

Moving Sentinels for a Nonlinear Heat Equation with Memory

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Abstract. We consider a nonlinear heat equation with memory terms and missing data. Motivated by its numerous applications in captivating physical phenomena and engineering fields and other complex systems, the problem presents considerable challenges due to the presence of a memory integral term and a moving control, and in addition, due to the presence of incomplete data (pollution terms and incomplete data). This paper investigates the existence of insensitizing controls *sentinel* for this equation. We show that finding a moving sentinel is equivalent to solving a memory-type null-controllability problem. Our approach relies on the application of duality arguments to obtain an observability inequality, using Carleman estimates.

1. INTRODUCTION

Equations with memory arise naturally in the modeling of various physical phenomena, such as viscoelasticity and heat conduction. They are used in industrial and engineering applications because of their capacity to dissipate mechanical energy, mitigate noise and vibrations, and improve structural durability. Their origin can be traced back to the seminal works of Volterra [25], Maxwell [17] and Boltzmann [6]. Other complex problems present pollution and missing terms. In forestry, Jyotsna and Tandon [14] present a model considering soil and air pollution due to mining activities. Non-local terms appear also in many diffusion models in biology. They can describe biological systems on scales that are convenient for observation and data collection. The

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chemotaxis model is a classical example (see Biccari et al. [4]). In [7], Burger et al. present a non-local Keller-Segel model for chemotaxis of a cell population in which the chemicals are produced by the cells themselves.

Motivated by many applications, non-local models raised the interest of the mathematical community. For example, one can refer to the paper of Vazquez [24] for a mathematical survey containing different examples of non-local diffusion models. The primary challenge in analyzing such equations stems from their nonlocal nature, which arises from the inclusion of the memory term. These memory-based equations have been the focus of extensive research (see for instance the works by Amendola et al. [2], Fabrizio and Giorgi [11], Renardy [21], Yong and Zhang [27] and the references therein).

In the present paper, we will consider the case where perturbations and incomplete data occur in the same problem. Indeed, some of the parameters of the nonlinear model are not directly observable, and the model is often disturbed by uncertainties in several parameters. Then, prediction of the unknown data is necessary. We here use the sentinel method introduced by Lions [15]. The method is well adapted to missing data problems. The sentinel method is a least-squares type method, used for the identification of parameters with incomplete data. The sentinel concept relies on the following three objects : a state equation, an observation function, and a control function to be determined. Finding a sentinel is the same as studying the null-controllability.

Many authors used the sentinel method of Lions in several situations. See, for example, Ainseba et al. [1], or the work by Bodart and Fabre [5]. A lot of articles followed since then, but in all these works, the control function and the observation function have their supports in the same set called the observatory. In this case there always exist a sentinel.

More recently, Miloudi et al. [18] generalize the notion of sentinels to the case where the observation and control functions have their supports in two different sets, which makes the problem of finding a sentinel non trivial. It is also a more realistic hypothesis. We also mention others works where instantaneous sentinels are obtained. In [22] Segueni et al. study the case of approximate sentinels (see also the references therein). In the present article, we show how to apply this method to non-local diffusion problems of missing data.

Since insensitivity problems are closely related to controllability problems, we refer to Liu [16] and [22] for the existence of insensitizing controls for some linear and semilinear heat equations, respectively.

In this paper, the study of insensitizing control problems for the heat equation with memory also represents a promising research direction. Therefore, we focus on the null controllability property under certain conditions, which we will call the memory-type null controllability (a precise definition will be given later).

In classical PDE control problems, the control function is defined over a fixed region $\omega \subset \Omega$, constant in time. However, in some complex systems with memory, this approach presents fundamental limitations:

- Nonlocal effects: Memory induces global temporal dependencies.
- Insufficient propagation: Local action cannot compensate for global memory effects.
- Established counterexamples: Even for the simple heat equation with memory, controllability fails under fixed control.

As far as we know, our study represents the first contribution to the problem of insensitizing controls for the heat equation with memory, requiring the control and observation supports $\omega(t)$ and $O(t)$ respectively, to evolve over time $t \in [0, T]$, moving in such a way that they collectively cover the entire domain over the given time interval.

To the best of our knowledge, only a limited number of results exist concerning the controllability of parabolic equations with memory. For parabolic problems, one can however mention the articles by Barbu and Iannelli [3], Guerrero and Imanuvilov [12], Halanay and Pandolfi [13], or by Taoa and Gaob [23]. In particular, Chaves-Silva et al. [9] (see also [10]), focused on the control of viscoelastic systems. Here, a key challenge arises: to ensure null controllability, the control support $\omega(t)$ that varies over time and must move and cover the entire domain. It is worth noting that all existing Carleman inequalities for the heat equation in the literature are established under the assumption that the control region is fixed. In our setting, however, the control region is time-dependent.

The article is organized as follows. In section 2, we present the problem for which we recall the existence result. We also introduce the definition of the sentinel. In section 3, we show that the existence of a sentinel is equivalent to a memory-type null-controllability problem. We then prove the existence of a sentinel in section 4. We also give some complementary results and Carleman estimates in the fifth section. Some concluding remarks are then presented in section 6.

