3\to\mathbb{R}$ satisfying $(\mathcal{M}_3)$, and let $\mathcal{K}\colon J\to\mathbb{R}$ be a bounded continuous mapping. If S is an equicontinuous set of functions. Then,*

$$\chi\left(\left\{\int_J \mathcal{K}(t,s)\Psi(t,x(s))ds\colon x\in S\right\}\right)\leq \int_J \mathcal{K}(t,s)\varrho_i(s)\chi(\{x(s)\colon x\in S\})ds$$

Theorem 5.1. *Suppose that the assumptions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ are satisfied. Then, the tripled system* (1.1)-(1.2) *has a solution in $\mathbf{B}_\kappa$.*

Proof. Let $(\omega_n)=(\omega_n^1,\omega_n^2,\omega_n^3)$ be a sequence in $\mathbb{Z}^3$ such that $\omega_n^i\to\omega_i$ as $n\to\infty$ and $i=1,2,3$. As in Theorem 4.1, we get

$$\begin{aligned}\|(\Phi_i\omega_n)-(\Phi_i\omega)\|\leq{}& \frac{\mathfrak{C}_u}{\Gamma(r_i+s_i+1)}\psi(\|\omega_n-\omega\|)+\frac{|l_i|}{\Gamma(s_i+1)}\|\omega_n^i-\omega_i\|\\ &+\frac{\Upsilon_{B_i}|}{|\Delta_i|}\|(\Pi_i\omega_n^i)-(\Pi_i\omega_i)\|+\frac{\Upsilon_{A_i}}{|\Delta_i|}\|(\Lambda_i\omega_n^i)-(\Lambda_i\omega_i)\|\end{aligned}$$

which implies that

$$\begin{aligned}\|(\Phi\omega_n)-(\Phi\omega)\|\leq{}& \sum_{i=1}^{3}\Bigg(\frac{1}{\Gamma(r_i+s_i+1)}+\frac{\Upsilon_{B_i}}{|\Delta_i|}\delta_i(r_i+s_i)\\ &+\frac{\Upsilon_{A_i}}{|\Delta_i|}\left[\tau_i(r_i)+|l_i|\tau_i(r_i+s_i)\right]\Bigg)\mathfrak{C}_u\psi(\|\omega_n-\omega\|)\\ &+\left(\frac{\ell}{\Gamma(s+1)}+\frac{\Upsilon_B}{\Delta}\ell\delta+\frac{\Upsilon_A}{\Delta}\ell^2\tau\right)\|\omega_n-\omega\|\\ \leq{}& \mathcal{H}\|\omega_n-\omega\|\end{aligned}$$

which leads to the continuity of the operator Φ. Suppose that $\omega\in\mathbf{B}_\kappa$, then as in the previous theorem, we get

$$\begin{aligned}\|(\Phi\omega)\|\leq{}& \kappa\mathfrak{C}_u\sum_{i=1}^{3}\frac{\|\omega_i\|}{\Gamma(r_i+s_i+1)}+\sum_{i=1}^{3}\frac{\aleph_i}{\Gamma(r_i+s_i+1)}+\frac{\ell}{\Gamma(s+1)}\sum_{i=1}^{3}\sup_{t\in[0,1]}|\omega_i(\rho_i t)|\\ &+\frac{\Upsilon_B}{\Delta}\sum_{i=1}^{3}|\Pi_i|+\frac{\Upsilon_A}{\Delta}\sum_{i=1}^{3}|\Lambda_i|\end{aligned}$$

which implies that Φ is bounded on $\mathbf{B}_\kappa$.

