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Boundedness of Sublinear Operators on Grand Mixed Herz Spaces

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Abstract. We establish the block-atomic decomposition of grand mixed Herz spaces and, via this atomic characterization, prove the boundedness of a class of sublinear operators on these scales. As an application, we further establish boundedness criteria for sublinear operators on grand mixed Herz-Morrey spaces.

1. Introduction

The germ of Herz spaces can be traced back to Beurling [1] weighted algebras; Herz [2] later crystallized them into homogeneous and inhomogeneous scales, using them to capture the local-to-global behavior of Fourier series and transforms. In the 1990s, Lu-Yang [3,4] formally introduced the spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$, extending the L^p-boundedness results of convolution operators, singular integrals, and multipliers to the Herz scale, thereby anchoring these spaces at the heart of harmonic analysis.

In recent years, mixed-norm Lebesgue spaces-viewed as the canonical lift of the classical Lebesgue scale have returned to the spotlight. Benedek-Panzone [5] laid the groundwork by verifying that $L^p(\mathbb{R}^n)$ inherits the usual pillars completeness, Hölder and Minkowski inequalities thereby providing the functional backbone for subsequent developments. The mixed paradigm quickly ramified: on the Hardy-space side, [6,7] delivered both real-variable and atomic portraits; in 2019, Nogayama [8] opened a new chapter with mixed Morrey spaces, demonstrating novel boundedness criteria for singular integrals and convolution operators, and further widening the mixed-norm universe within harmonic analysis.

Since their introduction by Iwaniec and Sbordone in [9] for studying the integrability of the Jacobian, grand Lebesgue spaces have been extensively studied and generalizations have been obtained [10–12]. The idea of grand variable Herz spaces was introduced in [13] and proved

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the boundedness of sublinear operators in these spaces. Then boundedness of Marcinkiewicz integrals, Multlinear Calderón-Zygmund integrals can be checked in [14]. Moreover the grand variable Herz-Morrey spaces are the generalization of grand variable Herz spaces and it was introduced in [15].

Anchored in both mixed-norm and grand Lebesgue scales, we introduce the grand mixed-Herz space $\dot{K}^{\alpha,\vec{q}}_{p),\theta}(\mathbb{R}^n)$, which in limit cases simultaneously recovers mixed Lebesgue, grand Lebesgue and classical Herz spaces. On this new scale we establish a central block-atomic characterization, providing a direct route to boundedness criteria for a broad class of sublinear operators. As an application, we further establish boundedness criteria for sublinear operators on grand mixed Herz-Morrey spaces.

2. Preliminaries

Throughout this paper we adopt the following harmonic-analysis conventions. Vector exponents: $\vec{q} = (q_1, \ldots, q_n) \in (0,\infty]^n$; $0 < \vec{q} < \infty$ means $0 < q_i < \infty$ for every i. We set $\frac{1}{\vec{q}} = \left(\frac{1}{q_1}, \ldots, \frac{1}{q_n}\right)$ and let $\vec{q}' = (q_1', \ldots, q_n')$ denote the conjugate exponents with $q_i' = q_i/(q_i - 1)$. $|B|$ stands for the Lebesgue measure of the ball B; χ_E is the indicator of the set E. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$; $[a]$ is the integer part of a. Dyadic annuli: $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $A_k = B_k \setminus B_{k-1}$ ($k \in \mathbb{Z}$); we write $\chi_k = \chi_{A_k}$ and set $\widetilde{\chi}_k = \chi_k$ for $k \geq 1$, $\widetilde{\chi}_0 = \chi_{B_0}$.

Definition 2.1 (Mixed Lebesgue spaces) [5] Let $\vec{p} = (p_1, p_2, \ldots, p_n) \in (0,\infty]^n$. Then the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left|f(x_1, x_2, \ldots, x_n)\right|^{p_1} dx_1\right)^{\frac{p_2}{p_1}} dx_2\right)^{\frac{p_3}{p_2}} \ldots dx_n\right)^{\frac{1}{p_n}} < \infty$$

If $p_j = \infty$, then we have to make appropriate modifications.

Definition 2.2 [16] Let $\alpha \in \mathbb{R}, 0 < p \leqslant \infty, 0 < \vec{q} \leqslant \infty$. The mixed homogeneous Herz space $\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n) := \left\{ f \in L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p} \left\|f\chi_k\right\|^p_{L^{\vec{q}}(\mathbb{R}^n)}\right)^{1/p} < \infty \right\}.$$

Definition 2.3 [17] Let $\alpha \in \mathbb{R}, \theta > 0, 0 < p \leqslant \infty, 0 < \vec{q} \leqslant \infty$. The grand mixed homogeneous Herz space $\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n) := \left\{ f \in L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} = \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha p(1+\varepsilon)} \left\|f\chi_k\right\|^{p(1+\varepsilon)}_{L^{\vec{q}}(\mathbb{R}^n)}\right)^{1/p(1+\varepsilon)} < \infty \right\}.$$

Now, we give an example in grand mixed Herz spaces.