2. POSITION OF THE PROBLEM

Let Ω be a non empty open and bounded connected subset of $\mathbb{R}^N$ ($N = 2$ or 3) of class C^∞ with a smooth boundary $\partial\Omega$ and let $T > 0$. We will use the notation $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$. We consider the following nonlinear heat equation with memory:

$$\left\{ \begin{array}{ll} \frac{\partial y}{\partial t} - \Delta y + f(y) + \int_0^t M(t-s)y(s, x) ds = \xi + \lambda \widehat{\xi} & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 + \mu \widehat{y}_0 & \text{in } \Omega, \end{array} \right. \quad (2.1)$$

where $y = y(t, x)$ is the state variable, $M(\cdot, \cdot)$ is a smooth memory kernel. We assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a globally Lipschitz-continuous function. The function $\xi := \xi(t, x)$ represents a given source in $L^2(Q)$. We take ξ and $y_0 \in L^2(\Omega)$ as known. Here, λ and τ are small real parameters. The term $\lambda \widehat{\xi}$ is called the pollution term, while $\tau \widehat{y}_0$ is the missing term; both are considered unknown functions. We suppose that we have:

$$\|\widehat{\xi}\|_{L^2(Q)} \leq 1, \quad \|\widehat{y}_0\|_{L^2(\Omega)} \leq 1.$$

Using standard methods, the problem (2.1) has a unique solution:

$$y \in C(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)).$$

For a proof of this classical result, one can see for example, Cannarsa and Sforza [8], Pazy [20] or Wang et al. [26].

2.1. The insensitizing control problem or the moving sentinel. Let $T > 0$ and $t \in [0, T]$ be fixed. We introduce an open set $O(t) \subset \Omega$, which is called the observatory set, and let h be some function of $L^2((0, T) \times O(t))$. Now, let $\omega \subset \Omega$ be a nonempty open subset, that is, the control support. We suppose that the control and observation sets $\omega(t)$ and $O(t)$, respectively, vary on time, and that they move collectively to cover the entire domain during the control time horizon $[0, T]$. For any control function $u \in L^2((0, T) \times \omega(t))$, we define the following functional:

$$\Phi(\lambda, \tau) = \int_0^T \int_{\Omega} (h\chi_{O(t)} + u\chi_{\omega(t)})y(\lambda, \tau) dxdt, \quad (2.2)$$

where $y(\lambda, \tau) := y(t, x; \lambda, \tau)$ is the corresponding solution of the problem (2.1) we are looking at, associated with λ and τ . The mobility of $O(t)$ allows a better coordination with the mobile control $\omega(t)$.

Remark 2.1. *Because of the moving sets $w(t)$ and $O(t)$, we adopt the notion of moving control and later on the notion of moving sentinel.*

Now, we introduce the following definition of insensitizing controls for (2.1).

Definition 2.1. *We shall say that the functional Φ is a moving sentinel if the following conditions are satisfied:*

$$\left. \frac{\partial \Phi(\lambda, \tau)}{\partial \tau} \right|_{\lambda=\tau=0} = 0, \quad (2.3)$$

$$\left. \frac{\partial \Phi(\lambda, \tau)}{\partial \lambda} \right|_{\lambda=\tau=0} = c, \quad (2.4)$$

$$\|u\|_{L^2((0,T) \times \omega(t))} \longrightarrow \min. \quad (2.5)$$

Remark 2.2. *If condition (2.3) is satisfied, we will say that the sentinel is insensitive to the missing term. The minimum energy condition on L^2 given by (2.5), expresses the fact that we deviate as little as possible from the average.*

Remark 2.3. *The insensitizing control problem "sentinel" is to find u such that the uncertainty in the initial data does not affect the measurement. If the sentinel Φ is determined, we obtain information on the pollution term $\lambda \widehat{y}_0$ since we can write:*

$$\lambda \simeq \frac{1}{c} (\Phi_{obs} - \Phi(0, 0))$$

where

$$\Phi_{obs}(\lambda, \tau) = \int_0^T \int_{\Omega} (h\chi_{O(t)} + u\chi_{\omega(t)}) y_{obs}(\lambda, \tau) dx dt, \quad (2.6)$$

$y_{obs}(t, x)$ being the observed solution over the domain Q .

3. EQUIVALENCE BETWEEN THE MOVING SENTINEL PROBLEM AND THE MEMORY-TYPE NULL CONTROLLABILITY ONE

The concept of insensitizing controls was first introduced by Lions in [15], sparking a wide range of studies on the subject of elliptic, parabolic, and hyperbolic equations. The existence of such controls for parabolic equations with Dirichlet or Neumann boundary conditions has been extensively investigated by many authors (see [1], [5], [18] or Nakoulima et al. [19]).

