

Copula-Based Bivariate Sine-Transformed Nakagami Distribution: Properties, Estimation, and Applications

Ibrahim Abdullahi¹, Wikanda Phaphan^{2,3,*}

¹*Department of Mathematics and Statistics, Faculty of Science, Yobe State University, Damaturu, 500501, Nigeria*

²*Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand*

³*Research Group in Statistical Learning and Inference, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand*

*Corresponding author: wikanda.p@sci.kmutnb.ac.th

Abstract. Modeling dependence in bivariate continuous data is fundamental in reliability, survival analysis, and environmental studies. In this paper, we propose a new copula-based bivariate sine-transformed Nakagami (CB-SSN) distribution by combining flexible sine-transformed Nakagami marginals with the Clayton copula to capture lower-tail dependence and nonlinear association structures. Key statistical properties of the proposed model are derived, including the joint and marginal distributions, moment expressions, dependence measures, and distributions of the minimum and maximum. Parameter estimation is performed via maximum likelihood, and a Monte Carlo simulation study is conducted to evaluate the finite-sample performance of the estimators. The results demonstrate that the estimators are consistent and asymptotically efficient, with bias and RMSE decreasing as the sample size increases. The applicability of the proposed model is illustrated using the faithful dataset. Comparative results based on log-likelihood, AIC, and BIC show that the CB-SSN model provides a better fit than the classical Clayton–Nakagami model. These findings highlight the flexibility and practical relevance of the proposed distribution for modeling dependent lifetime data.

1. INTRODUCTION

Modeling dependence structures in multivariate data remains a fundamental challenge in modern statistical analysis, with important applications in reliability engineering, survival analysis,

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finance, and wireless communications. In many practical situations, classical multivariate distributions impose restrictive assumptions that limit their ability to capture nonlinear dependence, asymmetry, and tail behavior observed in real datasets.

In recent decades, copula theory has emerged as a powerful and flexible framework for constructing multivariate distributions. By virtue of Sklar's theorem, any multivariate distribution can be decomposed into its marginal distributions and a copula function that fully characterizes the dependence structure [1,2]. This decomposition allows for independent modeling of marginals and dependence, thereby providing substantial flexibility and interpretability. Copula-based models have found wide applications in survival analysis, actuarial science, and reliability modeling; see [3,4].

Parallel to these developments, there has been increasing interest in the construction of generalized and transformed distributions to enhance modeling flexibility. Among these approaches, trigonometric transformations have attracted considerable attention due to their boundedness and their ability to induce diverse shapes in probability density and hazard rate functions. In particular, sine-based transformations have proven effective in generating flexible families of distributions while preserving analytical tractability.

The Nakagami distribution is widely used in reliability theory and wireless fading channel modeling owing to its ability to represent a broad range of distributional shapes. However, its classical form may be inadequate for capturing complex hazard rate behaviors encountered in practice. To address this limitation, a sine-transformed Nakagami (SS-Nakagami) distribution can be constructed, which introduces additional flexibility through a bounded trigonometric transformation of the baseline cumulative distribution function.

Despite the significant progress in univariate distribution theory, the extension of such transformed models to multivariate settings remains relatively underexplored. In many real-world applications, lifetimes or survival times are inherently dependent, and neglecting such dependence may lead to inefficient inference and misleading conclusions.

Motivated by these considerations, this paper proposes a new *Copula-Based Bivariate Sine-Transformed Nakagami (CB-SSN)* distribution. The proposed model is constructed by combining SS-Nakagami marginals with an appropriate copula function, thereby allowing flexible modeling of both marginal behavior and dependence structure.

The main contributions of this paper are summarized as follows:

- Introduction of a novel copula-based bivariate extension of the SS-Nakagami distribution;
- Derivation of key statistical properties, including joint and conditional distributions;
- Development of likelihood-based estimation procedures;
- Demonstration of the model's flexibility through simulation and real data applications.

The remainder of the paper is organized as follows. Section 2 introduces the SS-Nakagami distribution and presents the copula-based construction of the proposed model. Subsequent sections discuss statistical properties, estimation methods, and applications.

2. METHODOLOGY

2.1. The Sine-Transformed Nakagami Distribution. Let X be a positive continuous random variable following the Nakagami distribution with shape parameter $m > 0$ and spread parameter $\Omega > 0$. The baseline cumulative distribution function is given by

$$F_0(x; m, \Omega) = \frac{\gamma\left(m, \frac{m}{\Omega}x^2\right)}{\Gamma(m)}, \quad x > 0, \quad (2.1)$$

where $\gamma(\cdot, \cdot)$ denotes the lower incomplete gamma function and $\Gamma(\cdot)$ is the gamma function.

The corresponding probability density function is

$$f_0(x; m, \Omega) = \frac{2m^m}{\Gamma(m)\Omega^m} x^{2m-1} \exp\left(-\frac{m}{\Omega}x^2\right), \quad x > 0. \quad (2.2)$$

To enhance the flexibility of the baseline model, we define the sine-transformed Nakagami distribution through

$$F_X(x; m, \Omega) = \sin\left(\frac{\pi}{2} \frac{\gamma\left(m, \frac{m}{\Omega}x^2\right)}{\Gamma(m)}\right), \quad x > 0. \quad (2.3)$$

Since $0 \leq F_0(x; m, \Omega) \leq 1$, it follows that $F_X(x; m, \Omega)$ is a valid cumulative distribution function. Furthermore, the sine transformation is strictly increasing on $[0, \pi/2]$, thereby preserving the stochastic ordering of the baseline Nakagami distribution while introducing additional flexibility.

Differentiating (2.3) with respect to x , the corresponding probability density function is obtained as

$$f_X(x; m, \Omega) = \frac{\pi}{2} \cos\left(\frac{\pi}{2} \frac{\gamma\left(m, \frac{m}{\Omega}x^2\right)}{\Gamma(m)}\right) \times \frac{2m^m}{\Gamma(m)\Omega^m} x^{2m-1} \exp\left(-\frac{m}{\Omega}x^2\right), \quad (2.4)$$

The survival function is given by

$$S_X(x; m, \Omega) = 1 - \sin\left(\frac{\pi}{2} \frac{\gamma\left(m, \frac{m}{\Omega}x^2\right)}{\Gamma(m)}\right), \quad (2.5)$$

and the corresponding hazard rate function is

$$h_X(x; m, \Omega) = \frac{\frac{\pi}{2} \cos\left(\frac{\pi}{2} \frac{\gamma\left(m, \frac{m}{\Omega}x^2\right)}{\Gamma(m)}\right) \cdot \frac{2m^m}{\Gamma(m)\Omega^m} x^{2m-1} \exp\left(-\frac{m}{\Omega}x^2\right)}{1 - \sin\left(\frac{\pi}{2} \frac{\gamma\left(m, \frac{m}{\Omega}x^2\right)}{\Gamma(m)}\right)}. \quad (2.6)$$

2.2. Copula Construction. Let X and Y be two continuous random variables with marginal distribution functions $F_X(x)$ and $F_Y(y)$, respectively. Copula theory provides a flexible framework for constructing their joint distribution by modeling the marginals and the dependence structure separately.

