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# On Tripolar Fuzzy Soft Sets and Their Application in Ordered Semigroups

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**Abstract.** In this paper, we introduces and studies tripolar fuzzy soft sets in ordered semigroups. We define several key concepts, including tripolar fuzzy soft subsemigroups, left (right) ideals, quasi-ideals, and bi-ideals, and provide examples to clarify them. We analyze the structural properties of these ideals using a soft product operation. We show that the collection of all tripolar fuzzy soft sets forms a tripolar fuzzy soft ordered semigroup under this product. A main result of this study is the characterization of quasi-ideals: we prove that a tripolar fuzzy soft set is a quasi-ideal if and only if it is the intersection of a left ideal and a right ideal. Furthermore, we show that in regular ordered semigroups, the concepts of quasi-ideals and bi-ideals are identical.

## 1. Introduction

The study of uncertainty and vagueness has become a pivotal aspect of modern mathematics and decision science. Traditional set theory often falls short when dealing with problems where data is incomplete or multi-faceted. To address these challenges, Zadeh [1] introduced the concept of fuzzy sets, which allows for a degree of membership in a interval $[0, 1]$. This groundbreaking idea was later extended by Atanassov [2] into intuitionistic fuzzy sets (IFS), incorporating both a membership degree and a non-membership degree to better capture the hesitation of human judgment.

However, even IFS may not be sufficient when a problem involves "neutrality" or "conflicting negative information." This led to the development of tripolar fuzzy sets, which categorize information into three distinct aspects: positive membership, neutral membership, and negative

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membership. This tripolar approach provides a more granular framework for modeling complex real-world scenarios, such as evaluation systems where an expert might have a positive opinion, a neutral stance, or a strictly negative disagreement.

Parallel to these fuzzy developments, Molodtsov [3] introduced soft set theory as a general mathematical tool for dealing with uncertainties. Unlike fuzzy sets, soft sets are defined by a parameterization function, making them highly flexible for decision-making applications. The hybridization of these concepts—such as fuzzy soft sets [4] and intuitionistic fuzzy soft sets [5]—has opened new avenues in algebraic research.The application of fuzzy and soft structures to algebraic systems, particularly Semigroups, has been a fertile area of research. Semigroups serve as a fundamental structure in computer science, control engineering, and theoretical physics. Specifically, the study of ordered semigroups adds a layer of relational complexity, where the interaction between the algebraic operation and the partial order plays a crucial role. Ideal theory in semigroups, including left ideals, right ideals, quasi-ideals, and bi-ideals, provides deep insights into the structural properties of these systems.

In 2007, Kehayopulu and Tsingelis [7] demonstrated that in ordered groupoids, fuzzy right (resp. left) ideals are fuzzy quasi-ideals. Furthermore, they showed that in ordered semigroups, fuzzy quasi-ideals are fuzzy bi-ideals and that these two concepts coincide in the case of regular ordered semigroups. It was also established that fuzzy quasi-ideals in an ordered semigroup are precisely the intersections of fuzzy right and fuzzy left ideals. The concept of tripolar fuzzy set theory was first introduced by Murali Krishna Rao and Venkateswarlu [8]. Subsequently, Wattanasiripong et al. applied this theory to the study of ordered semigroups (see [9–12]). Tripolar complex fuzzy sets serve as a generalization of tripolar fuzzy sets. Based on the ideas in [7], in this paper, we extend the theory of ideals to the framework of tripolar fuzzy soft sets over ordered semigroups. We define the notions of tripolar fuzzy soft subsemigroups, left (right) ideals, quasi-ideals, and bi-ideals. A significant portion of our work is dedicated to characterizing these ideals through their operational products. Furthermore, we investigate the conditions under which these classes of ideals coincide, particularly within the context of regular ordered semigroups. Our main result provides a complete characterization of tripolar fuzzy soft quasi-ideals as the intersection of tripolar fuzzy soft left and right ideals, establishing a robust theoretical bridge between multi-polar fuzzy logic and algebraic ideal theory.

## 2. Preliminaries

In this section, we recall some fundamental definitions and results regarding ordered semigroups and tripolar fuzzy soft set theory that are used throughout this paper.

Let $S$ be a nonempty set together with a binary operation denoted by "$\cdot$". The structure $\langle S; \cdot\rangle$ is referred to as a groupoid. If the binary operation "$\cdot$" satisfies the associative property, meaning that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

holds for all $x, y, z \in S$, then the groupoid $\langle S; \cdot \rangle$ is called a semigroup.

A binary relation $\preceq$ on a set $S$ is called a partial order if it satisfies the following three conditions:

(1) *Reflexivity*: For all $a \in S$, $a \preceq a$.
(2) *Antisymmetry*: For all $a, b \in S$, if $a \preceq b$ and $b \preceq a$, then $a = b$.
(3) *Transitivity*: For all $a, b, c \in S$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

The structure $\langle S; \preceq \rangle$ is called a partially ordered set (or simply a poset).

**Definition 2.1.** *[7] The structure $\langle S; \cdot, \preceq \rangle$ is called an ordered semigroup if it satisfies the following conditions:*

(1) *$\langle S; \cdot \rangle$ is a semigroup.*
(2) *$\langle S; \preceq \rangle$ is a partially ordered set.*
(3) *For every $a, b, c \in S$ if $a \preceq b$, then $a \cdot c \preceq b \cdot c$ and $c \cdot a \preceq c \cdot b$.*

For the sake of simplicity, we denote an ordered semigroup $\langle S; \cdot, \preceq \rangle$ simply by its carrier set **S**. For any $a, b \in S$, the product $a \cdot b$ is denoted by $ab$. Let $A$ and $B$ be nonempty subsets of $S$. The product $AB$ is defined by

$$AB := \{ab : a \in A \text{ and } b \in B\}.$$

Let **S** be an ordered semigroup. A nonempty subset $A$ of $S$ is said to be a subsemigroup of **S** if $AA \subseteq A$.

**Definition 2.2.** *[7] Let* **S** *be an ordered semigroup. A nonempty subset $A$ of $S$ is called a left ideal (resp., right ideal) of* **S** *if it satisfies the following conditions:*

(1) *$SA \subseteq A$ (resp., $AS \subseteq A$).*
(2) *For any $x, y \in S$, if $x \preceq y$ and $y \in A$, then $x \in A$.*

A nonempty subset of $S$ is called a two-sided ideal, or simply an ideal, of **S** if it is both a left ideal and a right ideal of **S**.

Let $\langle S; \preceq \rangle$ be an ordered set, and let $(\ ] : \mathcal{P}(S) \to \mathcal{P}(S)$ be a mapping define as follows,

$$(A] := \{x \in S : x \preceq a \text{ for some } a \in A\}.$$

**Definition 2.3.** *[7] Let* **S** *be an ordered semigroup. A nonempty subset $A$ of $S$ is called a quasi-ideal of* **S** *if it satisfies the following conditions:*

(1) *$(SA] \cap (AS] \subseteq A$.*
(2) *For $x, y \in S$, if $x \preceq y$ and $y \in A$, then $x \in A$.*

**Definition 2.4.** *[7] Let* **S** *be an ordered semigroup. A nonempty subset $A$ of $S$ is called a bi-ideal of* **S** *if it satisfies the following conditions:*

(1) *$ASA \subseteq A$.*
(2) *For $x, y \in S$, if $x \preceq y$ and $y \in A$, then $x \in A$.*

In classical set theory, the membership of an element in a set is binary: an element either belongs to the set or it does not. However, many concepts in real-world problems are vague and cannot be defined by precise criteria. To address this uncertainty, Zadeh [1] introduced the concept of fuzzy sets in 1965. Unlike classical sets, a fuzzy set allows elements to have partial membership degrees ranging between 0 and 1.

**Definition 2.5.** *Let $X$ be a nonempty set. A fuzzy set $A$ in $X$ is defined as an object of the form:*

$$A := \{\langle x, \mu_A(x)\rangle : x \in X\}$$

*where $\mu_A \colon X \to [0,1]$ is a mapping called the membership function of $A$, and $\mu_A(x)$ represents the degree of membership of $x$ in $A$.*

Although fuzzy sets introduced by Zadeh provide a powerful tool for dealing with uncertainty, they assume that the degree of non-membership is simply the complement of the degree of membership (i.e., $1 - \mu_A(x)$). This assumption may not always accurately reflect real-world situations where there is a degree of hesitation or lack of knowledge. To overcome this limitation, Atanassov [2] introduced the concept of intuitionistic fuzzy sets (IFSs) in 1986. An IFS is characterized by two functions: a membership function $\mu_A$ and a non-membership function $\nu_A$, with the condition that their sum does not exceed 1. The remaining part, $1 - (\mu_A(x) + \nu_A(x))$, represents the degree of hesitation or uncertainty.

**Definition 2.6.** *Let $X$ be a nonempty set. An intuitionistic fuzzy set $A$ in $X$ is defined as an object of the form:*

$$A := \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

*where the mappings $\mu_A \colon X \to [0,1]$ and $\nu_A \colon X \to [0,1]$ denote the degree of membership and the degree of non-membership of the element $x \in X$ to the set $A$, respectively, satisfying the condition:*

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

*for all $x \in X$.*

In 1999, Molodtsov [3] introduced soft set theory as a general mathematical tool for dealing with uncertainty. However, the classical soft set theory is based on crisp sets. Recognizing that in many real-world problems, the parameters or the approximations themselves are fuzzy in nature, Maji et al. [4] combined the theories of fuzzy sets and soft sets. They introduced the concept of *fuzzy soft sets*, which allows for a gradation of membership in the approximation of parameters.

**Definition 2.7.** *Let $U$ be an initial universe set and $E$ a set of parameters. Let $\mathcal{F}(U)$ denote the set of all fuzzy sets in $U$. A pair $(F, E)$ is called a fuzzy soft set over $U$, where $F$ is a mapping given by:*

$$F \colon E \to \mathcal{F}(U).$$

*For any parameter $e \in E$, $F(e)$ is a fuzzy set in $U$ defined as an object of the form:*

$$F(e) := \{\langle x, \mu_{F(e)}(x)\rangle : x \in U\}$$

*where* $\mu_{F(e)} : U \to [0,1]$ *is the membership function.*

Later, Maji et al. [5] generalized the concept of fuzzy soft sets to intuitionistic fuzzy soft sets. This extension is based on the combination of the intuitionistic fuzzy set theory and the soft set theory, which allows for dealing with both the degree of membership and the degree of non-membership of the parameters.