Example 2.3 Let $\alpha \in \mathbb{R}, \theta > 0, 1 < \vec{q}, p < \infty$ such that $\alpha \geq n - \sum_{i=1}^{n} \frac{1}{q_i}$ and $\frac{n}{p} = \sum_{i=1}^{n} \frac{1}{q_i}$. For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \backslash \{0\}$, let

$$f(x)=\prod_{i=1}^{n}|x_i|^{-\frac{1}{2q_i}}\chi_{(-1,1)^n}$$

then

$$f\in \dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)$$

The grand mixed-Herz scale $\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)$ is intrinsically a quasi-Banach space; it upgrades to a Banach space as soon as $\min(q_1,\dots,q_n,p)\geq 1$. At the critical endpoint $\alpha=0$ with $0<p=q_i\leq\infty$ it collapses to the grand mixed-Lebesgue space $L^{p),\vec{q}}(\mathbb{R}^n)$. The scale expands monotonically with p:

$$\dot{K}^{p_1),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)\hookrightarrow \dot{K}^{p_2),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n),\qquad 0<p_1\leq p_2\leq\infty.$$

Also, we have the following embeddings.

Proposition 2.4 Let $\alpha,\beta\in\mathbb{R},\theta>0,0<\vec{q},\vec{r},p\leq\infty$.

(i) If $\vec{r}\leq\vec{q}$ and $\beta=\alpha+\sum_{i=1}^{n}\frac{1}{r_i}-\sum_{i=1}^{n}\frac{1}{q_i}$, then

$$\dot{K}^{p),\theta}_{\beta,\vec{q}}(\mathbb{R}^n)\hookrightarrow \dot{K}^{p),\theta}_{\alpha,\vec{r}}(\mathbb{R}^n).$$

(ii) If $1\leq q_n\leq\cdots\leq q_2\leq q_1<\infty$, then

$$\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)\hookrightarrow \dot{K}^{p),\theta}_{\alpha,\vec{q}_t}(\mathbb{R}^n),\quad \text{where}\quad \vec{q}_t=(q_n,\dots,q_2,q_1).$$

Proof. (i) drops out immediately from a routine application of Hölder's inequality

$$\begin{aligned}2^{k\alpha p(1+\varepsilon)}\left\|f\chi_k\right\|_{\vec{r}}^{p(1+\varepsilon)}&\lesssim 2^{k\alpha p(1+\varepsilon)}\left\|f\chi_k\right\|_{\vec{q}}^{p(1+\varepsilon)}\left\|\chi_k\right\|_{\vec{s}}^{p(1+\varepsilon)}\\&\lesssim 2^{k\beta p(1+\varepsilon)}\left\|f\chi_k\right\|_{\vec{q}}^{p(1+\varepsilon)}\end{aligned}$$

where $\vec{s}=(s_1,\dots,s_n)$ is defined by $\frac{1}{r_i}=\frac{1}{q_i}+\frac{1}{s_i}$ for any $i\in\{1,\dots,n\}$.

(ii) can be deduced from the Minkowski's inequality. □

Definition 2.5 [19] Let $\alpha\in\mathbb{R},\theta>0,0\leq\lambda<\infty$ and $1<p,\vec{q}\leq\infty$. Homogeneous grand mixed Herz-Morrey spaces $M\dot{K}^{\alpha,p),\theta}_{\vec{q},\lambda}(\mathbb{R}^n)$ are defined by

$$M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n):=\left\{f\in L^{\vec{q}}_{loc}(\mathbb{R}^n):\|f\|_{M\dot{K}^{\alpha,p,\theta}_{\vec{q},\lambda}(\mathbb{R}^n)}<\infty\right\},$$

where

$$\|f\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)}=\sup_{\varepsilon>0}\sup_{k_0>0}2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0}2^{k\alpha p(1+\varepsilon)}\left\|f\chi_k\right\|_{L_{\vec{q}}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}.$$

Lemma 2.6 [20] Let $0<b<1$ and $0<p\leq\infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^p(\mathbb{Z})}=I<\infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\leq k} b^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ and $\left\{\eta_k : \eta_k = \sum_{j\geq k} b^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ belong to $\ell^p(\mathbb{Z})$, and

$$\left\|\{\delta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^p(\mathbb{Z})} + \left\|\{\eta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^p(\mathbb{Z})} \leq cI,$$

with $c > 0$ only depending on b and p.

Now, we introduce the basic notation of block decomposition.

Definition 2.7 [21] Let $0 < \alpha < \infty$ and $1 < \vec{q} < \infty$.