According to Sklar's theorem [1, 2], for any bivariate distribution function $F_{X,Y}(x, y)$ with continuous marginals $F_X(x)$ and $F_Y(y)$, there exists a unique copula function $C : [0, 1]^2 \rightarrow [0, 1]$ such that

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)). \quad (2.7)$$

In this study, we adopt the Clayton copula to model the dependence between X and Y . This copula is well suited for reliability and survival applications because it captures positive dependence together with lower-tail association. It is defined by

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad \theta > 0, \quad (2.8)$$

where

$$u = F_X(x), \quad v = F_Y(y), \quad (2.9)$$

and θ is the dependence parameter. Hence,

$$F_{X,Y}(x, y) = C_\theta(F_X(x), F_Y(y)) = [F_X(x)^{-\theta} + F_Y(y)^{-\theta} - 1]^{-1/\theta}. \quad (2.10)$$

The parameter θ controls the strength of dependence. As $\theta \rightarrow 0^+$, the Clayton copula reduces to the independence copula $C(u, v) = uv$, whereas larger values of θ correspond to stronger positive dependence.

If the copula is twice continuously differentiable, then the joint density is

$$f_{X,Y}(x, y) = c_\theta(u, v) f_X(x) f_Y(y), \quad (2.11)$$

where the Clayton copula density is

$$c_\theta(u, v) = (\theta + 1)(uv)^{-(\theta+1)} (u^{-\theta} + v^{-\theta} - 1)^{-(2+\frac{1}{\theta})}, \quad \theta > 0. \quad (2.12)$$

Therefore,

$$f_{X,Y}(x, y) = (\theta + 1)(F_X(x)F_Y(y))^{-(\theta+1)} \times (F_X(x)^{-\theta} + F_Y(y)^{-\theta} - 1)^{-(2+\frac{1}{\theta})} \times f_X(x)f_Y(y). \quad (2.13)$$

2.3. Copula-Based Bivariate Sine-Transformed Nakagami Distribution. Assume that the marginals of X and Y follow sine-transformed Nakagami distributions with parameter sets (m_1, Ω_1) and (m_2, Ω_2) , respectively. Their cumulative distribution functions are

$$F_X(x) = \sin\left(\frac{\pi}{2} \frac{\gamma(m_1, \frac{m_1}{\Omega_1} x^2)}{\Gamma(m_1)}\right), \quad x > 0, \quad (2.14)$$

and

$$F_Y(y) = \sin\left(\frac{\pi}{2} \frac{\gamma(m_2, \frac{m_2}{\Omega_2} y^2)}{\Gamma(m_2)}\right), \quad y > 0. \quad (2.15)$$

The corresponding marginal densities are

$$f_X(x) = \frac{\pi}{2} \cos\left(\frac{\pi}{2} \frac{\gamma(m_1, \frac{m_1}{\Omega_1} x^2)}{\Gamma(m_1)}\right) \frac{2m_1^{m_1}}{\Gamma(m_1)\Omega_1^{m_1}} x^{2m_1-1} \times \exp\left(-\frac{m_1}{\Omega_1} x^2\right), \quad (2.16)$$

and

$$f_Y(y) = \frac{\pi}{2} \cos\left(\frac{\pi}{2} \frac{\gamma(m_2, \frac{m_2}{\Omega_2} y^2)}{\Gamma(m_2)}\right) \frac{2m_2^{m_2}}{\Gamma(m_2)\Omega_2^{m_2}} y^{2m_2-1} \times \exp\left(-\frac{m_2}{\Omega_2} y^2\right). \quad (2.17)$$

Substituting (2.14) and (2.15) into (2.10), the joint cumulative distribution function of the proposed copula-based bivariate sine-transformed Nakagami (CB-SSN) distribution is given as:

$$F_{X,Y}(x, y) = \left[F_X(x)^{-\theta} + F_Y(y)^{-\theta} - 1 \right]^{-1/\theta}, \quad \theta > 0. \quad (2.18)$$

Similarly, substituting (2.14), (2.15), (2.16), and (2.17) into (2.13), we obtain the joint density function

$$f_{X,Y}(x, y) = (\theta + 1) \left(F_X(x) F_Y(y) \right)^{-(\theta+1)} \times \left(F_X(x)^{-\theta} + F_Y(y)^{-\theta} - 1 \right)^{-(2+\frac{1}{\theta})} \times f_X(x) f_Y(y).$$

Thus, the CB-SSN model combines the marginal flexibility of the sine-transformed Nakagami distribution with the lower-tail dependence induced by the Clayton copula, yielding a tractable model for dependent lifetime data.

2.4. Linear Representations of the Joint CDF and PDF. In this section, we derive linear (series) representations for the joint cumulative distribution function (CDF) and probability density function (PDF) of the proposed copula-based bivariate sine-transformed Nakagami (CB-SSN) distribution. These representations facilitate theoretical analysis, numerical approximation, and the derivation of distributional properties.

2.5. Linear Representation of the Joint CDF. Using the series expansion of the sine function, the marginal distribution functions defined in (2.14) and (2.15) admit the representations

$$F_X(x) = \sum_{k=0}^{\infty} a_k G_1(x)^{2k+1}, \quad (2.19)$$

and

$$F_Y(y) = \sum_{l=0}^{\infty} a_l G_2(y)^{2l+1}, \quad (2.20)$$

where

$$G_1(x) = \frac{\gamma\left(m_1, \frac{m_1}{\Omega_1} x^2\right)}{\Gamma(m_1)}, \quad G_2(y) = \frac{\gamma\left(m_2, \frac{m_2}{\Omega_2} y^2\right)}{\Gamma(m_2)},$$

and

$$a_k = \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{2} \right)^{2k+1}.$$

Substituting these expansions into the Clayton copula-based joint CDF given in (2.18), we obtain

$$F_{X,Y}(x, y) = \left[\left(\sum_{k=0}^{\infty} a_k G_1(x)^{2k+1} \right)^{-\theta} + \left(\sum_{l=0}^{\infty} a_l G_2(y)^{2l+1} \right)^{-\theta} - 1 \right]^{-1/\theta}. \quad (2.21)$$

This representation expresses the joint CDF explicitly in terms of infinite series involving the baseline Nakagami distribution functions.