We show here that the existence of a control function u and, consequently, a sentinel Φ is equivalent to a particular null-controllability problem. To begin with, it is straightforward to verify that condition (2.3) is equivalent to:

$$\left. \frac{\partial \Phi(\lambda, \tau)}{\partial \tau} \right|_{\lambda=\tau=0} = \int_0^T \int_{\Omega} (h\chi_{O(t)} + u\chi_{\omega(t)}) y_{\tau}(\lambda, \tau) dx dt = 0, \quad (3.1)$$

where $y_{\tau} := \frac{\partial y}{\partial \tau}(0, 0) = \lim_{\tau \rightarrow 0} \left(\frac{y(0, \tau) - y(0, 0)}{\tau} \right)$ is the solution to:

$$\begin{cases} \frac{\partial y_{\tau}}{\partial t} - \Delta y_{\tau} + f'(y_0) y_{\tau} + \int_0^t M(t-s) y_{\tau}(s, x) ds = 0 & \text{in } Q, \\ y_{\tau} = 0 & \text{on } \Sigma, \\ y_{\tau}(0) = \widehat{y}_0 & \text{in } \Omega. \end{cases} \quad (3.2)$$

Lemma 3.1. Let $M(\cdot) \in L^1(0, T)$ be a memory kernel, and let $y_{\tau}, q, \in L^2(0, T; L^2(\Omega))$ be. Then the following equality holds

$$\begin{aligned} \int_0^T \int_{\Omega} \left(\int_0^t M(t-s) y_{\tau}(s, x) ds \right) q(t, x) dx dt \\ = \int_0^T \int_{\Omega} \left(\int_t^T M(s-t) q(s, x) ds \right) y_{\tau}(t, x) dx dt. \end{aligned}$$

Proof - For $0 \leq s \leq t \leq T$, let us consider:

$$g(s, t, x) = M(t-s) y_{\tau}(s, x) q(t, x).$$

Then we have:

$$\begin{aligned} \int_{\Omega} \int_0^T \int_0^t g(s, t, x) ds dt dx &= \int_{\Omega} \int_0^T \int_0^T \mathbf{1}_{[0,t]}(s) g(s, t, x) ds dt dx \\ &= \int_{\Omega} \int_0^T \int_0^T \mathbf{1}_{[s,T]}(t) g(s, t, x) dt ds dx \\ &= \int_{\Omega} \int_0^T \int_s^T g(s, t, x) dt ds dx. \end{aligned} \quad (3.3)$$

Doing the change of variable $t = s$ in the integral, we obtain the desired equality. ■

The following results on the existence of insensitizing control are based on the associated adjoint system. Let $q := q(t, x; u)$ be the solution to the adjoint of problem (2.1) with a moving control region $\omega(\cdot)$ and a moving source when the observation $O(\cdot) (\subset \Omega)$. The adjoint problem is then defined by:

$$\begin{cases} -\frac{\partial q}{\partial t} - \Delta q + f'(y_0)q + \int_t^T M(s-t) q(s, x) ds = h\chi_{O(t)} + u\chi_{\omega(t)} & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = 0 & \text{in } \Omega. \end{cases} \quad (3.4)$$

Going a step further, we introduce the following concept of memory-type null controllability.

Definition 3.1. For a given memory kernel $\tilde{M}(\cdot) \in L^1(0, T)$, which is not necessarily identical to $M(\cdot)$, the system is called **memory-type null-controllable** if there exists a control $u(\cdot) \in L^2(0, T; U)$ such that the corresponding solution $q(\cdot)$ to (3.4) satisfies the following:

$$q(0) = 0 \quad \text{and} \quad \int_0^T \tilde{M}(T-s) q(s, x) ds = 0.$$

Remark 3.1. The notion of memory-type null controllability for evolution equations incorporating memory terms was first introduced by Zuazua et al. [9], specifically in the context of controlled ordinary differential equations (ODEs) and parabolic equations with memory.

Remark 3.2. Accordingly, for a retrograde system, condition

$$\int_0^T \tilde{M}(T-s) q(s, x) ds = 0$$

guarantees that this information is correctly used to bring the primal system to a full rest.

Let us now specify our equivalence result from the following proposition:

Proposition 3.1. The existence of a moving sentinel to the problem (2.1) is equivalent to the existence of a control u and a condition:

$$\int_0^T \tilde{M}(T-s) q(s, x) ds = 0,$$

such that the solution of (3.4) verifies:

$$q(0) = 0, \quad \text{in } \Omega.$$

Proof - We multiply the first equation in (3.2) by y_τ . Using the Green formula and the result in Lemma 3.1, we obtain:

$$\begin{aligned}
\int_0^T \int_{\Omega} (h\chi_{O(t)} + u\chi_{\omega(t)}) y_{\tau} dx dt &= \int_0^T \int_{\Omega} \left(-\frac{\partial q}{\partial t} - \Delta q + f'(y_0)q \right) y_{\tau} dx dt \\
&+ \int_0^T \int_{\Omega} \left(\int_t^T M(s-t)q(s,x)ds \right) y_{\tau} dx dt \\
&= \int_0^T \int_{\Omega} \left(\frac{\partial y_{\tau}}{\partial t} - \Delta y_{\tau} + f'(y_0)y_{\tau} \right) q dx dt \\
&+ \int_0^T \int_{\Omega} \left(\int_0^t M(t-s)y_{\tau}(s,x)ds \right) q dx dt \\
&- \int_{\Omega} q(T)y_{\tau}(T) dx + \int_{\Omega} q(0)y_{\tau}(0) dx \\
&- \int_{\partial\Omega} y_{\tau} \frac{\partial q}{\partial \nu} d\sigma + \int_{\partial\Omega} q \frac{\partial y_{\tau}}{\partial \nu} d\sigma.
\end{aligned} \tag{3.5}$$