**Definition 2.8.** *Let* $U$ *be an initial universe set and let* $E$ *be a set of parameters. Let* $\mathcal{IF}(U)$ *denote the set of all intuitionistic fuzzy sets in* $U$*. A pair* $(F, E)$ *is called an intuitionistic fuzzy soft set over* $U$*, where* $F$ *is a mapping given by:*

$$F\colon E \to \mathcal{IF}(U).$$

*For any parameter* $e \in E$*,* $F(e)$ *is an intuitionistic fuzzy set in* $U$ *defined as an object of the form:*

$$F(e) := \{\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x)\rangle : x \in U\}$$

*where* $\mu_{F(e)}, \nu_{F(e)}\colon U \to [0,1]$ *are the membership and non-membership functions, respectively, satisfying the condition:*

$$0 \le \mu_{F(e)}(x) + \nu_{F(e)}(x) \le 1$$

*for all* $x \in U$*.*

The concept of tripolar fuzzy sets was originally introduced by M. Murali Krishna Rao [8] as a significant generalization of classical fuzzy sets, intuitionistic fuzzy sets, and bipolar fuzzy sets. This framework was specifically developed to capture and model tripolar decision-making information, as formally presented in the following definition.

**Definition 2.9.** *Let* $X$ *be a nonempty set. A tripolar fuzzy set* $A$ *in* $X$ *is an object of the form:*

$$A := \{\langle x, \mu_A(x), \nu_A(x), \delta_A(x)\rangle : x \in X\}$$

*where* $\mu_A, \nu_A : X \to [0,1]$ *and* $\delta_A : X \to [-1,0]$ *are mappings such that* $0 \le \mu_A(x) + \nu_A(x) \le 1$ *for all* $x \in X$*.*

Motivated by these definitions, the integration of soft set and fuzzy set theories will be developed and applied to ordered semigroups in the main section.

## 3. Main Results

In this section, we provide the core algebraic results of our investigation into tripolar fuzzy soft sets over ordered semigroups. We first establish the fundamental relationships between various classes of tripolar fuzzy soft ideals—including left, right, quasi, and bi-ideals—and characterize them through the tripolar fuzzy soft product. The section is organized as follows: we begin with preliminary lemmas that describe the behavior of tripolar fuzzy soft products in relation to the ordering of the semigroup. These tools are then employed to prove the equivalence of subsemigroups and ideals with their respective product-based inclusions. Finally, we present a characterization theorem for tripolar fuzzy soft quasi-ideals, showing that they coincide with the intersection of left and right ideals, and discuss these structures within the framework of regular ordered semigroups.

**Definition 3.1.** *Let $U$ be an initial universe set and $E$ a set of parameters. Let $\mathcal{TF}(U)$ denote the collection of all tripolar fuzzy sets in $U$. A pair $(F, E)$ is called a tripolar fuzzy soft set over $U$, where $F$ is a mapping given by:*

$$F : E \to \mathcal{TF}(U)$$

*For each parameter $e \in E$, $F(e)$ is a tripolar fuzzy set in $U$ defined as:*

$$F(e) := \{\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x), \delta_{F(e)}(x)\rangle : x \in U\}$$

*where $\mu_{F(e)}, \nu_{F(e)} : U \to [0,1]$ and $\delta_{F(e)} : U \to [-1,0]$ are the positive, neutral, and negative membership functions, respectively, satisfying the condition:*

$$0 \leq \mu_{F(e)}(x) + \nu_{F(e)}(x) \leq 1$$

*for all $x \in U$.*

To better understand the concept of a tripolar fuzzy soft set, let us consider a practical application in decision-making problems. In the following example, we illustrate how a tripolar fuzzy soft set can represent uncertain data involving positive, neutral, and negative aspects simultaneously.

**Example 3.1.** *Let $U = \{x_1, x_2\}$ be the set of two land plots and let $E = \{e_1, e_2\}$ be the set of parameters representing nutrients Nitrogen and Potassium, respectively. A tripolar fuzzy soft set $(F, E)$ describing the soil quality is given by:*

$$F(e_1) = \{\langle x_1, 0.7, 0.2, -0.1\rangle, \langle x_2, 0.4, 0.5, -0.3\rangle\}$$
$$F(e_2) = \{\langle x_1, 0.6, 0.1, -0.2\rangle, \langle x_2, 0.8, 0.1, -0.1\rangle\}$$

*The tabular representation of $(F, E)$ is shown below:*

| $U \setminus E$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $(0.7, 0.2, -0.1)$ | $(0.6, 0.1, -0.2)$ |
| $x_2$ | $(0.4, 0.5, -0.3)$ | $(0.8, 0.1, -0.1)$ |

We denote by $\mathcal{TFSS}(U, E)$ the set of all tripolar fuzzy soft sets over $U$. In the following definition, we introduce the concepts of inclusion relation, intersection and union operations on $\mathcal{TFSS}(U, E)$.

**Definition 3.2.** *Let $(F, E)$ and $(G, E)$ be two tripolar fuzzy soft sets over $U$.*

(1) *We say that $(F, E)$ is a tripolar fuzzy soft subset of $(G, E)$, denoted by $(F, E) \tilde{\subseteq} (G, E)$, if for all $e \in E$ and $x \in U$, the following conditions hold:*

$$\mu_{F(e)}(x) \leq \mu_{G(e)}(x),$$
$$\nu_{F(e)}(x) \geq \nu_{G(e)}(x),$$

$$\delta_{F(e)}(x) \geq \delta_{G(e)}(x).$$

(2) *Two tripolar fuzzy soft sets* $(F,E)$ *and* $(G,E)$ *are said to be equal, denoted by* $(F,E) = (G,E)$, *if* $(F,E) \tilde{\subseteq} (G,E)$ *and* $(G,E) \tilde{\subseteq} (F,E)$.

(3) *The intersection of* $(F,E)$ *and* $(G,E)$, *denoted by* $(F,E) \tilde{\cap} (G,E)$, *is defined as a tripolar fuzzy soft set* $(H,E)$ *over* $U$, *where for each* $e \in E$ *and* $x \in U$:

$$\mu_{H(e)}(x) := \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\},$$
$$\nu_{H(e)}(x) := \max\{\nu_{F(e)}(x), \nu_{G(e)}(x)\},$$
$$\delta_{H(e)}(x) := \max\{\delta_{F(e)}(x), \delta_{G(e)}(x)\}.$$

(4) *The union of* $(F,E)$ *and* $(G,E)$, *denoted by* $(F,E) \tilde{\cup} (G,E)$, *is defined as a tripolar fuzzy soft set* $(H,E)$ *over* $U$, *where for each* $e \in E$ *and* $x \in U$:

$$\mu_{H(e)}(x) := \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\},$$
$$\nu_{H(e)}(x) := \min\{\nu_{F(e)}(x), \nu_{G(e)}(x)\},$$
$$\delta_{H(e)}(x) := \min\{\delta_{F(e)}(x), \delta_{G(e)}(x)\}.$$

From the above definition, it is easy to verify that the relation $\tilde{\subseteq}$ is a partial order. Consequently, the structure $\langle \mathcal{TIFSS}(U,E), \tilde{\subseteq}\rangle$ is a partially ordered set.

Let $\mathbf{S}$ be an ordered semigroup and let $a \in S$. We define the set $\mathbf{S}_a$ by

$$\mathbf{S}_a := \{(x,y) \in S \times S : a \leq xy\}.$$

We now introduce additional operations on $\mathcal{TFSS}(S,E)$ as follows.

**Definition 3.3.** *Let* $\mathbf{S}$ *be an ordered semigroup. Let* $(F,E)$ *and* $(G,E)$ *be two tripolar fuzzy soft sets over* $S$. *The product of* $(F,E)$ *and* $(G,E)$, *denoted by* $(F,E)\tilde{\diamond}(G,E)$, *is defined as a tripolar fuzzy soft set* $(H,E)$ *over* $S$, *where for each* $e \in E$ *and* $x \in S$:

$$\mu_{H(e)}(x) := \begin{cases} \bigvee_{(y,z)\in\mathbf{S}_x} \{\min\{\mu_{F(e)}(y), \mu_{G(e)}(z)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{H(e)}(x) := \begin{cases} \bigwedge_{(y,z)\in\mathbf{S}_x} \{\max\{\nu_{F(e)}(y), \nu_{G(e)}(z)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

$$\delta_{H(e)}(x) := \begin{cases} \bigwedge_{(y,z)\in\mathbf{S}_x} \{\max\{\delta_{F(e)}(y), \delta_{G(e)}(z)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

By Definition 3.3, it is easy to see that the structure $\langle \mathcal{TFSS}(S,E), \tilde{\diamond}\rangle$ is a semigroup, and it is straightforward to verify that the operation $\tilde{\diamond}$ is compatible with the inclusion relation $\tilde{\subseteq}$. That is, for any $(A,E), (B,E), (C,E) \in \mathcal{TFSS}(S,E)$, if $(A,E)\tilde{\subseteq}(B,E)$, then

$$(A,E)\tilde{\diamond}(C,E)\tilde{\subseteq}(B,E)\tilde{\diamond}(C,E) \quad \text{and} \quad (C,E)\tilde{\diamond}(A,E)\tilde{\subseteq}(C,E)\tilde{\diamond}(B,E).$$

Consequently, the structure $\langle \mathcal{TFSS}(S,E), \tilde{\diamond}, \tilde{\subseteq} \rangle$ forms an ordered semigroup, which is called a tripolar fuzzy soft ordered semigroup.