(i)A function a on $\mathbb{R}^n$ is called a central $(\alpha, \vec{q}\,)$-block, if it satisfies

(1) $\operatorname{supp} a \subset \overline{Q(0,r)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq r \text{ for } i = 1, \ldots, n\}, \quad r > 0.$

(2) $\|a\|_{\vec{q}} \leq |\overline{Q(0,r)}|^{-\alpha/n}, \quad r > 0.$

(ii) A function a on $\mathbb{R}^n$ is called a central $(\alpha, \vec{q}\,)$-block of restrict type, if it satisfies

(1) $\operatorname{supp} a \subset Q(0,r), \quad r \geq 1.$

(2) $\|a\|_{\vec{q}} \leq |\overline{Q(0,r)}|^{-\alpha/n}, \quad r > 0.$

3. Atomic characterization of grand mixed Herz spaces

We next establish an atomic characterization of the spaces $\dot{K}^{\alpha,\vec{q}}_{p),\theta}(\mathbb{R}^n)$ via central block decompositions, thereby obtaining boundedness of sublinear operators on these scales.

Theorem 3.1 Let $0 < \alpha < \infty, \theta > 0, 0 < \vec{q}, p < \infty$. The following two statements are equivalentes

(i)$f \in \dot{K}^{p),\theta}_{\alpha,\vec{q}}\,(\mathbb{R}^n)$

(ii)f can be represented by

$$f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x), \tag{3.1}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each a_k is a central $(\alpha, \vec{q})$-block with support contained in Q_k and

$$\left(\sum_{k=-\infty}^{\infty} \lambda_k^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \lesssim \|f\|_{\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)}.$$

Moreover, the grand mixed Herz space $\|f\|_{\dot{K}^{p),\theta}_{\alpha,\vec{q}}}$is equivalent to the atomic (quasi-)norm

$$\inf\left\|\{\lambda_k\}\right\|_{\ell^{p(1+\varepsilon)}} = \inf\left(\sum_{k\in\mathbb{Z}} \lambda_k^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)},$$

where the infimum runs over all atomic decompositions $f = \sum_{k\in\mathbb{Z}} \lambda_k a_k$ as in (3.1), hence $\dot{K}^{p),\theta}_{\alpha,\vec{q}}$ carries the canonical atomic structure.

Proof of Theorem 3.1. The proof follows the same line of argument as [[18], Theorem 3.8], where the variable-exponent case is treated. We first prove (i) implies (ii). For every $f \in \dot{K}^{\alpha,p}_{\vec{q}}\,(\mathbb{R}^n)$, we decompose

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} f(x)\chi_k(x) \\ &= \sum_{k=-\infty}^{\infty} 2^{k\alpha} \left\| f\chi_k \right\|_{\vec{q}} \frac{f(x)\chi_k(x)}{2^{k\alpha} \left\| f\chi_k \right\|_{\vec{q}}} \\ &= \sum_{k=-\infty}^{\infty} \lambda_k a_k(x) \end{aligned}$$

with coefficients $\lambda_k := 2^{k\alpha}\|f\chi_k\|_{\vec{q}}$ and atoms $a_k(x) := \frac{f(x)\chi_k(x)}{2^{k\alpha}\|f\chi_k\|_{\vec{q}}}$,one immediately has $\operatorname{supp} a_k \subseteq Q_k$ and

$$\|a_k\|_{\vec{q}} \le 2^{-k\alpha}.$$

Hence each a_k qualifies as a central $(\alpha, \vec{q})$-block carried by Q_k, and

$$\sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{\infty} \lambda_k^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left\| f\chi_k \right\|_{\vec{q}}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} = \|f\|_{K^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)}.$$

Now, we prove (ii) implies (i). Take an atomic representation $f = \sum_{k\in\mathbb{Z}} \lambda_k a_k$ furnished by hypothesis (ii) of Theorem 3.1. Fix any $j \in \mathbb{Z}$, an application of the Minkowski inequality on the dyadic annulus A_j gives

$$\left\| f\chi_j \right\|_{\vec{q}} \lesssim \sum_{k=j}^{\infty} |\lambda_k| \, \|a_k\|_{\vec{q}}. \tag{3.2}$$

Under the condition $0 < \alpha < \infty$, an appeal to (3.2) combined with Lemma 2.6 (with the geometric decay factor ($b = 2^{-\alpha} \in (0,1)$) yields

$$\begin{aligned} \|f\|_{\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)} &\lesssim \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k}^{\infty} \left|\lambda_j\right| \left\| a_j \right\|_{\vec{q}} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k}^{\infty} \left|\lambda_j\right| 2^{-(j-k)\alpha} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{\infty} \lambda_k^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}}. \end{aligned}$$

Accordingly, Theorem 3.1 has been proven. □

Remark 3.1 The building blocks in Theorem 3.1 may be chosen with supports lying inside dyadic annuli, thus yielding a genuine annular-atomic decomposition.