Using the joint density representation in (2.13), together with the series expansions of the marginal densities, we write

$$f_X(x) = \sum_{k=0}^{\infty} \alpha_k G_1(x)^{2k} f_{01}(x), \quad (2.22)$$

$$f_Y(y) = \sum_{l=0}^{\infty} \alpha_l G_2(y)^{2l} f_{02}(y), \quad (2.23)$$

where

$$\alpha_k = \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1}.$$

Hence, the joint density admits the representation

$$f_{X,Y}(x, y) = c_{\theta}(F_X(x), F_Y(y)) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \alpha_k \alpha_l G_1(x)^{2k} G_2(y)^{2l} f_{01}(x) f_{02}(y). \quad (2.24)$$

Remark. The dependence structure is fully governed by the Clayton copula density $c_{\theta}(u, v)$ defined in (2.12). Consequently, the above series representation shows that the CB-SSN model is composed of baseline Nakagami components, while the parameter θ controls the strength of lower-tail dependence.

3. MOMENTS, DEPENDENCE PROPERTIES AND MOMENT GENERATING FUNCTION

In this section, we derive the moments, dependence measures, and moment generating function (MGF) of the proposed CB-SSN distribution using the linear representations established in Section 2.5.

3.1. Moments of the Marginal Distributions. Let $X \sim \text{SS-Nakagami}(m, \Omega)$. Using the linear representation of the marginal density obtained in Section 2.5, we write

$$f_X(x) = \sum_{k=0}^{\infty} \alpha_k G(x)^{2k} f_0(x),$$

where

$$G(x) = \frac{\gamma\left(m, \frac{m}{\Omega}x^2\right)}{\Gamma(m)}, \quad \alpha_k = \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1},$$

and $f_0(x)$ denotes the Nakagami density defined in (2.2).

The r -th raw moment of X is given by

$$\begin{aligned} E(X^r) &= \int_0^{\infty} x^r f_X(x) dx \\ &= \sum_{k=0}^{\infty} \alpha_k \int_0^{\infty} x^r G(x)^{2k} f_0(x) dx \\ &= \sum_{k=0}^{\infty} \alpha_k E_0[X^r G(X)^{2k}], \end{aligned} \quad (3.1)$$

where $E_0(\cdot)$ denotes expectation taken with respect to the baseline Nakagami distribution. This representation expresses the marginal moments as infinite weighted expectations under the Nakagami model, facilitating both analytical derivations and numerical evaluation.

3.2. Joint Moments. Let (X, Y) follow the CB-SSN distribution. The joint moment of order (r, s) is defined as

$$E(X^r Y^s) = \int_0^\infty \int_0^\infty x^r y^s f_{X,Y}(x, y) dx dy.$$

Using the joint density representation in (2.13), we obtain

$$E(X^r Y^s) = \int_0^\infty \int_0^\infty x^r y^s c_\theta(F_X(x), F_Y(y)) f_X(x) f_Y(y) dx dy. \quad (3.2)$$

Substituting the linear representations of the marginal densities from Section 2.5, we have

$$\begin{aligned} f_X(x) &= \sum_{k=0}^{\infty} \alpha_k G_1(x)^{2k} f_{01}(x), \\ f_Y(y) &= \sum_{l=0}^{\infty} \alpha_l G_2(y)^{2l} f_{02}(y). \end{aligned}$$

Hence, the joint moment becomes

$$E(X^r Y^s) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \alpha_k \alpha_l \int_0^\infty \int_0^\infty x^r y^s G_1(x)^{2k} G_2(y)^{2l} \times c_\theta(F_X(x), F_Y(y)) f_{01}(x) f_{02}(y) dx dy. \quad (3.3)$$

Equation (3.3) provides a series representation of the joint moments in terms of weighted integrals involving the baseline Nakagami densities and the Clayton copula dependence structure.

Remark. In the special case where $\theta \rightarrow 0^+$, the Clayton copula reduces to independence, and (3.3) simplifies to

$$E(X^r Y^s) = E(X^r) E(Y^s),$$

which is consistent with the independence property.

3.3. Dependence Properties. The dependence structure of the CB-SSN model is governed entirely by the Clayton copula. Since the sine transformation is strictly increasing, it preserves the copula structure; therefore, all rank-based dependence measures depend only on the copula parameter θ .

For the Clayton copula, Kendall's tau and Spearman's rho admit closed-form expressions given by

$$\tau = \frac{\theta}{\theta + 2}, \quad \theta > 0, \quad (3.4)$$

and

$$\rho = \frac{3\theta}{\theta + 3}, \quad \theta > 0. \quad (3.5)$$

Both measures are increasing functions of θ , indicating that the strength of dependence increases as θ increases.

The lower and upper tail dependence coefficients are given by

$$\lambda_L = 2^{-1/\theta}, \quad \lambda_U = 0. \quad (3.6)$$

Thus, the CB-SSN distribution exhibits positive dependence and lower-tail dependence, while upper-tail dependence is absent. This property makes the model particularly suitable for reliability and survival applications where joint extreme events tend to occur in the lower tail.

Remark. As $\theta \rightarrow 0^+$, we have $\tau \rightarrow 0$, $\rho \rightarrow 0$, and $\lambda_L \rightarrow 0$, corresponding to the independence case. Conversely, larger values of θ lead to stronger dependence and increased lower-tail association.

3.4. Moment Generating Function. Using the linear representation of the marginal density, the moment generating function (MGF) of X is given by

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{k=0}^{\infty} \alpha_k \int_0^{\infty} e^{tx} G(x)^{2k} f_0(x) dx \\ &= \sum_{k=0}^{\infty} \alpha_k E_0[e^{tX} G(X)^{2k}], \end{aligned} \quad (3.7)$$

where $E_0(\cdot)$ denotes expectation with respect to the baseline Nakagami distribution.

Similarly, the joint moment generating function of (X, Y) is defined as

$$M_{X,Y}(t_1, t_2) = E(e^{t_1 X + t_2 Y}).$$

Using the joint density representation and the linear expansions of the marginal densities, we obtain

$$\begin{aligned} M_{X,Y}(t_1, t_2) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \alpha_k \alpha_l \int_0^{\infty} \int_0^{\infty} e^{t_1 x + t_2 y} G_1(x)^{2k} G_2(y)^{2l} \\ &\quad \times c_\theta(F_X(x), F_Y(y)) f_{01}(x) f_{02}(y) dx dy. \end{aligned} \quad (3.8)$$

Existence Condition. The MGF exists for values of t (and t_1, t_2) such that the corresponding expectations under the Nakagami distribution are finite. In particular, since the Nakagami distribution has finite exponential moments in a neighborhood of zero, the MGF of the CB-SSN distribution exists for sufficiently small $|t|$.

