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Lower Bounds and Maximal Clique Structure in Matrix Commuting Graphs

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Abstract. Let R be a finite commutative ring with unity and let $M(m,R)$ denote the ring of $m \times m$ matrices over R. We investigate the commuting graph $\Gamma(M(m,R))$ whose vertices are the non-central matrices in $M(m,R)$ and two distinct vertices are adjacent if and only if they commute. In this paper, we establish the lower bound of the clique number $\omega(\Gamma(M(m,R)))$. Our main results we characterize the structure of maximal cliques.

1. Introduction

The systematic study of commuting graphs associated with algebraic structures has emerged as a vibrant area of research at the intersection of algebra and graph theory. While the commuting graph of a group, first introduced by Brauer and Fowler in the context of finite simple groups, has been extensively investigated [1, 15], the analogous construction for rings and their matrix extensions presents unique challenges and opportunities due to the richer underlying algebraic structure.

Let R be a finite commutative ring with unity and $M(m,R)$ denote the ring of $m \times m$ matrices over R. The center $Z(M(m,R))$ of $M(m,R)$ consists precisely of scalar matrices cI_m where $c \in R$ and I_m is the $m \times m$ identity matrix. The commuting graph $\Gamma(M(m,R))$ has vertex set $M(m,R) \setminus Z(M(m,R))$ and two distinct vertices A, B are adjacent if and only if $AB = BA$.

The motivation for studying the clique structure of $\Gamma(M(m,R))$ stems from several interconnected mathematical areas. First, understanding maximal sets of pairwise commuting matrices is fundamentally connected to the classical problem of simultaneous diagonalization in linear algebra [9, 13]. A set of matrices forms a clique in $\Gamma(M(m,R))$ if and only if they pairwise commute, which occurs precisely when they can be simultaneously diagonalized over an appropriate extension of R (when such extensions exist).

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Second, the study of $\omega(\Gamma(M(m,R)))$ provides valuable insights into the structure of commuting families of matrices, which plays a central role in representation theory, algebraic geometry, and the theory of linear operators [10, 16]. The finite ring setting introduces arithmetic constraints that lead to phenomena not present in the classical field case, offering new perspectives on matrix commutativity.

Third, from a graph-theoretic standpoint, matrix commuting graphs exhibit rich structural properties that differ significantly from commuting graphs of groups. The interplay between the ring-theoretic properties of R (such as nilpotent elements, units, and zero divisors) and the graph-theoretic invariants of $\Gamma(M(m,R))$ reveals deep connections between algebra and combinatorics.

Previous research on commuting graphs of matrix rings has primarily focused on connectivity and diameter properties. Akbari, Bidkhori, and Mohammadian [7, 8] established fundamental results on the connectivity of $\Gamma(M(m,F))$ for fields F, showing that for $m \geq 3$, the graph is connected if and only if every extension of F of degree m contains a proper intermediate subfield. They also proved diameter bounds $4 \leq \mathrm{diam}(\Gamma(M(m,F))) \leq 6$ for fields. Giudici and Pope [11] extended these results to matrix rings over Z_m, providing diameter estimates for commuting graphs of linear groups.

However, the systematic study of clique numbers in this context appears to be largely unexplored, particularly for matrices over finite commutative rings beyond fields. The clique number $\omega(G)$ of a graph G is a fundamental invariant that measures the size of the largest complete subgraph, and its determination often requires deep understanding of the underlying algebraic structure.

In this paper, we provide a comprehensive analysis of $\omega(\Gamma(M(m,R)))$ for finite commutative rings R. Our approach combines techniques from linear algebra, ring theory, and extremal graph theory to establish both structural results and explicit bounds. We show that the problem naturally decomposes according to the local structure of R, with different behaviors emerging for fields, local rings, and more general finite commutative rings.

The paper is organized as follows. Section 2 establishes preliminary results and basic properties of matrix commuting graphs. Section 3 characterizes the structure of maximal cliques through diagonalizability criteria and provides our main theorems on clique numbers. Section 4 presents applications to specific classes of rings and computational results for small cases.