Since y_{τ} and q are solutions of equations (3.2) and (3.4), respectively, we obtain:

$$\int_0^T \int_{\Omega} (h\chi_{O(t)} + u\chi_{\omega(t)}) y_{\tau} dx dt = \int_{\Omega} q(0)\widehat{y}_{\tau}(0) dx. \tag{3.6}$$

Finally,

$$q(0) = 0, \quad \text{in } \Omega \text{ for all } \widehat{y}_{\tau}(0) \in L^2(\Omega).$$

This shows the existence of a moving sentinel. ■

4. EXISTENCE OF A MOVING SENTINEL

As previously shown, the existence of a moving sentinel problem is equivalent to that of memory-type null controllability. We aim to prove that there exists a solution to problem (3.4) subject to (2.5). For this purpose, we introduce the following constrained optimization problem:

$$(\mathcal{P}) \left\{ \begin{array}{l} \min \|u\|_{L^2((0,T) \times \omega(t))} \\ (\mathcal{A}) \left\{ \begin{array}{ll} -\frac{\partial q}{\partial t} - \Delta q + f'(y_0)q + \int_t^T M(s-t)q(s,x)ds &= h\chi_{O(t)} + u\chi_{\omega(t)} \quad \text{in } Q, \\ q &= 0 \quad \text{on } \Sigma, \\ q(T) &= q_T \quad \text{in } \Omega, \\ q(0) = 0 \text{ and } \int_0^T \widetilde{M}(T-s)q(s,x)ds &= 0. \end{array} \right. \end{array} \right. \tag{4.1}$$

Our objective in this section is to establish that the constraint set $(\mathcal{A})$ is non-empty, thereby guaranteeing the existence of a moving sentinel. The main analytical tool employed to address this issue is an observability inequality for the solution of the following backward dual problem:

$$\left\{ \begin{array}{ll} w_t - \Delta w + f'(y_0)w + \int_0^t M(t-s)w(s,x)ds &= \widetilde{M}(T-t)z_0 \quad \text{in } Q, \\ w &= 0 \quad \text{on } \Sigma, \\ w(0) &= w_0 \quad \text{in } \Omega, \end{array} \right. \tag{4.2}$$

where w denotes the adjoint state variable of problem (4.2) where $\widetilde{M}(\cdot)$ is a given memory kernel and where $w_0, z_0 \in L^2(\Omega)$.

Remark 4.1. In the adjoint equation (4.2), an additional term $\tilde{M}(T-t)z_0$ appears. This term arises because we consider not only the final condition on q (i.e. $q(0) = 0$), but also the condition involving the memory term (i.e. $\int_0^T \tilde{M}(T-s)q(s, x) ds = 0$). Consequently, the adjoint system takes into account two conditions. We obtain an adjoint formulation involving two parameters w_0 and z_0

To prove the memory-type null-controllability of problem (3.4) when $O(t)$ constitutes a moving observatory domain distinct from the control domain $\omega(t)$, it is necessary to assume that the control domain satisfies the inequality of observability. Then we have the proposition:

Proposition 4.1. The problem (4.1) is memory-type null-controllable (with the memory kernel $\tilde{M}$) if and only if there exists a constant $C > 0$ such that for any $h \in L^2((0, T) \times O)$ verifying $\|h\chi_{O(t)}\|_{L^2(Q)} < +\infty$, the solution of (4.2) satisfies:

$$\|w(T)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\omega(t)} |w(t, x)|^2 dx dt, \quad \forall w_0, z_0 \in L^2(\Omega). \quad (4.3)$$

Proof - The proof is standard. For clarity, we present the details here. We first establish the “if” part by employing the duality method (see Proposition 2.1 [10]). We proceed by defining the space:

$$\mathfrak{L} = \{w\chi_\omega(\cdot) \mid w \text{ solution of (4.2), for all } w_0, z_0 \in L^2(\Omega)\}.$$

Define the linear functional $\mathfrak{F}$ on $\mathfrak{L}$ by:

$$\mathfrak{F}(w\chi_\omega) = -(w(T), q(T))_{L^2(\Omega)} - \int_0^T \int_\Omega w h \chi_{O(t)} dx dt. \quad (4.4)$$

We aim to show that $\mathfrak{F}$ is bounded on $\mathfrak{L}$ endowed with the norm $\|w\|_{L^2(0, T; L^2(\omega(t)))}$. We consider the following:

$$|\mathfrak{F}(w\chi_\omega)| \leq |(w(T), q(T))_{L^2(\Omega)}| + \left| \int_0^T \int_\Omega w h \chi_{O(t)} dx dt \right| \quad (4.5)$$