3.5. Distribution of the Minimum and Maximum of the Two Random Variables. Let X and Y be two random variables, and define

$$T_1 = \min(X, Y), \quad T_2 = \max(X, Y),$$

where (X, Y) follows the proposed copula-based bivariate sine-transformed Nakagami (CB-SSN) distribution.

The marginal cumulative distribution functions are

$$F_X(t) = \sin\left(\frac{\pi}{2} \frac{\gamma\left(m_1, \frac{m_1}{\Omega_1} t^2\right)}{\Gamma(m_1)}\right), \quad t > 0, \quad (3.9)$$

and

$$F_Y(t) = \sin\left(\frac{\pi}{2} \frac{\gamma\left(m_2, \frac{m_2}{\Omega_2} t^2\right)}{\Gamma(m_2)}\right), \quad t > 0. \quad (3.10)$$

The corresponding marginal densities are

$$f_X(t) = \frac{\pi}{2} \cos\left(\frac{\pi}{2} \frac{\gamma\left(m_1, \frac{m_1}{\Omega_1} t^2\right)}{\Gamma(m_1)}\right) \frac{2m_1^{m_1}}{\Gamma(m_1)\Omega_1^{m_1}} t^{2m_1-1} \exp\left(-\frac{m_1}{\Omega_1} t^2\right), \quad (3.11)$$

and

$$f_Y(t) = \frac{\pi}{2} \cos\left(\frac{\pi}{2} \frac{\gamma\left(m_2, \frac{m_2}{\Omega_2} t^2\right)}{\Gamma(m_2)}\right) \frac{2m_2^{m_2}}{\Gamma(m_2)\Omega_2^{m_2}} t^{2m_2-1} \exp\left(-\frac{m_2}{\Omega_2} t^2\right). \quad (3.12)$$

Under the Clayton copula, the joint cumulative distribution function evaluated at (t, t) is

$$F_{X,Y}(t, t) = \left[F_X(t)^{-\theta} + F_Y(t)^{-\theta} - 1\right]^{-1/\theta}, \quad \theta > 0. \quad (3.13)$$

3.6. Distribution of the Minimum. For $t > 0$, the cumulative distribution function of $T_1 = \min(X, Y)$ is

$$\begin{aligned} F_{T_1}(t) &= P(\min(X, Y) \leq t) \\ &= F_X(t) + F_Y(t) - F_{X,Y}(t, t). \end{aligned} \quad (3.14)$$

Hence, using (3.9)–(3.13), we obtain

$$F_{T_1}(t) = F_X(t) + F_Y(t) - \left[F_X(t)^{-\theta} + F_Y(t)^{-\theta} - 1\right]^{-1/\theta}. \quad (3.15)$$

Therefore, the survival function of T_1 is

$$S_{T_1}(t) = 1 - F_X(t) - F_Y(t) + \left[F_X(t)^{-\theta} + F_Y(t)^{-\theta} - 1\right]^{-1/\theta}. \quad (3.16)$$

To derive the density of T_1 , note that for the Clayton copula

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta},$$

its first-order partial derivatives are

$$C_1(u, v) = u^{-\theta-1} (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta-1}, \quad (3.17)$$

and

$$C_2(u, v) = v^{-\theta-1} (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta-1}. \quad (3.18)$$

Differentiating (3.15) with respect to t , the density of T_1 is

$$\begin{aligned} f_{T_1}(t) &= f_X(t) + f_Y(t) - C_1(F_X(t), F_Y(t))f_X(t) - C_2(F_X(t), F_Y(t))f_Y(t) \\ &= \left[1 - C_1(F_X(t), F_Y(t))\right]f_X(t) + \left[1 - C_2(F_X(t), F_Y(t))\right]f_Y(t). \end{aligned} \quad (3.19)$$

Accordingly, the hazard rate function of the minimum is

$$h_{T_1}(t) = \frac{f_{T_1}(t)}{S_{T_1}(t)}. \quad (3.20)$$

3.7. Distribution of the Maximum. For $t > 0$, the cumulative distribution function of $T_2 = \max(X, Y)$ is

$$F_{T_2}(t) = P(\max(X, Y) \leq t) = F_{X,Y}(t, t). \quad (3.21)$$

Therefore, from (3.13),

$$F_{T_2}(t) = [F_X(t)^{-\theta} + F_Y(t)^{-\theta} - 1]^{-1/\theta}, \quad \theta > 0. \quad (3.22)$$

The corresponding survival function is

$$S_{T_2}(t) = 1 - F_{T_2}(t) = 1 - [F_X(t)^{-\theta} + F_Y(t)^{-\theta} - 1]^{-1/\theta}. \quad (3.23)$$

Differentiating (3.22) with respect to t , the density of T_2 is

$$\begin{aligned} f_{T_2}(t) &= C_1(F_X(t), F_Y(t))f_X(t) + C_2(F_X(t), F_Y(t))f_Y(t) \\ &= F_X(t)^{-\theta-1} [F_X(t)^{-\theta} + F_Y(t)^{-\theta} - 1]^{-1/\theta-1} f_X(t) \\ &\quad + F_Y(t)^{-\theta-1} [F_X(t)^{-\theta} + F_Y(t)^{-\theta} - 1]^{-1/\theta-1} f_Y(t). \end{aligned} \quad (3.24)$$

Accordingly, the hazard rate function of the maximum is

$$h_{T_2}(t) = \frac{f_{T_2}(t)}{S_{T_2}(t)}. \quad (3.25)$$

3.8. Special Case: Independence. As $\theta \rightarrow 0^+$, the Clayton copula reduces to the independence copula $C(u, v) = uv$. Consequently,

$$F_{T_1}(t) = F_X(t) + F_Y(t) - F_X(t)F_Y(t), \quad (3.26)$$

and

$$F_{T_2}(t) = F_X(t)F_Y(t). \quad (3.27)$$

Using (3.9) and (3.10), these may be written as

$$\begin{aligned} F_{T_1}(t) &= \sin\left(\frac{\pi}{2} \frac{\gamma(m_1, \frac{m_1}{\Omega_1} t^2)}{\Gamma(m_1)}\right) + \sin\left(\frac{\pi}{2} \frac{\gamma(m_2, \frac{m_2}{\Omega_2} t^2)}{\Gamma(m_2)}\right) \\ &\quad - \sin\left(\frac{\pi}{2} \frac{\gamma(m_1, \frac{m_1}{\Omega_1} t^2)}{\Gamma(m_1)}\right) \sin\left(\frac{\pi}{2} \frac{\gamma(m_2, \frac{m_2}{\Omega_2} t^2)}{\Gamma(m_2)}\right), \end{aligned} \quad (3.28)$$

and

$$F_{T_2}(t) = \sin\left(\frac{\pi}{2} \frac{\gamma(m_1, \frac{m_1}{\Omega_1} t^2)}{\Gamma(m_1)}\right) \sin\left(\frac{\pi}{2} \frac{\gamma(m_2, \frac{m_2}{\Omega_2} t^2)}{\Gamma(m_2)}\right). \quad (3.29)$$