**Definition 3.4.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F,E)$ *be a tripolar fuzzy soft set over* $S$*. Then* $(F,E)$ *is called a tripolar fuzzy soft subsemigroup over* $\mathbf{S}$ *if for all* $e \in E$ *and* $x, y \in S$*, the following conditions hold:*

$$\mu_{F(e)}(xy) \geq \min\{\mu_{F(e)}(x), \mu_{F(e)}(y)\},$$
$$\nu_{F(e)}(xy) \leq \max\{\nu_{F(e)}(x), \nu_{F(e)}(y)\},$$
$$\delta_{F(e)}(xy) \leq \max\{\delta_{F(e)}(x), \delta_{F(e)}(y)\}.$$

**Example 3.2.** *Let* $S = \{a, b, c\}$ *be a semigroup with the multiplication table defined as follows:*

| $*$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $c$ |

*We define an order relation* $\preceq$ *on* $S$ *by:*

$$\preceq := \{(a,b), (a,c), (b,c)\} \cup \Delta_S,$$

*where* $\Delta_S$ *is the identity relation on* $S$*. It is easy to verify that* $\mathbf{S} := \langle S, *, \preceq \rangle$ *is an ordered semigroup.*

*Let* $E = \{e\}$ *be a set of parameters. We define a tripolar fuzzy soft set* $(F,E)$ *over* $S$ *by defining the values for* $x \in S$ *as follows:*

$$F(e) = \{\langle a, 0.8, 0.1, -0.9 \rangle, \langle b, 0.6, 0.3, -0.4 \rangle, \langle c, 0.4, 0.5, -0.2 \rangle\}.$$

*It is easy to verify that,* $(F,E)$ *is a tripolar fuzzy soft subsemigroup over* $\mathbf{S}$*.*

**Definition 3.5.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F,E)$ *be a tripolar fuzzy soft set over* $S$*. Then* $(F,E)$ *is called a tripolar fuzzy soft left ideal over* $\mathbf{S}$ *if for all* $e \in E$ *and* $x, y \in S$*, the following conditions hold:*

(1) *The left ideal condition:*

$$\mu_{F(e)}(xy) \geq \mu_{F(e)}(y),$$
$$\nu_{F(e)}(xy) \leq \nu_{F(e)}(y),$$
$$\delta_{F(e)}(xy) \leq \delta_{F(e)}(y).$$

(2) *The order condition: If* $x \preceq y$*, then*

$$\mu_{F(e)}(x) \geq \mu_{F(e)}(y),$$
$$\nu_{F(e)}(x) \leq \nu_{F(e)}(y),$$
$$\delta_{F(e)}(x) \leq \delta_{F(e)}(y).$$

**Example 3.3.** *Consider an ordered semigroup* $S = \{a, b, c\}$ *with the multiplication table defined by:*

| $*$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $c$ |

*and the order relation $\leq$ defined by:*

$$\leq := \{(a,b),(b,c),(a,c)\} \cup \Delta_S,$$

*where $\Delta_S$ is an identity relation on $S$. It is easy to calculate that $\mathbf{S} := \langle S; *, \leq \rangle$ is an ordered semigroup.*

*Let $E = \{e\}$ be a set of parameter. We define a tripolar fuzzy soft set $(F,E)$ over $S$ as follows:*

$$F(e) = \{\langle a, 0.9, 0.0, -0.8\rangle, \langle b, 0.5, 0.4, -0.5\rangle, \langle c, 0.2, 0.7, -0.1\rangle\}.$$

*Hence, $(F,E)$ is a tripolar fuzzy soft left ideal over $\mathbf{S}$.*

**Definition 3.6.** *Let $\mathbf{S}$ be an ordered semigroup, and let $(F,E)$ be a tripolar fuzzy soft set over $S$. Then $(F,E)$ is called a tripolar fuzzy soft right ideal over $\mathbf{S}$ if for all $e \in E$ and $x, y \in S$, the following conditions hold:*

(1) *The right ideal condition:*

$$\mu_{F(e)}(xy) \geq \mu_{F(e)}(x),$$
$$\nu_{F(e)}(xy) \leq \nu_{F(e)}(x),$$
$$\delta_{F(e)}(xy) \leq \delta_{F(e)}(x).$$

(2) *The order condition: If $x \leq y$, then*

$$\mu_{F(e)}(x) \geq \mu_{F(e)}(y),$$
$$\nu_{F(e)}(x) \leq \nu_{F(e)}(y),$$
$$\delta_{F(e)}(x) \leq \delta_{F(e)}(y).$$

**Example 3.4.** *Let $S = \{a,b,c\}$ be an ordered semigroup with the multiplication table defined by:*

| $*$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $c$ |

*and the order relation $\leq$ defined by:*

$$\leq := \{(a,b),(b,c),(a,c)\} \cup \Delta_S,$$

*where $\Delta_S := \{(x,x) : x \in S\}$. Therefore $\mathbf{S} := \langle S; *, \leq \rangle$ is an ordered semigroup.*

*Let $E = \{e\}$ be a set of parameters. We define a tripolar fuzzy soft set $(F,E)$ over $S$ as follows:*

$$F(e) = \{\langle a, 0.7, 0.2, -0.9\rangle, \langle b, 0.5, 0.4, -0.6\rangle, \langle c, 0.3, 0.6, -0.2\rangle\}.$$

*Thus, $(F,E)$ is a tripolar fuzzy soft right ideal over $\mathbf{S}$.*

A tripolar fuzzy soft set $(F,E)$ over $S$ is called a tripolar fuzzy soft two-sided ideal (simply, tripolar fuzzy soft ideal) over $\mathbf{S}$ if it is both a tripolar fuzzy soft left and right ideal over $\mathbf{S}$.

**Definition 3.7.** *Let $A$ be a subset of $S$, and let $E$ be a set of parameters. The characteristic tripolar fuzzy soft set over $S$, denoted by $(\chi_A, E)$, is defined by the mapping $\chi_A : E \to \mathcal{TF}(S)$ where for each $e \in E$, the tripolar fuzzy set $\chi_A(e)$ is given by:*

$$\chi_A(e) = \{\langle x, \mu_{\chi_A(e)}(x), \nu_{\chi_A(e)}(x), \delta_{\chi_A(e)}(x)\rangle : x \in S.\}$$

*The positive, neutral, and negative membership functions are defined for each $x \in S$ and $e \in E$ as follows:*

(1) *Positive membership function:* $\mu_{\chi_A(e)}(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$

(2) *Neutral membership function:* $\nu_{\chi_A(e)}(x) := \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{otherwise.} \end{cases}$

(3) *Negative membership function:* $\delta_{\chi_A(e)}(x) := \begin{cases} -1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$

If $A = S$, then the characteristic tripolar fuzzy soft set $(\chi_A, E)$ is denoted by $(\mathbf{1}, E)$. For each $e \in E$ and $x \in S$, the tripolar fuzzy set $\mathbf{1}(e)$ is defined by:

$$\mu_{\mathbf{1}(e)}(x) = 1, \quad \nu_{\mathbf{1}(e)}(x) = 0, \quad \text{and} \quad \delta_{\mathbf{1}(e)}(x) = -1.$$

**Definition 3.8.** *Let $\mathbf{S}$ be an ordered semigroup, and let $(F,E)$ be a tripolar fuzzy soft set over $S$. Then $(F,E)$ is called tripolar fuzzy soft quasi-ideal over $\mathbf{S}$ if for all $e \in E$ and $x, y \in S$, the following conditions hold:*

(1) *The quasi-ideal condition:*

$$[(F,E)\tilde{\diamond}(\mathbf{1},E)] \,\tilde{\cap}\, [(\mathbf{1},E)\tilde{\diamond}(F,E)] \,\tilde{\subseteq} (F,E).$$

(2) *The order condition: If $x \leq y$, then*

$$\begin{aligned} \mu_{F(e)}(x) &\geq \mu_{F(e)}(y), \\ \nu_{F(e)}(x) &\leq \nu_{F(e)}(y), \\ \delta_{F(e)}(x) &\leq \delta_{F(e)}(y). \end{aligned}$$

**Note that.** The condition (1) of Definition 3.8, means that, for all $e \in E$ and $x \in S$,

$$\min\{\mu_{(F\widetilde{\diamond}\mathbf{1})(e)}(x), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\} \leq \mu_{F(e)}(x),$$

$$\max\{\nu_{(F\widetilde{\diamond}\mathbf{1})(e)}(x), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\} \geq \nu_{F(e)}(x),$$

$$\max\{\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(x), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\} \geq \delta_{F(e)}(x).$$

**Definition 3.9.** *Let $\mathbf{S}$ be an ordered semigroup, and let $(F,E)$ be a tripolar fuzzy soft set over $S$. Then $(F,E)$ is called tripolar fuzzy soft bi-ideal over $\mathbf{S}$ if for all $e \in E$ and $x, y, z \in S$, the following conditions hold:*

(1) *The bi-ideal condition:*

$$\mu_{F(e)}(xyz) \geq \min\{\mu_{F(e)}(x), \mu_{F(e)}(z)\},$$
$$\nu_{F(e)}(xyz) \leq \max\{\nu_{F(e)}(x), \nu_{F(e)}(z)\},$$
$$\delta_{F(e)}(xyz) \leq \max\{\delta_{F(e)}(x), \delta_{F(e)}(z)\}.$$

(2) *The order condition: If $x \leq y$, then*

$$\mu_{F(e)}(x) \geq \mu_{F(e)}(y),$$
$$\nu_{F(e)}(x) \leq \nu_{F(e)}(y),$$
$$\delta_{F(e)}(x) \leq \delta_{F(e)}(y).$$

**Example 3.5.** *Let $S = \{0, 1, 2\}$ be an ordered semigroup with the standard multiplication and the usual order $\leq$. Let $E = \{e\}$ be the set of parameters. We define a tripolar fuzzy soft set $(F, E)$ as follows:*

$$F(e) = \{(0, 0.8, 0.1, -0.9), (1, 0.4, 0.5, -0.3), (2, 0.4, 0.5, -0.3)\}.$$

*It is easy to verify that $(F, E)$ satisfies the bi-ideal condition $\mu_{F(e)}(xyz) \geq \min\{\mu_{F(e)}(x), \mu_{F(e)}(z)\}$ and the order condition for any $x, y, z \in S$. Thus, $(F, E)$ is a tripolar fuzzy soft bi-ideal over* **S**.

Next, we establish the fundamental relationships between tripolar fuzzy soft sets and their algebraic operations in ordered semigroups. The following results characterize subsemigroups and ideals through the tripolar fuzzy soft product, providing a more operational perspective on these structures.