2. Preliminaries and Basic Properties

Throughout this paper, R denotes a finite commutative ring with unity 1_R, and $M(m,R)$ denotes the ring of $m \times m$ matrices over R. We denote by I_n the $m \times m$ identity matrix and by E_{ij} the matrix with 1 in position (i, j) and 0 elsewhere.

Definition 2.1. *The commuting graph $\Gamma(M(m,R))$ of $M(m,R)$ is the simple graph with vertex set*

$$V(\Gamma(M(m,R))) = M(m,R) \setminus Z(M(m,R)) = M(m,R) \setminus \{cI_m : c \in R\}$$

and edge set $E(\Gamma(M(m,R))) = \{\{A,B\} : A,B \in V(\Gamma(M(m,R))), A \neq B, AB = BA\}$.

Definition 2.2. *[14] Let $G = (V,E)$ be a simple undirected graph. A subset $S \subseteq V$ is called a* ***clique*** *if every pair of distinct vertices in S is adjacent, i.e., for all $u,v \in S$ with $u \neq v$, we have $\{u,v\} \in E$.*

Equivalently, S is a clique if and only if the induced subgraph $G[S]$ is a complete graph.

Definition 2.3. *[14] A clique S in a graph $G = (V,E)$ is called* ***maximal*** *if there exists no clique S' in G such that $S \subsetneq S'$. In other words, S is maximal if it cannot be extended by adding any additional vertex while preserving the clique property.*

Formally, S is a maximal clique if:

(1) *S is a clique, and*
(2) *For every $v \in V \setminus S$, the set $S \cup \{v\}$ is not a clique.*

Definition 2.4. *[14] The* ***clique number*** *of a graph G, denoted $\omega(G)$, is the size of the largest clique in G:*

$$\omega(G) = \max\{|S| : S \text{ is a clique in } G\}.$$

Lemma 2.1. *The center of $M(m,R)$ is $Z(M(m,R)) = \{cI_m : c \in R\}$, and $|Z(M(m,R))| = |R|$.*

Proof. Since R is commutative, any scalar matrix cI_m commutes with all matrices in $M(m,R)$. Conversely, if $A = (a_{ij}) \in Z(M(m,R))$, then A commutes with all elementary matrices E_{kl}. The condition $AE_{12} = E_{12}A$ implies $a_{ij} = 0$ for $i \neq j$ and $a_{11} = a_{22}$. Similarly, $AE_{23} = E_{23}A$ implies $a_{22} = a_{33}$. Continuing this process shows $A = a_{11}I_n$ for some $a_{11} \in R$. □

Proposition 2.1. *The commuting graph $\Gamma(M(m,R))$ has $|R|^{m^2} - |R|$ vertices.*

Proof. This follows immediately from $|M(m,R)| = |R|^{m^2}$ and Lemma 2.1. □

Giudici [11], studied the commuting graph of $M(2,R)$, where R is an integral domain, denoted by $\Gamma(M(2,R))$. He showed among other things that this graph is disconnected. The following figure for the graph $\Gamma(M(2,Z_2))$.

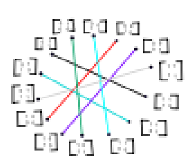

Figure 1. $\Gamma(M(2,Z_2))$.

3. Structure of Maximal Cliques

We now establish our principal results on the clique number of matrix commuting graphs.

Lemma 3.1. *Let $D(m,R) = \{diag(d_1,d_2,\ldots,d_m) : d_i \in R\} \setminus Z(M(m,R))$ be the set of non-central diagonal matrices. Then $D(m,R)$ is a maximal cliques in $\Gamma(M(m,R))$.*

Proof. Any two diagonal matrices commute since

$$\begin{aligned}(diag(a_1,\ldots,a_m))(diag(b_1,\ldots,b_m)) &= diag(a_1b_1,\ldots,a_mb_m)\\ &= (diag(b_1,\ldots,b_m))(diag(a_1,\ldots,a_m))\\ &\quad \text{(since } a_i, b_i \in R, R \text{ is a commutative ring)}\\ &= (diag(b_1,\ldots,b_m))(diag(a_1,\ldots,a_m)).\end{aligned}$$