If, in addition, X and Y are identically distributed, then

$$F_{T_1}(t) = 2F_X(t) - F_X(t)^2, \quad F_{T_2}(t) = F_X(t)^2. \quad (3.30)$$

4. PARAMETER ESTIMATION

In this section, the unknown parameters of the proposed copula-based bivariate sine-transformed Nakagami (CB-SSN) distribution are estimated using the method of maximum likelihood.

Let

$$(x_i, y_i), \quad i = 1, 2, \dots, n,$$

be a random sample drawn from the CB-SSN distribution with parameter vector

$$\Theta = (m_1, \Omega_1, m_2, \Omega_2, \theta)^\top,$$

where (m_1, Ω_1) and (m_2, Ω_2) are the marginal parameters, and $\theta > 0$ is the dependence parameter of the Clayton copula.

Define

$$u_i = F_X(x_i), \quad v_i = F_Y(y_i), \quad i = 1, \dots, n,$$

where $F_X(\cdot)$ and $F_Y(\cdot)$ are given in (2.14) and (2.15), respectively.

The joint density of (X, Y) is expressed as

$$f_{X,Y}(x, y) = c_\theta(u, v) f_X(x) f_Y(y),$$

where the Clayton copula density is

$$c_\theta(u, v) = (\theta + 1)(uv)^{-(\theta+1)} (u^{-\theta} + v^{-\theta} - 1)^{-(2+\frac{1}{\theta})}.$$

Hence, the likelihood function based on the sample is

$$L(\Theta) = \prod_{i=1}^n c_\theta(u_i, v_i) f_X(x_i) f_Y(y_i). \quad (4.1)$$

Taking logarithms, the log-likelihood function becomes

$$\begin{aligned} \ell(\Theta) &= n \log(\theta + 1) - (\theta + 1) \sum_{i=1}^n \log(u_i v_i) \\ &\quad - \left(2 + \frac{1}{\theta}\right) \sum_{i=1}^n \log(u_i^{-\theta} + v_i^{-\theta} - 1) + \sum_{i=1}^n \log f_X(x_i) + \sum_{i=1}^n \log f_Y(y_i). \end{aligned} \quad (4.2)$$

Substituting the marginal densities given in (2.16) and (2.17), the log-likelihood function is obtained explicitly in terms of the model parameters.

The maximum likelihood estimates (MLEs) of the parameter vector Θ are obtained by solving the system of nonlinear equations

$$\frac{\partial \ell}{\partial m_1} = 0, \quad \frac{\partial \ell}{\partial \Omega_1} = 0, \quad \frac{\partial \ell}{\partial m_2} = 0, \quad \frac{\partial \ell}{\partial \Omega_2} = 0, \quad \frac{\partial \ell}{\partial \theta} = 0.$$

In particular, the score function for the copula parameter θ is given by

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta + 1} - \sum_{i=1}^n \log(u_i v_i) + \frac{1}{\theta^2} \sum_{i=1}^n \log(u_i^{-\theta} + v_i^{-\theta} - 1)$$

$$-\left(2 + \frac{1}{\theta}\right) \sum_{i=1}^n \frac{u_i^{-\theta} \log u_i + v_i^{-\theta} \log v_i}{u_i^{-\theta} + v_i^{-\theta} - 1}. \quad (4.3)$$

Due to the nonlinear nature of the likelihood equations, closed-form solutions are not available. Therefore, the MLEs are obtained via numerical optimization techniques such as the Broyden–Fletcher–Goldfarb–Shanno (BFGS) or L-BFGS-B algorithms.

The parameter space is defined by

$$m_1 > 0, \quad \Omega_1 > 0, \quad m_2 > 0, \quad \Omega_2 > 0, \quad \theta > 0.$$

Initial values for the marginal parameters are obtained from separate fits of the sine-transformed Nakagami distribution, while the copula parameter is initialized using the inversion of Kendall's tau,

$$\theta = \frac{2\tau}{1-\tau}.$$

4.1. Simulation Algorithm. To assess the finite-sample performance of the maximum likelihood estimators for the proposed CB-SSN distribution, the following Monte Carlo simulation algorithm was employed.

Algorithm 1 Monte Carlo simulation algorithm for the CB-SSN distribution

- 1: Specify $(m_1, \Omega_1, m_2, \Omega_2, \theta)$, sample size n , and number of replications R
 - 2: **for** $r = 1$ to R **do**
 - 3: **for** $i = 1$ to n **do**
 - 4: Generate $W_i \sim \text{Gamma}(1/\theta, 1)$
 - 5: Generate $E_{1i}, E_{2i} \sim \text{Exp}(1)$ independently
 - 6: Set $U_i = \left(1 + \frac{E_{1i}}{W_i}\right)^{-1/\theta}$ and $V_i = \left(1 + \frac{E_{2i}}{W_i}\right)^{-1/\theta}$
 - 7: Compute $X_i = F_X^{-1}(U_i)$ and $Y_i = F_Y^{-1}(V_i)$
 - 8: **end for**
 - 9: Estimate $(m_1, \Omega_1, m_2, \Omega_2, \theta)$ by maximum likelihood
 - 10: Store the resulting estimates
 - 11: **end for**
 - 12: Compute Mean, Bias and RMSE
-

The simulation results presented in Tables 1 and 2 demonstrate that the maximum likelihood estimators possess satisfactory finite-sample properties across different levels of dependence. In particular, the bias and RMSE decrease systematically as the sample size increases, providing clear empirical evidence of consistency and asymptotic efficiency.