**Lemma 3.1.** *Let* **S** *be an ordered semigroup, and let $(F, E)$ be a tripolar fuzzy soft set over $S$ satisfying the order-reversing property: for any $e \in E$ and $x, y \in S$, $x \leq y$ implies $\mu_{F(e)}(x) \geq \mu_{F(e)}(y)$, $\nu_{F(e)}(x) \leq \nu_{F(e)}(y)$, and $\delta_{F(e)}(x) \leq \delta_{F(e)}(y)$. Then the following statements are equivalent:*

(1) *$(F, E)$ is a tripolar fuzzy soft subsemigroup over* **S**.
(2) *$(F, E)\tilde{\diamond}(F, E)\tilde{\sqsubseteq}(F, E)$.*

*Proof.* (1)⇒(2): Let $(F, E)$ be a tripolar fuzzy soft subsemigroup over **S**. For any $e \in E$ and $a \in S$, if $\mathbf{S}_a = \emptyset$, then it is clear that $(F, E)\tilde{\diamond}(F, E)\tilde{\sqsubseteq}(F, E)$, and if $\mathbf{S}_a \neq \emptyset$, then there exist $x, y \in S$ such that $a \leq xy$, that is $(x, y) \in \mathbf{S}_a$, then we obtain that

$$\begin{aligned}\mu_{(F\tilde{\diamond}F)(e)}(a) &= \bigvee_{(u,v)\in\mathbf{S}_a} \{\min\{\mu_{F(e)}(u), \mu_{F(e)}(v)\}\} \\ &\leq \min\{\mu_{F(e)}(x), \mu_{F(e)}(y)\} \\ &\leq \mu_{F(e)}(xy) \\ &\leq \mu_{F(e)}(a).\end{aligned}$$

Similarly, we have

$$\nu_{(F\tilde{\diamond}F)(e)}(a) = \bigwedge_{(u,v)\in\mathbf{S}_a} \{\max\{\nu_{F(e)}(u), \nu_{F(e)}(v)\}\}$$

$$\begin{aligned}&\geq \max\{\nu_{F(e)}(x), \nu_{F(e)}(y)\}\\ &\geq \nu_{F(e)}(xy)\\ &\geq \nu_{F(e)}(a),\end{aligned}$$

and

$$\begin{aligned}\delta_{(F\tilde{\diamond}F)(e)}(a) &= \bigwedge_{(u,v)\in\mathbf{S}_a}\{\max\{\delta_{F(e)}(u), \delta_{F(e)}(v)\}\}\\ &\geq \max\{\delta_{F(e)}(x), \delta_{F(e)}(y)\}\\ &\geq \delta_{F(e)}(xy)\\ &\geq \delta_{F(e)}(a).\end{aligned}$$

Thus $(F,E)\tilde{\diamond}(F,E)\tilde{\subseteq}(F,E)$. Therefore for any two cases, we obtain that $(F,E)\tilde{\diamond}(F,E)\tilde{\subseteq}(F,E)$.

(2)⇒(1): Suppose $(F,E)\tilde{\diamond}(F,E)\tilde{\subseteq}(F,E)$. For any $x, y \in S$, let $a = xy$. Since $xy \leq xy$, we have

$$\begin{aligned}\mu_{F(e)}(a) &= \mu_{F(e)}(xy)\\ &\geq \mu_{(F\tilde{\diamond}F)(e)}(xy)\\ &= \bigvee_{(u,v)\in\mathbf{S}_{xy}}\{\min\{\mu_{F(e)}(u), \mu_{F(e)}(v)\}\}\\ &\geq \min\{\mu_{F(e)}(x), \mu_{F(e)}(y)\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu_{F(e)}(a) &= \nu_{F(e)}(xy)\\ &\leq \nu_{(F\tilde{\diamond}F)(e)}(xy)\\ &= \bigwedge_{(u,v)\in\mathbf{S}_{xy}}\{\max\{\nu_{F(e)}(u), \nu_{F(e)}(v)\}\}\\ &\leq \max\{\nu_{F(e)}(x), \nu_{F(e)}(y)\},\end{aligned}$$

and

$$\begin{aligned}\delta_{F(e)}(a) &= \delta_{F(e)}(xy)\\ &\leq \delta_{(F\tilde{\diamond}F)(e)}(xy)\\ &= \bigwedge_{(u,v)\in\mathbf{S}_{xy}}\{\max\{\delta_{F(e)}(u), \delta_{F(e)}(v)\}\}\\ &\leq \max\{\delta_{F(e)}(x), \delta_{F(e)}(y)\}.\end{aligned}$$

Therefore, $(F,E)$ is a tripolar fuzzy soft subsemigroup over $\mathbf{S}$. □

**Lemma 3.2.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F,E)$ *be a tripolar fuzzy soft set over* $S$*. Then the following statements are equivalent:*

(1) $(F,E)$ *is a tripolar fuzzy soft left ideal over* $\mathbf{S}$.

(2) $(F,E)$ *satisfies the following conditions:*

(2.1) *For any* $e \in E$ *and* $x,y \in S$, $x \preceq y$ *implies* $\mu_{F(e)}(x) \geq \mu_{F(e)}(y)$, $\nu_{F(e)}(x) \leq \nu_{F(e)}(y)$, *and* $\delta_{F(e)}(x) \leq \delta_{F(e)}(y)$.

(2.2) $(\mathbf{1},E)\tilde{\diamond}(F,E)\tilde{\subseteq}(F,E)$.

*Proof.* (1)⇒(2): Let $(F,E)$ be a tripolar fuzzy soft left ideal over $\mathbf{S}$. The condition (2.1) follows directly from the definition of a tripolar fuzzy soft left ideal over $\mathbf{S}$. For (2.2), let $e \in E$ and $a \in S$. If $\mathbf{S}_a = \emptyset$, then it is clear that $(\mathbf{1},E)\tilde{\diamond}(F,E)\tilde{\subseteq}(F,E)$, and if $\mathbf{S}_a \neq \emptyset$, we have that $a \leq xy$ for some $x,y \in S$, that is $(x,y) \in \mathbf{S}_a$, then we have that

$$\begin{aligned}
\mu_{(\mathbf{1}\tilde{\diamond}F)(e)}(a) &= \bigvee_{(u,v)\in\mathbf{S}_a} \{\min\{\mu_{\mathbf{1}(e)}(u), \mu_{F(e)}(v)\}\} \\
&\leq \min\{\mu_{\mathbf{1}(e)}(x), \mu_{F(e)}(y)\} \\
&= \mu_{F(e)}(y) \\
&\leq \mu_{F(e)}(xy) \\
&\leq \mu_{F(e)}(a).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu_{(\mathbf{1}\tilde{\diamond}F)(e)}(a) &= \bigwedge_{(u,v)\in\mathbf{S}_a} \{\max\{\nu_{\mathbf{1}(e)}(u), \nu_{F(e)}(v)\}\} \\
&\geq \max\{\nu_{\mathbf{1}(e)}(x), \nu_{F(e)}(y)\} \\
&= \nu_{F(e)}(y) \\
&\geq \nu_{F(e)}(xy) \\
&\geq \nu_{F(e)}(a),
\end{aligned}$$

and

$$\begin{aligned}
\delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(a) &= \bigwedge_{(u,v)\in\mathbf{S}_a} \{\max\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\}\} \\
&\geq \max\{\delta_{\mathbf{1}(e)}(x), \delta_{F(e)}(y)\} \\
&= \delta_{F(e)}(y) \\
&\geq \delta_{F(e)}(xy) \\
&\geq \delta_{F(e)}(a).
\end{aligned}$$

Hence, $(\mathbf{1},E)\tilde{\diamond}(F,E)\tilde{\subseteq}(F,E)$. For any two cases, we obtain (2.2) holds.

(2)⇒(1): Suppose (2.1) and (2.2) hold. For any $x,y \in S$, since $xy \leq xy$, we have

$$\begin{aligned}
\mu_{F(e)}(xy) &\geq \mu_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy) \\
&= \bigvee_{(u,v)\in\mathbf{S}_{xy}} \{\min\{\mu_{\mathbf{1}(e)}(u), \mu_{F(e)}(v)\}\}
\end{aligned}$$

$$\geq \min\{\mu_{\mathbf{1}(e)}(x), \mu_{F(e)}(y)\}$$
$$= \mu_{F(e)}(y).$$

Similarly, we have

$$\begin{aligned}
\nu_{F(e)}(xy) &\leq \nu_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy) \\
&= \bigwedge_{(u,v)\in \mathbf{S}_{xy}} \{\max\{\nu_{\mathbf{1}(e)}(u), \nu_{F(e)}(v)\}\} \\
&\leq \max\{\nu_{\mathbf{1}(e)}(x), \nu_{F(e)}(y)\} \\
&= \nu_{F(e)}(y),
\end{aligned}$$

and

$$\begin{aligned}
\delta_{F(e)}(xy) &\leq \delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy) \\
&= \bigwedge_{(u,v)\in \mathbf{S}_{xy}} \{\max\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\}\} \\
&\leq \max\{\delta_{\mathbf{1}(e)}(x), \delta_{F(e)}(y)\} \\
&= \delta_{F(e)}(y).
\end{aligned}$$

Combined with the order property (2.1), we obtain that $(F, E)$ is a tripolar fuzzy soft left ideal over $\mathbf{S}$. □

Similarly to Lemma 3.2, we obtain the following results for a tripolar fuzzy soft right ideal over $\mathbf{S}$.

**Lemma 3.3.** *Let $\mathbf{S}$ be an ordered semigroup, and let $(F, E)$ be a tripolar fuzzy soft set over $S$. Then the following statements are equivalent:*

(1) *$(F, E)$ is a tripolar fuzzy soft right ideal over $\mathbf{S}$.*
(2) *$(F, E)$ satisfies the following conditions:*
  (2.1) *For any $e \in E$ and $x, y \in S$, $x \leq y$ implies $\mu_{F(e)}(x) \geq \mu_{F(e)}(y)$, $\nu_{F(e)}(x) \leq \nu_{F(e)}(y)$, and $\delta_{F(e)}(x) \leq \delta_{F(e)}(y)$.*
  (2.2) *$(F, E)\tilde{\diamond}(\mathbf{1}, E)\tilde{\subseteq}(F, E)$.*

By combining Lemma 3.2 and Lemma 3.3, we obtain the following corollary.

**Corollary 3.1.** *Let $\mathbf{S}$ be an ordered semigroup, and let $(F, E)$ be a tripolar fuzzy soft set over $S$. Then the following statements are equivalent:*

(1) *$(F, E)$ is a tripolar fuzzy soft ideal over $\mathbf{S}$.*
(2) *$(F, E)$ satisfies the following conditions:*
  (2.1) *For any $e \in E$ and $x, y \in S$, $x \leq y$ implies $\mu_{F(e)}(x) \geq \mu_{F(e)}(y)$, $\nu_{F(e)}(x) \leq \nu_{F(e)}(y)$, and $\delta_{F(e)}(x) \leq \delta_{F(e)}(y)$.*
  (2.2) *$(\mathbf{1}, E)\tilde{\diamond}(F, E)\tilde{\subseteq}(F, E)$ and $(F, E)\tilde{\diamond}(\mathbf{1}, E)\tilde{\subseteq}(F, E)$.*

Next, we investigate the relationships among these tripolar fuzzy soft ideals.

**Theorem 3.1.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F, E)$ *be a tripolar fuzzy soft set over* $S$. *If* $(F, E)$ *is a tripolar fuzzy soft left ideal over* $\mathbf{S}$, *then* $(F, E)$ *is a tripolar fuzzy soft quasi-ideal over* $\mathbf{S}$.