Armed with the decomposition just established, we proceed to derive boundedness criteria for a class of sublinear operators on the Herz scale.

Theorem 3.2 Let $\alpha \in \mathbb{R}, \theta > 0, 0 < p < \infty, 1 < \vec{q} < \infty$ such that $0 < \alpha < n \sum_{i=1}^{n} \frac{1}{q_i}$. If a sublinear operator T satisfies

$$|Tf(x)| \lesssim \frac{\|f\|_1}{|x|^n}, \quad \text{if} \quad \operatorname{dis}(x, \operatorname{supp} f) > \frac{|x|}{2}, \tag{3.3}$$

for every compactly supported integrable function f, once the operator T is known to be bounded on the mixed-Lebesgue scale $L^{\vec{q}}(\mathbb{R}^n)$, its boundedness on the grand homogeneous Herz space $\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)$ follows at once.

Proof of Theorem 3.2. It remains to verify that

$$\|Tf\|_{\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)}.$$

Fix any $f \in \dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)$, invoking the atomic flavour of Theorem 3.1 we can safely work with an atomic representation of f and carry out the required estimate

$$f = \sum_{i=-\infty}^{\infty} \lambda_i a_i,$$

where $\lambda_i \geq 0$ and a_i 's are ($\alpha, \vec{q}$)- block with $\operatorname{supp} a_i \subseteq Q_i$. We get

$$\begin{aligned}
\|Tf\|_{\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)} &= \sup_{\varepsilon>0} \left\{ \varepsilon^\theta \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left\| Tf \cdot \chi_k \right\|_{\vec{q}}^{p(1+\varepsilon)} \right\}^{1/p(1+\varepsilon)} \\
&\lesssim \sup_{\varepsilon>0} \left\{ \varepsilon^\theta \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{i=-\infty}^{k-2} \lambda_i \|Ta_i \cdot \chi_k\|_{\vec{q}} \right)^{p(1+\varepsilon)} \right\}^{1/p(1+\varepsilon)} \\
&\quad + \sup_{\varepsilon>0} \left\{ \varepsilon^\theta \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{i=k-1}^{\infty} \lambda_i \|Ta_i \cdot \chi_k\|_{\vec{q}} \right)^{p(1+\varepsilon)} \right\}^{1/p(1+\varepsilon)} \\
&= F_1 + F_2.
\end{aligned}$$

Let us estimate F_1. Observing that $\operatorname{dist}(x, \operatorname{supp} a_i) > 2^{k-1}/2 > |x|/2$, we invoke (3.3) to immediately obtain

$$\begin{aligned}
\|Ta_i \cdot \chi_k\|_{\vec{q}} &\lesssim \left(\int_{R_k^n} \cdots \left(\int_{R_k^2} \left(\int_{R_k^1} |x|^{-nq_1} \|a_i\|_1^{q_1} \, \mathrm{d}x_1 \right)^{\frac{q_2}{q_1}} \mathrm{d}x_2 \right)^{\frac{q_3}{q_2}} \cdots \, \mathrm{d}x_n \right)^{\frac{1}{q_n}} \\
&\lesssim 2^{k\left(\sum_{i=1}^{n} \frac{1}{q_i} - n\right)} \|a_i \chi_i\|_1 .
\end{aligned}$$

Applying Hölder inequality and Deffnition 2.7, we have

$$\begin{aligned}
\|Ta_i \cdot \chi_k\|_{\vec{q}} &\lesssim 2^{k\left(\sum_{i=1}^{n} \frac{1}{q_i} - n\right)} \|a_i\|_{\vec{q}} \|\chi_i\|_{\vec{q}'} \\
&\lesssim 2^{k\left(\sum_{i=1}^{n} \frac{1}{q_i} - n\right)} 2^{i\left(n - \sum_{i=1}^{n} \frac{1}{q_i}\right)} \|a_i\|_{\vec{q}} \\
&\lesssim 2^{-i\alpha} 2^{(i-k)\left(n - \sum_{i=1}^{n} \frac{1}{q_i}\right)}.
\end{aligned}$$

Therefore,

$$F_1 \lesssim \sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{i=-\infty}^{k-2}\lambda_i 2^{-i\alpha}2^{(i-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)}$$

$$=\sup_{\varepsilon>0}\left\{\sum_{k=-\infty}^{\infty}\left(\varepsilon^{\theta}\sum_{i=-\infty}^{k-2}\lambda_i 2^{(k-i)\left(\alpha-n+\sum_{i=1}^{n}\frac{1}{q_i}\right)}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)}$$

$$=\sup_{\varepsilon>0}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\varepsilon^{\theta}\sum_{i=-\infty}^{k-2}\lambda_i 2^{-i\alpha}2^{(i-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)},$$