A comparison between the two dependence scenarios shows that stronger dependence ($\theta = 3.0$) leads to slightly higher variability, especially in the estimation of the copula parameter, as observed

TABLE 1. Monte Carlo simulation results for the CB-SSN model under the Clayton copula with $\theta = 2.0$.

n	Parameter	Mean	Bias	RMSE
50	m_1	1.5758	0.0758	0.2673
	Ω_1	0.9952	-0.0048	0.1078
	m_2	1.8898	0.0898	0.3302
	Ω_2	1.1970	-0.0030	0.1143
	θ	2.0688	0.0688	0.5376
100	m_1	1.5388	0.0388	0.1840
	Ω_1	0.9943	-0.0057	0.0774
	m_2	1.8428	0.0428	0.2165
	Ω_2	1.1973	-0.0027	0.0779
	θ	2.0199	0.0199	0.3660
250	m_1	1.5156	0.0156	0.1109
	Ω_1	0.9981	-0.0019	0.0466
	m_2	1.8205	0.0205	0.1369
	Ω_2	1.2003	0.0003	0.0511
	θ	2.0076	0.0076	0.2316

in Table 2. This behavior is expected since larger values of θ correspond to stronger lower-tail dependence in the Clayton copula, which increases estimation complexity in small samples. Nevertheless, this effect diminishes as the sample size increases.

Overall, the CB-SSN model exhibits stable and reliable estimation performance under varying dependence structures, confirming its flexibility and suitability for modeling dependent lifetime data.

5. REAL DATA APPLICATION

5.1. Data Description. To demonstrate the practical applicability of the proposed model, we consider the well-known `faithful` dataset available in the R statistical software [5]. This dataset contains observations from the Old Faithful geyser located in Yellowstone National Park, USA.

The dataset comprises $n = 272$ paired observations of two continuous variables:

- X_1 : Eruption duration (in minutes),
- X_2 : Waiting time between successive eruptions (in minutes).

TABLE 2. Monte Carlo simulation results for the CB-SSN model under the Clayton copula with $\theta = 3.0$.

n	Parameter	True	Mean	Bias	RMSE
50	m_1	1.5	1.5688	0.0688	0.2698
	Ω_1	1.0	0.9925	-0.0075	0.1060
	m_2	1.8	1.8835	0.0835	0.3271
	Ω_2	1.2	1.1940	-0.0060	0.1139
	θ	3.0	3.0999	0.0999	0.7253
100	m_1	1.5	1.5286	0.0286	0.1719
	Ω_1	1.0	0.9954	-0.0046	0.0728
	m_2	1.8	1.8398	0.0398	0.2106
	Ω_2	1.2	1.1929	-0.0071	0.0801
	θ	3.0	3.0694	0.0694	0.4909
250	m_1	1.5	1.5212	0.0212	0.1107
	Ω_1	1.0	0.9983	-0.0017	0.0464
	m_2	1.8	1.8212	0.0212	0.1355
	Ω_2	1.2	1.2001	0.0001	0.0509
	θ	3.0	3.0077	0.0077	0.3199

Five-Panel Mean Estimates for CB-SSN Parameters under Clayton Copula ($\theta = 2.0$)

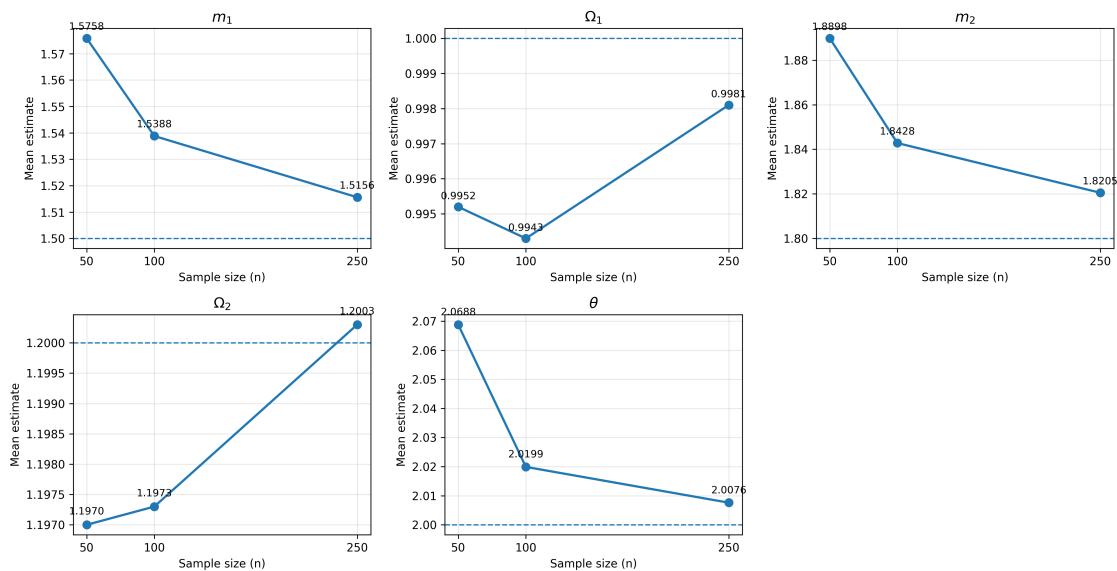


FIGURE 1. Mean estimates of the CB-SSN model parameters for different sample sizes.

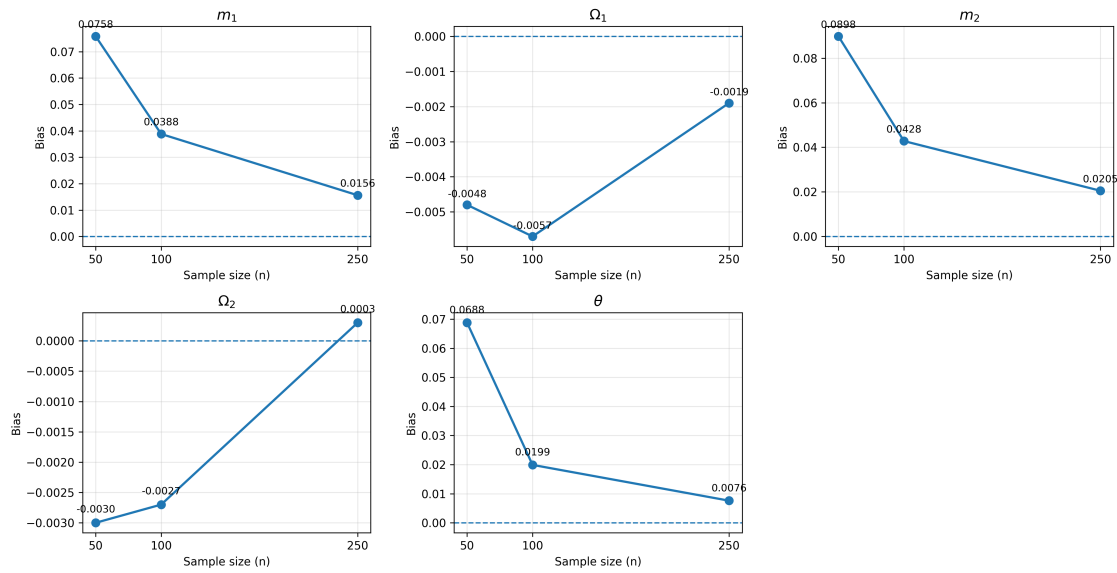
Five-Panel Bias for CB-SSN Parameters under Clayton Copula ($\theta = 2.0$)

FIGURE 2. Bias of the CB-SSN model parameter estimates for different sample sizes.