Using the de Cauchy-Schwarz inequality,

$$|(w(T), q(T))_{L^2(\Omega)}| \leq \|w(T)\|_{L^2(\Omega)} \|q(T)\|_{L^2(\Omega)}$$

and

$$\left| \int_0^T \int_\Omega w h \chi_{O(t)} dx dt \right| \leq \|h\chi_{O(t)}\|_{L^2(Q)} \|w\|_{L^2(Q)},$$

as $O(t) \subset \Omega$. Using a standard energy estimate and the inequality (4.3), we obtain:

$$|\mathfrak{F}(w\chi_\omega)| \leq C \|w(T)\|_{L^2(\Omega)}.$$

Thus, $\mathfrak{F}$ is bounded on $\mathfrak{L}$. By the Hahn-Banach theorem, $\mathfrak{F}$ extends to a bounded linear functional on $L^2(0, T; L^2(\omega(t)))$. By the Riesz representation theorem, there exists $\eta \in L^2(0, T; L^2(\omega(t)))$ such that:

$$\int_0^T \int_\Omega w\chi_{\omega(t)} \eta dx dt = -(w(T), q(T))_{L^2(\Omega)} - \int_0^T \int_\Omega w h \chi_{O(t)} dx dt. \quad (4.6)$$

We chose the control

$$u(\cdot) = \eta(\cdot). \quad (4.7)$$

Let q be the solution of (4.1) with this control u . We want to show that:

$$\int_0^T \tilde{M}(T-s) q(s, x) ds = 0 \text{ and } q(0) = 0.$$

For any $w_0, z_0 \in L^2(\Omega)$, it follows from (4.1) and (4.2) that:

$$\begin{aligned} & (w_T, q(T))_{L^2(\Omega)} - (w(0), q(0))_{L^2(\Omega)} \\ &= \int_0^T \left[\left(\frac{\partial w}{\partial t}, q \right) + \left(w, \frac{\partial q}{\partial t} \right) \right] dt \\ &= \int_0^T \left(\Delta w - f'(y_0)w - \int_0^t M(t-s) w(s, x) ds + \tilde{M}(T-t)z_0, q \right)_{L^2(\Omega)} dt \\ &+ \int_0^T \left(w, -\Delta q + f'(y_0)q + \int_s^t M(t-s) y(s, x) ds - u\chi_{\omega(t)} - h\chi_{O(t)} \right)_{L^2(\Omega)} dt \\ &= \left(z_0, \int_0^T \tilde{M}(T-t)q(t) dt \right)_{L^2(\Omega)} - \int_0^T \int_{\Omega} w h \chi_{O(t)} dx dt - \int_0^T \int_{\Omega} w u \chi_{\omega(t)} dx dt. \end{aligned} \quad (4.8)$$

Combining equations (4.6), (4.7), and (4.8), we derive:

$$-(w_0, q(0))_{L^2(\Omega)} + \left(z_0, \int_0^T \tilde{M}(T-t) q(t) dt \right) = 0, \quad \forall w_0, z_0 \in L^2(\Omega). \quad (4.9)$$

This implies that:

$$q(0) = 0 \quad \text{and} \quad \int_0^T \tilde{M}(T-t) q(t) dt = 0.$$

Next, we establish the "only if" direction. For any $w_0, z_0 \in L^2(\Omega)$. Equation (4.2) enables us to define a bounded linear operator $F : L^2(\Omega) \times L^2(\Omega) \rightarrow L^2(\Omega)$ as follows.

$$F(w_0, z_0) = w(T).$$

We proceed by contradiction, assuming that (4.3) is false. Then there exist sequences $(w_0^k, z_0^k)_{k \in \mathbb{N}}$ such that the corresponding solutions w^k satisfy:

$$\int_0^T \int_{\omega} |w^k(t)|^2 dx dt < \frac{1}{k^2} \|w^k(T)\|_{L^2(\Omega)}^2, \quad \forall k \in \mathbb{N}. \quad (4.10)$$

Set

$$\tilde{w}_0^k = \frac{\sqrt{k}}{\|w^k(T)\|_{L^2(\Omega)}} w_0^k, \quad \tilde{z}_0^k = \frac{\sqrt{k}}{\|z^k(T)\|_{L^2(\Omega)}} z_0^k.$$

and let $\tilde{w}^k(\cdot)$ be the solution of (4.2) corresponding to $(\tilde{w}_0^k, \tilde{z}_0^k)_{k \in \mathbb{N}}$, then we have:

$$\|\tilde{w}(T)\|_{L^2(\Omega)} = \|F(\tilde{w}_0, \tilde{z}_0)\|_{L^2(\Omega)} = \sqrt{k} \quad \text{and} \quad \int_0^T \int_{\omega} |\tilde{w}^k(t)|^2 dx dt < \frac{1}{k}. \quad (4.11)$$

Define $v^k = \tilde{w}^k / \sqrt{k}$ so that $\|v^k(T)\| = 1$ and v^k are bounded. By a compactness argument, we can extract a subsequence (still called v^k) such that:

$$v^k \rightarrow v^* \quad \text{strongly in } L^2(Q), \quad \text{and} \quad v^k(T) \rightharpoonup v^*(T) \text{ weakly in } L^2(\Omega),$$

for some $v^* \in L^2(Q)$.