Since $\alpha < n-\sum_{i=1}^{n}\frac{1}{q_i}$, Lemma 2.6 applies with the geometric decay factor$b = 2^{\alpha-n+\sum_{i=1}^{n}\frac{1}{q_i}} \in (0,1)$, we get

$$\begin{aligned} F_1 &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{i=-\infty}^{\infty}\lambda_i^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)} \\ &\lesssim \|f\|_{\dot K^{p),\theta}_{\alpha,\vec q}(\mathbb{R}^n)}. \end{aligned}$$

Next, we estimate F_2. Invoking the $L^{\vec q}(\mathbb{R}^n)$-boundedness of T, condition (ii) in Definition 2.7 and Lemma 2.6 with the geometric decay factor $b = 2^{-\alpha} \in (0,1)$, we obtain

$$\begin{aligned} F_2 &= \sup_{\varepsilon>0}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\varepsilon^{\theta}\sum_{i=k-1}^{\infty}\lambda_i\,\|Ta_i\cdot\chi_k\|_{\vec q}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\ &\lesssim \sup_{\varepsilon>0}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\varepsilon^{\theta}\sum_{i=k-1}^{\infty}\lambda_i\sup_{\varepsilon>0}\|a_i\|_{\vec q}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\ &\lesssim \sup_{\varepsilon>0}\left\{\sum_{k=-\infty}^{\infty}\left(\varepsilon^{\theta}\sum_{i=k-1}^{\infty}\lambda_i 2^{-(i-k)\alpha}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\ &\lesssim \|f\|_{\dot K^{p),\theta}_{\alpha,\vec q}(\mathbb{R}^n)}. \end{aligned}$$

Accordingly, Theorem 3.2 has been proven. □

4. Boundedness for a class of sublinear operators on grand mixed Herz spaces

As an application of Theorem 3.1, we now examine a class of sublinear operators satisfying a size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y, \quad x \notin \operatorname{supp} f \tag{4.1}$$

for integrable and compactly supported functions f. The following result present the $\dot K^{p),\theta}_{\alpha,\vec q}(\mathbb{R}^n)$-boundedness of the sublinear operators T.

Theorem 4.1 Let $\alpha \in \mathbb{R}, \theta > 0, 0 < p < \infty$ and $1 < \vec{q} < \infty$ such that $0 < \alpha < n - \sum_{i=1}^{n} \frac{1}{q_i}$. If a sublinear operator T satisfies the condition (4.1) and is bounded on $L^{\vec{q}}(\mathbb{R}^n)$, then T is bounded on $\dot{K}_{\alpha,\vec{q}}^{p),\theta}(\mathbb{R}^n)$.

Proof of Theorem 4.1. It remains to verify that

$$\|Tf\|_{\dot{K}_{\alpha,\vec{q}}^{p),\theta}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{K}_{\alpha,\vec{q}}^{p),\theta}(\mathbb{R}^n)}.$$

for all $f \in \dot{K}_{\alpha,\vec{q}}^{p),\theta}(\mathbb{R}^n)$. Using Theorem 3.1, we get

$$f = \sum_{i=-\infty}^{\infty} \lambda_i a_i,$$

where $\lambda_i \geq 0$ and a_i 's are $(\alpha, \vec{q})$- block with $\operatorname{supp} a_i \subseteq Q_i$. We obtain

$$\begin{aligned}
\|Tf\|_{\dot{K}_{\alpha,\vec{q}}^{p),\theta}(\mathbb{R}^n)} &= \sup_{\varepsilon>0} \left\{ \varepsilon^{\theta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{i=-\infty}^{\infty} \lambda_i \|Ta_i \cdot \chi_k\|_{\vec{q}} \right)^{p(1+\varepsilon)} \right\}^{1/p(1+\varepsilon)} \\
&\lesssim \sup_{\varepsilon>0} \left\{ \varepsilon^{\theta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{i=-\infty}^{k-2} \lambda_i \|Ta_i \cdot \chi_k\|_{\vec{q}} \right)^{p(1+\varepsilon)} \right\}^{1/p(1+\varepsilon)} \\
&\quad + \sup_{\varepsilon>0} \left\{ \varepsilon^{\theta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{i=k-1}^{\infty} \lambda_i \|Ta_i \cdot \chi_k\|_{\vec{q}} \right)^{p(1+\varepsilon)} \right\}^{1/p(1+\varepsilon)} \\
&= G_1 + G_2.
\end{aligned}$$