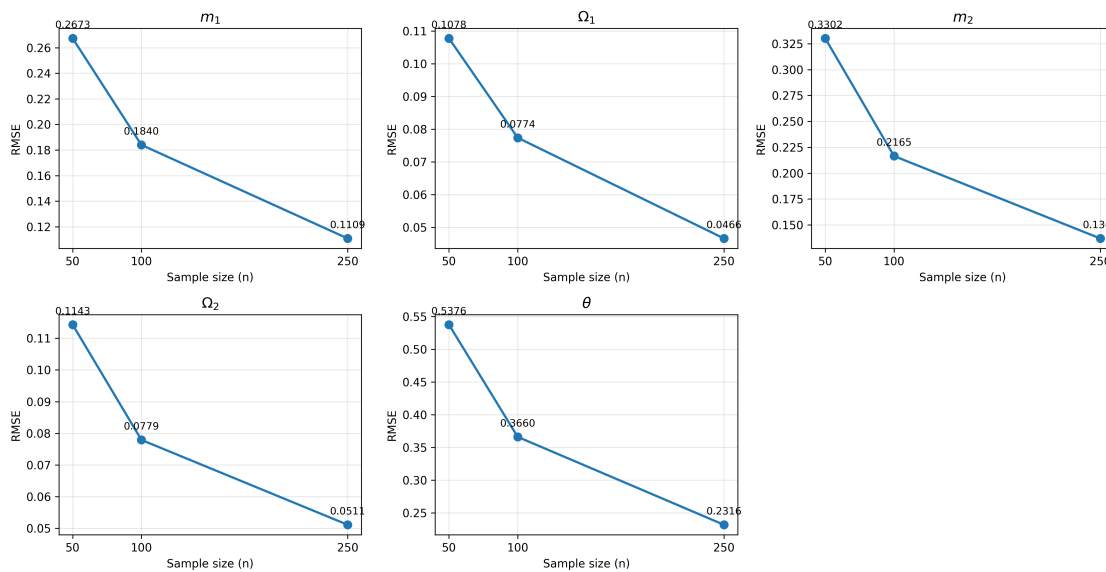
Five-Panel RMSE for CB-SSN Parameters under Clayton Copula ($\theta = 2.0$)

FIGURE 3. Root mean squared error (RMSE) of the parameter estimates of the CB-SSN model for different sample sizes.

Both variables are strictly positive and continuous, making them suitable for modeling using continuous probability distributions. Moreover, the variables exhibit a clear dependence structure. In particular, longer eruption durations are generally associated with longer waiting times, indicating a strong positive relationship.

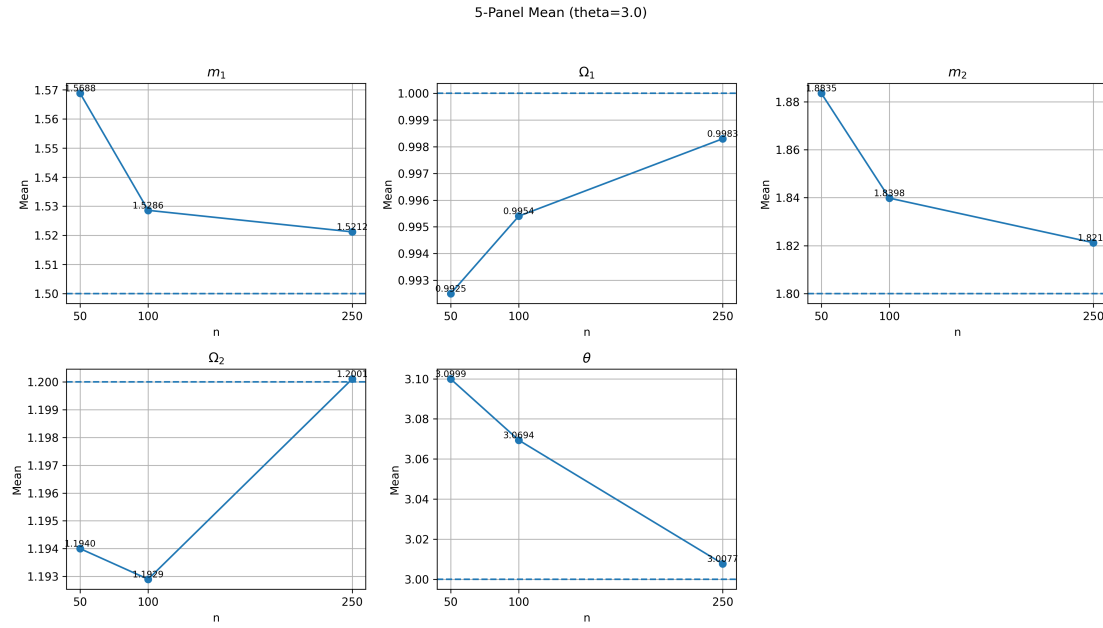


FIGURE 4. Mean estimates of the CB-SSN model parameters for different sample sizes.