*Proof.* Let $(F, E)$ be a tripolar fuzzy soft left ideal over $\mathbf{S}$. For any $e \in E$ and $x \in S$, since $(F, E)$ is a tripolar fuzzy soft left ideal over $\mathbf{S}$, we have $(\mathbf{1}, E)\tilde{\diamond}(F, E)\tilde{\subseteq}(F, E)$, this means that

$$\mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) \leq \mu_{F(e)}(x),\ \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) \geq \nu_{F(e)}(x),\ \text{and } \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) \geq \delta_{F(e)}(x).$$

For any $x \in S$, we observe that:

$$\begin{aligned}\min\{\mu_{(F\tilde{\diamond}\mathbf{1})(e)}(x), \mu_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\} &\leq \mu_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\\ &\leq \mu_{F(e)}(x).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\max\{\nu_{(F\tilde{\diamond}\mathbf{1})(e)}(x), \nu_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\} &\geq \nu_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\\ &\geq \nu_{F(e)}(x),\end{aligned}$$

and

$$\begin{aligned}\max\{\delta_{(F\tilde{\diamond}\mathbf{1})(e)}(x), \delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\} &\geq \delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\\ &\geq \delta_{F(e)}(x).\end{aligned}$$

Hence, the condition for a quasi-ideal holds. Furthermore, the order property is inherited directly from the definition of a tripolar fuzzy soft left ideal. Consequently, $(F, E)$ is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$. □

Similarly to Theorem 3.1, we obtain the following theorem.

**Theorem 3.2.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F, E)$ *be a tripolar fuzzy soft set over* $S$. *If* $(F, E)$ *is a tripolar fuzzy soft right ideal over* $\mathbf{S}$, *then* $(F, E)$ *is a tripolar fuzzy soft quasi-ideal over* $\mathbf{S}$.

By combining Theorem 3.1 and Theorem 3.2, we obtain the following corollary.

**Corollary 3.2.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F, E)$ *be a tripolar fuzzy soft set over* $S$. *If* $(F, E)$ *is a tripolar fuzzy soft ideal over* $\mathbf{S}$, *then* $(F, E)$ *is a tripolar fuzzy soft quasi-ideal over* $\mathbf{S}$.

The following theorem provides a fundamental relationship between two important classes of tripolar fuzzy soft ideals, showing that every tripolar fuzzy soft quasi-ideal over an ordered semigroup $\mathbf{S}$ is, in fact, a tripolar fuzzy soft bi-ideal over $\mathbf{S}$.

**Theorem 3.3.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F, E)$ *be a tripolar fuzzy soft set over* $S$. *If* $(F, E)$ *is a tripolar fuzzy soft quasi-ideal over* $\mathbf{S}$, *then* $(F, E)$ *is a tripolar fuzzy soft bi-ideal over* $\mathbf{S}$.

*Proof.* Assume that $(F, E)$ is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$. For any $e \in E$ and $x, y, z \in S$, let $a = xyz$. Since $(F, E)$ is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$, we have

$$\mu_{F(e)}(a) \geq \min\{\mu_{(F\tilde{\circ}\mathbf{1})(e)}(a), \mu_{(\mathbf{1}\tilde{\circ}F)(e)}(a)\},$$

$$\nu_{F(e)}(a) \leq \max\{\nu_{(F\tilde{\circ}\mathbf{1})(e)}(a), \nu_{(\mathbf{1}\tilde{\circ}F)(e)}(a)\},$$

and

$$\delta_{F(e)}(a) \leq \max\{\delta_{(F\tilde{\circ}\mathbf{1})(e)}(a), \delta_{(\mathbf{1}\tilde{\circ}F)(e)}(a)\}.$$

Since $a = x(yz)$, it follows from the definition of soft product that

$$\begin{aligned}
\mu_{(F\tilde{\circ}\mathbf{1})(e)}(a) &= \bigvee_{(u,v)\in\mathbf{S}_a} \{\min\{\mu_{F(e)}(u), \mu_{\mathbf{1}(e)}(v)\}\} \\
&\geq \min\{\mu_{F(e)}(x), \mu_{\mathbf{1}(e)}(yz)\} \\
&= \mu_{F(e)}(x).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu_{(F\tilde{\circ}\mathbf{1})(e)}(a) &= \bigwedge_{(u,v)\in\mathbf{S}_a} \{\max\{\nu_{F(e)}(u), \nu_{\mathbf{1}(e)}(v)\}\} \\
&\leq \max\{\nu_{F(e)}(x), \nu_{\mathbf{1}(e)}(yz)\} \\
&= \nu_{F(e)}(x),
\end{aligned}$$

and

$$\begin{aligned}
\delta_{(F\tilde{\circ}\mathbf{1})(e)}(a) &= \bigwedge_{(u,v)\in\mathbf{S}_a} \{\max\{\delta_{F(e)}(u), \delta_{\mathbf{1}(e)}(v)\}\} \\
&\leq \max\{\delta_{F(e)}(x), \delta_{\mathbf{1}(e)}(yz)\} \\
&= \delta_{F(e)}(x).
\end{aligned}$$

Similarly, since $a = (xy)z$, we have

$$\begin{aligned}
\mu_{(\mathbf{1}\tilde{\circ}F)(e)}(a) &= \bigvee_{(u,v)\in\mathbf{S}_a} \{\min\{\mu_{\mathbf{1}(e)}(u), \mu_{F(e)}(v)\}\} \\
&\geq \min\{\mu_{\mathbf{1}(e)}(xy), \mu_{F(e)}(z)\} \\
&= \mu_{F(e)}(z).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu_{(\mathbf{1}\tilde{\circ}F)(e)}(a) &= \bigwedge_{(u,v)\in\mathbf{S}_a} \{\max\{\nu_{\mathbf{1}(e)}(u), \nu_{F(e)}(v)\}\} \\
&\leq \max\{\nu_{\mathbf{1}(e)}(xy), \nu_{F(e)}(z)\} \\
&= \nu_{F(e)}(z),
\end{aligned}$$

and

$$\begin{aligned}\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a) &= \bigwedge_{(u,v)\in\mathbf{S}_a}\{\max\{\delta_{\mathbf{1}(e)}(u),\delta_{F(e)}(v)\}\}\\ &\leq \max\{\delta_{\mathbf{1}(e)}(xy),\delta_{F(e)}(z)\}\\ &= \delta_{F(e)}(z).\end{aligned}$$

Finally, the order-reversing property is satisfied by the definition of a tripolar fuzzy soft quasi-ideal over **S**. Hence, $(F,E)$ is a tripolar fuzzy soft bi-ideal over **S**. □

However, the converse of Theorem 3.3 is not true in general, as shown by the following counterexample.

**Example 3.6.** *Let $S=\{a,b,c,d\}$ be an ordered semigroup with the following multiplication table:*

| $*$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $a$ | $b$ | $a$ |
| $d$ | $a$ | $a$ | $b$ | $b$ |

*and the order relation $\preceq$ is the equality relation. It is easy to calculate that $\mathbf{S}:=\langle S;*,\preceq\rangle$ is an ordered semigroup. Let $E=\{e\}$. Define a tripolar fuzzy soft set $(F,E)$ by:*

$$F(e)=\{(a,0.8,0.1,-0.9),(b,0.4,0.5,-0.2),(c,0.5,0.3,-0.6),(d,0.5,0.3,-0.6)\}.$$

*It can be verified that $(F,E)$ is a tripolar fuzzy soft bi-ideal over **S**. However, for $b\in S$, we have*

$$\begin{aligned}\mu_{F(e)}(b) &= 0.4\\ &\ngeq 0.5\\ &= \min\{\mu_{(F\widetilde{\diamond}\mathbf{1})(e)}(b),\mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(b),\}\end{aligned}$$

*it follows that the quasi-ideal condition is not satisfied. Thus, $(F,E)$ is not a tripolar fuzzy soft quasi-ideal over **S**.*

An ordered semigroup **S** is called regular if for each $a\in S$, there exists an element $x\in S$ such that $a\preceq axa$. The following theorem investigates the relationship between tripolar fuzzy soft bi-ideals and quasi-ideals within the context of regular ordered semigroups. We prove that regularity acts as a bridging condition that allows a bi-ideal to be characterized as a quasi-ideal.

**Theorem 3.4.** *Let **S** be a regular ordered semigroup, and let $(F,E)$ be a tripolar fuzzy soft set over $S$. If $(F,E)$ is a tripolar fuzzy soft bi-ideal over **S**, then $(F,E)$ is a tripolar fuzzy soft quasi-ideal over **S**.*

*Proof.* Let $(F,E)$ be a tripolar fuzzy soft bi-ideal over a regular ordered semigroup **S**. For any $e\in E$ and $a\in S$, since **S** is regular, there exists $x\in S$ such that $a\preceq axa$. It follows that $a\preceq axaxa=(ax)a(xa)$. To prove the quasi-ideal condition, let $a\preceq uv$ and $a\preceq yz$ for some $u,v,y,z\in S$. Then

$$a\preceq axa$$

$$\begin{aligned}&\preceq (uv)x(yz)\\&= u(vxy)z,\end{aligned}$$

and we obtain that

$$\begin{aligned}\mu_{F(e)}(a) &\geq \mu_{F(e)}(u(vxy)z)\\&\geq \min\{\mu_{F(e)}(u), \mu_{F(e)}(z)\}\\&= \min\{\bigvee_{(u,v)\in \mathbf{S}_a} \min\{\mu_{F(e)}(u), \mu_{\mathbf{1}(e)}(v)\}, \bigvee_{(u,v)\in \mathbf{S}_a} \min\{\mu_{\mathbf{1}(e)}(y), \mu_{F(e)}(z)\}\}\\&= \min\{\mu_{(F\widetilde{\diamond}\mathbf{1})(e)}(a), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a)\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu_{F(e)}(a) &\leq \nu_{F(e)}(u(vxy)z)\\&\leq \max\{\nu_{F(e)}(u), \nu_{F(e)}(z)\}\\&= \max\{\bigwedge_{(u,v)\in \mathbf{S}_a} \max\{\nu_{F(e)}(u), \nu_{\mathbf{1}(e)}(v)\}, \bigwedge_{(u,v)\in \mathbf{S}_a} \max\{\nu_{\mathbf{1}(e)}(y), \nu_{F(e)}(z)\}\}\\&= \max\{\nu_{(F\widetilde{\diamond}\mathbf{1})(e)}(a), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a)\},\end{aligned}$$

and

$$\begin{aligned}\delta_{F(e)}(a) &\leq \delta_{F(e)}(u(vxy)z)\\&\leq \max\{\delta_{F(e)}(u), \delta_{F(e)}(z)\}\\&= \max\{\bigwedge_{(u,v)\in \mathbf{S}_a} \max\{\delta_{F(e)}(u), \delta_{\mathbf{1}(e)}(v)\}, \bigwedge_{(u,v)\in \mathbf{S}_a} \max\{\delta_{\mathbf{1}(e)}(y), \delta_{F(e)}(z)\}\}\\&= \max\{\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(a), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a)\}.\end{aligned}$$

Finally, the order-reversing property is satisfied by the definition of a tripolar fuzzy soft bi-ideal over $\mathbf{S}$. Thus, $(F, E)$ is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$. □

Combining the results of Theorem 3.3 and Theorem 3.4, we immediately obtain the following corollary, which shows that these two classes of tripolar fuzzy soft ideals coincide in the presence of regularity.