Let us estimate G_2. Invoking the $L^{\vec{q}}(\mathbb{R}^n)$-boundedness of T, condition (ii) in Definition 2.7 and Lemma 2.6 with the geometric decay factor $b = 2^{-\alpha} \in (0,1)$, we arrive at

$$\begin{aligned}
G_2 &\lesssim \sup_{\varepsilon>0} \left\{ \varepsilon^{\theta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{i=k-1}^{\infty} \lambda_i \|a_i\|_{\vec{q}} \right)^{p(1+\varepsilon)} \right\}^{1/p(1+\varepsilon)} \\
&\lesssim \sup_{\varepsilon>0} \left\{ \varepsilon^{\theta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{i=k-1}^{\infty} \lambda_i 2^{(k-i)\alpha} \right)^{p(1+\varepsilon)} \right\}^{1/p(1+\varepsilon)} \\
&\lesssim \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{i=-\infty}^{\infty} \lambda_i^{p(1+\varepsilon)} \right)^{1/p(1+\varepsilon)} \\
&\lesssim \|f\|_{\dot{K}_{\alpha,\vec{q}}^{p),\theta}(\mathbb{R}^n)}.
\end{aligned}$$

Let us estimate G_1. By the condition (4.1) and the fact if $x \in R_k$, $y \in Q_i$ and $i \leq k-2$, then $|x-y| \approx |x|$, we get

$$\begin{aligned}\left|Ta_i(x)\right| &\lesssim \int_{Q_i} \frac{\left|a_i(y)\right|}{|x|^n}\,\mathrm{d}y \\ &\lesssim 2^{-kn}\int_{Q_i}\left|a_i(y)\right|\mathrm{d}y\end{aligned}$$

Using (4.1), a routine application of Hölder's inequality (with $1 = \frac{1}{\vec{q}} + \frac{1}{\vec{q}'}$) and condition (ii) in Definition 2.7, we obtain

$$\begin{aligned}\left|Ta_i(x)\right| &\lesssim 2^{-kn}\,\|a_i\|_{\vec{q}}\left\|\chi_{Q_i}\right\|_{\vec{q}'} \\ &\lesssim 2^{-kn-\alpha i+ni-i\sum_{i=1}^{n}\frac{1}{q_i}},\end{aligned}$$

which obtain

$$\|Ta_i\cdot\chi_k\|_{\vec{q}} \lesssim 2^{-kn-\alpha i+ni-i\sum_{i=1}^{n}\frac{1}{q_i}}\,\|\chi_k\|_{\vec{q}} \quad \lesssim 2^{k\left(-n+\sum_{i=1}^{n}\frac{1}{q_i}\right)+i\left(-\alpha+n-\sum_{i=1}^{n}\frac{1}{q_i}\right)},$$

Since $\alpha - n + \sum_{i=1}^{n}\frac{1}{q_i} < 0$, Lemma 2.6 applies with the geometric decay factor $b = 2^{\alpha-n+\sum_{i=1}^{n}\frac{1}{q_i}} \in (0,1)$, yielding

$$\begin{aligned}G_1 &\lesssim \sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}\left(\sum_{i=-\infty}^{k-2}\lambda_i 2^{(k-i)\left(\alpha-n+\sum_{i=1}^{n}\frac{1}{q_i}\right)}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\ &\lesssim \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}\lambda_k^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)} \\ &\lesssim \|f\|_{\dot{K}^{p),\theta}_{\alpha,\vec{q}}(\mathbb{R}^n)}.\end{aligned}$$

Accordingly, Theorem 4.1 has been proven. □

Remake 4.1

(1) Condition (4.1) is fulfilled by a host of classical operators—Calderón–Zygmund and Hardy–Littlewood maximal among them.

(2) Since [22, 23] already establish the $L^{\vec{q}}$-boundedness of these maps, an immediate check of Theorem 4.1 upgrades their boundedness to the mixed-Herz scale.

5. Some applications

The principal aim of this section is to transplant boundedness results for sublinear operators into the framework of grand mixed Herz-Morrey spaces.

Theorem 5.1 Let $\alpha \in \mathbb{R}, \theta > 0, \lambda > 0, 0 < p < \infty, 1 < \vec{q} < \infty$ such that $0 < \alpha < n\sum_{i=1}^{n}\frac{1}{q_i}$. If a sublinear operator T satisfies

$$|Tf(x)| \lesssim \frac{\|f\|_1}{|x|^n}, \quad \text{if} \quad \mathrm{dis}(x, \operatorname{supp} f) > \frac{|x|}{2}, \tag{5.1}$$

for every compactly supported integrable function f, once the operator T is known to be bounded on the mixed-Lebesgue scale $L^{\vec{q}}(\mathbb{R}^n)$, its boundedness on the grand homogeneous Herz-Morrey space $M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)$ follows at once.