Any matrix commute with all diagonal matrices must be a diagonal matrix, therefore $D(m,R) = \{diag(d_1,d_2,\ldots,d_m) : d_i \in R\}$ is a maximal cliques in $\Gamma(M(m,R))$. □

Theorem 3.1. *For any finite commutative ring R with unity and positive integer $m \geq 3$,*

$$\omega(\Gamma(M(m,R))) \geq |R|^m - |R|. \tag{3.1}$$

Proof. By Lemma 3.1, the diagonal matrices form a clique. The number of diagonal matrices is $|R|^m$, and the number of scalar matrices (which are central) is $|R|$. Therefore, $|D(m,R)| = |R|^m - |R|$, giving the desired lower bound. □

The following theorem is proved in [12].

Theorem 3.2. *If R is a finite field, then for $m \geq 2$,*

$$\omega(\Gamma(M(m,R))) = |R|^{\lceil \frac{m}{2} \rceil + 1} - |R|. \tag{3.2}$$

Now, we give other examples of maximal cliques in $\Gamma(M(m,Z_{p^r}))$ when $r \geq 3$ is odd. We start with the following remark.

Remark 3.1. *Any matrix* $A = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix}$ *commutes with the matrix*

$$A + cI = \begin{pmatrix} x_{1,1}+c & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2}+c & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m}+c \end{pmatrix}.$$

Let $S_1 = \{x_j p^t : x_j, \in Z_{p^r} \text{ and } t \geq \frac{r+1}{2}\}$ and $O_1 = \{x_j p^t : x_j \in Z_{p^r} \text{ and } t \geq \frac{t-1}{2}\}$.

Let L be the set of all matrices A of the form

$$\begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2} & \cdots & x_{m-1,m-1}+c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix},$$

such that $x_{1,1},\ldots,x_{m-1,m-1} \in O_1$, $c_1 \in Z_{p^r}$, and $x_{i,j} \in S_1$, $i \neq j$.

Lemma 3.2. *Suppose that r is an odd number, L and S_1 are defined as above. Then L induces a maximal clique in Z_{p^r}.*

Proof. Let X and Y be any two matrices in the set L. Then

$$X = \begin{pmatrix} a_{1,1}+c_1 & a_{1,2} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{2,1} & a_{2,2}+c_1 & \cdots & a_{2,m-1} & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & \vdots & \cdots & a_{m-1,m-1}+c_1 & a_{m-1,m} \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m-1} & c_1 \end{pmatrix},$$

$a_{i,j} \in O_1$, $a_{i,i} \in S_1$, $i \neq j$, $c_1 \in Z_{p^r}$. One can write $X = (J + c_1 I + G)$, where

$$G = \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & 0 & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & 0 \end{pmatrix}$$

is a matrix with all elements in S_1, and the matrix

$$J = \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix}$$

is a diagonal matrix where the main diagonal consists of elements that belong to O_1.

Similarly

$$Y = \begin{pmatrix} b_{1,1}+c_2 & b_{1,2} & \cdots & b_{1,m-1} & b_{1,m} \\ b_{2,1} & b_{2,2}+c_2 & \cdots & b_{2,m-1} & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{m-1,1} & b_{m-1,2} & \cdots & b_{m-1,m-1}+c_2 & b_{m-1,m} \\ b_{m,1} & b_{m,2} & \cdots & b_{m,m-1} & c_2 \end{pmatrix}$$

and $Y = (J^* + c_2 I + G^*)$, where

$$G^* = \begin{pmatrix} 0 & b_{1,2} & \cdots & b_{1,m} \\ b_{2,1} & 0 & \cdots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & 0 \end{pmatrix}$$

is a matrix with all elements in S_1

$$J^* = \begin{pmatrix} b_{1,1} & 0 & \cdots & 0 \\ 0 & b_{2,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{m,m} \end{pmatrix}$$

is a diagonal matrix where the main diagonal consists elements belonging to O. Then $XY = (J + c_1I + G)(J^* + c_2I + G^*) = JJ^* + c_2J + 0 + c_1J^* + c_1c_2I + c_1G^* + 0 + c_2G + 0 = J^*J + c_1J^* + 0 + c_2J + c_2c_1I + c_2G + 0 + c_1G^* + 0 = (J^* + c_2I + G^*)(J + c_1I + G) = YX$. Hence X and Y commute and the induced subgraph on L is complete.