Let $0\leq t_1<t_2\leq 1$ and there exists $\epsilon>$ such that $t_2-t_1<\epsilon$, by using the results in Lemma 3.2, we get

$$\begin{aligned}|(\Phi_i\omega)(t_2)-(\Phi_i\omega)(t_1)|={}& \frac{1}{\Gamma(r_i+s_i)}\left|\int_0^{t_2}(t_2-s)^{r_i+s_i-1}\Psi_i(s)ds-\int_0^{t_1}(t_1-s)^{r_i+s_i-1}\Psi_i(s)ds\right|\\ &+\frac{|l_i|}{\Gamma(s_i)}\left|\int_0^{t_2}(t_2-s)^{s_i-1}\omega_i(s)ds-\int_0^{t_1}(t_1-s)^{s_i-1}\omega_i(s)ds\right|\\ &+\frac{|\Upsilon_{B_i}(t_2)-\Upsilon_{B_i}(t_1)|}{|\Delta_i|}|\Pi_i|+\frac{|\Upsilon_{A_i}(t_2)-\Upsilon_{A_i}(t_1)|}{|\Delta_i|}|\Lambda_i|\end{aligned}$$

$$\leq \frac{\psi_i(\|\omega\|)}{\Gamma(r_i+s_i)} \int_0^{t_1} \left[(t_2-s)^{r_i+s_i-1} - (t_1-s)^{r_i+s_i-1}\right] u(s)ds$$
$$+ \frac{\psi_i(\|\omega\|)}{\Gamma(r_i+s_i)} \int_{t_1}^{t_2} (t_2-s)^{r_i+s_i-1} u(s)ds + \frac{|l_i|}{\Gamma(s_i)} \int_{t_1}^{t_2} (t_2-s)^{s_i-1} |\omega_i(s)|ds$$
$$+ \frac{|l_i|}{\Gamma(s_i)} \int_0^{t_1} \left[(t_1-s)^{s_i-1} - (t_2-s)^{s_i-1}\right] |\omega_i(s)|ds$$
$$+ \frac{|\Pi_i||B_i(1)| + |\Lambda_i||A_i(1)|}{|\Delta_i|} \epsilon^{s_i} + \frac{|\Pi_i||B_i(0)| + |\Lambda_i||A_i(0)|}{|\Delta_i|} (s_i+1)\epsilon.$$

From the assumption that u is nondecreasing, we obtain

$$\begin{aligned}
|(\Phi_i\omega)(t_2) - (\Phi_i\omega)(t_1)| &\leq \frac{\psi_i(\|\omega\|)u(t_1)}{\Gamma(r_i+s_i+1)} \left[t_2^{r_i+s_i} - t_1^{r_i+s_i} - (t_2-t_1)^{r_i+s_i}\right] \\
&+ \frac{\psi_i(\|\omega\|)u(t_2)}{\Gamma(r_i+s_i+1)} (t_2-t_1)^{r_i+s_i} + \frac{|l_i|\|\omega_i\|}{\Gamma(s_i+1)} \left[2(t_2-t_1)^{s_i} + t_1^{r_i} - t_2^{r_i}\right] \\
&+ \frac{|\Pi_i||B_i(1)| + |\Lambda_i||A_i(1)|}{|\Delta_i|} \epsilon^{s_i} + \frac{|\Pi_i||B_i(0)| + |\Lambda_i||A_i(0)|}{|\Delta_i|} (s_i+1)\epsilon \\
&\leq \frac{\psi_i(\|\omega\|)u(t_2)}{\Gamma(r_i+s_i+1)} (r_i+s_i)\epsilon + \frac{2|l_i|\|\omega_i\|}{\Gamma(s_i+1)} \epsilon^{s_i} \\
&+ \frac{|\Pi_i||B_i(1)| + |\Lambda_i||A_i(1)|}{|\Delta_i|} \epsilon^{s_i} + \frac{|\Pi_i||B_i(0)| + |\Lambda_i||A_i(0)|}{|\Delta_i|} (s_i+1)\epsilon.
\end{aligned}$$