Proof of Theorem 5.1. It remains to verify that

$$\|Tf\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)}.$$

Fix any $f \in M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)$, invoking the atomic flavour of Theorem 3.1 we can safely work with an atomic representation of f and carry out the required estimate

$$f = \sum_{i=-\infty}^{\infty} \lambda_i a_i,$$

where $\lambda_i \geq 0$ and a_i 's are ($\alpha, \vec{q}$)- block with $\operatorname{supp} a_i \subseteq Q_i$. We get

$$\begin{aligned}
\|Tf\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)} =& \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)}\left\|Tf\cdot\chi_k\right\|_{\vec{q}}^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\
\lesssim& \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)}\left(\sum_{i=-\infty}^{k-2}\lambda_i\|Ta_i\cdot\chi_k\|_{\vec{q}}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\
&+ \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)}\left(\sum_{i=k-1}^{\infty}\lambda_i\|Ta_i\cdot\chi_k\|_{\vec{q}}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\
=& J_1 + J_2.
\end{aligned}$$

Let us estimate J_1. Observing that $\operatorname{dist}(x, \operatorname{supp} a_i) > 2^{k-1}/2 > |x|/2$, we invoke (3.3) to immediately obtain

$$\begin{aligned}
\|Ta_i\cdot\chi_k\|_{\vec{q}} &\lesssim \left(\int_{R_k^n}\cdots\left(\int_{R_k^2}\left(\int_{R_k^1}|x|^{-nq_1}\|a_i\|_1^{q_1}\,\mathrm{d}x_1\right)^{\frac{q_2}{q_1}}\,\mathrm{d}x_2\right)^{\frac{q_3}{q_2}}\cdots\,\mathrm{d}x_n\right)^{\frac{1}{q_n}} \\
&\lesssim 2^{k\left(\sum_{i=1}^n \frac{1}{q_i}-n\right)}\|a_i\chi_i\|_1 .
\end{aligned}$$

Applying Hölder inequality and Deffnition 2.7, we have

$$\begin{aligned}
\|Ta_i\cdot\chi_k\|_{\vec{q}} &\lesssim 2^{k\left(\sum_{i=1}^n \frac{1}{q_i}-n\right)}\|a_i\|_{\vec{q}}\|\chi_i\|_{\vec{q}'} \\
&\lesssim 2^{k\left(\sum_{i=1}^n \frac{1}{q_i}-n\right)}2^{i\left(n-\sum_{i=1}^n \frac{1}{q_i}\right)}\|a_i\|_{\vec{q}} \\
&\lesssim 2^{-i\alpha}2^{(i-k)\left(n-\sum_{i=1}^n \frac{1}{q_i}\right)}.
\end{aligned}$$

Therefore,

$$J_1 \lesssim \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)}\left(\sum_{i=-\infty}^{k-2}\lambda_i 2^{-i\alpha}2^{(i-k)\left(n-\sum_{i=1}^n \frac{1}{q_i}\right)}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)}$$

$$= \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{\infty}\left(\varepsilon^{\theta}\sum_{i=-\infty}^{k-2}\lambda_i 2^{(k-i)\left(\alpha-n+\sum_{i=1}^{n}\frac{1}{q_i}\right)}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)}$$

$$= \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\varepsilon^{\theta}\sum_{i=-\infty}^{k-2}\lambda_i 2^{-i\alpha}2^{(i-k)\left(n-\sum_{i=1}^{n}\frac{1}{q_i}\right)}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)},$$

Since $\alpha < n-\sum_{i=1}^{n}\frac{1}{q_i}$, Lemma 2.6 applies with the geometric decay factor$b = 2^{\alpha-n+\sum_{i=1}^{n}\frac{1}{q_i}} \in (0,1)$, we get

$$\begin{aligned} J_1 &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{i=-\infty}^{\infty}\lambda_i^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)} \\ &\lesssim \|f\|_{\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)}. \end{aligned}$$

Next, we estimate J_2. Invoking the $L^{\vec{q}}(\mathbb{R}^n)$-boundedness of T, condition (ii) in Definition 2.7 and Lemma 2.6 with the geometric decay factor $b = 2^{-\alpha} \in (0,1)$, we obtain

$$\begin{aligned} J_2 &= \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\varepsilon^{\theta}\sum_{i=k-1}^{\infty}\lambda_i\|Ta_i\cdot\chi_k\|_{\vec{q}}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\varepsilon^{\theta}\sum_{i=k-1}^{\infty}\lambda_i\sup_{\varepsilon>0}\|a_i\|_{\vec{q}}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{\infty}\left(\varepsilon^{\theta}\sum_{i=k-1}^{\infty}\lambda_i 2^{-(i-k)\alpha}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)} \\ &\lesssim \|f\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)}. \end{aligned}$$

Accordingly, Theorem 5.1 has been proven. □

Theorem 5.2 Let $\alpha\in\mathbb{R}, \theta>0, \lambda>0, 0<p<\infty$ and $1<\vec{q}<\infty$ such that $0<\alpha<n-\sum_{i=1}^{n}\frac{1}{q_i}$. If a sublinear operator T satisfies the condition (4.1) and is bounded on $L^{\vec{q}}(\mathbb{R}^n)$, then T is bounded on $M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)$.