Let $A = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix}$; $x_{i,j} \in Z_{p^r}$ be any matrix that commutes with all elements in L. The matrix A commute with $p^{\frac{r-1}{2}}E_{1,1}$ and hence $Ap^{\frac{r-1}{2}}E_{1,1} = p^{\frac{r-1}{2}}E_{1,1}A$. From that we get $x_{1,2}p^{\frac{r-1}{2}} = 0 = \cdots = x_{1,m}p^{\frac{r-1}{2}} = 0$, and $x_{2,1}p^{\frac{r-1}{2}} = 0 = \cdots = x_{m,1}p^{\frac{r-1}{2}} = 0$. Similarly A commutes with all the matrices $p^{\frac{r-1}{2}}E_{2,2} \cdots p^{\frac{r-1}{2}}E_{m,m}$. From that we get $x_{1,m}p^{\frac{r-1}{2}} = 0 = \cdots = x_{m-1,m}p^{\frac{r-1}{2}}$, and $x_{m,1}p^{\frac{r-1}{2}} = 0 = \cdots = x_{m,m-1}p^{\frac{r-1}{2}}$. So, $x_{l,j} = (c_{l,j} + id_{l,j})p^e$, $p^e \in \{0, p^{\frac{r+1}{2}}, \ldots, p^{r-1}\}$, $l \neq j$. So

$$A = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix};$$

where $x_{1,1}, x_{1,2}, \ldots, x_{m,m} \in Z_{p^r}$, $x_{i,j} \in S_1$ $\forall i \neq j$. Now, we need to show that $A \in L$. Since A commutes with $p^{\frac{r+1}{2}}E_{1,2}, \ldots, p^{\frac{r+1}{2}}E_{m-1,m}$, we have $A(p^{\frac{r+1}{2}}E_{1,2}) = (p^{\frac{r+1}{2}}E_{1,2})A, \ldots, A(p^{\frac{r+1}{2}}E_{m-1,m}) = (p^{\frac{r+1}{2}}E_{m-1,m})A$. So,

$$A(p^{\frac{r+1}{2}}E_{1,2}) = \begin{pmatrix} 0 & x_{1,1}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = (p^{\frac{r+1}{2}}E_{1,2})A = \begin{pmatrix} 0 & x_{2,2}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

So, we get $x_{1,1}p^{\frac{r+1}{2}} = x_{2,2}p^{\frac{r+1}{2}}$ and hence $(x_{1,1} - x_{2,2})p^{\frac{r+1}{2}} = 0$. From that we get $x_{1,1} = x_{2,2} + u_1p^{\frac{r-1}{2}}$; $u_1 \in Z_{p^r}$. Similarly since

$$A(p^{\frac{r+1}{2}}E_{2,3}) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_{2,2}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix} = (p^{\frac{r+1}{2}}E_{2,3})A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_{3,3}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix},$$

we get $x_{2,2} = x_{3,3} + u_2p^{\frac{r-1}{2}}$; $u_2 \in Z_{p^r}$. So $x_{1,1} = x_{2,2} + u_1p^{\frac{r-1}{2}}$ then $x_{1,1} = x_{3,3} + u_1^{(1)}p^{\frac{r-1}{2}}$.