Let Y be a bounded nonempty subset of Banach space $\mathbb{E}$. Then, the operator $\Phi \in Y$ and the modulus of continuity of Φ_i can be calculated as

$$\begin{aligned}
\varpi(\Phi_i\omega, \epsilon) &= \sup\{|(\Phi\omega)(t_2) - (\Phi\omega)(t_1)|\colon t_1, t_2 \in J,\ t_2 - t_1 \leq \epsilon\} \\
&\leq \frac{\psi_i(\|\omega\|)u(t_2)}{\Gamma(r_i+s_i+1)} (r_i+s_i)\epsilon + \frac{|l_i|\|\omega_i\|}{\Gamma(s_i+1)} \epsilon^{s_i} \\
&+ \frac{|\Pi_i||B_i(1)| + |\Lambda_i||A_i(1)|}{|\Delta_i|} \epsilon^{s_i} + \frac{|\Pi_i||B_i(0)| + |\Lambda_i||A_i(0)|}{|\Delta_i|} (s_i+1)\epsilon
\end{aligned}$$

which implies that

$$\begin{aligned}
\varpi(\Phi\omega, \epsilon) &= \sum_{i=1}^{3} \varpi(\Phi_i\omega, \epsilon) \\
&\leq \sum_{i=1}^{3} \left[\frac{\psi_i(\|\omega\|)u(t_2)}{\Gamma(r_i+s_i+1)} (r_i+s_i) + \frac{|\Pi_i||B_i(0)||\Lambda_i||A_i(0)|}{|\Delta_i|} (s_i+1)\right] \epsilon \\
&+ \sum_{i=1}^{3} \left[\frac{|l_i|\|\omega_i\|}{\Gamma(s_i+1)} + \frac{|\Pi_i||B_i(1)| + |\Lambda_i||A_i(1)|}{|\Delta_i|}\right] \epsilon^{s_i}.
\end{aligned}$$

In addition, the modulus of continuity of a set Y can be calculated as

$$\varpi(Y, \epsilon) = \sup\{\varpi(\Phi\omega, \epsilon) : \Phi\omega \in Y\}.$$

Finally, the measure of noncompactness ϖ_0 can be calculated as

$$\varpi_0(Y) = \lim_{\epsilon \to 0} \varpi(Y, \epsilon) = 0.$$

The previous steps tell that Φ is continuous compact mapping from $\mathbf{B}_\kappa$ to it self. Applying Schauder fixed point Theorem 2.2, the fractional tripled system (1.1)-(1.2) has a solution in $\mathbf{B}_\kappa \in \ker \varpi_0$. □

Theorem 5.2. *Suppose that the assumptions* $(\mathcal{M}_1)$-$(\mathcal{M}_4)$ *are satisfied. Then, the tripled system* (1.1)-(1.2) *has a solution in* $\mathbf{B}_\kappa$.

Proof. The operator $\Phi\colon \mathbf{B}_\kappa \to \mathbf{B}_\kappa$ is implied to be a continuous operator on the closed convex bounded subset $\mathbf{B}_\kappa$ of $\mathbb{Z}^3$ by the criteria $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$. Additionally, $\overline{\mathbf{B}_\kappa}$ is a compact subset of $\mathbf{B}_\kappa$. If $\chi(\mathbf{B}_\kappa) = 0$, then by (Ω_2), we obtain $\chi(S) = 0$ for any $S \in \mathbf{B}_\kappa$, therefore the result is as in Theorem 5.1. In the event when $\chi(S) > 0$, with using the properties (Ω_4) and (Ω_5) and the results in Lemma 5.1, we obtain