Proof of Theorem 5.2. It remains to verify that

$$\|Tf\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)}.$$

for all $f\in M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)$. Using Theorem 3.1, we get

$$f = \sum_{i=-\infty}^{\infty}\lambda_i a_i,$$

where $\lambda_i \geq 0$ and a_i 's are ($\alpha,\vec{q}$)- block with $\operatorname{supp} a_i \subseteq Q_i$. We obtain

$$
\begin{aligned}
\|Tf\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)} =& \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{i=-\infty}^{\infty}\lambda_i\|Ta_i\cdot\chi_k\|_{\vec{q}}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)}\\
\lesssim& \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{i=-\infty}^{k-2}\lambda_i\|Ta_i\cdot\chi_k\|_{\vec{q}}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)}\\
&+\sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{i=k-1}^{\infty}\lambda_i\|Ta_i\cdot\chi_k\|_{\vec{q}}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)}\\
=&I_1+I_2.
\end{aligned}
$$

Let us estimate I_2. Invoking the $L^{\vec{q}}(\mathbb{R}^n)$-boundedness of T, condition (ii) in Definition 2.7 and Lemma 2.6 with the geometric decay factor $b=2^{-\alpha}\in(0,1)$, we arrive at

$$
\begin{aligned}
I_2 &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{i=k-1}^{\infty}\lambda_i\|a_i\|_{\vec{q}}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)}\\
&\lesssim \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{i=k-1}^{\infty}\lambda_i 2^{(k-i)\alpha}\right)^{p(1+\varepsilon)}\right\}^{1/p(1+\varepsilon)}\\
&\lesssim \sup_{\varepsilon>0}\sup_{k_0>0} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{i=-\infty}^{\infty}\lambda_i^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\
&\lesssim \|f\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)}.
\end{aligned}
$$

Let us estimate I_1. By the condition (4.1) and the fact if $x\in R_k$, $y\in Q_i$ and $i\le k-2$, then $|x-y|\approx|x|$, we get

$$
\begin{aligned}
\left|Ta_i(x)\right| &\lesssim \int_{Q_i}\frac{\left|a_i(y)\right|}{|x|^n}\,\mathrm{d}y\\
&\lesssim 2^{-kn}\int_{Q_i}\left|a_i(y)\right|\mathrm{d}y
\end{aligned}
$$

Using (4.1), a routine application of Hölder's inequality (with $1=\frac{1}{\vec{q}}+\frac{1}{\vec{q}'}$) and condition (ii) in Definition 2.7, we obtain

$$
\begin{aligned}
\left|Ta_i(x)\right| &\lesssim 2^{-kn}\|a_i\|_{\vec{q}}\left\|\chi_{Q_i}\right\|_{\vec{q}'}\\
&\lesssim 2^{-kn-\alpha i+ni-i\sum_{i=1}^{n}\frac{1}{q_i}},
\end{aligned}
$$

which obtain

$$
\|Ta_i\cdot\chi_k\|_{\vec{q}}\lesssim 2^{-kn-\alpha i+ni-i\sum_{i=1}^{n}\frac{1}{q_i}}\|\chi_k\|_{\vec{q}} \quad \lesssim 2^{k\left(-n+\sum_{i=1}^{n}\frac{1}{q_i}\right)+i\left(-\alpha+n-\sum_{i=1}^{n}\frac{1}{q_i}\right)},
$$

Since $\alpha - n + \sum_{i=1}^{n} \frac{1}{q_i} < 0$, Lemma 2.6 applies with the geometric decay factor $b = 2^{\alpha - n + \sum_{i=1}^{n} \frac{1}{q_i}} \in (0,1)$, yielding

$$\begin{aligned} I_1 &\lesssim \sup_{\varepsilon>0} \sup_{k_0>0} 2^{-k_0\lambda} \left\{ \varepsilon^{\theta} \sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{k-2} \lambda_i 2^{(k-i)\left(\alpha - n + \sum_{i=1}^{n} \frac{1}{q_i}\right)} \right)^{p(1+\varepsilon)} \right\}^{1/p(1+\varepsilon)} \\ &\lesssim \sup_{\varepsilon>0} \sup_{k_0>0} 2^{-k_0\lambda} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{\infty} \lambda_k^{p(1+\varepsilon)} \right)^{1/p(1+\varepsilon)} \\ &\lesssim \|f\|_{M\dot{K}^{p),\theta}_{\alpha,\vec{q},\lambda}(\mathbb{R}^n)}. \end{aligned}$$

Accordingly, Theorem 5.2 has been proven. □

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