By the same technique we get

$$A(p^{\frac{r+1}{2}}E_{m-1,m}) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}}x_{m-1,m-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = (p^{\frac{r+1}{2}}E_{m-1,m})A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}}x_{m,m} \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

then $x_{m-1,m-1} = x_{m,m} + u_{m-1}p^{\frac{r-1}{2}}$; $u_{m-1} \in Z_{p^r}$. So $x_{1,1} = x_{m,m} + u_{m-1}^{(m)}p^{\frac{r-1}{2}}$; $u_{m-1}^{(m)} \in Z_{p^r}$.

Then

$$A = \begin{pmatrix} d_{1,1} + x_{m,m} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & d_{2,2} + x_{m,m} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix};$$

$d_{1,1}, d_{1,2}, \ldots, d_{m-1,m-1} \in O_1$, $x_{m,m} \in Z_{p^r}$ and $x_{i,j} \in S_1$ $\forall i \neq j$. Then A is a matrix in L. So, the set L is a maximal clique of $\Gamma(M(m, Z_{p^r}))$. □

Observe that $|D(m, Z_{p^r})| = (p^r)^m - p^r$, since D contains diagonal matrices except scalar matrices. We have

$$L = \left\{ \begin{pmatrix} x_{1,1} + c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2} + c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2} + c_1 & \cdots & x_{m-1,m-1} + c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} : x_{ii} \in O_1, x_{i,j} \in S_1; i \neq j, c_1 \in Z_{p^r} \right\}.$$

Where $S_1 = \{x_j p^t : x_j \in Z_{p^r} \text{ and } t \geq \frac{r+1}{2}\}$. And $O_1 = \{x_j p^t : x_j \in Z_{p^r} \text{ and } t \geq \frac{r-1}{2}\}$. Observe that $|S_1| = p^{\frac{r-1}{2}}$, $|O_1| = p^{\frac{r+1}{2}}$. Hence

$$|L| = (p^{\frac{r-1}{2}})^{m^2-m}(p^{\frac{r+1}{2}})^{m-1}p^r - p^r. \tag{3.3}$$

Now, we give an example of a maximal clique of the graph $\Gamma(M(m, Z_{p^r}))$ where r is an even integer.

Let $S = \{x_j p^t : x_j \in Z_{p^r} \text{ and } t \geq \frac{r}{2}\}$. Let L be the set of all matrices A of the form

$$\begin{pmatrix} x_{1,1} + c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2} + c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2} + c_1 & \cdots & x_{m-1,m-1} + c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix},$$

such that $x_{i,j} \in S$, for all i, j, $c_1 \in Z_{p^r}$.

Lemma 3.3. *Let r be an even number, L and S are defined as above. Then L is a maximal clique in Z_{p^r}.*

Proof. The proof is similar to that one of Lemma 3.2. □

Now, we look at the size of the set L.

We have

$$L = \left\{ \begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2}+c_1 & \cdots & x_{m-1,m-1}+c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} : x_{i,j} \in S, \text{ for all } i,j,c_1 \in Z_{p^r} \right\}.$$

Where $S = \{x_j p^t : x_j \in Z_{p^r}\}$. Observe that $|S| = p^{\frac{r}{2}}$. Hence

$$|L| = (p^{\frac{r}{2}})^{m^2-1} p^r - p^r \tag{3.4}$$

.

Corollary 3.1. *For any finite commutative ring R isomorphic to Z_{p^r} with $r \geq 3$ is an odd number,*

$$\omega(\Gamma(M(m,R))) \geq max\{p^{rm} - p^r, (p^{\frac{r-1}{2}})^{m^2-m}(p^{\frac{r+1}{2}})^{m-1}p^r - p^r\}. \tag{3.5}$$

Proof. Using inequality (3.1) and equation (3.3), then we conclude the result. □

Corollary 3.2. *For any finite commutative ring R isomorphic to Z_{p^r} with $r \geq 2$ is an even number,*

$$\omega(\Gamma(M(m,R))) \geq max\{p^{rm} - p^r, (p^{\frac{r}{2}})^{m^2-1}p^r - p^r\}. \tag{3.6}$$

Proof. Using inequality (3.1) and equation (3.4), then we conclude the result. □

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