Let u be the control whose existence is guaranteed by the memory-type null controllability property. Following the same steps as in the derivation of (4.8) and using (4.1) together with (4.2), we obtain

$$(w(T), q(T))_{L^2(\Omega)} = - \int_0^T \int_{\Omega} w u \chi_{\omega(t)} dx dt - \int_0^T \int_{\Omega} w h \chi_{O(t)} dx dt. \quad (4.12)$$

Now apply the duality identity to v^k . For any q_T , the first term on the right-hand side in (4.12) tends to 0. Hence, passing to the limit gives

$$(v^*(T), q_T) = - \int_0^T \int_{\Omega} v^* h \chi_{O(t)} dx dt \quad \forall q_T.$$

The right-hand side is independent of q_T , so $v^*(T) = 0$ and $\iint_Q v^* h \chi_{O(t)} dx dt = 0$. By unique continuation, $v^* \equiv 0$; thus

$$v^k \rightarrow 0, \quad \int_0^T \int_{\Omega} v^k h \chi_{O(t)} dx dt \rightarrow 0.$$

Now $\tilde{w}^k = \sqrt{k} v^k$ gives :

$$\int_0^T \int_{\Omega} \tilde{w}^k h \chi_{O(t)} dx dt = o(\sqrt{k}).$$

Taking $q_T = \tilde{w}^k(T) / \sqrt{k}$ in the duality for $\tilde{w}^k$ from (4.12), we obtain:

$$-\sqrt{k} = \int_0^T \int_{\omega(t)} \tilde{w}^k u dx dt + \int_0^T \int_{\Omega} \tilde{w}^k h \chi_{O(t)} dx dt.$$

Consequently $\sqrt{k} = o(1/\sqrt{k}) + o(\sqrt{k})$, impossible. Thus (4.3) holds. ■

5. THE OBSERVABILITY INEQUALITY

5.1. Hypothesis an preliminary results. We next provide a detailed formulation of the assumptions regarding the moving control support and analyze their consequences. These will serve as a key foundation for addressing the controllability of the heat equation with memory.

In general, the motion of the control and observatory regions can be assumed to follow the dynamics of flow $X(x, t, t_0)$ (called admissible trajectories), governed by underlying vector fields $\eta \in C([0, T]; W^{2,\infty}(\mathbb{R}^n; \mathbb{R}^n))$ satisfying:

$$\begin{cases} \frac{\partial X(x, t, t_0)}{\partial t} = \eta(t, X(x, t, t_0)), & t \in [0, T], \\ X(x, T, T) = x \in \mathbb{R}^n, \end{cases} \quad (5.1)$$

We adopt the following structural condition from [10], inspired by [9].

Hypothesis 5.1. *There exist a flow $X(x, t, T)$ generated by a velocity field $\eta \in C([0, T]; W^{2,\infty}(\mathbb{R}^n; \mathbb{R}^n))$ a bounded, smooth, and open sets $\omega_0, O_0 \subset \Omega$, and a curve $\Gamma(\cdot) \in C^\infty[0, T]; \mathbb{R}^n$, and two numbers t_1, t_2 with $0 \leq t_1 < t_2 \leq T$ such that:*

- i) $\omega(t) = X(\omega_0, t, T)$, and $O(t) = X(O_0, t, T)$,
- ii) $O_0 \cap \omega_0 \neq \emptyset$

$$\left\{ \begin{array}{l} \Gamma(t) \in X(\omega_0, t, T) \cap \Omega, \quad \forall t \in [0, T], \\ \overline{\Omega} \subset \bigcup_{t \in [0, T]} (X(\omega_0, t, T)) = \{X(x, t, T) \mid x \in \omega_0, t \in [0, T]\}, \\ \Omega \setminus \overline{X(\omega_0, t, T)} \text{ is nonempty and connected for } t \in [0, t_1] \cup [t_2, T], \\ \Omega \setminus \overline{X(\omega_0, t, T)} \text{ has two (nonempty) connected components for } t \in (t_1, t_2), \\ \forall \gamma(\cdot) \in C([0, T]; \Omega), \exists \bar{t} \in [0, T] : \gamma(\bar{t}) \in X(\omega_0, \bar{t}, T). \end{array} \right.$$

We will need a known weight function. This is given by the following result.

Lemma 5.1. [9] *Consider two nonempty open subsets ω, ω_1 in $\mathbb{R}^n$ with the property that $\overline{\omega_0} \subset \omega_1$, $\overline{\omega_1} \subset \omega$. Then, under hypothesis 5.1, there exists a parameter $\delta \in (0, T/2)$ and a smooth weight function $\psi \in C^\infty(\overline{Q})$ with the following properties:*

$$\left\{ \begin{array}{ll} \nabla \psi(t, x) \neq 0, & (t, x) \in [0, T] \times (\overline{\Omega} \setminus X\omega_1, t, T), \\ \psi_t(t, x) \neq 0, & (t, x) \in [0, T] \times \overline{\Omega} \setminus X(\omega_1, t, T), \\ \psi_t(t, x) > 0, & (t, x) \in [0, \delta] \times \overline{\Omega} \setminus X(\omega_1, t, T), \\ \psi_t(t, x) < 0, & (t, x) \in [T - \delta, T] \times \overline{\Omega} \setminus X(\omega_1, t, T), \\ \frac{\partial \psi}{\partial \nu}(t, x) \leq 0, & (t, x) \in [0, T] \times \partial \Omega, \\ \psi(t, x) > \frac{3}{4} \|\psi\|_{L^\infty(Q)}, & (t, x) \in [0, T] \times \overline{\Omega}. \end{array} \right.$$