**Corollary 3.3.** *Let $\mathbf{S}$ be a regular ordered semigroup, and $(F, E)$ be a tripolar fuzzy soft set over $S$. Then the following statements are equivalent:*

(1) *$(F, E)$ is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$.*
(2) *$(F, E)$ is a tripolar fuzzy soft bi-ideal over $\mathbf{S}$.*

In the following part, we show that the tripolar fuzzy soft quasi-ideals of an ordered semigroup are characterized by the intersection of tripolar fuzzy soft left ideals and tripolar fuzzy soft right ideals. To achieve this, we introduce some necessary preliminary tools as follows.

**Lemma 3.4.** *Let* $\mathbf{S}$ *be an ordered semigroup and let* $(F, E)$ *be a tripolar fuzzy soft set over* $S$*. Then for any* $e \in E$ *and* $x, y \in S$*, the following properties hold:*

(1) $\mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \geq \mu_{F(e)}(y)$, $\nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \nu_{F(e)}(y)$, *and* $\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \delta_{F(e)}(y)$.

(2) $\mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \geq \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$, $\nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$, *and* $\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$.

*Proof.* (1) By the definition of the soft product, for any $e \in E$ and $x, y \in S$, we have

$$\begin{aligned}
\mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) &= \bigvee_{(u,v)\in \mathbf{S}_{xy}} \{\min\{\mu_{\mathbf{1}(e)}(u), \mu_{F(e)}(v)\}\} \\
&\geq \min\{\mu_{\mathbf{1}(e)}(x), \mu_{F(e)}(y)\} \\
&= \min\{1, \mu_{F(e)}(y)\} \\
&= \mu_{F(e)}(y).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) &= \bigwedge_{(u,v)\in \mathbf{S}_{xy}} \{\max\{\nu_{\mathbf{1}(e)}(u), \nu_{F(e)}(v)\}\} \\
&\leq \max\{\nu_{\mathbf{1}(e)}(x), \nu_{F(e)}(y)\} \\
&= \max\{0, \nu_{F(e)}(y)\} \\
&= \nu_{F(e)}(y),
\end{aligned}$$

and

$$\begin{aligned}
\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) &= \bigwedge_{(u,v)\in \mathbf{S}_{xy}} \{\max\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\}\} \\
&\leq \max\{\delta_{\mathbf{1}(e)}(x), \delta_{F(e)}(y)\} \\
&= \max\{-1, \delta_{F(e)}(y)\} \\
&= \delta_{F(e)}(y).
\end{aligned}$$

This completes the proof that (1) holds.

(2) For any $u, v \in S$ such that $y \leq uv$, it follows that $xy \leq x(uv) = (xu)v$. Then

$$\begin{aligned}
(\mu_{(\mathbf{1}\widetilde{\diamond}F)(e)})(xy) &= \bigvee_{(p,q)\in \mathbf{S}_{y}} \{\min\{\mu_{\mathbf{1}(e)}(p), \mu_{F(e)}(q)\}\} \\
&\geq \min\{\mu_{\mathbf{1}(e)}(xu), \mu_{F(e)}(v)\} \\
&= \mu_{F(e)}(v) \\
&= \min\{\mu_{\mathbf{1}(e)}(u), \mu_{F(e)}(v)\} \\
&= \bigvee_{(p,q)\in \mathbf{S}_{y}} \{\min\{\mu_{\mathbf{1}(e)}(p), \mu_{F(e)}(q)\}\} \\
&= \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(\nu_{(\mathbf{1}\widetilde{\diamond}F)(e)})(xy) &= \bigwedge_{(p,q)\in\mathbf{S}_y}\{\max\{\nu_{\mathbf{1}(e)}(p),\nu_{F(e)}(q)\}\}\\
&\leq \max\{\nu_{\mathbf{1}(e)}(xu),\nu_{F(e)}(v)\}\\
&= \nu_{F(e)}(v)\\
&= \max\{\nu_{\mathbf{1}(e)}(u),\nu_{F(e)}(v)\}\\
&= \bigwedge_{(p,q)\in\mathbf{S}_y}\{\max\{\nu_{\mathbf{1}(e)}(p),\nu_{F(e)}(q)\}\}\\
&= \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y),
\end{aligned}$$

and

$$\begin{aligned}
(\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)})(xy) &= \bigwedge_{(p,q)\in\mathbf{S}_y}\{\max\{\delta_{\mathbf{1}(e)}(p),\delta_{F(e)}(q)\}\}\\
&\leq \max\{\delta_{\mathbf{1}(e)}(xu),\delta_{F(e)}(v)\}\\
&= \delta_{F(e)}(v)\\
&= \max\{\delta_{\mathbf{1}(e)}(u),\delta_{F(e)}(v)\}\\
&= \bigwedge_{(p,q)\in\mathbf{S}_y}\{\max\{\delta_{\mathbf{1}(e)}(p),\delta_{F(e)}(q)\}\}\\
&= \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y).
\end{aligned}$$

This completed the proof that (2) holds. □

The following lemma can be established analogously to Lemma 3.4.

**Lemma 3.5.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F,E)$ *be a tripolar fuzzy soft set over* $S$*. Then for any* $e \in E$ *and* $x, y \in S$*, the following properties hold:*

(1) $\mu_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \geq \mu_{F(e)}(x)$, $\nu_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \leq \nu_{F(e)}(x)$, *and* $\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \leq \delta_{F(e)}(x)$.

(2) $\mu_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \geq \mu_{(F\widetilde{\diamond}\mathbf{1})(e)}(x)$, $\nu_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \leq \nu_{(F\widetilde{\diamond}\mathbf{1})(e)}(x)$, *and* $\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \leq \delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(x)$.

**Lemma 3.6.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F,E)$ *be a tripolar fuzzy soft set over* $S$*. For any* $e \in E$ *and* $x, y \in S$*, if* $x \preceq y$*, then the following properties hold:*

(1) $\mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) \geq \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$.

(2) $\nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) \leq (\nu_{\mathbf{1}\widetilde{\diamond}F)(e)}(y)$.

(3) $\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) \leq \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$.

*Proof.* Let $e \in E$ and $x, y \in S$ such that $x \preceq y$. For the positive membership part, consider the sets $\mathbf{S}_y = \{(u,v) \in S \times S : y \preceq uv\}$ and $\mathbf{S}_x = \{(u,v) \in S \times S : x \preceq uv\}$. Since $x \preceq y$, it follows from the transitivity of $\preceq$ that $y \preceq uv$ implies $x \preceq uv$. Hence, $\mathbf{S}_y \subseteq \mathbf{S}_x$. Thus, we have

$$\mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) = \bigvee_{(p,q)\in\mathbf{S}_x}\{\min\{\mu_{\mathbf{1}(e)}(p),\mu_{F(e)}(q)\}\}$$

$$\begin{aligned}&\geq \min\{\mu_{\mathbf{1}(e)}(u), \mu_{F(e)}(v)\}\\ &\geq \bigvee_{(p,q)\in \mathbf{S}_y} \{\min\{\mu_{\mathbf{1}(e)}(p), \mu_{F(e)}(q)\}\}\\ &= \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) &= \bigwedge_{(p,q)\in \mathbf{S}_x} \{\max\{\nu_{\mathbf{1}(e)}(p), \nu_{F(e)}(q)\}\}\\ &\leq \max\{\nu_{\mathbf{1}(e)}(u), \nu_{F(e)}(v)\}\\ &\leq \bigwedge_{(p,q)\in \mathbf{S}_y} \{\max\{\nu_{\mathbf{1}(e)}(p), \nu_{F(e)}(q)\}\}\\ &= \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y),\end{aligned}$$

and

$$\begin{aligned}\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) &= \bigwedge_{(p,q)\in \mathbf{S}_x} \{\max\{\delta_{\mathbf{1}(e)}(p), \delta_{F(e)}(q)\}\}\\ &\leq \max\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\}\\ &\leq \bigwedge_{(p,q)\in \mathbf{S}_y} \{\max\{\delta_{\mathbf{1}(e)}(p), \delta_{F(e)}(q)\}\}\\ &= \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y).\end{aligned}$$

The proof is now complete. □

Similar to Lemma 3.6, we obtain the following result.

**Lemma 3.7.** *Let* $\mathbf{S}$ *be an ordered semigroup and let* $(F, E)$ *be a tripolar fuzzy soft set over* $S$*. For any* $e \in E$ *and* $x, y \in S$*, if* $x \preceq y$*, then the following properties hold:*

(1) $\mu_{(F\widetilde{\diamond}\mathbf{1})(e)}(x) \geq \mu_{(F\widetilde{\diamond}\mathbf{1})(e)}(y)$.
(2) $\nu_{(F\widetilde{\diamond}\mathbf{1})(e)}(x) \leq \nu_{(F\widetilde{\diamond}\mathbf{1})(e)}(y)$.
(3) $\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(x) \leq \delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(y)$.

**Lemma 3.8.** *Let* $\mathbf{S}$ *be an ordered semigroup and let* $(F, E)$ *be a tripolar fuzzy soft set over* $S$*. Then for any* $e \in E$ *and* $x, y \in S$*, the following properties hold:*

(1) $\max\{\mu_{F(e)}(xy), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} \geq \max\{\mu_{F(e)}(y), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}$.
(2) $\min\{\nu_{F(e)}(xy), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} \leq \min\{\nu_{F(e)}(y), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}$.
(3) $\min\{\delta_{F(e)}(xy), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} \leq \min\{\delta_{F(e)}(y), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}$.

*Proof.* Let $e \in E$ and $x, y \in S$.