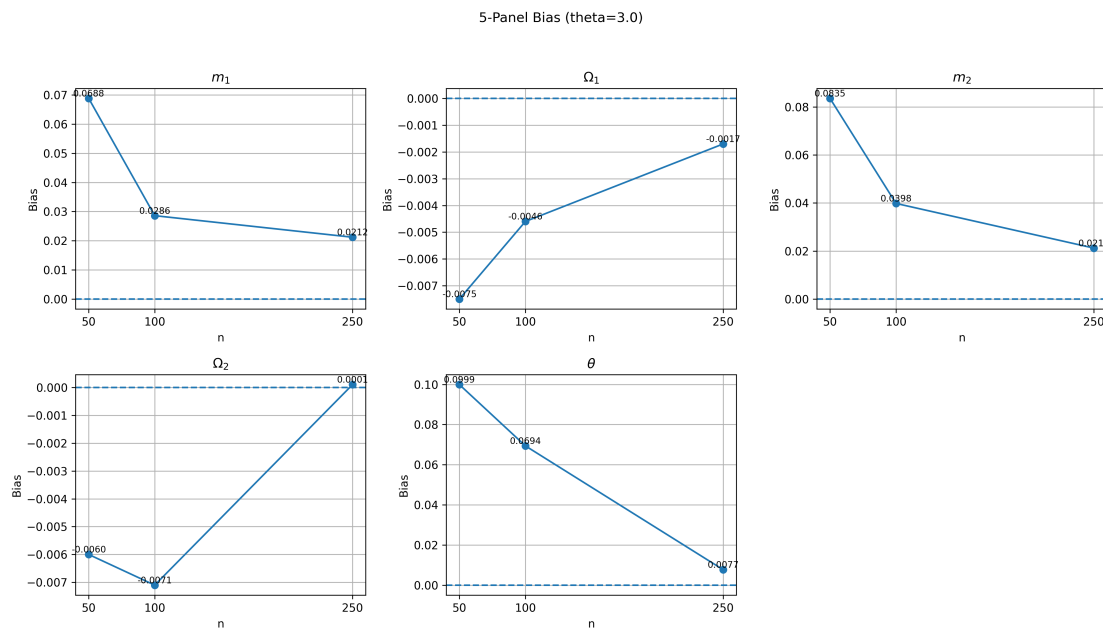


FIGURE 5. Bias of the CB-SSN model parameter estimates for different sample sizes.

5.2. Results and Discussion. To evaluate the performance of the proposed model, two copula-based bivariate models are fitted to the data:

- (1) Clayton copula with Nakagami marginals (Clayton–Nakagami model),
- (2) Clayton copula with sine-transformed Nakagami marginals (proposed CB-SSN model).

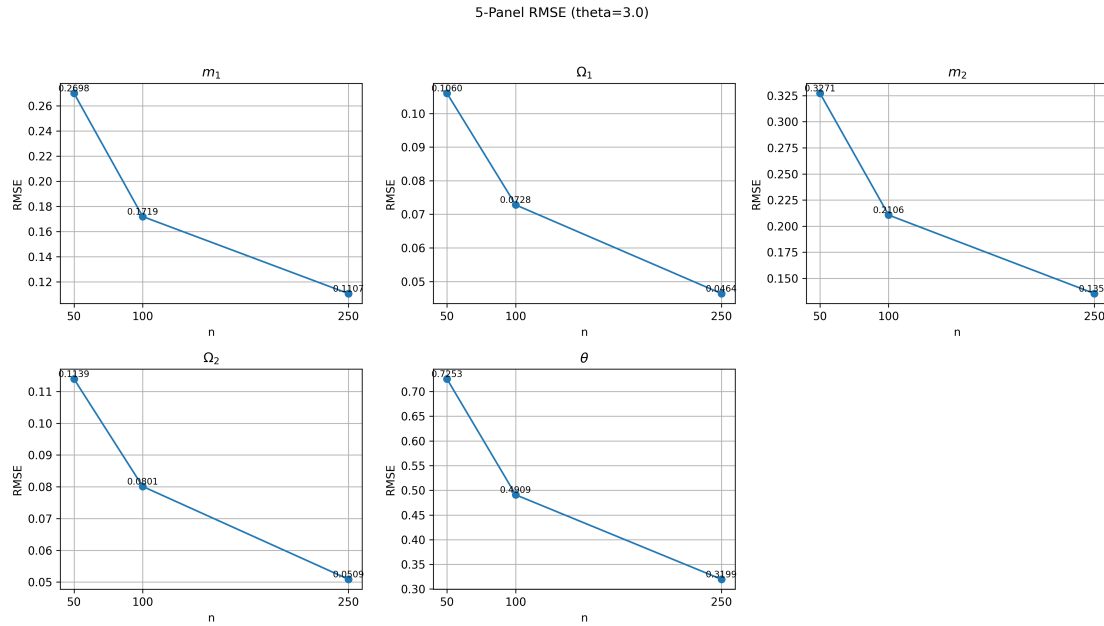


FIGURE 6. RMSE of the CB-SSN model parameter estimates for different sample sizes.

TABLE 3. Parameter estimates and model comparison for the faithful dataset

Model	Parameter	Estimate	LogLik	AIC	BIC	Kendall's τ
Clayton–Nakagami	m_1	2.5137	-1338.6249	2687.2498	2705.2788	0.5491
	Ω_1	13.2961				
	m_2	6.9601				
	Ω_2	5210.5408				
	θ	2.4361				
CB-SSN	m_1	2.2935	-1322.7185	2655.4370	2673.4660	0.5465
	Ω_1	19.6696				
	m_2	5.8752				
	Ω_2	6589.0449				
	θ	2.4104				

In both models, the dependence structure is modeled using the Clayton copula, which is well known for its ability to capture lower-tail dependence. The model parameters are estimated via the method of maximum likelihood.

The results presented in Table 3 show that the proposed CB-SSN model achieves the highest log-likelihood and the lowest AIC and BIC values among the competing models. This provides strong

evidence that the proposed model offers a superior fit to the data compared to the classical Clayton–Nakagami model. The estimated dependence parameter θ is relatively consistent across both models, indicating a stable moderate-to-strong positive dependence between eruption duration and waiting time. This observation is further supported by the corresponding Kendall's tau values, which confirm the presence of positive association. Importantly, the improvement in model performance is primarily attributed to the flexibility of the proposed sine-transformed Nakagami marginals. The differences in the estimated marginal parameters suggest that the CB–SSN model is better able to capture the underlying distributional characteristics of the data. Overall, the numerical results clearly demonstrate that the proposed copula-based bivariate sine-transformed Nakagami (CB–SSN) model provides enhanced flexibility and improved goodness-of-fit, making it a more suitable choice for modeling the joint behavior of the faithful dataset.

6. CONCLUSION

This paper introduced a new copula-based bivariate sine-transformed Nakagami (CB–SSN) distribution, combining flexible marginal modeling with the Clayton copula dependence structure. Key statistical properties were derived, and maximum likelihood estimation was developed for parameter inference. Simulation results confirmed that the estimators exhibit satisfactory finite-sample performance. The application to the faithful dataset demonstrated that the proposed model provides a superior fit compared to the classical Clayton–Nakagami model, as evidenced by improved likelihood and information criteria. Overall, the CB–SSN model offers a flexible and tractable framework for modeling dependent continuous data and represents a useful addition to the class of copula-based distributions.

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