As in [9], we introduce a function $g \in C^\infty(0, T)$ such that:

$$g(t) = \begin{cases} t^{-1} & 0 < t < \delta/2 \\ \text{strictly decreasing} & 0 < t \leq \delta \\ 1 & \delta \leq t \leq T/2 \\ g(T-t) & T/2 \leq t < T \end{cases}$$

For $\lambda > 0$, define the following weight functions, which will be used in the Carleman estimates.

$$\varphi(t, x) = g(t) \left[e^{\frac{3}{2}\lambda \|\psi\|_{L^\infty(Q)}} - e^{\lambda \psi(t, x)} \right], \quad \theta(t, x) = g(t) e^{\lambda \psi(t, x)},$$

Let $(p, \tilde{p}) \in H^{1,2}(Q) \times L^2(Q)$ be any function, and let $s > 0$ be a parameter; we introduce the following notation:

$$I(p) = \int_0^T \int_{\Omega} \left[(s\theta)^{-1} (|\Delta p|^2 + \left| \frac{\partial p}{\partial t} \right|^2) + \lambda^2 s \theta |\nabla p|^2 + \lambda^4 (s\theta)^3 |p|^2 \right] e^{-2s\varphi} dx dt, \quad (5.2)$$

and

$$\tilde{I}(\tilde{p}) = \lambda^2 s \int_0^T \int_{\Omega} \theta |\tilde{p}|^2 e^{-2s\varphi} dx dt. \quad (5.3)$$

In what follows, we use C to denote a generic positive constant that can change from one line to the next (unless specified otherwise). The next two results are established in [9].

Lemma 5.2. *Let ω_1 be a subset as found in Lemma 5.1. Then, under hypothesis 5.1, there exist constants $\lambda_0 > 0$, $s_0 > 0$, and $C > 0$ such that for any $\lambda \geq \lambda_0$, $s \geq s_0$, and any function $p \in C([0, T]; L^2(\Omega))$ with $p_t - \Delta p \in L^2(0, T; L^2(\Omega))$, the following estimate holds:*

$$I(p) \leq C \left(\int_Q \left| \frac{\partial p}{\partial t} - \Delta p \right|^2 e^{-2s\varphi} dx dt + \lambda^4 s^3 \int_0^T \int_{X(\omega_1, t, T)} \theta^3 |p|^2 e^{-2s\varphi} dx dt \right).$$

Lemma 5.3. *Let ω be as in Lemma 5.2. Then, under hypothesis 5.1, there exist constants $\lambda_1 \geq \lambda_0$, $s_1 \geq s_0$, and $C > 0$ such that for any $\lambda \geq \lambda_1$, $s \geq s_1$, and any $\tilde{p} \in H^1(0, T; L^2(\Omega))$, the following inequality holds:*

$$\tilde{I}(\tilde{p}) \leq C \left(\int_Q \left| \frac{\partial \tilde{p}}{\partial t} \right|^2 e^{-2s\varphi} dx dt + \lambda^2 s^2 \int_0^T \int_{X(\omega, t, T)} \theta^2 |\tilde{p}|^2 e^{-2s\varphi} dx dt \right).$$

The following result follows directly from Lemma 5.3.

Corollary 5.1. *Under the hypothesis of Lemma 5.3, the following estimate holds for any $m \in \mathbb{N}$, and for any $\tilde{p} \in H^m(0, T; L^2(\Omega))$:*

$$\tilde{I}(\tilde{p}) + \sum_{k=1}^{m-1} \tilde{I}(\partial_t^k \tilde{p}) \leq C \left(\int_Q |\partial_t^m \tilde{p}|^2 e^{-2s\varphi} dx dt + \int_0^T \int_{X(\omega, t, T)} (\lambda s \theta)^{P(m)} |\tilde{p}|^2 e^{-2s\varphi} dx dt \right),$$

where $P(m)$ is a polynomial in m .

5.2. The main result. We prove here our main result. We suppose that hypothesis 5.1 holds, then ω is any open set in Ω such that $\overline{\omega_0} \subset \omega$. We have the:

Theorem 5.1. *Assume the memory kernels $M(\cdot)$ and $\tilde{M}(\cdot)$ admit the following polynomial-exponential structure:*

$$M(t) = e^{at} \sum_{k=0}^K a_k t^k, \quad \tilde{M}(t) = e^{at} \sum_{k=0}^K b_k t^k,$$

where $K \in \mathbb{N}$ and $a, a_k, b_k \in \mathbb{R}$ are real coefficients. Then, for all $w_0, z_0 \in L^2(\Omega)$, there exists a constant $C > 0$ such that, for any solution w of (4.2), the observability inequalities are satisfied:

$$|w(T)|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\omega(t)} |w|^2 dx dt. \quad (5.4)$$

Proof - The proof of this theorem uses the ideas of [10]. We proceed in several steps. Without loss of generality, we assume $a = 0$ in the memory kernels. If $a \neq 0$, we perform the transformation $w(t, x) \rightarrow e^{-at}w(t, x)$, which preserves the structure of the memory terms while eliminating the exponential factor.