(1) For the positive membership part, by Lemma 3.4, we know that

$$\mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \geq \mu_{F(e)}(y) \quad \text{and} \quad \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \geq \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y).$$

It follows that

$$\begin{aligned}\max\{\mu_{F(e)}(y), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} &\geq \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\\ &\geq \max\{\mu_{F(e)}(y), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}.\end{aligned}$$

(2) Similarly, for the neutral part, from Lemma 3.4, we have

$$\nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \nu_{F(e)}(y) \quad \text{and} \quad \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y).$$

$$\begin{aligned}\min\{\nu_{F(e)}(y), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} &\leq \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\\ &\leq \min\{\nu_{F(e)}(y), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}.\end{aligned}$$

(3) Similarly, for the negative part, from Lemma 3.4, we have

$$\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \delta_{F(e)}(y) \quad \text{and} \quad \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y).$$

$$\begin{aligned}\min\{\delta_{F(e)}(y), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} &\leq \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\\ &\leq \min\{\delta_{F(e)}(y), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}.\end{aligned}$$

□

In a similar manner to Lemma 3.8, we obtain the following results for the right-hand composition.

**Lemma 3.9.** *Let* **S** *be an ordered semigroup and let* $(F, E)$ *be a tripolar fuzzy soft set over* $S$*. Then for any* $e \in E$ *and* $x, y \in S$*, the following properties hold:*

(1) $\max\{\mu_{F(e)}(xy), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} \geq \max\{\mu_{F(e)}(x), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\}$.
(2) $\min\{\nu_{F(e)}(xy), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} \leq \min\{\nu_{F(e)}(x), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\}$.
(3) $\min\{\delta_{F(e)}(xy), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} \leq \min\{\delta_{F(e)}(x), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\}$.

**Lemma 3.10.** *Let* **S** *be an ordered semigroup and let* $(F, E)$ *be a tripolar fuzzy soft set over* $S$ *satisfying the order-reversing property: for any* $e \in E$ *and* $x, y \in S$*,* $x \leq y$ *implies* $\mu_{F(e)}(x) \geq \mu_{F(e)}(y)$*,* $\nu_{F(e)}(x) \leq \nu_{F(e)}(y)$*, and* $\delta_{F(e)}(x) \leq \delta_{F(e)}(y)$*. Then the tripolar fuzzy soft set* $(G, E) = (F, E)\tilde{\cup}[(\mathbf{1}, E)\tilde{\diamond}(F, E)]$ *is a tripolar fuzzy soft left ideal over* **S**.

*Proof.* Let $(G, E) = (F, E)\tilde{\cup}[(\mathbf{1}, E)\tilde{\diamond}(F, E)]$. For any $e \in E$ and $x, y \in S$, by applying Lemma 3.6, we have

$$\begin{aligned}\mu_{G(e)}(xy) &= \max\{\mu_{F(e)}(xy), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\}\\ &\geq \max\{\mu_{F(e)}(y), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}\\ &= \mu_{G(e)}(y).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu_{G(e)}(xy) &= \min\{\nu_{F(e)}(xy), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\}\\ &\leq \min\{\nu_{F(e)}(y), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}\end{aligned}$$

$$= \nu_{G(e)}(y),$$

and

$$\begin{aligned}\delta_{G(e)}(xy) &= \min\{\delta_{F(e)}(xy), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\}\\ &\leq \min\{\delta_{F(e)}(y), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}\\ &= \delta_{G(e)}(y).\end{aligned}$$

This shows that $(G,E)$ satisfies the left ideal property. Next, suppose $x, y \in S$ with $x \preceq y$. By Lemma 3.6, we have that

$$\begin{aligned}\mu_{G(e)}(x) &= \max\{\mu_{F(e)}(x), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\}\\ &\geq \max\{\mu_{F(e)}(y), \mu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}\\ &= \mu_{G(e)}(y).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu_{G(e)}(x) &= \min\{\nu_{F(e)}(x), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\}\\ &\leq \min\{\nu_{F(e)}(y), \nu_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}\\ &= \nu_{G(e)}(y),\end{aligned}$$

and

$$\begin{aligned}\delta_{G(e)}(x) &= \min\{\delta_{F(e)}(x), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\}\\ &\leq \min\{\delta_{F(e)}(y), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}\\ &= \delta_{G(e)}(y).\end{aligned}$$

Hence, $(G,E)$ is a tripolar fuzzy soft left ideal over $\mathbf{S}$. □

Analogously to Lemma 3.10, we obtain the following result.

**Lemma 3.11.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F,E)$ *be a tripolar fuzzy soft set over* $S$ *satisfying the order-reversing property. Then the tripolar fuzzy soft set* $(H,E) = (F,E)\tilde{\cup}[(F,E)\tilde{\diamond}(\mathbf{1},E)]$ *is a tripolar fuzzy soft right ideal over* $\mathbf{S}$.

**Lemma 3.12.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F,E), (G,E)$*, and* $(H,E)$ *be tripolar fuzzy soft sets over* $S$*. Then the following distributive law holds:*

$$(F,E)\tilde{\cap}\left[(G,E)\tilde{\cup}(H,E)\right] = \left[(F,E)\tilde{\cap}(G,E)\right]\tilde{\cup}\left[(F,E)\tilde{\cap}(H,E)\right].$$

*Proof.* Let $e \in E$ and $x \in S$. For the positive membership part, we have

$$\begin{aligned}\mu_{(F\tilde{\cap}(G\tilde{\cup}H))(e)}(x) &= \min\{\mu_{F(e)}(x), \mu_{(G\tilde{\cup}H)(e)}(x)\}\\ &= \min\{\mu_{F(e)}(x), \max\{\mu_{G(e)}(x), \mu_{H(e)}(x)\}\}\\ &= \max\{\min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \min\{\mu_{F(e)}(x), \mu_{H(e)}(x)\}\}\end{aligned}$$

$$= \max\{\mu_{(F\tilde{\cap}G)(e)}(x), \mu_{(F\tilde{\cap}H)(e)}(x)\}$$
$$= \mu_{((F\tilde{\cap}G)\tilde{\cup}(F\tilde{\cap}H))(e)}(x).$$

Similarly, we have

$$\begin{aligned}
\nu_{(F\tilde{\cup}(G\tilde{\cap}H))(e)}(x) &= \max\{\nu_{F(e)}(x), \nu_{(G\tilde{\cap}H)(e)}(x)\} \\
&= \max\{\nu_{F(e)}(x), \min\{\nu_{G(e)}(x), \nu_{H(e)}(x)\}\} \\
&= \min\{\max\{\nu_{F(e)}(x), \nu_{G(e)}(x)\}, \max\{\nu_{F(e)}(x), \nu_{H(e)}(x)\}\} \\
&= \min\{\nu_{(F\tilde{\cup}G)(e)}(x), \nu_{(F\tilde{\cup}H)(e)}(x)\} \\
&= \nu_{((F\tilde{\cup}G)\tilde{\cap}(F\tilde{\cup}H))(e)}(x),
\end{aligned}$$

and

$$\begin{aligned}
\delta_{(F\tilde{\cup}(G\tilde{\cap}H))(e)}(x) &= \max\{\delta_{F(e)}(x), \delta_{(G\tilde{\cap}H)(e)}(x)\} \\
&= \max\{\delta_{F(e)}(x), \min\{\delta_{G(e)}(x), \delta_{H(e)}(x)\}\} \\
&= \min\{\max\{\delta_{F(e)}(x), \delta_{G(e)}(x)\}, \max\{\delta_{F(e)}(x), \delta_{H(e)}(x)\}\} \\
&= \min\{\delta_{(F\tilde{\cup}G)(e)}(x), \delta_{(F\tilde{\cup}H)(e)}(x)\} \\
&= \delta_{((F\tilde{\cup}G)\tilde{\cap}(F\tilde{\cup}H))(e)}(x).
\end{aligned}$$

Thus, the distributive law holds. □

**Lemma 3.13.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F, E)$ *and* $(G, E)$ *be tripolar fuzzy soft quasi-ideals over* $\mathbf{S}$*. Then their intersection* $(H, E) = (F, E)\tilde{\cap}(G, E)$ *is also a tripolar fuzzy soft quasi-ideal over* $\mathbf{S}$*.*

*Proof.* Let $(F, E)$ and $(G, E)$ be tripolar fuzzy soft quasi-ideals over $\mathbf{S}$. Let $(H, E) = (F, E)\tilde{\cap}(G, E)$, where $H(e) = F(e) \cap G(e)$ for all $e \in E$. For the sake of simplicity, for each parameter $e \in E$, we denote the components of $F(e)$, $G(e)$, and $H(e)$ at any element $x \in S$ by $(\mu_{F(e)}(x), \nu_{F(e)}(x), \delta_{F(e)}(x))$, $(\mu_{G(e)}(x), \nu_{G(e)}(x), \delta_{G(e)}(x))$, and $(\mu_{H(e)}(x), \nu_{H(e)}(x), \delta_{H(e)}(x))$, respectively. By the definition of soft intersection, we have:

$$\begin{aligned}
\mu_{H(e)}(x) &= \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \\
\nu_{H(e)}(x) &= \max\{\nu_{F(e)}(x), \nu_{G(e)}(x)\}, \\
\delta_{H(e)}(x) &= \max\{\delta_{F(e)}(x), \delta_{G(e)}(x)\}.
\end{aligned}$$