$$\begin{aligned}
\chi(\{\Pi_i\colon \omega \in S\}) &\leq \frac{1}{\Gamma(r_i+s_i)}\left[\alpha_i \int_0^{\eta_i} (\eta_i - s)^{r_i+s_i-1} + \int_0^1 (1-s)^{r_i+s_i-1}\right]\chi(\{\Psi_i(s,\omega(s))\colon \omega \in S\})ds \\
&+ \frac{|l_i|}{\Gamma(s_i)}\left[|\alpha_i| \int_0^{\eta_i} (\eta_i - s)^{s_i-1} + \int_0^1 (1-s)^{s_i-1}\right]\chi(\{\omega_i(s)\colon \omega \in S\})ds \\
&\leq \frac{1}{\Gamma(r_i+s_i)}\left[|\alpha_i| \int_0^{\eta_i} (\eta_i - s)^{r_i+s_i-1} + \int_0^1 (1-s)^{r_i+s_i-1}\right]\varrho_i(s)\chi(S)ds \\
&+ \frac{|l_i|}{\Gamma(s_i)}\left[\alpha_i \int_0^{\eta_i} (\eta_i - s)^{s_i-1} + \int_0^1 (1-s)^{s_i-1}\right]\chi(S)ds \\
&= \left[(I_0^{r_i+s_i}\varrho_i)(\eta_i) + (I_0^{r_i+s_i}\varrho_i)(1) + |l_i||\delta_i(s_i)\right]\chi(S)
\end{aligned}$$

and by the same way, we get

$$\chi(\{\Lambda_i\colon \omega \in S\}) \leq \left[|\beta_i|(I_0^{r_i}\varrho_i)(\xi_i) + |l_i||\beta_i|(I_0^{r_i+s_i}\varrho_i)(\xi_i) + l_i^2\tau_i(s_i)\right]\chi(S).$$

Therefore, we get

$$\begin{aligned}
\chi(\Phi_i S)(t)) &\leq \frac{1}{\Gamma(r_i+s_i)}\int_0^t (t-s)^{r_i+s_i-1}\chi(\{\Psi_i(s,\omega(s))\colon \omega \in S\})ds \\
&+ \frac{|l_i|}{\Gamma(s_i)}\int_0^t (t-s)^{s_i-1}\chi(\{\omega_i(s)\colon \omega \in S\})ds \\
&+ \frac{|\Upsilon_{B_i}(t)|}{|\Delta_i|}\chi(\{\Pi_i\colon \omega \in S\}) + \frac{|\Upsilon_{A_i}(t)|}{|\Delta_i|}\chi(\{\Lambda_i\colon \omega \in S\}) \\
&\leq \left[(I_0^{r_i+s_i}\varrho_i)(t) + \frac{|l_i|t^{s_i}}{\Gamma(s_i+1)}\right]\chi(S) \\
&+ \frac{\Upsilon_{B_i}}{|\Delta_i|}\left[(I_0^{r_i+s_i}\varrho_i)(\eta_i) + (I_0^{r_i+s_i}\varrho_i)(1) + |l_i||\delta_i(s_i)\right]\chi(S) \\
&+ \frac{\Upsilon_{A_i}|}{|\Delta_i|}\left[|\beta_i|(I_0^{r_i}\varrho_i)(\xi_i) + |l_i||\beta_i|(I_0^{r_i+s_i}\varrho_i)(\xi_i) + l_i^2\tau_i(s_i)\right]\chi(S) \\
&= \mathcal{H}_i\chi(S)
\end{aligned}$$

which implies, in view of (Ω_4), that

$$\chi(\Phi S)(t)) \leq \sum_{i=1}^{3} \chi(\Phi_i S)(t)) = \sum_{i=1}^{3} \mathcal{H}_i \chi(S), \qquad \sum_{i=1}^{3} \mathcal{H}_i < 1.$$

The required results are given by applying Darbo's fixed point Theorem 2.3. □

6. Conclusions

In this work, we created a tripled system of fractional Langiven equations with some specified operators and generalized boundary conditions. We examined the assumptions and conditions that require solutions utilizing the threefold fixed point and the continuity coefficient by introducing noncompactness measures. Additionally, the singularity property was verified using Banach's principal. Also, we demonstrated the validity of our findings for both the Hausdorff and Kuratowski noncompactness measures. We created a control problem example to verify this correctness.

Our results can be applied to a related problem with non-singular operators in subsequent research.

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