Define then the auxiliary variable:

$$Z(t, x) = - \int_0^t M(s-t)w(s)ds + \tilde{M}(T-t)z_0.$$

The adjoint system becomes:

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w = Z & \text{in } Q, \\ w = 0 & \text{on } \Sigma. \end{cases} \quad (5.5)$$

Under the polynomial assumption $M(t) = \sum_{k=0}^K a_k t^k$ and $\tilde{M}(t) = \sum_{k=0}^K b_k t^k$, we compute the $(K+1)$ time derivative of Z :

$$\partial_t^{K+1}Z = \sum_{k=0}^K (-1)^k k! a_k \partial_t^{K-k}w.$$

We then introduce $\hat{w} = \partial_t^{K+1}w$. Differentiating (5.5) $(K+1)$ times, gives the coupled system:

$$\begin{cases} \frac{\partial \hat{w}}{\partial t} - \Delta \hat{w} = \sum_{k=0}^K k! a_k \partial_t^{K-k}w & \text{in } Q, \\ \hat{w} = 0 & \text{on } \Sigma, \\ \partial_t^{K+1}w = \hat{w} & \text{in } Q. \end{cases} \quad (5.6)$$

Apply now Lemma 5.2 to $\hat{w}$ and Corollary 5.1 to w and its time derivatives. After absorbing lower-order terms for sufficiently large Carleman parameters s and λ , we obtain:

$$I(\hat{w}) + \tilde{I}(w) + \sum_{i=1}^K \tilde{I}(\partial_t^i w) \leq C \int_0^T \int_{\omega(t)} (\lambda s \theta)^{P(K)} |w|^2 e^{-2s\varphi} dx dt, \quad (5.7)$$

where $P(K)$ is a polynomial in K .

Finally, we get from inequality (5.7) and standard energy estimates for the backward heat equation, we deduce the existence of a constant $C > 0$ such that:

$$\|w(T)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\omega(t)} |w(t, x)|^2 dx dt.$$

.

■

The memory-type null-controllability result for equation (3.4), follows directly from Theorem 5.1 through the duality framework provided by Proposition 4.1. We have the:

Theorem 5.2. *Under hypothesis 5.1 and for memory kernels $M(\cdot)$, $\tilde{M}(\cdot)$ of polynomial-exponential type as in Theorem 5.1, the system (3.4) is memory-type null-controllable. That is, there exists a control $u \in L^2(Q)$ with $\omega(t) = X(w, t, T)$ for $t \in (0, T)$ such that the solution satisfies:*

$$q(T) = 0 \quad \text{and} \quad \int_0^T \tilde{M}(T-s)y(s)ds = 0 \quad \text{in } \Omega.$$

Remark 5.1. Many control functions having the property (i.e. $(u, q) \in \mathcal{A}$) may exist. We select in the proposition below the control of minimal norm. ■

In the following proposition, we prove the existence of an optimal control of $(\mathcal{P})$, under the additional condition $q(T) = 0$, which itself requires the existence of a Moving sentinel for observation without noisy terms.

Proposition 5.1. The problem $(\mathcal{P})$ has a unique solution $(\hat{u}, \hat{q})$.

Proof - Indeed, from proposition 4.1, we deduce that the domain $\mathcal{A}$ is non empty, closed, and convex. Moreover, the mapping $u \mapsto \|u\|_{L^2(Q)}$ is continuous, coercive, and strictly convex. Then, there exists a unique solution to the problem $(\mathcal{P})$ noticed by $(\hat{u}, \hat{q}) \in \mathcal{A}$ and which satisfy :

$$\|\hat{u}\|_{L^2(Q)} \leq \|u\|_{L^2(Q)}, \quad \forall (u, q) \in \mathcal{A}.$$

6. CONCLUSION

In this paper, we have established the null-controllability of a heat equation with a memory term. We presented a new definition of a sentinel and addressed the problem of retrieving information at a desired time, about the pollution term in the equation with memory and with incomplete initial data. We showed that the problem of finding a moving sentinel is equivalent to a memory null-controllability problem. The method can be applied to industrial (or complex) problems governed by memory-based diffusion equations. Here, a sentinel gives a precise identification of the pollution. We think that this method can give significant results from a computational point of view.

The mathematical framework developed here can be applied to engineering sectors as well as to complex systems in biology with diffusion, and to real-world industrial in general. For example, a relevant industrial application can be found in the thermal regulation of aerospace components during the curing process, wherein this theoretical model acts as a predictive tool for the operation of a mobile laser control system (controller $\omega(t)$) and the spatial configuration of infrared sensor arrays (observatory $O(t)$).

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