We need to verify that $(H, E)$ satisfies both the algebraic condition and the order condition of a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$. Since $(F, E)$ and $(G, E)$ are tripolar fuzzy soft quasi-ideals of $\mathbf{S}$, for any $x \in S$, we have:

$$\begin{aligned}
\mu_{H(e)}(x) &= \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\} \\
&\geq \min\{ \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu_{F(e)}(a), \mu_{\mathbf{1}(e)}(b)\}\}, \bigvee_{(c,d)\in \mathbf{S}_x} \{\min\{\mu_{\mathbf{1}(e)}(c), \mu_{F(e)}(d)\}\},
\end{aligned}$$

$$
\begin{aligned}
&\bigvee_{(a,b)\in \mathbf{S}_x}\{\min\{\mu_{G(e)}(a),\mu_{\mathbf{1}(e)}(b)\}\},\bigvee_{(c,d)\in \mathbf{S}_x}\{\min\{\mu_{\mathbf{1}(e)}(c),\mu_{G(e)}(d)\}\}\}\\
&=\min\{\bigvee_{(a,b)\in \mathbf{S}_x}\{\min\{\mu_{F(e)}(a),\mu_{\mathbf{1}(e)}(b)\}\},\bigvee_{(c,d)\in \mathbf{S}_x}\{\min\{\mu_{G(e)}(a),\mu_{\mathbf{1}(e)}(b)\}\},\\
&\bigvee_{(a,b)\in \mathbf{S}_x}\{\min\{\mu_{\mathbf{1}(e)}(c),\mu_{F(e)}(d)\}\},\bigvee_{(c,d)\in \mathbf{S}_x}\{\min\{\mu_{\mathbf{1}(e)}(c),\mu_{G(e)}(d)\}\}\}\\
&=\min\{\bigvee_{(a,b)\in \mathbf{S}_x}\{\min\{\min\{\mu_{F(e)}(a),\mu_{G(e)}(a)\},\mu_{\mathbf{1}(e)}(b)\}\},\\
&\bigvee_{(c,d)\in \mathbf{S}_x}\{\min\{\mu_{\mathbf{1}(e)}(c),\min\{\mu_{F(e)}(d),\mu_{G(e)}(d)\}\}\}\\
&=\min\{\bigvee_{(a,b)\in \mathbf{S}_x}\{\min\{\mu_{H(e)}(a),\mu_{\mathbf{1}(e)}(b)\}\},\bigvee_{(c,d)\in \mathbf{S}_x}\{\min\{\mu_{\mathbf{1}(e)}(c),\mu_{H(e)}(d)\}\}\\
&=\min\{\mu_{(H\widetilde{\diamond}\mathbf{1})(e)}(x),\mu_{(\mathbf{1}\widetilde{\diamond}H)(e)}(x)\}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\nu_{H(e)}(x)&=\max\{\nu_{F(e)}(x),\nu_{G(e)}(x)\}\\
&\leq\max\{\bigwedge_{(a,b)\in \mathbf{S}_x}\{\max\{\nu_{F(e)}(a),\nu_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in \mathbf{S}_x}\{\max\{\nu_{\mathbf{1}(e)}(c),\nu_{F(e)}(d)\}\},\\
&\bigwedge_{(a,b)\in \mathbf{S}_x}\{\max\{\nu_{G(e)}(a),\nu_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in \mathbf{S}_x}\{\max\{\nu_{\mathbf{1}(e)}(c),\nu_{G(e)}(d)\}\}\}\\
&=\max\{\bigwedge_{(a,b)\in \mathbf{S}_x}\{\max\{\nu_{F(e)}(a),\nu_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in \mathbf{S}_x}\{\max\{\nu_{G(e)}(a),\nu_{\mathbf{1}(e)}(b)\}\},\\
&\bigwedge_{(a,b)\in \mathbf{S}_x}\{\max\{\nu_{\mathbf{1}(e)}(c),\nu_{F(e)}(d)\}\},\bigwedge_{(c,d)\in \mathbf{S}_x}\{\max\{\nu_{\mathbf{1}(e)}(c),\nu_{G(e)}(d)\}\}\}\\
&=\max\{\bigwedge_{(a,b)\in \mathbf{S}_x}\{\max\{\max\{\nu_{F(e)}(a),\nu_{G(e)}(a)\},\nu_{\mathbf{1}(e)}(b)\}\},\\
&\bigwedge_{(c,d)\in \mathbf{S}_x}\{\max\{\nu_{\mathbf{1}(e)}(c),\max\{\nu_{F(e)}(d),\nu_{G(e)}(d)\}\}\}\\
&=\max\{\bigwedge_{(a,b)\in \mathbf{S}_x}\{\max\{\nu_{H(e)}(a),\nu_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in \mathbf{S}_x}\{\max\{\nu_{\mathbf{1}(e)}(c),\nu_{H(e)}(d)\}\}\}\\
&=\max\{\nu_{(H\widetilde{\diamond}\mathbf{1})(e)}(x),\nu_{(\mathbf{1}\widetilde{\diamond}H)(e)}(x)\},
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_{H(e)}(x)&=\max\{\delta_{F(e)}(x),\delta_{G(e)}(x)\}\\
&\leq\max\{\bigwedge_{(a,b)\in \mathbf{S}_x}\{\max\{\delta_{F(e)}(a),\nu_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in \mathbf{S}_x}\{\max\{\delta_{\mathbf{1}(e)}(c),\delta_{F(e)}(d)\}\},\\
&\bigwedge_{(a,b)\in \mathbf{S}_x}\{\max\{\delta_{G(e)}(a),\delta_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in \mathbf{S}_x}\{\max\{\delta_{\mathbf{1}(e)}(c),\delta_{G(e)}(d)\}\}\}
\end{aligned}
$$

$$= \max\{ \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\delta_{F(e)}(a), \delta_{\mathbf{1}(e)}(b)\}\}, \bigwedge_{(c,d)\in \mathbf{S}_x} \{\max\{\delta_{G(e)}(a), \delta_{\mathbf{1}(e)}(b)\}\},$$
$$\bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\delta_{\mathbf{1}(e)}(c), \delta_{F(e)}(d)\}\}, \bigwedge_{(c,d)\in \mathbf{S}_x} \{\max\{\delta_{\mathbf{1}(e)}(c), \delta_{G(e)}(d)\}\}\}$$
$$= \max\{ \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\max\{\delta_{F(e)}(a), \delta_{G(e)}(a)\}, \delta_{\mathbf{1}(e)}(b)\}\},$$
$$\bigwedge_{(c,d)\in \mathbf{S}_x} \{\max\{\delta_{\mathbf{1}(e)}(c), \max\{\delta_{F(e)}(d), \delta_{G(e)}(d)\}\}\}$$
$$= \max\{ \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\delta_{H(e)}(a), \delta_{\mathbf{1}(e)}(b)\}\}, \bigwedge_{(c,d)\in \mathbf{S}_x} \{\max\{\delta_{\mathbf{1}(e)}(c), \delta_{H(e)}(d)\}\}\}$$
$$= \max\{\delta_{(H\widetilde{\diamond}\mathbf{1})(e)}(x), \delta_{(\mathbf{1}\widetilde{\diamond}H)(e)}(x)\}.$$

Thus, the algebraic conditions for a tripolar fuzzy soft quasi-ideal are completely satisfied. Let $x, y \in S$ such that $x \leq y$. Since $(F, E)$ and $(G, E)$ are tripolar fuzzy soft quasi-ideals, they both satisfy the order condition. Thus, we have:

$$\begin{aligned} \mu_{H(e)}(x) &= \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\} \\ &\geq \min\{\mu_{F(e)}(y), \mu_{G(e)}(y)\} \\ &= \mu_{H(e)}(y). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \nu_{H(e)}(x) &= \max\{\nu_{F(e)}(x), \nu_{G(e)}(x)\} \\ &\leq \max\{\nu_{F(e)}(y), \nu_{G(e)}(y)\} \\ &= \nu_{H(e)}(y), \end{aligned}$$

and

$$\begin{aligned} \delta_{H(e)}(x) &= \max\{\delta_{F(e)}(x), \delta_{G(e)}(x)\} \\ &\leq \max\{\delta_{F(e)}(y), \delta_{G(e)}(y)\} \\ &= \delta_{H(e)}(y). \end{aligned}$$

This confirms that $(H, E)$ satisfies the order condition. Consequently, $(H, E) = (F, E)\tilde{\cap}(G, E)$ is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$. □

One of the most significant results in the theory of ideals is the ability to represent a complex structure through simpler, well-defined components. In the following theorem, we provide a complete characterization of tripolar fuzzy soft quasi-ideals by showing that they are exactly the intersections of tripolar fuzzy soft left and right ideals. This result simplifies the study of quasi-ideals by linking them directly to the properties of left and right ideals established in the previous lemmas.

**Theorem 3.5.** *Let* $\mathbf{S}$ *be an ordered semigroup, and let* $(F,E)$ *be a tripolar fuzzy soft set over* $S$*. Then the following conditions are equivalent.*

(1) $(F,E)$ *is a tripolar fuzzy soft quasi-ideal over* $\mathbf{S}$*.*
(2) *There exist a tripolar fuzzy soft left ideal* $(L,E)$ *and a tripolar fuzzy soft right ideal* $(R,E)$ *over* $\mathbf{S}$ *such that* $(F,E)=(L,E)\tilde{\cap}(R,E)$*.*

*Proof.* (1)⇒(2): Assume that $(F,E)$ is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$. Define

$$(L,E)=(F,E)\tilde{\cup}[(\mathbf{1},E)\tilde{\diamond}(F,E)] \quad \text{and} \quad (R,E)=(F,E)\tilde{\cup}[(F,E)\tilde{\diamond}(\mathbf{1},E)].$$

According to Lemma 3.10 and Lemma 3.11, $(L,E)$ and $(R,E)$ are a tripolar fuzzy soft left ideal and a tripolar fuzzy soft right ideal over $\mathbf{S}$, respectively. By Lemma 3.12, we obtain

$$\begin{aligned}(L,E)\tilde{\cap}(R,E)&=[(F,E)\tilde{\cup}[(\mathbf{1},E)\tilde{\diamond}(F,E)]]\tilde{\cap}[(F,E)\tilde{\cup}[(F,E)\tilde{\diamond}(\mathbf{1},E)]]\\&=(F,E)\tilde{\cup}\,[[(\mathbf{1},E)\tilde{\diamond}(F,E)]\tilde{\cap}[(F,E)\tilde{\diamond}(\mathbf{1},E)]]\\&=(F,E)\tilde{\cup}(F,E)\\&=(F,E).\end{aligned}$$

Therefore, $(L,E)\tilde{\cap}(R,E)=(F,E)$.

(2)⇒(1): Conversely, let $(F,E)=(L,E)\tilde{\cap}(R,E)$ for some tripolar fuzzy soft left ideal $(L,E)$ and tripolar fuzzy soft right ideal $(R,E)$. Since any left (right) ideal is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$, by Theorem 3.1, and by Lemma 3.13, the intersection of tripolar fuzzy soft quasi-ideals is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$, we conclude that $(F,E)$ is a tripolar fuzzy soft quasi-ideal over $\mathbf{S}$. □

## 4. Conclusion

In this paper, we have investigated the properties of tripolar fuzzy soft sets within the structure of ordered semigroups. We introduced the notions of tripolar fuzzy soft subsemigroups, tripolar fuzzy soft left (right) ideals, quasi-ideals, and bi-ideals, and explored their interrelationships.

We proved that the classes of tripolar fuzzy soft quasi-ideals and bi-ideals coincide when the underlying ordered semigroup is regular. Furthermore, the significant findings of this study include the characterization of tripolar fuzzy soft quasi-ideals as the intersection of tripolar fuzzy soft left and tripolar fuzzy soft right ideals. These results not only generalize existing findings in fuzzy soft set theory but also provide a more detailed framework by incorporating neutral and negative membership dimensions.

Future work may involve the study of tripolar fuzzy soft prime ideals and the application of the proposed tripolar fuzzy soft structures to specialized decision-making problems where neutral and conflicting information are prevalent.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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